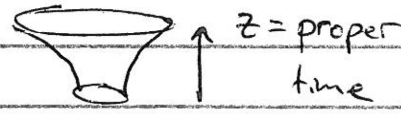
DW/C correspondence

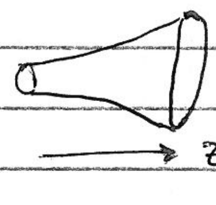
(ADM assumes existence of global 'time' (or radial) coord, can use to define foliation)

(1) ADM decomposition: $x^\mu = (z, x^i)$ $\quad i = 1,2,3$

$$ds^2 = \sigma N^2 dz^2 + g_{ij}(dx^i + N^i dz)(dx^j + N^j dz)$$

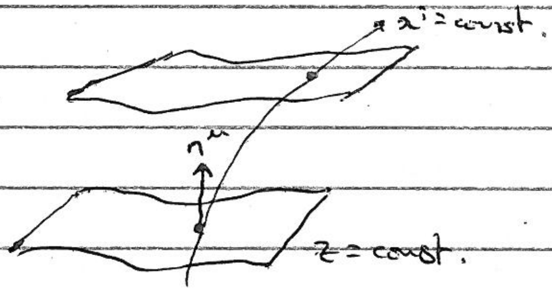
$$\sigma = \begin{cases} -1 \rightsquigarrow \text{(Lorentzian) cosmology} \\ +1 \rightsquigarrow \text{(Euclidean) "domain-wall" or holographic RG flow} \end{cases}$$





Handle $\sigma = \pm 1$ simultaneously.

$$\begin{cases} N = N(z, \underline{x}) \text{ "lapse" fn} \\ N^i = N^i(z, \underline{x}) \text{ "shift" fn} \end{cases}$$



Geometrically, unit normal

$$n^\mu = \frac{1}{N}(1, -N^i) \qquad (n^2 = \sigma \qquad n_\mu = -N\partial_\mu z)$$

$N_j = g_{jk}N^k$

Extrinsic curvature: $K_{ij} = \frac{1}{2}\mathcal{L}_n g_{ij} = \frac{1}{N}\left(\frac{1}{2}\dot{g}_{ij} - \nabla_{(i}N_{j)}\right)$

$\dot{} = \partial_z$ $\qquad$ wrt 3-metric g_{ij}

Action: gravity + scalar (= inflaton)

$$S = \frac{\sigma}{2\kappa^2}\int d^4x \sqrt{|g_{(4)}|}\left(-R^{(4)} + (\partial\Phi)^2 + 2\kappa^2 V(\Phi)\right)$$

$\kappa^2 = 8\pi G_N$

Φ dim'less

$$= \frac{1}{2\kappa^2}\int d^4x\, N\sqrt{g}\left\{K_{ij}K^{ij} - K^2 + N^{-2}(\dot{\Phi} - N^i\Phi_{,i})^2 + \sigma\left(-R + g^{ij}\Phi_{,i}\Phi_{,j} + 2\kappa^2 V(\Phi)\right)\right\}$$

$= -\sigma R^{(4)} + \text{divergence}$

of 3-metric g_{ij}

$g^{ik}g_{kj} = \delta^i_j$

Check signs: for scalar field

	σ	$\dot{\Phi}^2$	$(\underline{\nabla}\Phi)^2$	$V(\Phi)$	
(LC)	$-$	$+$	$-$	$-$	KE − grad. − ptl
(EDW)	$+$	$+$	$+$	$+$	KE + grad. + ptl.

will find same signs for metric pert$^{\underline{ns}}$ too ⇝ see later

N, N^i = Lagrange multipliers

Ex.

$\frac{\partial L}{\partial N}$ ⇝ Hamiltonian constraint (⊥⊥ cpt of Einstein eq$^{\underline{ns}}$)

$$0 = -K_{ij}K^{ij} + K^2 - N^{-2}(\dot{\Phi} - N^i \Phi_{,i})^2 + \sigma(-R + g^{ij}\Phi_{,i}\Phi_{,j} + 2\kappa^2 V)$$

$\frac{\partial L}{\partial N^i}$ ⇝ Momentum constraint (⊥ // cpt)

$$0 = \nabla_j K^j{}_i - K_{,i} - \frac{\Phi_{,i}}{N}(\dot{\Phi} - N^j \Phi_{,j})$$

(2) For DW/C

	Background	Perturbations
$N =$	1	$+\ \delta N$
$N_i =$	0	$+\ \delta N_i$
$g_{ij} =$	$a^2(z)\,(\delta_{ij}$	$+\ h_{ij}\,)$
$\Phi =$	$\varphi(z)$	$+\ \delta\varphi$

⇝ all f$^{\underline{ns}}$ of z, x

Bgd eom: $H = \dot{a}/a$ HJ or 1st order form:

Ex.
$$\left.\begin{aligned} 0 &= \tfrac{1}{2}\dot{\varphi}^2 - 3H^2 - 2\sigma\kappa^2 V \\ 0 &= \dot{H} + \tfrac{1}{2}\dot{\varphi}^2 \\ 0 &= \ddot{\varphi} + 3H\dot{\varphi} - \sigma\kappa^2 V' \end{aligned}\right\} \Longrightarrow \boxed{\begin{aligned} H &= -\tfrac{1}{2}W(\varphi) \\ \dot{\varphi} &= W' \\ 2\sigma\kappa^2 V &= W'^2 - \tfrac{3}{2}W^2 \end{aligned}}$$

If $\varphi(z)$ monotonic, invert to $z(\varphi)$ SGT (guarantees stable DW sol^n)

$$H(z) \to -\tfrac{1}{2}W(\varphi)$$

③ Perturbations:

"longitudinal" $\partial_i v \xrightarrow{FT} q_i v$ "transverse" $0 = v_{i,i} \xrightarrow{FT} 0 = q_i v_i$

* SVT decomposition:
$$\begin{cases} \delta N_i = a^2(v_{,i} + v_i) \\ h_{ij} = -2\psi\,\delta_{ij} + 2\chi_{,ij} + 2\omega_{(i,j)} + \gamma_{ij} \end{cases}$$

