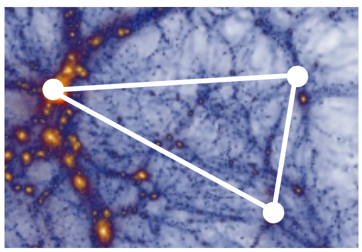
Lectures on

The Cosmological Bootstrap

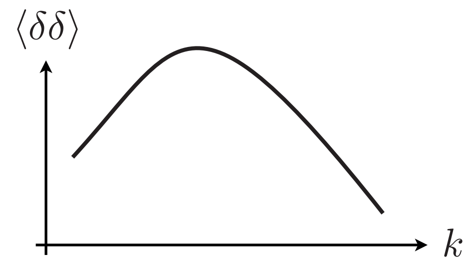
(with Arkani-Hamed, Lee and Pimentel)

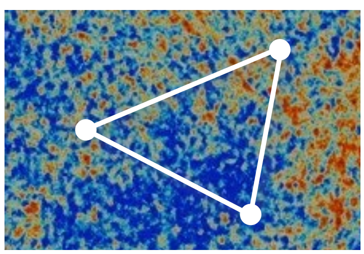
Motivation

The physics of the early universe is encoded in the spatial correlations between cosmological structures at late times:

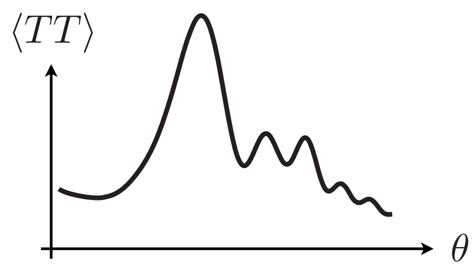


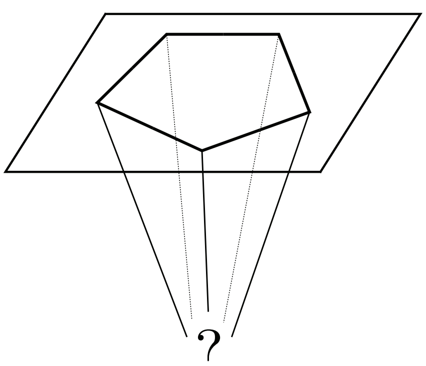
$$\langle \delta\rho(x_1) \cdots \delta\rho(x_N) \rangle$$





$$\langle \delta T(\theta_1) \cdots \delta T(\theta_N) \rangle$$





$$\langle \zeta(x_1) \cdots \zeta(x_N) \rangle$$

What are the rules for consistent correlators?
Which correlations can arise from a consistent history?

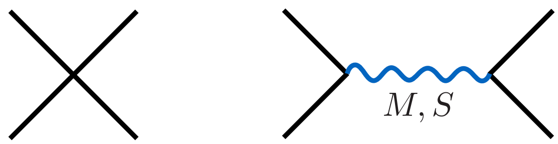
Cosmological Bootstrap

$\Rightarrow$ correlators are fixed by consistency requirements alone.

S-matrix Bootstrap

$\Rightarrow$ scattering amplitudes are fixed by Lorentz invariance, locality and unitarity:

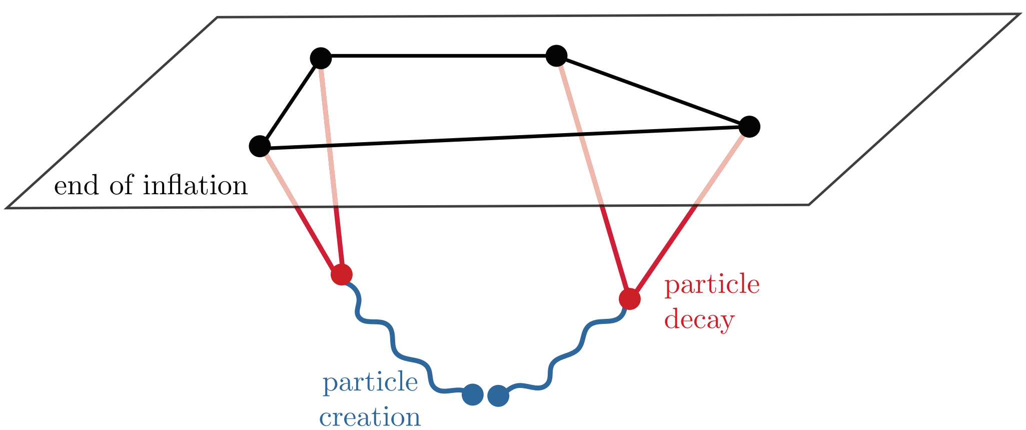
$$A(s,t) \;=\; \sum a_{nm} s^n t^m \;+\; \frac{g^2}{s-M^2} P_S\left(1+\frac{2t}{M^2}\right)$$



- No Lagrangian and Feynman diagrams are needed to derive this.
- Basic principles allow only a small menu of possibilities.

Cosmological Collider Physics

$\Rightarrow$ massive particles (up to 10^{14} GeV) can be created by the rapid expansion of the inflationary spacetime:



The cosmological bootstrap is a systematic way to study this physics.

Outline for the rest of the lectures:

I. Review of Cosmological Correlations

II. Bootstrapping Inflationary Correlators

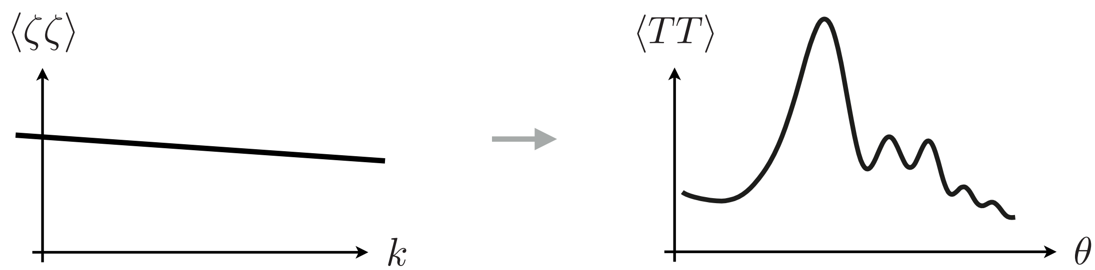
III. Summary and Future Directions

1 Review of Cosmological Correlations

1.1 Observed Correlations

Two important facts:

