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Holography: from QFT to cosmos.

Worked solutions to exercises.

1.1) For the Hamiltonian constraint, note the powers of N:

$$S = \frac{1}{2\kappa^2}\int dz\, d^3x \sqrt{g}\,\Big\{ \frac{1}{N}(E_{ij}E^{ij} - E^2) + \frac{1}{N}(\dot{\Phi} - N^i \Phi_{,i})^2 + \sigma N\big(-R + g^{ij}\Phi_{,i}\Phi_{,j} + 2\kappa^2 V(\Phi)\big)\Big\}$$

where $E_{ij} = \frac{1}{2}\dot{g}_{ij} - \nabla_{(i}N_{j)}$.

For the extrinsic curvature, using $n^\mu = \frac{1}{N}(1, -N^i)$ and $g_{0i} = g_{ij}N^j = N_i$

$$\mathcal{L}_n g_{ij} = n^\mu g_{ij,\mu} + 2g_{\mu(i}n^\mu{}_{,j)}$$

$$= \frac{1}{N}\dot{g}_{ij} - \frac{N^k}{N}g_{ij,k} + \underbrace{2N_{(i}\left(\frac{1}{N}\right)_{,j)} - 2g_{k(i}\left(\frac{N^k}{N}\right)_{,j)}}_{-\frac{2}{N}g_{k(i}N^k{}_{,j)} = -\frac{2}{N}\left(N_{(i,j)} - N^k g_{k(i,j)}\right)}$$

$$= \frac{1}{N}\dot{g}_{ij} - \frac{2}{N}N_{(i,j)} + \underbrace{\frac{N^k}{N}\left(2g_{k(i,j)} - g_{ij,k}\right)}_{\frac{2}{N}N_k\Gamma^k_{ij}}$$

$$= \frac{1}{N}\dot{g}_{ij} - \frac{2}{N}\nabla_{(i}N_{j)}$$

and K_{ij} is then half of this.

1.2) The background action is

$$S = \frac{1}{2\kappa^2}\int dz\, d^3x\, a^3\left(K_{ij}K^{ij} - K^2 + \dot{\varphi}^2 + \sigma\big(-\overset{0}{\cancel{R}} + 2\kappa^2 V\big)\right)$$

and $K_{ij} = \frac{1}{2}\dot{g}_{ij} = a^2 H\delta_{ij}$ where $H = \dot{a}/a$. The 3-curvature $R = 0$ since $g_{ij} = a^2(z)\delta_{ij}$, and for constant-$z$ slices this is just a rescaling of flat space.

This gives $S = \frac{1}{2\kappa^2}\int dz\, d^3x \left(-6\dot{a}^2 a + a^3\dot{\varphi}^2 + 2\sigma a^3\kappa^2 V\right)$

Varying wrt a and φ $\rightsquigarrow$ $\begin{cases} 0 = H^2 + 2\frac{\ddot{a}}{a} + \frac{1}{2}\dot{\varphi}^2 + \sigma\kappa^2 V \\ 0 = \ddot{\varphi} + 3H\dot{\varphi} - \sigma\kappa^2 V' \quad \text{(KG eq}^{\text{n}}\text{)} \end{cases}$

The Hamiltonian constraint gives also $0 = 6H^2 - \dot{\varphi}^2 + 2\sigma\kappa^2 V$.

1st Friedmann eq$^{\text{n}}$

$\rightsquigarrow 0 = \underbrace{\frac{\ddot{a}}{a} - H^2}_{\dot{H}} + \frac{1}{2}\dot{\varphi}^2$ 2nd Friedmann eq$^{\text{n}}$

Assuming monotonicity, we can express $H = H(z)$ as a function of φ, let's say $H = -\frac{1}{2}W(\varphi)$. Then

$$\dot{H} = -\frac{1}{2}W'\dot{\varphi} = -\frac{1}{2}\dot{\varphi}^2 \quad \text{so} \quad \dot{\varphi} = W'$$

$$\text{and} \quad 2\sigma\kappa^2 V = \dot{\varphi}^2 - 6H^2 = W'^2 - \frac{3}{2}W^2.$$

1.3) a) To $O(h^2)$, $g^{ik}g_{kj} = \delta^i_j$.