10 cpts of $\delta g_{\mu\nu}$ $\Longrightarrow$
$$\begin{cases} 4 \text{ scalars } \delta N, v, \psi, \chi \\ 2 \text{ transverse vectors } v_i, \omega_i \\ 1 \text{ transverse-traceless tensor } \gamma_{ij} \end{cases}$$

$v_{i,i} = 0$, $\omega_{i,i} = 0$

$\gamma_{ij,j} = 0$, $\gamma_{ii} = 0$

$$10 = 4 + \underbrace{2 \times 2}_{v_i, \omega_i \text{ have 2 dof each} = 3-1} + \underbrace{2}_{\text{ie. } \gamma_{ij} \text{ has } 6-3-1 = 2 \text{ dof}}$$

infinitesimal coord. transf or "gauge

* Gauge freedom: $x^\mu \to x'^\mu = x^\mu - \xi^\mu$ sends

$$\delta g_{\mu\nu} \to \delta g'_{\mu\nu} = \delta g_{\mu\nu} + \underbrace{\mathcal{L}_\xi g^{(0)}_{\mu\nu}}_{2\nabla_{(\mu}\xi_{\nu)}} \qquad \delta\varphi \to \delta\varphi' = \delta\varphi + \underbrace{\mathcal{L}_\xi \varphi^{(0)}}_{\xi^\mu \nabla_\mu \varphi^{(0)}}$$

ie. Lie deriv. of bgd metric wrt ξ^μ

SVT decompose ⇝ eg. $\Psi \to \Psi' = \Psi - H\xi^0$

$\delta\varphi \to \delta\varphi' = \delta\varphi + \dot{\varphi}\,\xi^0$ etc.

⇒ Gauge-invariant variables:

$$\begin{cases} \zeta = -\Psi - \dfrac{H}{\dot{\varphi}}\,\delta\varphi & \text{"comoving curvature" pert}^n \\ & \rightsquigarrow \Delta T \text{ on CMB} \\ \gamma_{ij} \quad (\text{to linear order}) & \text{graviton} \rightsquigarrow \text{B-mode polarisation} \end{cases}$$

* Relevant gauges:

Holographic analysis ⇒ Synchronous (or Fefferman-Graham)
(asymptotic)

$N = 1, \; N_i = 0 \;\Rightarrow\; \delta N, \nu, \nu_i \rightsquigarrow 0$

Residual gauge freedom ⇒ QFT Ward id.

Cosmological analysis ⇒ Comoving gauge (i.e. with matter)

$$\begin{cases} g_{ij} = a^2 e^{2\zeta}[e^{\gamma}]_{ij} \\ \Phi = \varphi \end{cases} \Rightarrow \quad \delta\varphi, \kappa, \omega_i \rightsquigarrow 0$$

$h_{ij} = 2\zeta\delta_{ij} + \gamma_{ij}$

Fully fixed ⇝ identify true dynamical dof as ζ, γ_{ij}

Ex. Solving constraints for N, N_i and inserting into ADM action:

$$S^{(2)} = \frac{1}{\kappa^2}\int d^4x \left\{ a^3\epsilon\dot{\zeta}^2 + \sigma a\epsilon\,\zeta_{,i}\zeta_{,i} + \frac{1}{8}a^3\dot{\gamma}_{ij}\dot{\gamma}_{ij} + \frac{1}{8}\sigma a\,\gamma_{ij,k}\gamma_{ij,k} \right\}$$

$$\epsilon = \frac{\dot{\varphi}^2}{2H^2} = \left(\frac{1}{H}\right)^{\cdot} = \frac{2w'^2}{w^2}$$

$$\begin{cases} \sigma = -1 & \text{KE} - \text{gradients} \quad \checkmark \text{ (comoving)} \\ \sigma = +1 & \text{KE} + \text{gradients} \end{cases}$$

(spatial 3-

Eom:

$$\boxed{\begin{aligned} 0 &= \ddot{\zeta} + (3H + \dot{\epsilon}/\epsilon)\dot{\zeta} - \sigma\frac{q^2}{a^2}\zeta \\ 0 &= \ddot{\gamma}_{ij} + 3H\dot{\gamma}_{ij} - \sigma\frac{q^2}{a^2}\gamma_{ij} \end{aligned}}$$

σ

q_i = momentum (FT wrt x^i)

(5) Asymptotics:

* As $z \to \infty$, (late time / near bdy)

$$\begin{cases} \text{power-law} \sim \text{see Kostas's lectures} \\ \text{(A)dS:} \quad a \to e^{z/\ell}, \quad \varphi \to 0 \end{cases} \quad W(\varphi) = -2H = \frac{1}{\ell}\underbrace{\left(-2 + \#\varphi^2 + \dots\right)}_{\omega(\varphi)}$$

$\dot{\varphi} = W' \to$

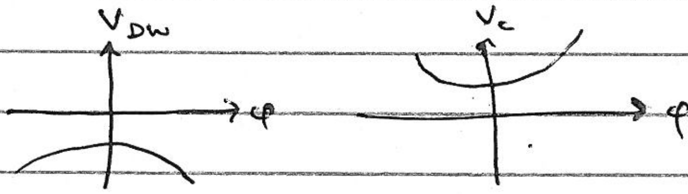
dmlless f^n

$\zeta, \gamma_{ij} \to$ const.

$$V(\varphi) = -\frac{\sigma}{\kappa^2 \ell^2}\left(3 + \#\varphi^2 + \dots\right)$$

i.e. $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$

$= -\kappa^2 V g_{\mu\nu} = \frac{\#}{\ell^2} g_{\mu\nu}$



i.e. $\int \frac{d\varphi}{\dot{\varphi} a}$

* As $z \to 0$, conformal 'time' $\tau(z) = \int \frac{dz}{a} = \ell \int \frac{d\varphi}{\omega'} \exp\left(\int \frac{d\varphi\, \omega}{2\omega'}\right)$

(early times / deep interior)