1. The CMB is correlated over superhorizon scales:

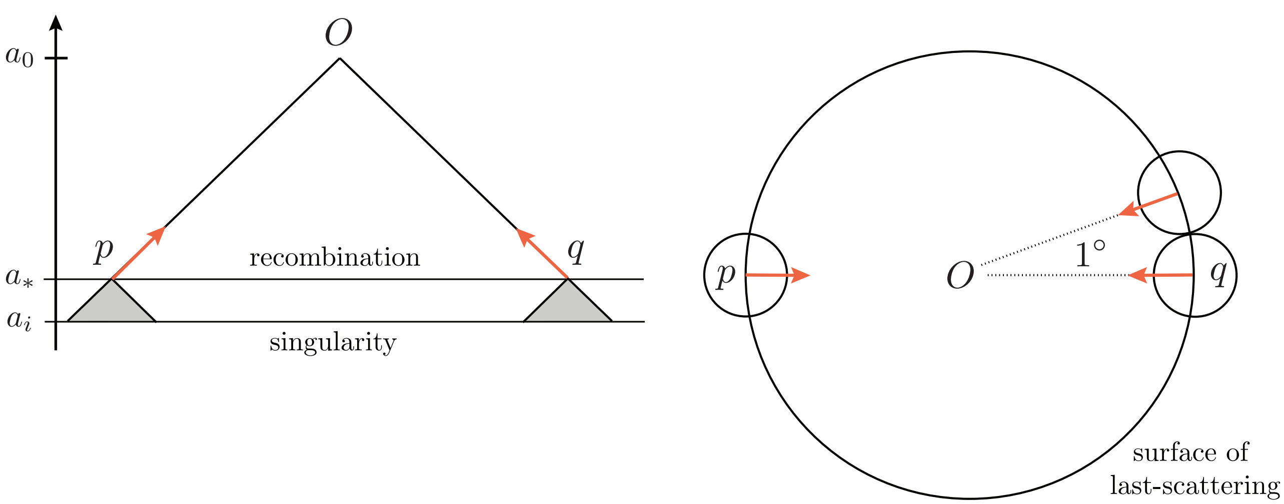


2. The initial conditions are approximately scale-invariant:

$$P(k) \equiv \frac{k^3}{2\pi^2} \langle \zeta_{\mathbf{k}} \zeta_{-\mathbf{k}} \rangle' \approx const.$$

1.2 Horizon Problem

In the standard Big Bang, we can't explain the observed correlations:



The CMB (naively) consists of 10^4 disconnected regions.

- Why is it so uniform?
- Why is it correlated?

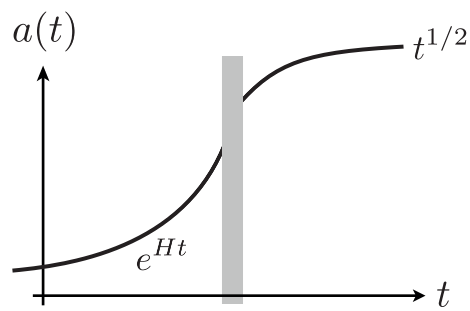
$=$ **horizon problem**

1.3 Inflation

The horizon problem is solved if the early universe went through an extended period of **quasi-de Sitter expansion** (= inflation):

$$H(t) \equiv \frac{1}{a}\frac{da}{dt} \approx const.$$

$$\varepsilon(t) \equiv -\frac{\dot{H}}{H^2} \ll 1\,.$$



The comoving horizon then becomes

$$\eta = \int \frac{\mathrm{d}t}{a(t)} \approx -\frac{1}{aH}\,,$$

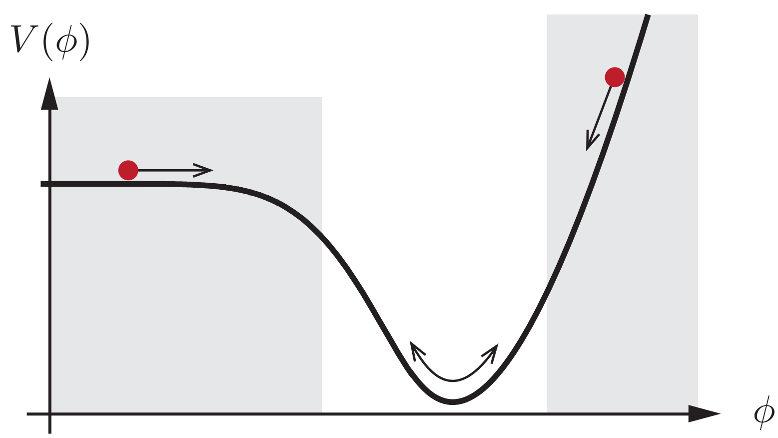
which receives large contributions from early times ($t \to 0$ or $\eta \to -\infty$). This solves the horizon problem.

1.4 Slow-Roll Inflation

Consider

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(M_{\mathrm{pl}}^2 R - (\nabla\phi)^2 - 2V(\phi)\right).$$

This supports inflation in regions where the potential is flat



1.5 Quantum Fluctuations

Inflaton fluctuations satisfy

$$\ddot{\delta\phi} - \frac{2}{\eta}\dot{\delta\phi} + k^2\delta\phi \approx 0\,.$$

Write

$$\hat{\phi}_{\mathbf{k}}(\eta) = f_k(\eta)\,\hat{a}_{\mathbf{k}} + f_k^*(\eta)\,\hat{a}^\dagger_{-\mathbf{k}}\,, \qquad \text{where} \qquad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}')\,,$$