In general, $\delta^\mu_\nu = (g^{\mu\lambda}_{(0)} + \delta g^{\mu\lambda})(g_{(0)\lambda\nu} + \delta g_{\lambda\nu})$ so $\delta g^{\mu\nu} = -g^{\mu\lambda}_{(0)}g^{\nu\sigma}_{(0)}\delta g_{\lambda\sigma}$

Thus, $\delta g^{\mu\nu}$ is <u>minus</u> what we get by raising $\delta g_{\mu\nu}$ with two background inverse metrics.

b) $K_{ij} = \frac{1}{N}\left(\frac{1}{2}\dot{g}_{ij} - \nabla_{(i}N_{j)}\right)$

N_k already 1st order

$$= (1-\delta N)\left(\frac{1}{2}\left(a^2(\delta_{ij}+h_{ij})\right)^{\cdot} - \partial_{(i}N_{j)} + \overset{(0)}{\Gamma}{}^k_{ij}N_k\right)$$

$$= (1-\delta N)\left[a^2 H\delta_{ij} + a^2 H h_{ij} + \frac{1}{2}a^2\dot{h}_{ij} - a^2(\nu_{,ij} + \nu_{(i,j)})\right]$$

since $\overset{(0)}{\Gamma}{}^k_{ij} = \frac{1}{2}a^{-2}\delta^{kl}\left[(a^2\delta_{li})_{,j} + (a^2\delta_{lj})_{,i} - (a^2\delta_{ij})_{,l}\right] = 0$

$$K_{ij} = a^2\left[H\delta_{ij} - H\delta N\delta_{ij} + Hh_{ij} + \tfrac{1}{2}\dot h_{ij} - \nu_{,ij} - \nu_{(i,j)}\right]$$

Now $K^i{}_j = g^{ik}K_{kj} = a^{-2}(\delta_{ik} - h_{ik})K_{kj}$

$$= H\delta_{ij} - H\delta N\delta_{ij} + \cancel{Hh_{ij}} + \tfrac{1}{2}\dot h_{ij} - \nu_{,ij} - \nu_{(i,j)}$$

$$- \cancel{Hh_{ij}}$$

and, $K = 3H - 3H\delta N + 3\dot\zeta - \partial^2\nu$ $\qquad (\partial^2 \equiv \partial_i\partial_i)$

c) Momentum constraint is $0 = \nabla_j K^j{}_i - K_{,i} - \frac{\cancelto{0}{\Phi_{,i}}}{N}(\dot\Phi - N^j\Phi_{,j})$

$$\leadsto 0 = -H\delta N_{,i} + \tfrac{1}{2}\dot h_{ij,j} - \cancel{\partial^2\nu_{,i}} - \tfrac{1}{2}\partial^2\nu_i + \underbrace{\Gamma^j_{jk}\overset{(0)}{K^k{}_i} - \Gamma^k_{ji}\overset{(0)}{K^j{}_k}}$$

$$+ 3H\delta N_{,i} - 3\dot\zeta_{,i} + \cancel{\partial^2\nu_{,i}}$$

as $\overset{(0)}{\Gamma}{}^i_{jk} = 0$, Γ^i_{jk} is 1st order, so only need $\overset{(0)}{K^i{}_j}$

and $\frac{1}{2}\dot h_{ij,j} = \dot\zeta_{,i}$ *

Also, $\Phi_{,i} = \delta\varphi_{,i} = 0$ in comoving gauge.

$$0 = 2H\delta N_{,i} - 2\dot\zeta_{,i} - \tfrac{1}{2}\partial^2\nu_i + H\Big(\underbrace{\overset{(1)}{\Gamma}{}^j_{ji} - \overset{(1)}{\Gamma}{}^j_{ji}}_{0}\Big)$$

so $\quad 0 = 2\partial_i(H\delta N - \dot\zeta) - \frac{1}{2}\partial^2\nu_i$

Acting with $\partial^{-2}\partial_i$ $\leadsto \delta N = \dot\zeta/H$

Acting with transverse projector $\Pi_{ij} = \delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}$ $\leadsto \nu_i = 0$

d) Hamiltonian constraint is

$$0 = \underbrace{-K^i{}_jK^j{}_i + K^2}_{\text{use results above, substituting } \delta N = \dot\zeta/H \text{ and } \nu_i = 0} - N^{-2}\underbrace{(\dot\Phi - N^i\cancelto{0}{\Phi_{,i}})^2}_{\dot\varphi^2} + \sigma\left(-R + g^{ij}\cancelto{0}{\Phi_{,i}}\Phi_{,j} + 2\kappa^2 V\right)$$