$\to -\infty$

sets scale of τ

(also $\varphi = \varphi(z/\ell)$ $\to z = \ell z(\varphi)$)

DW: $q = +\sqrt{q^2}$ magnitude

Cosmology:

$\zeta, \gamma_{ij} \sim e^{+q\tau}$

$\zeta, \gamma_{ij} \sim e^{-iq\tau}$

(WKB, focussing on phase part)

decaying in DW interior

Bunch-Davies vacuum

(6) DW/C correspondence: $(q\ell)_{DW} = -i\,(q\ell)_c$

(Also holds at nonlinear order, i.e. for non-Gaussianities) $(\kappa^2\ell^2)_{DW} = -(\kappa^2\ell^2)_c$

Either: Planck units $\kappa^2_{DW} = \kappa^2_c = 1$ $\Rightarrow$ $\ell_{DW} = -i\ell_c$, $q_{DW} = q_c$ } Ψ_{univ} literature $\Rightarrow$ Re par

Or: (A)dS units $l_{DW} = l_c = 1$

$$\rightsquigarrow \boxed{\begin{array}{l} \kappa^2_{DW} = -\kappa^2_c \\ q_{DW} = -i\, q_c \end{array}}$$

our papers

$\Rightarrow$ lwr parts

(bar for DW quantities)

In dual QFT, $\kappa^2/\ell^2 \propto N^{-2}$ # colours = rank of SU(N) gauge group

In 3d, $g^2_{eff} = \frac{g^2_{YM} N}{q}$ dim'less

Correlators non-analytic in g^2_{eff} $\Rightarrow$ $\boxed{N_{DW} = -i\, N_c}$

The holographic power spectrum

① Response f$^{\underline{n}}$s $\Omega(q), E(q)$ — magnitudes, not vectors
$\quad\quad\quad\quad\quad\quad\quad z, \quad\quad z,$

($\kappa^2 \times$) canonical momenta wrt ζ, γ_{ij}

$$\begin{cases} \Pi^{(\zeta)} \equiv \kappa^2 \dfrac{\partial L}{\partial \dot\zeta} = 2a^3\varepsilon\dot\zeta = \Omega\zeta \\ \Pi^{(\gamma)}_{ij} \equiv \kappa^2 \dfrac{\partial L}{\partial \dot\gamma_{ij}} = \dfrac{1}{4}a^3\dot\gamma_{ij} = E\gamma_{ij} \end{cases}$$

Obey 1st order Riccatti eq$^{\underline{n}}$s

$$\begin{cases} 0 = \dot\Omega + \dfrac{\Omega^2}{2a^3\varepsilon} - 2\sigma a\varepsilon q^2 \\ 0 = \dot E + \dfrac{4}{a^3}E^2 - \dfrac{\sigma a}{4}q^2 \end{cases}$$

(see ζ, γ_{ij} eom)

Initial cond$^{\underline{n}}$s as $z \to 0$

$$\begin{cases} \Omega_{DW} \to 2a^3\varepsilon q_{DW} \\ \Omega_c \to -2ia^3\varepsilon q_c \end{cases} \qquad \begin{cases} E_{DW} \to \dfrac{a^2}{4}q_{DW} \\ E_c \to -i\dfrac{a^2}{4}q_c \end{cases}$$

a/c: $\Omega_{DW}(q_{DW}) = \Omega_{DW}(-iq_c) = \Omega_c(q_c)$ etc.

② Cosmological power spectrum $(\sigma = -1)$

Quantise pert$^{\underline{n}}$s:

$$\hat\zeta(z, x) = \int \frac{d^3q}{(2\pi)^3}\Big(\zeta_q(z)\hat a(q)e^{iq\cdot x} + \underbrace{\zeta^*_q(z)\hat a^\dagger(q)e^{-iq\cdot x}}_{\text{h.c.}}\Big)$$

$$= \int \frac{d^3q}{(2\pi)^3}\hat\zeta(z, q)\, e^{iq\cdot x} \qquad q \to -q$$

$$\hookrightarrow \hat\zeta(z, q) = \zeta_q(z)\hat a(q) + \zeta^*_q(z)\hat a^\dagger(-q)$$

$$\hat\gamma_{ij}(z, q) = \sum_{s=\pm}\hat\gamma^{(s)}(z, q)\,\varepsilon^{(s)}_{ij}(q)$$ helicity "tensor"

$$\hookrightarrow \hat\gamma^{(s)}(z, q) = \gamma_q(z)\hat b^{(s)}(q) + \gamma^*_q(z)\hat b^{\dagger(s)}(-q)$$

$\zeta_q(z)$, $\gamma_q(z)$ $\rightsquigarrow$ "mode $f^{\underline{n}}$s" obeying classical eom

magnitudes

(tell us about spacetime curvature of bgd)

Creation / annihilation ops:

commute unless $q = q'$

$$\begin{cases} [\hat{a}(q), \hat{a}^+(q')] = (2\pi)^3 \delta(q - q') \\ [\hat{b}^{(s)}(q), \hat{b}^{(s')+}(q')] = (2\pi)^3 \delta(q - q') \delta^{ss'} \end{cases}$$

$[\hat{a}, \hat{a}] = 0$ etc.

Canonical commutation relations:

($\hat{\Pi}^{(\zeta)}$ on LHS is κ^2 times real can. mom.)

$$\begin{cases} [\hat{\zeta}(z, q), \hat{\Pi}^{(\zeta)}(z, q')] = i\kappa^2 (2\pi)^3 \delta(q + q') \\ [\hat{\gamma}^{(s)}(z, q), \hat{\Pi}^{(\gamma)(s')}(z, q')] = \frac{i}{2} \kappa^2 (2\pi)^3 \delta(q + q') \delta^{ss'} \end{cases}$$

equal times

(helicity tensor conventions)

$\delta(x - x')$ in position space

but $\hat{\Pi}^{(\zeta)} = 2a^3 \varepsilon \dot{\hat{\zeta}} = \Omega(q) \zeta_q \hat{a}(q) + \Omega^*(q) \zeta_q^* \hat{a}^+(-q)$ etc.