$$f_k(\eta) = \frac{H}{\sqrt{2k^3}}\,(1+ik\eta)\,e^{-ik\eta}\,.$$

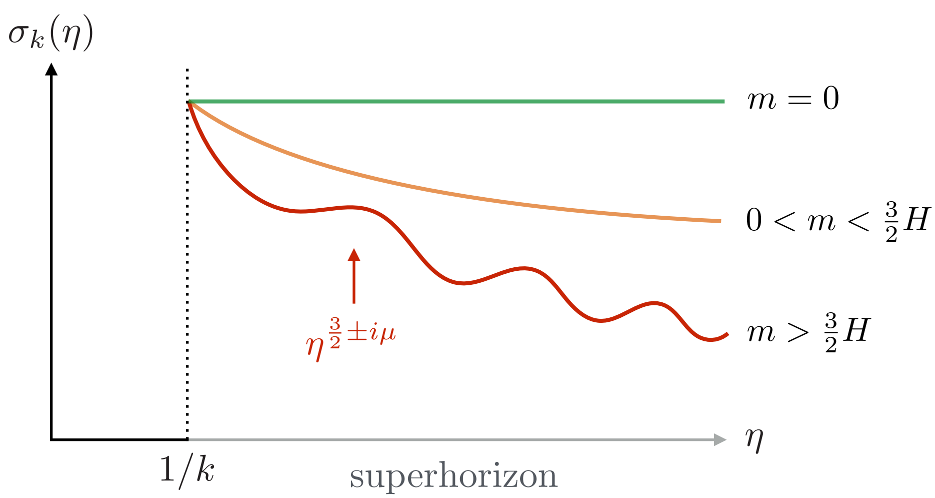
We then find $\quad \langle 0|\,\hat{\phi}_{\mathbf{k}}\hat{\phi}_{-\mathbf{k}}\,|0\rangle = |f_k(\eta)|^2$

$$P_\phi(k,\eta) \equiv \frac{k^3}{2\pi^2}|f_k(\eta)|^2 \xrightarrow{k\eta\to 0} \left(\frac{H}{2\pi}\right)^2 .$$

Using $\zeta = (H/\dot{\phi})\delta\phi$, we get

$$P_\zeta(k,\eta) = \left(\frac{H}{\dot{\phi}}\right)^2 P_\phi(k,\eta) \;\approx\; \frac{1}{4\pi^2}\left(\frac{H^2}{\dot{\phi}}\right)^2\Bigg|_{-k\eta=1} \equiv A_s k^{n_s-1}\,.$$

Massive fields are also produced during inflation, but don't survive:



The imprints of new massive particles can be found in higher-order correlations of the inflaton.

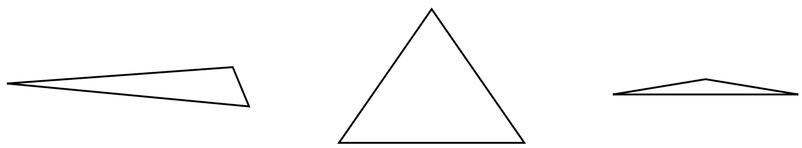
1.6 Non-Gaussianity

The main diagnostic of primordial non-Gaussianity is the bispectrum:

$$\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle = \frac{(2\pi^2)^2}{(k_1k_2k_3)^2}\, B(k_1,k_2,k_3)\,\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,.$$

amplitude: $\quad f_{\rm NL} \equiv \dfrac{5}{18}\dfrac{B(k,k,k)}{P^2(k)}$

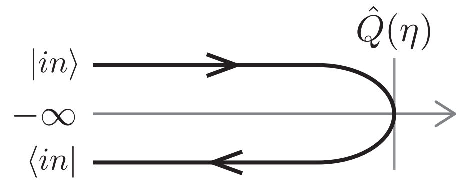
shape:



	local	equilateral	flattened
Planck constraints:	$f_{\rm NL}^{\rm loc} < 5$	$f_{\rm NL}^{\rm equil} < 40$	$f_{\rm NL}^{\rm flat} < 20$

Inflationary correlators are computed in the in-in formalism

$$\langle \hat Q(\eta)\rangle \equiv \langle in|\,\hat Q(\eta)\,|in\rangle =$$



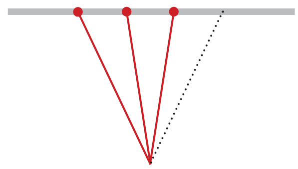
$$= \langle 0|\left[\bar T e^{i\int_{-\infty}^{\eta}{\rm d}\eta'\,H_{\rm int}^I(\eta')}\right]\hat Q^I(\eta)\left[T e^{i\int_{-\infty}^{\eta}{\rm d}\eta''\,H_{\rm int}^I(\eta'')}\right]|0\rangle$$

$$= -i\int_{-\infty}^{\eta}{\rm d}\eta'\langle 0|\left[\hat Q^I(\eta), H_{\rm int}^I(\eta')\right]|0\rangle \;+\;\cdots$$

where $\hat Q \equiv \hat\zeta_{\mathbf{k}_1}\hat\zeta_{\mathbf{k}_2}\cdots\hat\zeta_{\mathbf{k}_n}$.

Contact Interactions

$$\frac{(\partial_\mu \phi)^4}{8\Lambda^4} \quad \Rightarrow$$



$$\Leftrightarrow \quad \mathcal{L}_{\rm int} = -\frac{\dot{\phi}}{4\Lambda^4}\,\delta\dot{\phi}(\partial_\mu \delta\phi)^2 + \cdots \,,$$

The corresponding bispectrum is

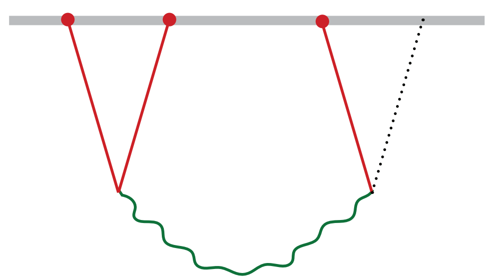
$$\frac{B(k_1,k_2,k_3)}{P^2} = \frac{8}{k_1k_2k_3}\frac{\dot{\phi}^2}{\Lambda^4}\frac{\text{Poly}[k^5]}{K^2}\,,$$

where $K \equiv k_1 + k_2 + k_3$.

- The signal peaks in the equilateral configuration, $k_1 = k_2 = k_3$.
- The squeezed limit, $\lim_{k_3 \to 0}\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle$, is an analytic function of k_3/k_1.

