Holography: from QFT to cosmos

Exercises

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These exercises are intended to provide practice with cosmological perturbation theory, and further background to the lectures. All exercises can be completed by hand in approx. 10 pages.

1 Lecture 1

1.1 ADM formalism

In the ADM formalism, the metric for domain-walls ($\sigma = 1$) and cosmologies ($\sigma = -1$) is

$$\mathrm{d}s^2 = \sigma N^2 \mathrm{d}z^2 + g_{ij}(\mathrm{d}x^i + N^i \mathrm{d}z)(\mathrm{d}x^j + N^j \mathrm{d}z), \tag{1}$$

and the action is

$$S = \frac{1}{2\kappa^2}\int \mathrm{d}z\, \mathrm{d}^3x\, N\sqrt{g}\, \big[K_{ij}K^{ij} - K^2 + N^{-2}(\dot{\Phi} - N^i\Phi_{,i})^2 + \sigma(-R + g^{ij}\Phi_{,i}\Phi_{,j} + 2\kappa^2 V(\Phi))\big], \tag{2}$$

where $\kappa^2 = 8\pi G_N$ and the extrinsic curvature

$$K_{ij} = \frac{1}{2}\pounds_n g_{ij} = \frac{1}{N}\Big(\frac{1}{2}\dot{g}_{ij} - \nabla_{(i}N_{j)}\Big). \tag{3}$$

We use dots as a shorthand for ∂_z, and commas for partial derivatives. All covariant derivatives, raised indices, and the scalar curvature R are evaluated using the spatial 3-metric g_{ij}.

Vary the ADM action with respect to the lapse N and shift N^i to obtain the Hamiltonian and momentum constraints:

$$0 = -K_{ij}K^{ij} + K^2 - N^{-2}(\dot{\Phi} - N^i\Phi_{,i})^2 + \sigma(-R + g^{ij}\Phi_{,i}\Phi_{,j} + 2\kappa^2 V(\Phi)), \tag{4}$$

$$0 = \nabla_j K^j_i - K_{,i} - \frac{\Phi_{,i}}{N}(\dot{\Phi} - N^j\Phi_{,j}). \tag{5}$$

[Optional: Verify (3) by evaluating the Lie derivative $\pounds_n g_{ij} = n^\mu g_{ij,\mu} + 2g_{\mu(i}n^\mu_{,j)}$, where $n^\mu = N^{-1}(1, -N^i)$ is the unit normal to constant-z slices.]

1.2 Background equations of motion

Derive the background equations of motion by inserting the ansatz

$$\Phi = \varphi(z), \qquad N = 1, \qquad N_i = 0, \qquad g_{ij} = a^2(z)\delta_{ij}, \tag{6}$$

into the ADM action, as well as evaluating the Hamiltonian constraint. For monotonic evolution, we can invert $\varphi = \varphi(z)$ to $z = z(\varphi)$. Show we then have:

$$H = \frac{\dot{a}}{a} = -\frac{1}{2}W(\varphi), \tag{7}$$

$$\dot{\varphi} = W'(\varphi), \tag{8}$$

$$2\sigma\kappa^2 V(\varphi) = W'^2(\varphi) - \frac{3}{2}W^2(\varphi). \tag{9}$$

1.3 Solving the constraints in comoving gauge

In comoving gauge, we have

$$\delta\varphi = 0, \qquad g_{ij} = a^2(\delta_{ij} + h_{ij}), \qquad h_{ij} = 2\zeta\delta_{ij} + \gamma_{ij}, \qquad N = 1 + \delta N, \qquad N_i = a^2(\nu_{,i} + \nu_i)$$

where the perturbations depend on both z and x_i. The graviton is transverse-traceless while ν_i is transverse (*i.e.*, $\gamma_{ij,j} = 0$, $\gamma_{ii} = 0$, and $\nu_{i,i} = 0$). We will work to linear order throughout.

(a) Show the inverse 3-metric is[1]

$$g^{ij} = a^{-2}(\delta_{ij} - h_{ij}). \tag{10}$$

(b) Show that to linear order

$$K^i_j = H\delta_{ij} - H\delta N\delta_{ij} + \frac{1}{2}\dot{h}_{ij} - \nu_{,ij} - \nu_{(i,j)} \tag{11}$$

(c) Evaluating the momentum constraint, show that

$$\nu_i = 0, \qquad \delta N = \frac{\dot{\zeta}}{H}. \tag{12}$$

(d) Evaluate the Hamiltonian constraint to show that

$$\nu = \epsilon\,\partial^{-2}\dot{\zeta} + \frac{\sigma}{a^2 H}\zeta, \tag{13}$$

where $\epsilon = \dot{\varphi}^2/2H^2 = (1/H)\dot{}$. Note that in momentum space, ∂^{-2} is simply $-q^{-2}$.