giving

$$\left. \begin{aligned} |\zeta_q|^2 &= \frac{-\kappa^2}{2 \operatorname{Im} \Omega(q)} \\ |\gamma_q|^2 &= \frac{-\kappa^2}{4 \operatorname{Im} E(q)} \end{aligned} \right\}$$

"Wronskian" relations

i.e. $\zeta \dot{\zeta}^* - \zeta^* \dot{\zeta}$

2-point $f^{\underline{n}}$s:

$$\langle 0 | \hat{\zeta}(z, q) \hat{\zeta}(z, q') | 0 \rangle = |\zeta_q(z)|^2 (2\pi)^3 \delta(q + q')$$

$$\langle 0 | \hat{\gamma}^{(s)}(z, q) \hat{\gamma}^{(s')}(z, q') | 0 \rangle = |\gamma_q(z)|^2 (2\pi)^3 \delta(q + q') \delta^{ss'}$$

same time

momentum conservation

Power spectra: $\langle \hat{\zeta}^2(x) \rangle = \int d\ln q \, \Delta_S^2(q)$ ← i.e. contribution to variance per log interval of q

$$\boxed{\begin{aligned} \Delta_S^2(q) &= \frac{q^3}{2\pi^2}|\zeta_q|^2 = \frac{-\kappa^2 q^3}{4\pi^2 \operatorname{Im}\Omega_{(0)}(q)} \\ \Delta_T^2(q) &= \frac{2q^3}{\pi^2}|\gamma_q|^2 = \frac{-\kappa^2 q^3}{2\pi^2 \operatorname{Im} E_{(0)}(q)} \end{aligned}}$$

← "scalar" ⇝ ΔT in CMB

← "tensor" ⇝ primordial B-mode polarisation

(0) ⇝ Evaluate at end of inflation = "late times" $(z \to \infty)$

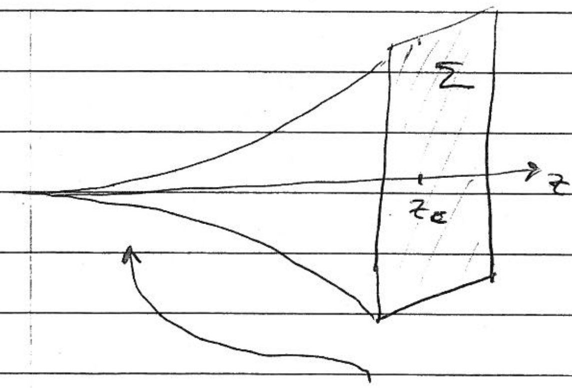
③ Holographic calc$^{\underline{n}}$ of $\langle T_{ij} T_{kl} \rangle$ for DW $(\sigma = +1)$

Synch. gauge $\left.\begin{array}{l} N = 1 \\ N_i = 0 \end{array}\right\}$ $ds^2 = dz^2 + g_{ij}\, dx^i dx^j$

$\hookrightarrow K_{ij} = \frac{1}{2}\dot{g}_{ij}$

Radial Hamiltonian formalism:

variables



$$\left\{ \begin{array}{ll} g_{ij},\ \Phi & \text{fields on } \Sigma \\ \Pi^{ij} = \dfrac{\partial L}{\partial \dot{g}_{ij}} = \dfrac{1}{2\kappa^2}\sqrt{g}\left(K^{ij} - K g^{ij}\right) & \\ \Pi = \dfrac{\partial L}{\partial \dot{\Phi}} = \dfrac{1}{\kappa^2}\sqrt{g}\,\dot{\Phi} & \end{array} \right.$$

On-shell + regularity in interior

canonical momenta on Σ (see ADM action)

$\Rightarrow \Pi^{ij}[g_{kl}, \Phi], \quad \Pi[g_{kl}, \Phi]$

(On-shell momenta nonlocal functionals of fields on Σ)

Holography: QFT data $\rightleftharpoons$ bulk asymptotics (as $z \to \infty$)

* works both ways!
* smooth asymptotic geometry even when curvatures large (ie. stringy) in interior

↳ see K's lectures

QFT data:

* sources: $g_{(0)ij}$ for T_{ij}, $\varphi_{(0)}$ for O_Δ

* 1-point f^ns: $\langle T_{ij} \rangle_s = \frac{-2}{\sqrt{g_{(0)}}} \frac{\delta W_{QFT}}{\delta g_{(0)}^{ij}}$

nonzero sources $s = (g_{(0)ij}, \varphi_{(0)})$

$$\langle O \rangle_s = \frac{-1}{\sqrt{g_{(0)}}} \frac{\delta W_{QFT}}{\delta \varphi_{(0)}}$$

⇝ differentiate to get higher-pt. f^ns

Bulk asymptotics: for (locally) AdS ($\ell = 1$)

as $z \to \infty$, $\begin{cases} g_{ij} \to e^{2z} g_{(0)ij} \\ \Phi \to e^{-(3-\Delta)z} \varphi_{(0)} \end{cases}$ ← QFT sources

Idea:

$W_{QFT} = -S_{grav.\,onshell}$ and vary wrt sources

$$\begin{cases} \langle T^i_j \rangle_s = \left(\frac{-2}{\sqrt{g}} \Pi^i_j \right)_{(3)} = \kappa^{-2} \left(K \delta^i_j - K^i_j \right)_{(3)} \\ \langle O \rangle_s = \left(\frac{1}{\sqrt{g}} \Pi \right)_{(\Delta)} = \kappa^{-2} \left(\dot{\Phi} \right)_{(\Delta)} \end{cases}$$

↑ dilatation weights

Bare on-shell momenta diverge as $z_i \to \infty$

⇝ holographic renormalisation.