Graviton Exchange

The non-Gaussianity in slow-roll inflation comes from graviton exchange:



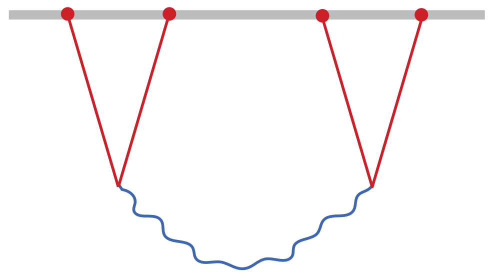
The corresponding bispectrum is

$$\frac{B(k_1,k_2,k_3)}{P^2} = \frac{\varepsilon}{k_1k_2k_3}\left[\sum_{n\neq m} k_n k_m^2 + \frac{8}{K}\sum_{n>m} k_n^2 k_m^2\right] + \frac{n_s - 1}{k_1k_2k_3}\sum_n k_n^3\,.$$

- The signal is still analytic in the squeezed limit.

Massive Particles

Non-analyticity in the squeezed limit arises from massive particles:

$F \;=$ 

$$\sim\; -g^2 \int \frac{\mathrm{d}\eta}{\eta^2}\frac{\mathrm{d}\eta'}{\eta'^2}\, \underset{\substack{\uparrow \\ \text{written for conformally} \\ \text{coupled scalars}}}{e^{i(k_1+k_2)\eta} e^{i(k_3+k_4)\eta'}}\; \underset{\substack{\uparrow \\ \text{complicated function} \\ \text{of Hankel functions}}}{G(|\mathbf{k}_1+\mathbf{k}_2|,\eta,\eta')}\,.$$

Instead of trying to compute the integral, we note that G satisfies

$$\left(\eta^2\partial_\eta^2 - 2\eta\partial_\eta + k_I^2\eta^2 + m^2\right) G(k_I,\eta,\eta') = -i\eta^2\eta'^2\,\delta(\eta-\eta')\,,$$

which implies

$$\frac{1}{k_I}\left(k_I^2\partial_{k_I}^2 - 2k_I\partial_{k_I} - k_I^2\partial_{k_1+k_2}^2 + m^2\right)\left(k_I^2 F\right) = g^2\frac{k_I}{E}\,.$$

Using

$$u^{-1} \equiv \frac{k_1+k_2}{k_I} \qquad\qquad \hat{F}(u,v) \equiv k_I F\,,$$
$$v^{-1} \equiv \frac{k_3+k_4}{k_I}$$

we can write this as

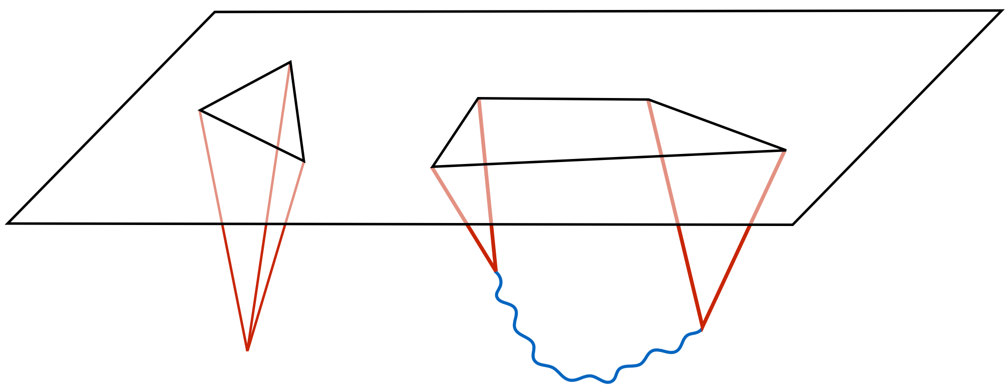
$$\boxed{\left(u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + m^2 - 2\right)\hat{F} = g^2\frac{uv}{u+v}}\,.$$

As we will see, this equation can also be derived from the symmetries of the boundary theory.

2 Bootstrapping Inflationary Correlators

2.1 Time Without Time

All cosmological correlations can be traced back to the spacelike boundary of the inflationary quasi-de Sitter spacetime:



Time dependence in the bulk = momentum dependence on the boundary.

Is there a purely boundary way to derive these correlators?

2.2 De Sitter Space

The metric of de Sitter space is

$$ds^2 = \frac{-d\eta^2 + d\mathbf{x}^2}{(H\eta)^2}\,.$$

In the limit $\eta \to 0$, these isometries of the metric act as conformal transformations on $\mathbb{R}^3$.

Consider a massive scalar field in de Sitter space:

$$\ddot{\phi} - \frac{2}{\eta}\dot{\phi} - \nabla^2\phi + \frac{m^2}{H^2}\frac{\phi}{\eta^2} = 0\,.$$

At late times, the solution is

$$\phi(\eta, \mathbf{x}) \approx \eta^{\Delta_+} O_+(\mathbf{x}) + \eta^{\Delta_-} O_-(\mathbf{x})\,,$$

where

$$\Delta_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \equiv \frac{3}{2} \pm i\mu\,.$$

Define $O \equiv O_+$ (with $\Delta \equiv \Delta_+$) and its shadow $\tilde{O} \equiv O_-$ (with $\tilde{\Delta} \equiv \Delta_-$). Correlators of O and $\tilde{O}$ are related by

$$\langle \tilde{O}(\mathbf{k}_1)\tilde{O}(\mathbf{k}_2)\cdots\tilde{O}(\mathbf{k}_N)\rangle' = \frac{\langle O(\mathbf{k}_1)O(\mathbf{k}_2)\cdots O(\mathbf{k}_N)\rangle'}{(k_1k_2\cdots k_N)^{2\Delta-3}}\,.$$

The form of the boundary correlators is constrained by conformal symmetry.

2.3 Conformal Field Theory

A conformal transformation leaves the metric invariant up to a scale change:

$$\begin{aligned} x^i &\to \tilde{x}^i\,, \\ g_{ij}(x) &\to \tilde{g}_{ij}(\tilde{x}) = \Omega^2(x)g_{ij}(x)\,. \end{aligned}$$

The elements of the conformal group are:

$$\begin{array}{rll} \text{T:} & \tilde{x}^i = a^i & \Omega(x) = 1 \\ \text{R:} & \tilde{x}^i = R^{ij}x_j & \Omega(x) = 1 \\ \text{D:} & \tilde{x}^i = \lambda x^i & \Omega(x) = \lambda^{-1} \\ \text{SCT:} & \tilde{x}^i = \dfrac{x^i - b^i x^2}{1 - 2\,b\cdot x + b^2x^2} & \Omega(x) = 1 - 2\,b\cdot x + b^2x^2\,. \end{array}$$

Acting on scalar primary operators O, we have

$$O(x) \to \tilde{O}(\tilde{x}) = \Omega(x)^{\Delta}O(x)\,,$$

where Δ is the scaling dimension of the operator.