Using results from part (b) plus $\delta N = \dot{\zeta}/H$ and $\nu_i = 0$,

$$\left.\begin{aligned} K &= 3H - \partial^2 \nu \\ K^i_j &= H\delta_{ij} + \tfrac{1}{2}\dot{\gamma}_{ij} - \nu_{,ij} \end{aligned}\right\} \; -K^i_j K^j_i + K^2 = 6H^2 - 4H\partial^2\nu$$

The Hamiltonian constraint is then

$$0 = \underbrace{6H^2 - \dot{\varphi}^2 + 2\sigma\kappa^2 V(\varphi)}_{\text{background Ham. constr.}} \underbrace{- \sigma R + 2\dot{\varphi}^2 \overset{\dot{\zeta}/H}{\delta N} - 4H\partial^2\nu}_{\text{1st order piece.}}$$

Using $R = -\frac{4}{a^2}\partial^2\zeta$ we find $\nu = \epsilon\partial^{-2}\dot{\zeta} + \frac{\sigma}{a^2 H}\zeta$

where $\epsilon = \frac{\dot{\varphi}^2}{2H^2} = -\frac{\dot{H}}{H^2} = (1/H)^{\dot{}}$.

Finally, we can check $R_{ij} = \Gamma^k{}_{ij,k} - \Gamma^k{}_{kj,i} + \cancel{\Gamma\Gamma - \Gamma\Gamma}$ (quadratic order)

and $\Gamma^k{}_{ij} = \frac{1}{2} g^{kl}(g_{il,j} + g_{jl,i} - g_{ij,l})$

$$= \tfrac{1}{2}(a^{-2}\delta_{kl})\, a^2 (h_{il,j} + h_{jl,i} - h_{ij,l})$$

$$= \tfrac{1}{2}(h_{ik,j} + h_{jk,i} - h_{ij,k})$$

$$= \zeta_{,j}\delta_{ik} + \zeta_{,i}\delta_{jk} - \zeta_{,k}\delta_{ij} + \tfrac{1}{2}\gamma_{ik,j} + \tfrac{1}{2}\gamma_{jk,i} - \tfrac{1}{2}\gamma_{ij,k}.$$

$$R_{ij} = 2\zeta_{,ij} - \delta_{ij}\partial^2\zeta - \tfrac{1}{2}\partial^2\gamma_{ij} - \partial_i(3\zeta_{,j} + \cancel{\zeta_{,j}} - \cancel{\zeta_{,j}})$$

$$= -\zeta_{,ij} - \delta_{ij}\partial^2\zeta - \tfrac{1}{2}\partial^2\gamma_{ij} \qquad \text{so } R = a^{-2}\delta^{ij}\overset{(1)}{R}_{ij}$$

$$= -\frac{4}{a^2}\partial^2\zeta$$

2.1) In synchronous gauge, the momentum constraint is

$$0 = \partial_j K^j_i + \overset{(1)}{\Gamma}{}^j_{jk} \overset{(0)}{K}{}^k_i - \overset{(1)}{\Gamma}{}^k_{ji} \overset{(0)}{K}{}^j_k - K_{,i} - \delta\varphi_{,i}\,\dot\varphi$$

as $\overset{(0)}{\Gamma}{}^i_{jk} = 0$, only need background $\overset{(0)}{K}{}^i_j$ here

$$K_{ij} = \tfrac{1}{2}\dot g_{ij} = a^2(H\delta_{ij} + Hh_{ij} + \tfrac{1}{2}\dot h_{ij})$$

so $K^i_j = a^{-2}(\delta_{ik} - h_{ik})K_{kj} = H\delta_{ij} + \tfrac{1}{2}\dot h_{ij}$ and $K = 3H + \tfrac{1}{2}\dot h$

$$\rightarrow 0 = \tfrac{1}{2}\dot h_{ij,j} + H\underbrace{(\overset{(1)}{\Gamma}{}^j_{ji} - \overset{(1)}{\Gamma}{}^j_{ji})}_{0} - \tfrac{1}{2}\dot h_{,i} - \dot\varphi\,\delta\varphi_{,i}$$