[sign for $\langle T^i_j \rangle$ because used $\delta g^{ij}_{(0)}$ for LHS but δg_{ij} (ie. Π^{ij}) for RHS]

remove lower weight pieces with (local, covariant) counterterms, on Σ
built from fields

Steps: 1) Regulate: cut off at finite z_c

2) Add counterterms = local, covariant f^{n}s of fields g_{ij}, Φ on Σ

* preserves symmetries (diff. cov. on Σ) hence QFT Ward ids

* bulk dynamics unaffected

Organise by dilatation weight: $\delta_D A_{(n)} = -n A_{(n)}$

$$\delta_D = \int_\Sigma d^3x \left(-2 g_{ij} \frac{\delta}{\delta g_{ij}} - (3-\Delta)\Phi \frac{\delta}{\delta \Phi} \right)$$

(covariant unlike radial expansion in z)

Obtain from eom: $\partial_z = \delta_D + \ldots$ as $z \to \infty$

3) Renormalise by sending $z_c \to \infty$

2-point $f^{\underline{n}}$:

momentum conservation

$$\langle T_{ij}(q)\,T_{kl}(q')\rangle = \underbrace{\langle\langle T_{ij}(q)\,T_{kl}(-q)\rangle\rangle}\,(2\pi)^3\,\delta(q+q')$$

$$= A(q)\,\Pi_{ijkl} + B(q)\,\pi_{ij}\pi_{kl}$$

(Ward ids $\Rightarrow$ transverse)

where $\begin{cases} \Pi_{ijkl} = \pi_{i(k}\pi_{l)j} - \frac{1}{2}\pi_{ij}\pi_{kl} & \text{TT projector} \\ \pi_{ij} = \delta_{ij} - q_i q_j/q^2 & \text{transverse projector} \end{cases}$

i.e. $\begin{cases} \langle\langle T(q)\,T(-q)\rangle\rangle = 4B(q) \\ \langle\langle T^{(s)}(q)\,T^{(s')}(-q)\rangle\rangle = \frac{1}{2}A(q)\,\delta^{ss'} \end{cases}$

Can show (Varying source) $\boxed{\delta\langle T_{ij}(q)\rangle_s = \frac{1}{2}A(q)\,\gamma_{(0)ij} - 2B(q)\,\psi_{(0)}\,\pi_{ij}}$

i.e. set $g_{(0)ij} = \delta_{ij} + h_{(0)ij}$

$h_{(0)ij} = -2\psi_{(0)}\delta_{ij} + \dots + \gamma_{(0)ij}$

In bulk, find (see exercises)

$$\delta\langle T^i_j(q)\rangle_s = \kappa^{-2}\left[-\frac{2E(q)}{a^3}\gamma_{ij} + \left(\frac{-q^2}{a^2H} + \frac{\Omega(q)}{2a^3}\right)\psi\,\pi_{ij}\right]_{(3)}$$

$$= -2\kappa^{-2}E_{(0)}(q)\,\gamma_{(0)ij} + \frac{1}{2}\kappa^{-2}\Omega_{(0)}(q)\,\psi_{(0)}\pi_{ij}$$

wt 2 piece $\Rightarrow$ removed by counterterm $\left(R = -4\frac{q^2}{a^2}\psi\right)$

~~as $\delta_D a$~~ since a has weight -1

$$\left(\begin{array}{l} \delta_D = a\partial_a - (d-\Delta)\varphi\partial_\varphi \\ \text{and } \delta_D A_{(n)} = -n\,A_{(n)} \end{array}\right)$$

$$\boxed{\begin{array}{l} A(q) = -4\kappa^{-2}E_{(3)}(q) \\ B(q) = -\frac{1}{4}\kappa^{-2}\Omega_{(0)}(q) \end{array}}$$

wt 0 pieces: $E_{(0)}, \Omega_{(0)}$ = limit as $z\to\infty$ after subtracting counterterms

Holographic power spectra:

$\begin{cases} \text{Barred} = \text{DW} \\ \text{Unbarred} = \text{Cosmology} \end{cases}$

$$\Delta_S^2(q) = \frac{-\kappa^2 q^3}{4\pi^2 \,\mathrm{Im}\, \Omega_{(0)}(q)} \qquad \bar{B}(\bar{q}) = -\tfrac{1}{4}\bar{\kappa}^{-2}\bar{\Omega}_{(0)}(\bar{q})$$

response f$^{\text{n}}$s
fix both.

DW/C $\rightsquigarrow$ $\bar{q} = -iq$ $\qquad \bar{\Omega}(\bar{q}) = \bar{\Omega}(-iq) = \Omega(q)$

$\bar{\kappa}^2 = -\kappa^2 \quad (\bar{N} = -iN)$

$$\rightsquigarrow \quad \boxed{\Delta_S^2(q) = \frac{-q^3}{16\pi^2\,\mathrm{Im}\, B(-iq)} \qquad \Delta_T^2(q) = \frac{-2q^3}{\pi^2\,\mathrm{Im}\, A(-iq)}}$$

+ve, eg. free fields

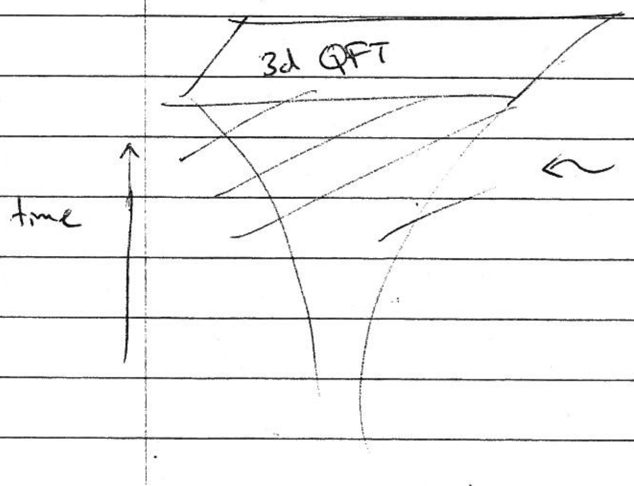
$B(q) \propto \bar{N}^2 \bar{q}^3$

$\mathrm{Im}\, B = \mathrm{Im}\left(-N^2(-iq)^3\right)$
$= -N^2 q^3$

Counterterm contributions local
ie, $\bar{q}^{2n}$ for $n \in \mathbb{Z}^+$
$\Rightarrow$ Drop out of Im part

$$\text{QFT } \langle T_{ij} T_{kl} \rangle \;\overset{\text{DW/C}}{\rightleftharpoons}\; \text{cosmological power spectra}$$

(similarly
$\langle T_{ij} T_{kl} T_{mn} \rangle \rightleftharpoons$ 3-point non-Gaussianities)



$\leftarrow$ holographically reconstruct late-time metric perturbations

Inflation from the OPE

Slow-roll inflation $\longleftrightarrow$ slow RG flow $\beta, \beta' \ll 1$

$\approx dS$

* nearly a CFT
* strongly interacting (= ordinary GR)
  $\rightsquigarrow$ how to compute?

