Turnaround size of non-spherical structures

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Plan of the talk

- What is the turnaround radius? Why is it useful?
- Quasi-local energy in General Relativity;
- A rigorous definition of turnaround radius for spherical structures;
- What happens if we introduce small deviations from the spherical symmetry?

Turnaround Radius: An Astronomer's View

Assumptions

- Accelerated FLRW universe;
- One spherical inhomogeneity, $\phi = \phi(r)$.

Line element:

$$\mathrm{d}s^2 = a^2(\eta)\Big[-(1+2\,\phi)\,\mathrm{d}\eta^2 + (1-2\,\phi)\,(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2)\Big]$$

Areal radius $R(\eta, r) := a(\eta)\, r\, \sqrt{1 - 2\,\phi(r)}$.

The radial timelike geodesics read

$$\ddot{R} = \frac{\ddot{a}}{a}\, R - \frac{G_{\mathrm{N}}\, \mathcal{M}}{R^2}\,, \quad \mathcal{M} = 4\pi \int_0^R \mathrm{d}x\, x^2\, \rho(x)\,,$$

($\rho =$ DE + the inhomogeneous density)

So, $\exists R_{\rm c}$ such that massive particles with $\dot{R}(0) = 0$ cannot collapse if $R(0) \geq R_{\rm c}$, but they only expand.

Hence, the **Turnaround Radius** is defined by the condition

$$\ddot{R} = 0$$

that leads to

$$R_{\rm c} = \left(\frac{G_{\rm N}\,\mathcal{M}}{\ddot{a}/a} \right)^{1/3}$$

Example: Schwarschild-de Sitter

Schwarschild-de Sitter spacetime

- Static spherically symmetric (exact) solution of the "vacuum" Einstein equations with $\Lambda > 0$;
- Line element (static coordinates)

$$\mathrm{d}s^2 = -f(R)\,\mathrm{d}t^2 + \frac{\mathrm{d}R^2}{f(R)} + R^2\,\mathrm{d}\Omega^2$$

with

$$f(R) = 1 - \frac{2\,G_\mathrm{N}\,M}{R} - H^2\,R^2\,, \quad H^2 = \Lambda/3$$

- *Turnaround Radius*

$$\boxed{R_\mathrm{c} = \left(\frac{3\,G_\mathrm{N}\,M}{\Lambda}\right)^{1/3}}$$

[1] Faraoni *et al.*, *JCAP* **1510** (2015) 013.

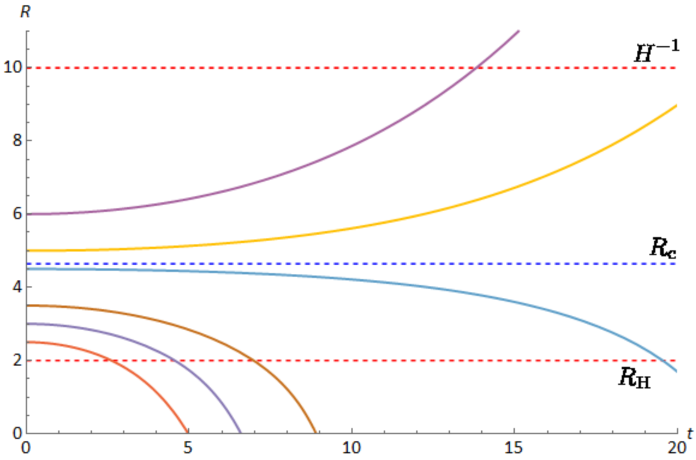

Figure: Radial geodesic of a massive particle with zero initial velocity and initial position $R(0) = R_*$ ($>$ or $<$ than R_c).

Problems with this definition

- Gauge-invariance of the result?
- What does $\mathcal{M}(R)$ mean?
  - Should it contain $\rho_{\rm DE}$?
  - Should it include only $\rho_{\rm pert}$?
- Can we relate the turnaround radius with the notion of *quasi-local energy*?

Quasi-local energy in General Relativity

Hawking-Hayward mass

$\mathcal{S}$ spacelike, closed, orientable, 2-surface

$$M_{\mathrm{HH}} \equiv \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}}\int_S \mu\left(\mathcal{R} + \theta_{(+)}\theta_{(-)} - \frac{1}{2}\sigma^{(+)}_{ab}\sigma^{ab}_{(-)} - 2\omega_a\omega^a\right)$$

- μ area 2-form of $\mathcal{S}$, $A := \int_{\mathcal{S}} \mu$;
- $\theta_{(\pm)}$ expansion scalars of outgoing/ingoing null geodesics;
- $\sigma^{(\pm)}_{ab}$ shear tensors, outgoing/ingoing null geodesics;
- ω^a projection onto $\mathcal{S}$ of the commutator of the null normal vectors to $\mathcal{S}$ (*anholonomicity*).