Correlators must then satisfy

$$\langle O_1(\tilde{x}_1)\ldots O_N(\tilde{x}_N)\rangle = \Omega(x_1)^{\Delta_1}\cdots\Omega(x_N)^{\Delta_N}\langle O_1(x_1)\ldots O_N(x_N)\rangle\,.$$

Two- and three-point functions are uniquely fixed

$$\begin{aligned} \langle O_1O_2\rangle &= \frac{1}{x_{12}^{2\Delta_1}}\,\delta_{\Delta_1,\Delta_2}\,, \\ \langle O_1O_2O_3\rangle &= \frac{c_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_1+\Delta_3-\Delta_2}}\,, \end{aligned}$$

where $O_n \equiv O_n(x_n)$ and $x_{nm} \equiv |x_n - x_m|$.

The four-point function of (identical) scalar operators is

$$\langle OOOO\rangle = \frac{f(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\,, \quad \text{where} \quad \begin{aligned} u &\equiv \left(\frac{x_{12}x_{34}}{x_{13}x_{24}}\right)^2\,, \\ v &\equiv \left(\frac{x_{23}x_{14}}{x_{13}x_{24}}\right)^2\,. \end{aligned}$$

These constraints can also be expressed as Ward identities:

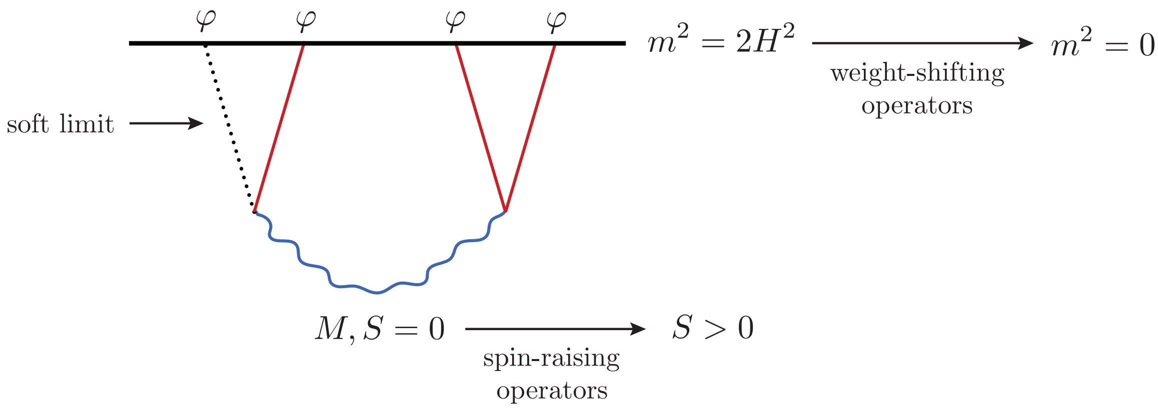
$$\begin{aligned} \mathrm{D}:\quad 0 &= \sum_{n=1}^{N}\left(\Delta_n + x_n^j\frac{\partial}{\partial x_n^j}\right)\langle O_1\cdots O_N\rangle\,, \\ \mathrm{SCT}:\quad 0 &= \sum_{n=1}^{N}\left(\Delta_n x_n^i + x_n^i x_n^j\frac{\partial}{\partial x_n^j} - \frac{x_n^2}{2}\frac{\partial}{\partial x_{n,i}}\right)\langle O_1\cdots O_N\rangle\,. \end{aligned}$$

In cosmology, we are interested in these constraints in Fourier space:

$$\begin{aligned} \mathrm{D}:\quad 0 &= \sum_{n=1}^{N}\left((\Delta_n-3) - k_n^j\frac{\partial}{\partial k_n^j}\right)\langle O_1\cdots O_N\rangle'\,, \\ \mathrm{SCT}:\quad 0 &= \sum_{n=1}^{N}\left((\Delta_n-3)\frac{\partial}{\partial k_{n,i}} - k_n^j\frac{\partial^2}{\partial k_n^j k_{n,i}} + \frac{k_n^i}{2}\frac{\partial^2}{\partial k_n^j k_{n,j}}\right)\langle O_1\cdots O_N\rangle'\,. \end{aligned}$$

In the following, we will study the solutions to these equations.

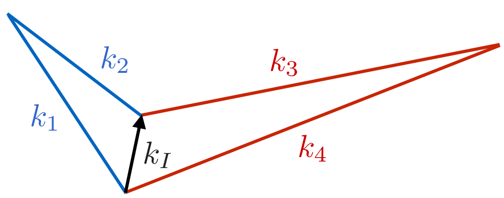
2.4 De Sitter Four-Point Functions



Kinematics

The four-point function of conformally coupled scalars can be written as

$$\langle \varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\varphi_{\mathbf{k}_4}\rangle' =$$



$$= \frac{1}{k_I}\hat{F}(u, v)\,,$$

where we have introduced

$$u^{-1} = \frac{k_1 + k_2}{k_I}\,, \quad v^{-1} = \frac{k_3 + k_4}{k_I}\,.$$

This ansatz solves the dilatation Ward identity.

Conformal Symmetry

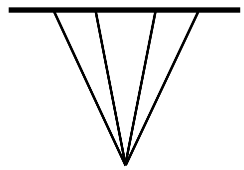
After some work, the conformal Ward identity can be written as

$$\boxed{(\nabla_u - \nabla_v)\hat{F} = 0}\,,$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$ (hypergeometric).

Contact Interactions

The simplest solutions correspond to contact interactions:

$$\hat{F}_c \equiv$$  $$= \sum_n \frac{c_n(u,v)}{\underset{\substack{\uparrow \\ \varphi^4,\, (\partial_\mu\varphi)^4,\, \cdots}}{E^{2n+1}}}\,,$$

which have poles at vanishing total energy

$$E \equiv \sum_n k_n = \frac{u+v}{uv}\, k_I\,.$$

Note that $F_c^{(n)} = \Delta_u^n F_c^{(0)}$, where $F_c^{(0)} \equiv uv/(u+v)$.