Alternative expansion parameter:

spectral tilt $n_s - 1 \ll 1$

(Δ_s^2 nearly scale invariant:
$\Delta_s^2(q) = \Delta_s^2(q_0)\left(\frac{q}{q_0}\right)^{n_s-1}$
CMB $\Rightarrow n_s \simeq 0.96$)

Dual QFT:

$$S = S_{CFT} + \int \varphi\, \Lambda^{-\lambda}\, O_\Delta$$

scalar operator dual to inflaton

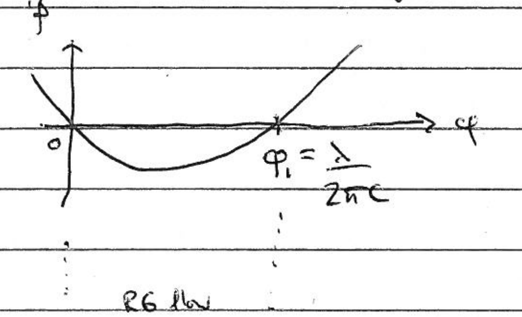
coupling, Λ = length scale

$\Delta_{UV} = 3 - \lambda$

$\lambda \ll 1$

nearly marginal

$\rightsquigarrow$ slow RG flow



$$\beta(\varphi) = -\lambda\varphi + 2\pi C \varphi^2$$

UV CFT $\xrightarrow{\text{RG flow}}$ IR CFT

$(\Delta_{IR} = 3 + \lambda)$

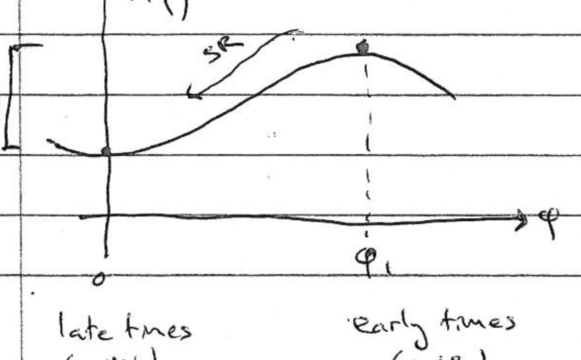
for $C \simeq 1$, $\varphi_1 \ll 1$

* nearby fixed points
* collide as $\lambda \to 0$
  $\hookrightarrow$ impose regularity

Cosmology:

$\Delta V \sim \lambda^3$

(toy model still)



* leading order results (for all q) follow from OPE
* universal ~ depend only on λ, C
  (also hold for weakly interacting QFT)
* analogous to ε-expansion for Gaussian $\to$ Wilson Fisher

① Operator product expansion (OPE)

Inside correlator, as $|x_{12}| = |\underline{x}_1 - \underline{x}_2| \to 0$

$$\mathcal{O}(x_1)\mathcal{O}(x_2) = \frac{\alpha}{|x_{12}|^{2\Delta}} + \frac{C}{|x_{12}|^{\Delta}}\mathcal{O}(x_1) + \dots$$

contributions from other operators, descendants, etc. won't be relevant here.

In CFT, $\langle \mathcal{O}(x)\rangle_0 = 0$ no vev (breaks conf. inv.)

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle_0 = \frac{\alpha}{|x_{12}|^{2\Delta}} \qquad \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \frac{\alpha C}{|x_{12}|^{\Delta}|x_{23}|^{\Delta}|x_{31}|^{\Delta}}$$

α = operator normalisation $\qquad$ C = OPE coeff't (or structure const.)

also: (2pt f$^{\underline{n}}$ vanishes for non-identical operators)

(check: as $|x_{12}| \to 0$

$\langle \mathcal{O}\mathcal{O}\mathcal{O}\rangle \to \frac{C}{|x_{12}|^{\Delta}}\langle \mathcal{O}\mathcal{O}\rangle$)

② Conformal pert$^{\underline{n}}$ theory:

$$Z = \int \underbrace{\mathcal{D}\Phi}_{\text{fields in CFT}}\, e^{-S_{CFT}[\Phi] - \underbrace{\int \varphi \Lambda^{-\lambda}\mathcal{O}}_{\text{external source}}} = \langle e^{-\int \varphi\Lambda^{-\lambda}\mathcal{O}}\rangle_0$$

evaluate in UV CFT i.e. for $\varphi = 0$

Expand "deformed CFT" correlators in terms of true CFT correlators

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle = \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\, e^{-\int\varphi\Lambda^{-\lambda}\mathcal{O}}\rangle_0$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}(-\varphi\Lambda^{-\lambda})^n I_n \quad \text{where } I_n = \int dz_1 \dots dz_n \langle \mathcal{O}(x_1)\mathcal{O}(x_2) \cdot \mathcal{O}(z_1)\dots\mathcal{O}(z_n)\rangle_0$$

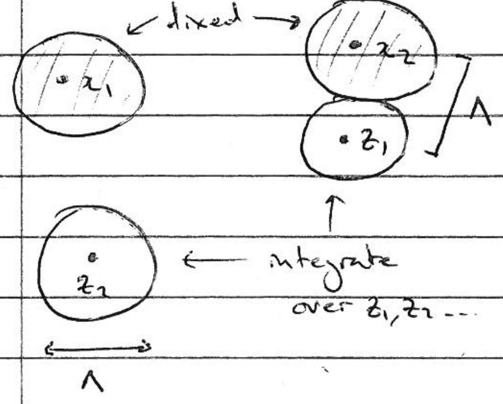
Need to sum entire series!