Since $\dot h_{ij,j} = -2\dot\psi_{,i} + 2\partial^2\dot\chi_{,i} + \partial^2\dot\omega_i$ and $\dot h = -6\dot\psi + 2\partial^2\dot\chi$

$$0 = \dot h_{ij,j} - \dot h_{,i} - 2\dot\varphi\,\delta\varphi_{,i} = \partial_i(-2\dot\psi + \cancel{2\partial^2\dot\chi} + 6\dot\psi - \cancel{2\partial^2\dot\chi} - \cancel{2}\dot\varphi\,\delta\varphi) + \partial^2\dot\omega_i$$

$$\rightsquigarrow \dot\psi = \tfrac{1}{2}\dot\varphi\,\delta\varphi \quad \text{and} \quad \dot\omega_i = 0$$

(b) The Hamiltonian constraint is

$$0 = -K^i_j K^j_i + K^2 - (\dot\varphi + \delta\dot\varphi)^2 + \sigma(-R + \underbrace{\cancel{g^{ij}\delta\varphi_{,i}\delta\varphi_{,i}}}_{2^{nd}\text{ order}} + 2\kappa^2 V(\varphi) + 2\kappa^2 V'(\varphi)\,\delta\varphi)$$

$$= -(3H^2 + H\dot h) + (9H^2 + 3H\dot h) - (\dot\varphi^2 + 2\dot\varphi\,\delta\dot\varphi) + \sigma(-R + 2\kappa^2 V + 2\kappa^2 V'\delta\varphi)$$

$$= \underbrace{(6H^2 - \dot\varphi^2 + 2\sigma\kappa^2 V)}_{\text{background}} + 2H\dot h - 2\dot\varphi\,\delta\dot\varphi - \sigma R + 2\kappa^2 V'\sigma\,\delta\varphi$$

From earlier, $\overset{(1)}{R}_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{kj,i} = \tfrac{1}{2}(h_{ik,jk} + h_{jk,ik} - \partial^2 h_{ij}) - \tfrac{1}{2}(h_{,ij} + \cancel{h_{jj,ij}} - \cancel{h_{jj,ij}})$

so $\overset{(1)}{R} = \tfrac{1}{\cancel{2}}a^{-2}(\cancel{2}h_{ij,ij} - \cancel{2}\partial^2 h) = a^{-2}(-2\partial^2\psi + \cancel{2\partial^4\chi} - \partial^2(-6\psi + \cancel{2\partial^2\chi}))$
$= \frac{4}{a^2}\partial^2\psi$

$$\Rightarrow \quad \dot{h} = \frac{2\sigma}{a^2 H}\partial^2\psi + \frac{\dot{\varphi}}{H}\delta\dot{\varphi} - \sigma\frac{\kappa^2 V'(\varphi)\delta\varphi}{H}$$

Since $\dot{h} = -6\dot{\psi} + 2\partial^2\dot{\chi} = -3\dot{\varphi}\delta\varphi + 2\partial^2\dot{\chi}$

$$\Rightarrow \quad \partial^2\dot{\chi} = \frac{\sigma}{a^2 H}\partial^2\psi + \frac{\dot{\varphi}}{2H}\delta\dot{\varphi} + \left(\frac{3}{2}\dot{\varphi} - \frac{\sigma\kappa^2 V'(\varphi)}{2H}\right)\delta\varphi$$

2.2) Holographic stress tensor $(\sigma = +1)$

$$\kappa^2\delta\langle T^i{}_j\rangle = \left(\delta K\delta_{ij} - \delta K^i{}_j\right)_{(3)} = \frac{1}{2}\left(\dot{h}\delta_{ij} - \dot{h}_{ij}\right)_{(3)}$$

$$= \frac{1}{2}\Bigg(\left(\frac{2}{a^2 H}\partial^2\psi + \frac{\dot{\varphi}}{H}\delta\dot{\varphi}\right)\delta_{ij} + (\dots)\delta\varphi$$

$$- \Big(\underbrace{-2\dot{\psi}\delta_{ij}}_{-\dot{\varphi}\delta\varphi} + \underbrace{2\dot{\chi}_{,ij}}_{} + \underbrace{2\dot{\omega}_{(i,j)}}_{0} + \dot{\gamma}_{ij}\Big)\Bigg)_{(3)}$$