Here, the curvature of the 3-metric g_{ij} is $R = -4a^{-2}\partial^2\zeta$, as one can show noting that

$$R_{ij} = \Gamma^k_{ij,k} + \Gamma^k_{kj,i} + O(h^2). \tag{14}$$

We now know the lapse and shift to linear order in comoving gauge:

$$N = 1 + \frac{\dot{\zeta}}{H}, \qquad N_i = a^2\epsilon\,\partial^{-2}\dot{\zeta}_{,i} + \frac{\sigma}{H}\zeta_{,i} \tag{15}$$

Substituting these into the ADM action, one can go on to evaluate the quadratic action for the perturbations, but we'll leave this for another time. Note, however, that to obtain the action to quadratic order, we only need evaluate the lapse and shift to *linear* order: in the action, the second-order pieces of N and N_i appear multiplying the zeroth order Hamiltonian and momentum constraints. As we saw above, the latter vanish for $a(z)$ and $\varphi(z)$ obeying the background equations of motion. This same approach can be extended to obtain the perturbed action to cubic order: see Maldacena's famous `astro-ph/0210603`. The analogous calculation for combined domain-walls/cosmologies, as well as the higher-order version of the exercises here, can be found in `1011.0452` and `1104.3894`.

[1] For compactness, we write all indices that should be raised with δ^{ij} as lowered, thus (10) is shorthand for $g^{ij} = a^{-2}\delta^{ik}\delta^{jl}(\delta_{kl} - h_{kl})$. Any repeated lower indices are then summed over. With this convention, the only raised indices appearing are those raised with the full inverse 3-metric g^{ij}.

2 Lecture 2

2.1 Solving the constraints in synchronous gauge

In synchronous gauge,

$$\Phi = \varphi + \delta\varphi, \quad N = 1, \quad N_i = 0, \quad g_{ij} = a^2(\delta_{ij} + h_{ij}), \quad h_{ij} = -2\psi\delta_{ij} + 2\chi_{,ij} + 2\omega_{(i,j)} + \gamma_{ij} \tag{16}$$

where the perturbations depend on both z and x_i. The vector ω_i is transverse and γ_{ij} is transverse-traceless (*i.e.*, $\omega_{i,i} = 0$, $\gamma_{ij,j} = 0$, $\gamma_{ii} = 0$).

(a) Evaluating the momentum constraint, show that

$$0 = \dot{h}_{ij,j} - \dot{h}_{,i} - 2\dot{\varphi}\delta\varphi_{,i} \tag{17}$$

and hence

$$\dot{\psi} = \frac{1}{2}\dot{\varphi}\delta\varphi, \qquad \dot{\omega}_i = 0. \tag{18}$$

(b) Using $R = 4a^{-2}\partial^2\psi$, show the Hamiltonian constraint is

$$\dot{h} = \frac{2\sigma}{a^2 H}\partial^2\psi + \frac{\dot{\varphi}}{H}\delta\dot{\varphi} - \frac{\sigma\kappa^2 V'(\varphi)}{H}\delta\varphi, \tag{19}$$

and then using the solution of the momentum constraint above, that

$$\partial^2\dot{\chi} = \frac{\sigma}{a^2 H}\partial^2\psi + \frac{\dot{\varphi}}{2H}\delta\dot{\varphi} + \Big(\frac{3}{2}\dot{\varphi} - \frac{\sigma\kappa^2 V'(\varphi)}{2H}\Big)\delta\varphi. \tag{20}$$

2.2 The holographic stress tensor

Let's now focus on a domain wall for which $\sigma = +1$, staying in synchronous gauge.[2] We want to evaluate the 1-point function for the stress tensor of the dual QFT, in the presence of a nontrivial background metric which acts as a source. Hamiltonian holographic renormalisation gives

$$\langle T^i_j\rangle_g = \Big(\frac{-2}{\sqrt{g}}\Pi^i_j\Big)_{(3)} = \bar{\kappa}^{-2}\Big(K\delta^i_j - K^i_j\Big)_{(3)}, \tag{21}$$

where $\bar{\kappa}^2 = 8\pi G_N$ in AdS units and the subscript (3) indicates selecting the piece with dilatation weight three. Our task is now to expand this expression to linear order in the perturbations, enabling us to read off the stress tensor 2-point function in the dual QFT as in the lectures.