Exchange Interactions

For tree exchange, we try

$$(\Delta_u + M^2)\hat{F} = \hat{F}_c\,,$$
$$(\Delta_v + M^2)\hat{F} = \hat{F}_c\,,$$

where $\hat{F}_c$ is a contact solution.

Using the simplest contact interaction as a source, we have

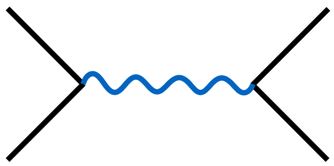
$$\boxed{\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + M^2\right]\hat{F} = g^2\,\frac{uv}{u+v}}\,, \qquad (\star)$$

where $M^2 = \mu^2 + \dfrac{1}{4}$.

Singularities

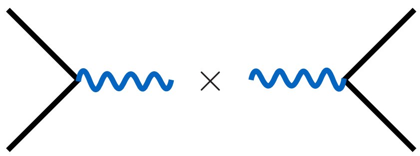
The equation has a number of interesting singularities:

- Flat-space limit: $\lim_{u \to -v} \hat{F} = A_4 \,(u+v) \log(u+v)$

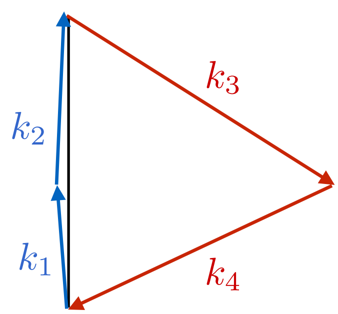


The correlator contains the scattering amplitude.

- Factorization limit: $\lim_{u,v \to -1} \hat{F} = A_3 \log(1+u) \times A_3 \log(1+v)$

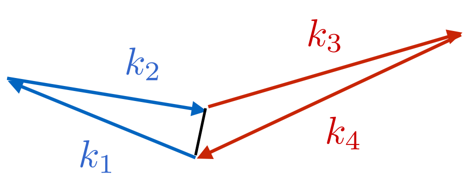


- Folded limit: $\lim_{u \to +1} \hat{F} \propto \log(1-u)$



This singularity should be absent in the standard vacuum.

- Collapsed limit: $\lim_{u \to 0} \hat{F} \propto u^{\frac{1}{2}+i\mu}$



This non-analyticity corresponds to spontaneous particle production.

Imposing regularity in the folded limit and the correct normalization in the factorization limit uniquely fixes the solution.

EFT Expansion

A formal solution of $(\star)$ is

$$\hat{F} = \frac{\hat{F}_c^{(0)}}{\Delta_u + M^2} = \sum_n \frac{1}{n!}\left(-\frac{\Delta_u}{M^2}\right)^n \frac{\hat{F}_c^{(0)}}{M^2}$$

$$= \underset{\substack{\uparrow \\ \varphi^4}}{\frac{\hat{F}_c^{(0)}}{M^2}} - \underset{\substack{\uparrow \\ \varphi^2(\partial_\mu\varphi)^2}}{\frac{\hat{F}_c^{(1)}}{M^4}} + \underset{\substack{\uparrow \\ (\partial_\mu\varphi)^4}}{\frac{1}{2}\frac{\hat{F}_c^{(2)}}{M^6}} + \cdots$$

This misses particle production!

Particle Production

Consider $v \to 0$. Writing $e^t \equiv u/v \equiv \xi$ and $f \equiv (uv)^{-1/2}\hat{F}$, eq. $(\star)$ becomes

$$\left[\frac{d^2}{dt^2} + \mu^2\right] f = \frac{1}{2\cosh(\frac{1}{2}t)} \quad \Leftarrow \quad \text{forced harmonic oscillator.}$$

The homogeneous solutions are $f_\pm = e^{\pm i\mu t} = \xi^{\pm i\mu}$.

Around $\xi = 0$, the inhomogeneous solution is

$$f_<(\xi) = \sqrt{\xi}\sum_{n=0}^\infty (-1)^n \frac{\xi^n}{(n+\frac{1}{2})^2 + \mu^2}\,.$$

Around $\xi = \infty$, we have

$$f_>(\xi) = \frac{1}{\sqrt{\xi}}\sum_{n=0}^\infty (-1)^n \frac{\xi^{-n}}{(n+\frac{1}{2})^2 + \mu^2}\,.$$

Matching the solutions at $\xi = 1$, we find

$$\tilde{F}_<(\xi) = \begin{cases} \displaystyle\sum_{n=0}^\infty (-1)^n \frac{\xi^{n+1}}{(n+\frac{1}{2})^2+\mu^2} & \xi \le 1\,, \\ \displaystyle\underbrace{\sum_{n=0}^\infty (-1)^n \frac{\xi^{-n}}{(n+\frac{1}{2})^2+\mu^2}}_{\text{EFT expansion}} + \underbrace{\frac{\pi}{\cosh \pi\mu}\frac{\xi^{\frac{1}{2}-i\mu} - \xi^{\frac{1}{2}+i\mu}}{2i\mu}}_{\text{particle production}} & \xi \ge 1\,. \end{cases}$$

General Solution

The homogeneous solutions are

$$\hat{F}_{\pm}(u) = \left(\frac{iu}{2\mu}\right)^{\frac{1}{2}\pm i\mu} {}_2F_1\left[\left.\begin{matrix}\frac{1}{4}\pm\frac{i\mu}{2}, \frac{3}{4}\pm\frac{i\mu}{2} \\ 1\pm i\mu\end{matrix}\right| u^2\right].$$

Around $u = 0$, the inhomogeneous solution is

$$\hat{F}_{<}(u,v) \sum_{m,n=0}^{\infty} c_{mn}(\mu) u^{2m+1}(u/v)^n .$$

Around $u = \infty$, we have $F_{>}(u,v) = F_{<}(v,u)$.

Matching at $u = v$, we find

$$\hat{F}_{<}(u,v) = \begin{cases} \displaystyle\sum_{m,n=0}^{\infty} c_{mn} u^{2m+1}(u/v)^n & u \leq v\,, \\ \displaystyle\sum_{m,n=0}^{\infty} c_{mn} v^{2m+1}(v/u)^n + \frac{\pi}{\cosh \pi\mu}\hat{F}_h(u,v) & u \geq v\,, \end{cases}$$

where $\hat{F}_h(u,v) \equiv \hat{F}_{+}(v)\hat{F}_{-}(u) - \hat{F}_{-}(v)\hat{F}_{+}(u)$.