Note $\mathcal{H}$ Hamiltonian 2-form, then $M_{\mathrm{HH}} \propto -\int_{\mathcal{S}} \mathcal{H}$.
$\Longrightarrow$ **Total energy** contained in $\mathcal{S}$

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[2]Hawking, *J. Math. Phys.* **9** (1968) 598

[3]Hayward, *Phys. Rev. D* **49** (1994) 831

Spherical Symmetry

Hawking-Hayward $=$ Misner-Sharp-Hernandez

$$M_{\mathrm{HH}} \equiv \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}} \int_S \mu \left(\mathcal{R} + \theta_{(+)}\theta_{(-)}\right) = 4\pi \int_0^R \rho(x)\, x^2 \,\mathrm{d}x = M_{\mathrm{MSH}}$$

$$1 - \frac{2\, G_{\mathrm{N}}\, M_{\mathrm{MSH}}}{R} = \nabla^a R \,\nabla_a R$$

What happens if we try to put together the notions of quasi-local energy and turnaround radius?

[4]Misner, Sharp, *Phys. Rev.* **136** (1964) B571

[5]Hernandez, Misner, *Astrophys. J.* **143** (1966) 452

Back to accelerated universes

Idea: the turnaround occurs when the DE contribution overcomes the local gravitational pull.

Perturbed FLRW metric ($|\phi(r)| \ll 1$)

$$\mathrm{d}\tilde{s}^2 = a^2(\eta)\Big[-(1+2\,\phi)\,\mathrm{d}\eta^2 + (1-2\,\phi)\,(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2)\Big]$$

conformal to ($\Omega^2 = a^2(\eta)$) a post-Newtonian, asymptotically flat, approximation of a static spacetime metric in isotropic coordinates

$$\mathrm{d}s^2 = -(1+2\,\phi)\,\mathrm{d}\eta^2 + (1-2\,\phi)\,(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2)$$

MSH Mass and Conformal Transformations

Einstein equations in the "static frame" to the first order in ϕ

$$\triangle\phi = 4\,\pi\,G_\mathrm{N}\,\rho(r)\,,$$

from which one can infer

$$m \equiv M_\mathrm{MSH} \simeq \frac{R^2}{G_\mathrm{N}}\,\frac{\mathrm{d}\phi}{\mathrm{d}R}\,.$$

Besides,

$$\phi \simeq -\frac{G_\mathrm{N}\,m}{R} \simeq -\frac{G_\mathrm{N}\,m}{r}$$

to the first order of approximation.

Conformal transformation $\widetilde{g}_{ab} = \Omega^2(x)\, g_{ab}$, one has

$$M_{\rm MSH} \;\longrightarrow\; \widetilde{M}_{\rm MSH} = \Omega\, M_{\rm MSH} - \frac{R^3}{2\Omega}\, \nabla^c\Omega\nabla_c\Omega - R^2\, \nabla^c\Omega\nabla_c R$$

Recalling that $\widetilde{R} = a(\eta)\, r\, \sqrt{1-2\,\phi} = a(\eta)\, R$ and that $\Omega = a(\eta)$,

$$\widetilde{M}_{\rm MSH} \simeq \underbrace{m\, a}_{\text{Local}} + \underbrace{\frac{H^2\, \widetilde{R}^3}{2\, G_{\rm N}}(1-\phi)}_{\text{Cosmological}}$$

Turnaround:

$$\text{Local} \simeq \text{Cosmological} \;\Longrightarrow\; \widetilde{R}_{\rm c}(t) \simeq \left(\frac{2\, G_{\rm N}\, m\, a}{H^2}\right)^{1/3}$$

(t cosmic time)

[6] Faraoni *et al.*, *Phys. Rev. D* **92** (2015) 023511

Turnaround radius and Dark Energy

Assuming: $P_{\rm DE} = w\,\rho_{\rm DE}$ with $w = const.$
Then,

$$w(z) = -1 + \frac{\ln\left[(3ma_0/4\pi\rho_0)^{1/3}\,R_{\rm c}(z)^{-1}\right]}{1+z}\,,$$

- ρ_0 dark energy density, today;
- $z = (a_0/a) - 1$ redshift parameter.

Measuring ma_0 and $R_{\rm c}(z)$ one could constrain the equation of state parameter w of dark energy.

Technical/Experimental problem
Deviations from the spherical symmetry cannot be neglected!