$-\dot{\varphi}\delta\varphi$ from momentum constraint

$\dot{\psi} \sim (\dots)\delta\varphi$

0 as momentum constraint $\Rightarrow \dot{\omega}_i = 0$

from Hamiltonian constraint

$$2\dot{\chi}_{,ij} = 2\partial^{-2}\partial_i\partial_j\left\{\frac{1}{a^2 H}\psi + \frac{\dot{\varphi}}{2H}\delta\dot{\varphi} + (\dots)\delta\varphi\right.$$

$$= \frac{1}{2}\left(\left(\frac{2}{a^2 H}\partial^2\psi + \frac{\dot{\varphi}}{H}\delta\dot{\varphi}\right)\underbrace{\pi_{ij}} + (\dots)\delta\varphi - \dot{\gamma}_{ij}\right)_{(3)}$$

transverse projector $\pi_{ij} = \delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}$

(or in momentum space, $\pi_{ij} = \delta_{ij} - \frac{\bar{q}_i\bar{q}_j}{\bar{q}^2}$)

$$\dot{\zeta} = \left(-\psi - \frac{H}{\dot{\varphi}}\delta\varphi\right)^{\cdot}$$

$$= (\dots)\delta\varphi - \frac{H}{\dot{\varphi}}\delta\dot{\varphi}$$

but also, $\dot{\zeta} = \frac{1}{2a^3\varepsilon}\Pi^{(\zeta)} = \frac{\bar{\Omega}\zeta}{2a^3\varepsilon} = -\frac{\bar{\Omega}}{2a^3\varepsilon}\psi + (\dots)\delta\varphi$

hence $\delta\dot{\varphi} = \frac{\dot{\varphi}}{2a^3\varepsilon H}\bar{\Omega}\psi + (\dots)\delta\varphi = \frac{H}{a^3\dot{\varphi}}\bar{\Omega}\psi + (\dots)\delta\varphi$

recalling that $\epsilon = \frac{\dot{\varphi}^2}{2H^2}$. Noting also that

$$\Pi^{(\gamma)}_{ij} = \frac{1}{4} a^3 \dot{\gamma}_{ij} = \bar{E}\,\gamma_{ij} \quad \Rightarrow \quad \dot{\gamma}_{ij} = \frac{4\bar{E}}{a^3}\gamma_{ij}$$

we have

$$\bar{\kappa}^2 \delta\langle T^i_{\ j}\rangle = \left[\left(\frac{1}{a^2 H}\partial^2\psi + \frac{\bar{\Omega}}{2a^3}\psi\right)\pi_{ij} - \frac{2\bar{E}}{a^3}\gamma_{ij} + (\dots)\delta\varphi\right]_{(3)}$$

or in momentum space,

$$\delta\langle T^i_{\ j}(\bar{q})\rangle = \bar{\kappa}^{-2}\left[\left(\underbrace{\frac{-\bar{q}^2}{a^2 H}}_{\substack{\text{weight 2 at leading order;}\\ \text{removed with counterterm}}} + \frac{\bar{\Omega}}{2a^3}\right)\psi\,\pi_{ij} - \frac{2\bar{E}}{a^3}\gamma_{ij} + (\dots)\delta\varphi\right]_{(3)}$$

Since the scale factor is weight minus one (ie. $a \to e^z$ as $z \to \infty$)

$$\delta\langle T^i_{\ j}(\bar{q})\rangle = \bar{\kappa}^{-2}\left[\underbrace{\frac{1}{2}\bar{\Omega}_{(0)}\psi_{(0)}\pi_{ij}}_{-2B(\bar{q})} \underbrace{- 2\bar{E}_{(0)}\gamma_{(0)ij}}_{\frac{1}{2}A(\bar{q})} + (\dots)\delta\varphi\right]$$

so $\langle\langle T(\bar{q})T(-\bar{q})\rangle\rangle = 4B(\bar{q}) = -\bar{\kappa}^{-2}\bar{\Omega}_{(0)}(\bar{q})$

$$\langle\langle T^{(s)}(\bar{q})\, T^{(s')}(-\bar{q})\rangle\rangle = \frac{1}{2}\delta^{ss'}A(\bar{q}) = -2\bar{\kappa}^{-2}\bar{E}_{(0)}(\bar{q})\,\delta^{ss'}$$

Key point: the stress tensor 2-point function in the dual QFT can be read off from the weight zero pieces of the response functions. These are just the values as $z \to \infty$ after removing ~~counter~~ any negative weight pieces with counterterms.