i.e. CFT correlator with two fixed + n integrated insertions

$\varphi \lesssim \varphi_1 \sim \lambda$ so $\varphi^n \sim \lambda^n$ <u>but</u>, $I_n \sim \frac{1}{\lambda^n}$

$\rightsquigarrow$ all terms same order

Operator 'gas' picture:



* Hard-sphere cut off removes singularities from collisions

$\Lambda =$ diameter $\quad (\sim \mu_{RG}^{-1})$

* ~~E~~valuate 2-body collisions using OPE

* Dilute gas: 3-body (+ higher) collisions subleading

Invariance of partition f^n Z as vary Λ $\rightsquigarrow \beta(\varphi) = -\lambda\varphi + 2\pi C\varphi^2$ (exercise)

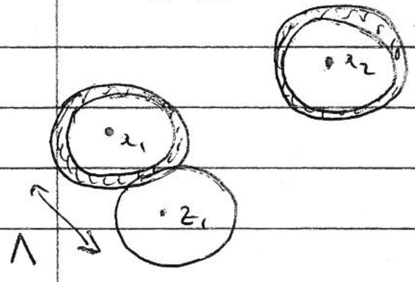
Consider $I_1 = \int dz_1 \, \langle O(x_1) O(x_2) O(z_1) \rangle_0$

Varying Λ,

$$\delta I_1 = -2 \, \underbrace{(4\pi\Lambda^2\delta\Lambda)}_{\text{shell volume}} \underbrace{\frac{C}{\Lambda^{3-\lambda}} \langle O(x_1) O(x_2) \rangle_0}_{\text{OPE: } OO \sim \frac{C}{\Lambda^{3-\lambda}} O + \ldots} + \ldots$$

lose this volume

two collisions



(no $\not{\!\!1}$ piece as 1pt vanishes)

[Overall $\delta I_1 \sim \Lambda^{\lambda-1}\delta\Lambda$ giving $\frac{1}{\lambda}$ pole when integrate:]

Integrate: $I_1 = -\frac{8\pi C}{\lambda} \left(\Lambda^\lambda - b|x_{12}|^\lambda \right) \langle O(x_1) O(x_2) \rangle_0 + \ldots$

contribution from descendants subleading

singular as $\lambda \to 0$

$b =$ integration constant

$|x_{12}|^\lambda \sim$ only dimensional scale (apart from Λ)

* Regularity as $\lambda \to 0$, fixes $b = 1$!

(i.e. as O becomes marginal get logarithm)

For I_n: $\quad \delta I_n = -B_n \left(4\pi\Lambda^2 \delta\Lambda\right) \frac{C}{\Lambda^{3-\lambda}} I_{n-1} + \ldots$

\# collisions

$= \binom{n}{2} + 2n = \frac{1}{2}n(n+3)$

$z_i \to z_j$ $\quad \begin{cases} z_i \to x_1 \\ z_i \to x_2 \end{cases}$

subleading after integrate:

eg. $OO \sim \frac{\alpha}{\Lambda^{6-2\lambda}} O$

$\Rightarrow \delta I_n \sim \Lambda^{2\lambda-4}\delta\Lambda\, I_{n-2}$

$\hookrightarrow$ no $\frac{1}{\lambda}$ pole

* Assuming O only operator becoming marginal as $\lambda \to 0$ $\quad \left(\frac{1}{\lambda}\text{ poles require } OO \sim \frac{1}{\Lambda^{3+n\lambda}} X\right)$

$\left(T_{ij} \sim \text{integral over shell vanishes}\right)$

Define $y = 1 - \left(\frac{|x_{12}|}{\Lambda}\right)^{-\lambda}$ $\quad (y \to 0 \text{ as } \lambda \to 0)$

$\rightsquigarrow \frac{dI_n}{dy} = \frac{2\pi C}{\lambda} n(n+3)|x_{12}|^{\lambda} I_{n-1} + \ldots$

$\rightsquigarrow$ gives $I_n \sim \frac{1}{\lambda^n}$ $\rightsquigarrow$ choose integration const. to cancel this leading singularity: $\lambda^n I_n \to 0$ as $\lambda \to 0$ (want I_n finite as $\lambda \to 0$ but only working to leading order)

$\rightsquigarrow$ all vanish since $y \to 0$ as $\lambda \to 0$

$$\Rightarrow I_n = \left(\frac{2\pi C}{\lambda}|x_{12}|^{\lambda}\right)^n \cancel{n!}\,\frac{(n+3)!}{3!}\,\frac{y^n}{\cancel{n!}} I_0 + \ldots$$

giving $$\langle O(x_1)O(x_2)\rangle = \frac{\alpha}{6}|x_{12}|^{-6+2\lambda}\sum_{n=0}^{\infty}(n+3)(n+2)(n+1) \times \left[-\frac{\varphi}{\varphi_1}\left(\frac{|x_{12}|^{\lambda}}{\Lambda^{\lambda}} - 1\right)\right]^n$$

$\left(\text{recall: } \sum_{n=0}^{\infty}\frac{(-\varphi\Lambda^{-\lambda})^n I_n}{n!}\right)$

$\rightsquigarrow$ binomial series

For $\lambda \neq 0$, can now remove cut off:

$$\beta = -\frac{d\varphi}{d\ln\Lambda} = -\lambda\varphi + 2\pi C\varphi^2 \rightsquigarrow \varphi = \frac{\varphi_1}{1 + \frac{\varphi_1}{\phi}\Lambda^{-\lambda}}$$

so $\varphi \to \phi\Lambda^{\lambda}$ as $\Lambda \to 0$

(i.e. $\int \varphi \Lambda^{-\lambda} O \to \int \phi O$)

So $-\frac{\varphi}{\varphi_1}\left(\frac{|x_{12}|^{\lambda}}{\Lambda^{\lambda}} - 1\right) \longrightarrow -\frac{\phi}{\varphi_1}|x_{12}|^{\lambda}$ as $\Lambda \to 0$

Resumming binomial series,

$$\boxed{\langle O(x_1) O(x_2)\rangle = \alpha |x_{12}|^{-6+2\lambda} \times \left[1 + \frac{\phi}{\varphi_1}|x_{12}|^{\lambda}\right]^{-4}}$$

↓ or:

FT term by term first

$$\boxed{\langle\langle O(q) O(-q)\rangle\rangle = \frac{\pi^2}{12}\alpha q^{3-2\lambda}\left[1 + \frac{\phi}{\varphi_1}q^{-\lambda}\right]^{-4}}$$

leading order 2-pt ~~fn in~~ def. th.