The freedom to add homogeneous solutions is fixed by the boundary conditions

$$\lim_{u\to+1} \hat{F} = \text{regular}$$
$$\lim_{u,v\to-1} \hat{F} = \frac{1}{2}\log(1+u)\log(1+v)\,.$$

The final result is

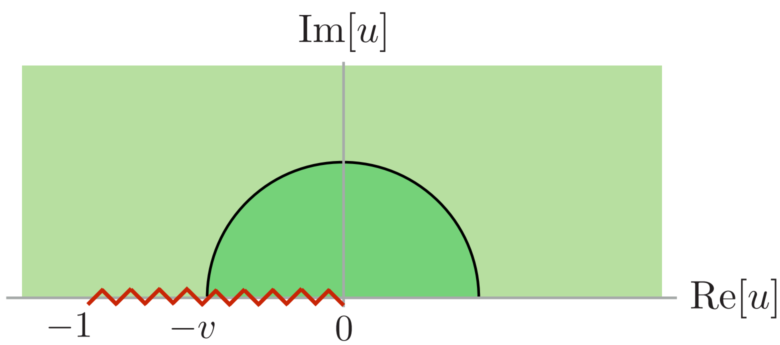
$$\hat{F}(u,v) = \begin{cases} \displaystyle\sum_{m,n=0}^{\infty} c_{mn} u^{2m+1}(u/v)^n + \frac{\pi}{2\cosh \pi\mu}\hat{g}(u,v) & u \leq v\,, \\ \displaystyle\sum_{m,n=0}^{\infty} c_{mn} v^{2m+1}(v/u)^n + \frac{\pi}{2\cosh \pi\mu}\hat{g}(v,u) & u \geq v\,, \end{cases}$$

where $\hat{g}(u,v)$ is a known function [arXiv:1811.00024].

Flat-Space Limit

An interesting limit is $u \to -v$ (or $E = \sum k_n \to 0$).

In this limit, the solution has a branch cut singularity:



The discontinuity across the cut is

$$\lim_{u \to -v} \frac{\mathrm{Disc}[\hat{F}']}{2\pi i} = \frac{1}{(k_1+k_2)^2 - (\mathbf{k}_1+\mathbf{k}_2)^2} = A_4\,.$$

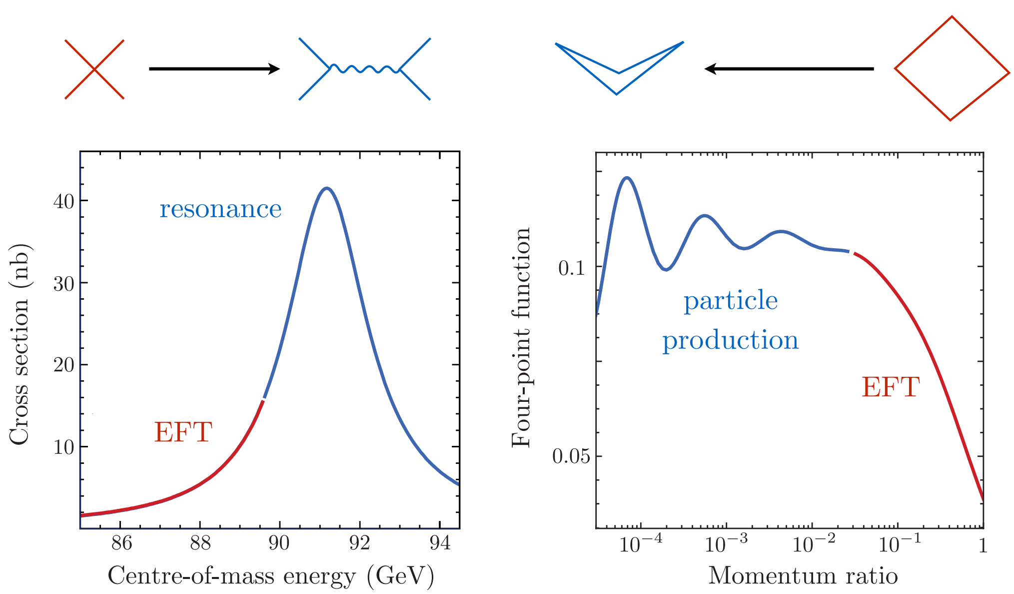
This relates curved-space particle production to flat-space scattering.

Soft Limit and Spectroscopy

The particle production piece dominates in the soft limit $u \to 0$:

$$\lim_{u \to 0} \hat{F} = g^2\, e^{-\pi\mu} \left(\frac{u}{v}\right)^{1/2} \frac{\sin[\mu \log(u/v)]}{\mu}\,.$$

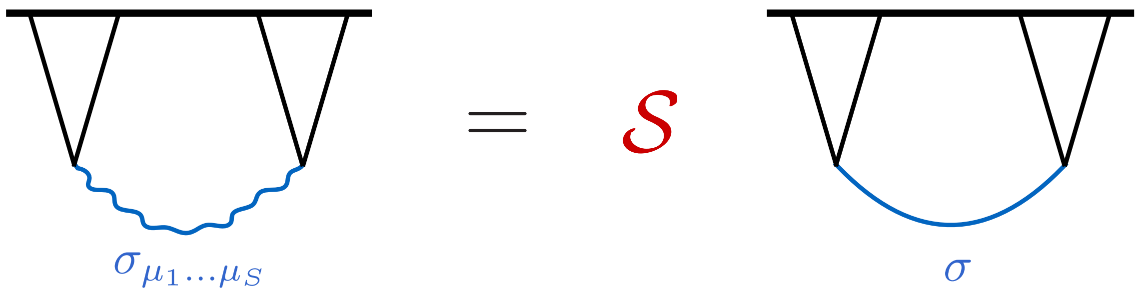
These oscillations are the analog of resonances in collider physics.



2.5 Exchange of Spinning Particles

Strategy

Find differential operators that relate scalar exchange to spin exchange:



It turns out that this is best implemented in embedding space and then Fourier transformed.