Turnaround size of (slightly) non-spherical structures

Let's go back to the post-Newtonian line element in isotropic coordinates

$$ds^2 = -(1 + 2\,\phi)\,d\eta^2 + (1 - 2\,\phi)\,(dr^2 + r^2\,d\Omega^2)$$

- $\phi = \phi(r, \theta, \varphi) = \phi_0(r) + \xi\, f(r, \theta, \varphi)$, with $\mathcal{O}(f) = \mathcal{O}(\phi_0) = \mathcal{O}(\phi)$ and $0 < \xi \ll 1$
- $\mathcal{S} = \{\phi(r, \theta, \varphi) = const.\}$ and $\mathcal{S}_0 = \{\phi_0(r) = const.\}$

[7] Giusti, Faraoni, arXiv:1905.04263

Hawking-Hayward mass

First order in both ξ and ϕ

$$M_{\mathrm{HH}} = \frac{1}{8\pi}\sqrt{\frac{A}{16\pi}}\int_S \mu\Bigg[\frac{4}{r}\,\frac{\mathrm{d}\phi_0}{\mathrm{d}r} + \xi\left(\frac{2\,\cot\theta}{r^2}\,\frac{\partial f}{\partial\theta} + \frac{4}{r}\,\frac{\partial f}{\partial r} + \frac{2}{r^2}\,\frac{\partial^2 f}{\partial\theta^2} + \frac{2}{r^2\sin^2\theta}\,\frac{\partial^2 f}{\partial\varphi^2}\right)\Bigg]$$

$\mathcal{S}$ as a small perturbation of $\mathcal{S}_0$

Equipotential surface of the unperturbed potential:

$$\mathcal{S}_0 \iff \phi_0(r) = const. \implies r = r_0\,.$$

Equipotential surface of the perturbed potential:

$$\mathcal{S} \iff \phi(r,\theta,\varphi) = \phi_0(r) + \xi\, f(r,\theta,\varphi) = const.$$

Assuming that the points of $\mathcal{S}$ are located within a distance δr from r_0, with

$$\delta r / r_0 = \mathcal{O}(\xi) \iff \text{small deformations,}$$

Then,

$$\begin{aligned} \mathcal{S} \iff \phi_0(r_0 + \delta r) + \xi\, f(r_0 + \delta r, \theta, \varphi) = const. \\ \implies f(r_0, \theta, \varphi) = const.' \text{ to } \mathcal{O}(\xi). \end{aligned}$$

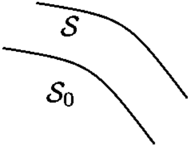
Hence,

$$\int_{\mathcal{S}} \mu\Big(\cdots\Big) = \int_{\mathcal{S}_0} \mu\Big(\cdots\Big) + \mathcal{O}(\xi^2)\,,$$

from which one can infer that

$$\boxed{M_{\mathrm{HH}} = M_{\mathrm{MSH}} + \mathcal{O}(\xi^2)}$$

Intuitively,



$$M = M_0 + \delta M$$

where

$$\delta M \simeq \delta V\,\delta\rho \sim r_0^2\,\delta r\,\delta\rho = \mathcal{O}(\xi^2)\,,$$

since $\delta r/r_0$ and $\delta\rho/\rho_0$ are $\mathcal{O}(\xi)$.

Back to Cosmology

Conformal transformation of the Hawking-Hayward mass

$$\widetilde{M}_{\mathrm{HH}} = \Omega\, M_{\mathrm{HH}} + \frac{R\,\Omega_{,\eta}}{4\pi}\left(\int_{\tilde{S}_0} \mu\,\phi_{,\eta} - \frac{\Omega_{,\eta}}{\Omega}\int_{\tilde{S}_0} \mu\,\phi\right) + \frac{R^3\,\Omega_{,\eta}^2}{2\Omega}$$

- $\phi_{,\eta} = 0$;
- Easy to see that

$$\frac{R\,\Omega_{,\eta}^2}{4\pi\Omega}\int_{\tilde{S}_0} \mu\phi \approx \frac{\dot{a}^2\, a\, R}{4\pi}\, 4\pi R^2 \phi \approx \left(H\,\widetilde{R}\right)^2 \widetilde{R}\,\phi = \mathcal{O}\left[\left(\widetilde{R}/H^{-1}\right)^2\right]$$

Negligible for structures of size $\widetilde{R} \ll H^{-1}$.

$$\Longrightarrow \quad \widetilde{M}_{\mathrm{HH}} = m\,a + \frac{H^2\,\widetilde{R}^3}{2}\,(1-\phi_0) \simeq m\,a + \frac{H^2\,\widetilde{R}^3}{2}$$

Conclusions

- Turnaround radius: astronomer's definition
  - Gauge-dependent;
  - Not explicitly covariant;
  - Physical meaning of $\mathcal{M}(R)$?
- Turnaround radius: alternative definition based on quasi-local energy in GR
  - MSH or HH mass;
  - Explicitly gauge-independent and covariant;
  - Physical meaning of $M_{\mathrm{HH}}/M_{\mathrm{MSH}}$;
  - Competition between *local* and *cosmological* effects.
- Turnaround size: non-spherical structures
  - HH mass!
  - Well... Cosmic structures are not spherically symmetric;
  - How can we extend this notion to these systems?
  - Small deviation from sphericity: nothing changes to $\mathcal{O}(\xi)$.

Thank You!