Trace Ward id:

$\langle\langle T(q) T(-q)\rangle\rangle$

$= \underbrace{\lambda^2\phi^2}_{\beta_\phi^2} \langle\langle O(q) O(-q)\rangle\rangle$

$$q \begin{cases} \text{In UV, } \sim q^{3-2\lambda} \text{ as } q \to \infty & \Delta_{UV} = 3-\lambda \\ \text{In IR, } \sim q^{3+2\lambda} \text{ as } q \to 0 & \Delta_{IR} = 3+\lambda \end{cases}$$

Cosmo. power spectrum $\Delta_s^2(q) = \dfrac{-q^3}{4\pi^2 \,\mathrm{Im}\langle\langle TT\rangle\rangle\big|_{\substack{q \to -iq \\ N^2 \to -N^2}}}$

$\mathrm{Im}\left(\bar{N}^2 q^{3+n\lambda}\right)$
$= -N^2 q^{3+n\lambda} + \ldots$
$= -\alpha q^{3+n\lambda} + \ldots$

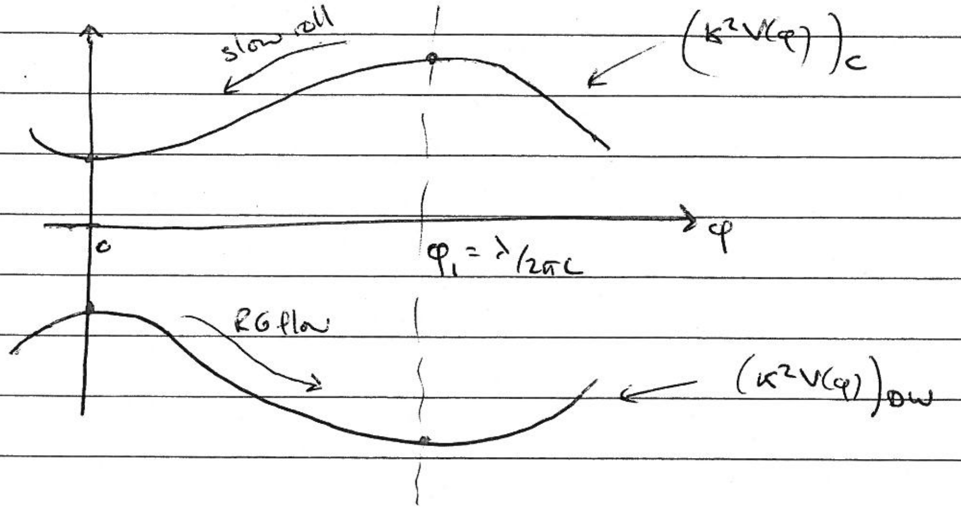
$(\alpha \sim N^2)$

$$\left(= \frac{3}{\pi^4 \alpha \lambda^2 \phi^2} q^{2\lambda}\left[1 + \frac{\phi}{\varphi_1}q^{-\lambda}\right]^4 \right)$$

$$\boxed{\Delta_s^2(q) = \Delta_s^2(q_0)\left(\frac{q}{q_0}\right)^{-2\lambda}\frac{1}{16}\left[1 + \left(\frac{q}{q_0}\right)^{\lambda}\right]^4}$$

$$\begin{cases} \Delta_s^2(q_0) = \dfrac{192 C^2}{\pi^2 \alpha \lambda^4} \\ q_0^{\lambda} = \phi/\varphi_1 \end{cases}$$

Bulk picture:

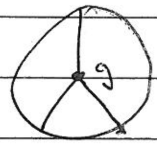


$$(\kappa^2 V(\varphi))_{DW} = -3 + \tfrac{1}{2} m^2 \varphi^2 - \tfrac{1}{3} g \varphi^3 + O(\varphi^4)$$

AdS/CFT $m^2 = \Delta(\Delta - d)$

$(\ell = 1)$ $\quad = -\lambda(3-\lambda) < 0$

$g = (\lambda - 1)\, 6\pi C$



(cubic coupling controls OPE coeff't = 3pt normalisation)

Also $\alpha = \dfrac{12}{\pi^2 \kappa^2}$

$\Rightarrow W(\varphi) = -2 - \tfrac{1}{2}\lambda\varphi^2 + \tfrac{2}{3}\pi C \varphi^3 + O(\varphi^4)$ for both DW/C

Solving cosmology: $H_* = 1 + O(\lambda^3)$

$\left(\epsilon = \dfrac{\dot\varphi^2}{2H^2}\right)$ $\quad \epsilon_* = \dfrac{1}{2}\lambda^2 \varphi_1^2 \left(\dfrac{q}{q_0}\right)^{2\lambda} \left[1 + \left(\dfrac{q}{q_0}\right)^{\lambda}\right]^{-4} + O(\lambda^7)$

SR power spectrum $\Delta_S^2(q) = \dfrac{\kappa^2 H_*^2}{8\pi^2 \epsilon_*} = \dfrac{\Delta_S^2(q_0)}{16} \left(\dfrac{q}{q_0}\right)^{-2\lambda} \left[1 + \left(\dfrac{q}{q_0}\right)^{\lambda}\right]^4 \times (1 + O(\lambda))$

$*$ = at horizon crossing $(q = aH)$

▲ Exact match!

▲ Can extend to 2nd order in SR, + to 3pt non-Gaussianities

$$\boxed{\text{Structure of inflationary correlators} \longleftrightarrow \text{perturbative breaking of conformal symmetry.}}$$

↳ cf. Daniel's talks