3) a) $\beta = -\frac{d\varphi}{d\ln\Lambda} = -\lambda\varphi + 2\pi C\varphi^2 = 2\pi C\,\varphi(\varphi-\varphi_1)$ $\varphi_1 = \frac{\lambda}{2\pi C}$

$$\ln\Lambda/\Lambda_0 = -\frac{1}{2\pi C}\int_{\varphi_0}^{\varphi} d\varphi\,\frac{1}{\varphi(\varphi-\varphi_1)} = -\frac{1}{\lambda}\int_{\varphi_0}^{\varphi} d\varphi\left(\frac{1}{\varphi-\varphi_1} - \frac{1}{\varphi}\right)$$

$$= -\frac{1}{\lambda}\ln\frac{(\varphi-\varphi_1)\varphi_0}{(\varphi_0-\varphi_1)\varphi} = -\frac{1}{\lambda}\ln\left(\frac{1-\varphi_1/\varphi}{1-\varphi_1/\varphi_0}\right)$$

$$\rightsquigarrow \varphi = \frac{\varphi_1}{1+\left(\frac{\varphi_1}{\varphi_0}-1\right)\left(\frac{\Lambda}{\Lambda_0}\right)^{-\lambda}} \longrightarrow \underbrace{\frac{\varphi_1}{\left(\frac{\varphi_1}{\varphi_0}-1\right)\Lambda_0^{\lambda}}}_{=\phi}\Lambda^{\lambda} \quad \text{as } \Lambda\to 0$$

hence $$\varphi = \frac{\varphi_1}{1+\frac{\varphi_1}{\phi}\Lambda^{-\lambda}}$$

b) Under a variation $\delta\Lambda$ of the cut-off lengthscale, assuming the 1-point function $\langle O\rangle_0$ in the UV CFT vanishes, expanding $Z = \langle e^{-\int\varphi\Lambda^{-\lambda}O}\rangle_0$ we find

$$\delta_\Lambda Z = \delta_\Lambda\Big(\tfrac{1}{2}\varphi^2\Lambda^{-2\lambda}\int d^3x_1\int d^3x_2\,\langle O(x_1)O(x_2)\rangle_0 - \tfrac{1}{6}\varphi^3\Lambda^{-3\lambda}\int d^3x_1\int d^3x_2\int d^3x_3\,\langle O(x_1)O(x_2)O(x_3)\rangle_0 + \dots\Big)$$

$$= \left(-\varphi\beta\frac{\delta\Lambda}{\Lambda}\Lambda^{-2\lambda} - 2\lambda\cdot\tfrac{1}{2}\varphi^2\Lambda^{-2\lambda}\frac{\delta\Lambda}{\Lambda} + \tfrac{1}{6}\varphi^3\Lambda^{-3\lambda}\frac{3C}{\Lambda^{3-\lambda}}4\pi\Lambda^2\delta\Lambda\right)$$
$$\times\int d^3x_1\int d^3x_2\,\langle O(x_1)O(x_2)\rangle_0 + \dots$$

variation of $\varphi(\Lambda)$ in 2-pt contribution
$\delta\varphi = -\beta\frac{\delta\Lambda}{\Lambda}$

variation of explicit $\Lambda^{-2\lambda}$ in 2-pt contribution

collision contribution from 3-point function: use $OO \sim \frac{C}{\Lambda^{3-\lambda}}O$ part of OPE.

The factor of 3 is because there are three distinct collisions ($1 \leftrightarrow 2$, $2 \leftrightarrow 3$, $3 \leftrightarrow 1$). The + sign is because we lose this contribution by increasing Λ (and we had $-\frac{1}{6}\varphi^3 \Lambda^{-3\lambda} \iiint \langle OOO \rangle_0$ to begin with).

The $4\pi \Lambda^2 \delta\Lambda$ is the shell volume.

$$\delta_\Lambda Z = \varphi \frac{\delta\Lambda}{\Lambda} \Lambda^{-2\lambda} \underbrace{\left(-\beta - \lambda\varphi + 2\pi C \varphi^2\right)} \int d^3x_1 \int d^3x_2 \langle O(x_1) O(x_2) \rangle_0 + \dots$$

for $\delta_\Lambda Z = 0$, $\beta = -\lambda\varphi + 2\pi C \varphi^2$.

See 1211.4550 and 1308.0331 for further discussion of this system.