(a) Using your results from the previous question, show that

$$\bar{\kappa}^2\delta\langle T^i_j\rangle = \frac{1}{2}(\dot{h}\delta_{ij} - \dot{h}_{ij})_{(3)} = \Big[\Big(\frac{1}{a^2 H}\partial^2\psi + \frac{\dot{\varphi}}{2H}\delta\dot{\varphi}\Big)\pi_{ij} - \frac{1}{2}\dot{\gamma}_{ij} + (\ldots)\delta\varphi\Big]_{(3)} \tag{22}$$

The omitted terms proportional to $\delta\varphi$ correspond to the mixed correlator $\langle T_{ij}\mathcal{O}\rangle$ and aren't of interest here. The transverse projector $\pi_{ij} = \delta_{ij} - \partial_i\partial_j\partial^{-2}$, which in momentum space (using now $\bar{q}_i$ for the domain-wall momenta) is $\pi_{ij} = \delta_{ij} - \bar{q}_i\bar{q}_j/\bar{q}^2$.

(b) Recall that the comoving curvature perturbation is

$$\zeta = -\psi - \frac{H}{\dot{\varphi}}\delta\varphi \tag{23}$$

[2] Also known as Fefferman-Graham gauge in the holographic literature.

and the domain-wall response function $\bar{\Omega}$ is defined as

$$\Pi^{(\zeta)} = 2a^3\epsilon\dot{\zeta} = \bar{\Omega}\zeta. \tag{24}$$

By evaluating $\dot{\zeta}$ directly, and comparing with its expression in terms of the response function, show that

$$\delta\dot{\varphi} = \frac{H}{a^3\dot{\varphi}}\bar{\Omega}\psi + (\ldots)\delta\varphi \tag{25}$$

(c) Show that

$$\bar{\kappa}^2\delta\langle T^i_j\rangle = \Big[\Big(\frac{1}{a^2H}\partial^2\psi + \frac{\bar{\Omega}}{2a^3}\psi\Big)\pi_{ij} - \frac{2E}{a^3}\gamma_{ij} + (\ldots)\delta\varphi\Big]_{(3)} \tag{26}$$

and hence, after removing the term proportional to $(a^2H)^{-1}\partial^2\psi$ with a counterterm,

$$\bar{\kappa}^2\delta\langle T^i_j\rangle = \frac{1}{2}\bar{\Omega}_{(0)}\psi_{(0)}\pi_{ij} - 2\bar{E}_{(0)}\gamma_{(0)ij} + (\ldots)\delta\varphi. \tag{27}$$

Using this result, one can read off the transverse-traceless and trace pieces of the stress tensor 2-point function:

$$A(\bar{q}) = -4\bar{\kappa}^{-2}\bar{E}_{(0)}(\bar{q}), \qquad B(\bar{q}) = -\frac{1}{4}\bar{\kappa}^{-2}\bar{\Omega}_{(0)}(\bar{q}) \tag{28}$$

where $\bar{\kappa}^{-2} \propto \bar{N}^2$, the square of the number of colours in the dual QFT, and

$$\langle\langle T_{ij}(\bar{q})T_{kl}(-\bar{q})\rangle\rangle = A(\bar{q})\Pi_{ijkl} + B(\bar{q})\pi_{ij}\pi_{kl}. \tag{29}$$

Here, $\Pi_{ijkl} = \pi_{i(k}\pi_{l)j} - \frac{1}{2}\pi_{ij}\pi_{kl}$ is the transverse-traceless projector.

3 Lecture 3

(a) Integrate the beta function

$$\beta(\varphi) = -\frac{\mathrm{d}\varphi}{\mathrm{d}\ln\Lambda} = -\lambda\varphi + 2\pi C\varphi^2, \tag{30}$$

to find φ as a function of the UV cutoff length scale Λ, fixing the constant of integration such that $\varphi \to \phi\Lambda^\lambda$ as $\Lambda \to 0$.

(b) Show this beta function follows from invariance of the partition function of the deformed CFT,

$$Z = \langle e^{-\int\varphi\Lambda^{-\lambda}\mathcal{O}}\rangle_0, \tag{31}$$

under changes of Λ. For this, note there are three types of contribution: those from the explicit factors of $\Lambda^{-\lambda}$, those from the variation of the coupling $\varphi = \varphi(\Lambda)$, and those from operator collisions which should be evaluated using the OPE,

$$\mathcal{O}(x_1)\mathcal{O}(x_2) = \frac{\alpha}{|x_{12}|^{2\Delta}} + \frac{C}{|x_{12}|^{\Delta}}\mathcal{O}(x_1) + \ldots, \tag{32}$$

where the UV dimension of $\mathcal{O}$ is $\Delta = 3 - \lambda$.