CFT in Embedding Space

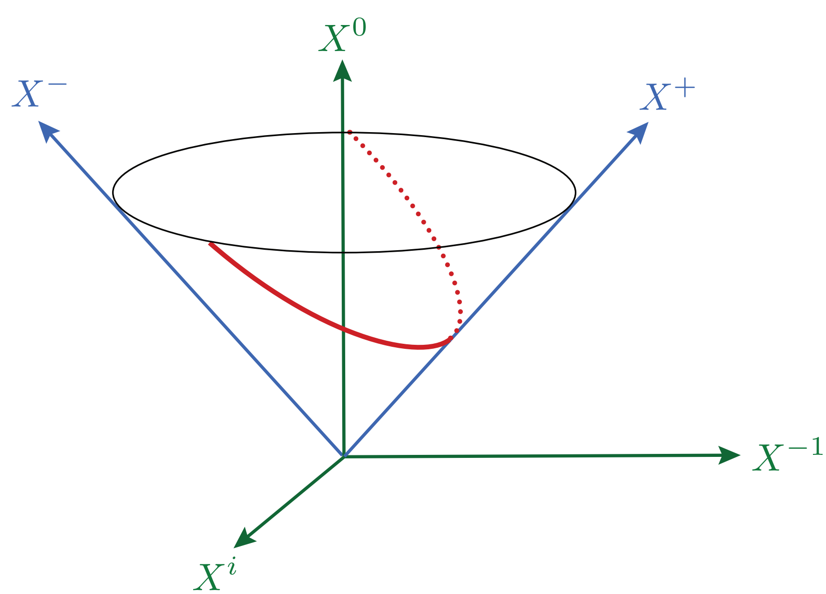
Consider $d+2$ dimensional Minkowski space, with coordinates

$$X^M, \;\; M=-1,0,1,\ldots,d\,.$$
$$X^{\pm} \equiv X^0 \pm X^{-1}$$

The embedding of $\mathbb{R}^d$ into $\mathbb{R}^{1,d+1}$ is defined by

- $X^2=0$ (null cone)
- $X^+=1$ (Euclidean section)

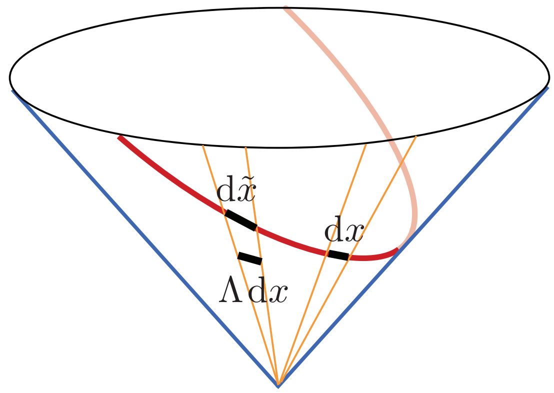
$$\Rightarrow \quad X^M = (X^+, X^-, X^i) = (1, x^2, x^i)$$



Lorentz transformations on $\mathbb{R}^{1,d+1}$ become conformal transformations on $\mathbb{R}^d$:

- $X^M \to \Lambda^M{}_N X^N$
- $X^M \to \lambda\, X^M$

$$\Rightarrow \quad g_{ij} \to \tilde{g}_{ij} = \Omega^2(x) g_{ij}, \ \text{ with } \ \Omega(x) = \lambda(X).$$



Conformal transformations of fields on $\mathbb{R}^d$ are scaling transformations on $\mathbb{R}^{1,d+1}$:

$$O(\lambda X) = \lambda^{-\Delta} O(X) \quad \Leftrightarrow \quad O(\tilde{x}) = \Omega(x)^{\Delta} O(x)\,.$$

Conformal correlators in embedding space are simply the most general Lorentz-invariant expressions with the correct scaling behavior.

Examples

- Two- and three-point functions of scalar operators:

$$\begin{aligned} \langle O_1 O_2 \rangle &= \frac{1}{X_{12}^{\Delta_1}} \delta_{\Delta_1,\Delta_2}\,, \\ \langle O_1 O_2 O_3 \rangle &= \frac{c_{123}}{X_{12}^{(\Delta_1+\Delta_2-\Delta_3)/2} X_{23}^{(\Delta_2+\Delta_3-\Delta_1)/2} X_{31}^{(\Delta_3+\Delta_1-\Delta_2)/2}}\,, \end{aligned}$$

  where $X_{nm} \equiv X_n \cdot X_m = -\frac{1}{2} x_{nm}^2$.

- Four-point function of identical scalars:

$$\langle OOOO \rangle = \frac{1}{X_{12}^{\Delta} X_{34}^{\Delta}} f(u,v)\,, \quad \text{where} \quad \begin{aligned} u &\equiv \frac{X_{12}X_{34}}{X_{13}X_{24}}\,, \\ v &\equiv u(2 \leftrightarrow 4)\,. \end{aligned}$$

Fields with Spin

$$O_{M_1M_2\ldots}(X) \quad \Rightarrow \quad O_{i_1i_2\ldots}(x) = O_{M_1M_2\ldots}(X)\, \underset{\substack{\uparrow \\ \frac{\partial X^M}{\partial x^i} = (0, 2x_i, \delta_i^j)}}{\frac{\partial X^{M_1}}{\partial x^{i_1}}} \frac{\partial X^{M_2}}{\partial x^{i_2}} \cdots .$$

Extra components are removed by

- $X^M O_{M\cdots}(X) = 0$ (transversality)
- $O_{M\cdots} + X_M(\cdots) \sim O_{M\cdots}$ ("gauge invariance")

Lorentz transformations on $\mathbb{R}^{1,d+1}$ become conformal transformations on $\mathbb{R}^d$.

Examples

- Two-point function of spin-S fields

$$\langle \Sigma_1^{(S)} \Sigma_2^{(S)} \rangle = \left(Z_1 \cdot Z_2 - \frac{Z_1 \cdot X_2\, Z_2 \cdot X_1}{X_{12}} \right)^S \langle \Sigma_1 \Sigma_2 \rangle ,$$

where $\Sigma_n^{(S)} \equiv Z_n^{M_1} \cdots Z_n^{M_S} \Sigma_{M_1 \ldots M_S}(X_n)$.

- Scalar-scalar-spin-S three-point function

$$\langle O_1 O_2 \Sigma_3^{(S)} \rangle = \left(\frac{(Z_3 \cdot X_1)(X_2 \cdot X_3) - (Z_3 \cdot X_2)(X_1 \cdot X_3)}{(X_{12} X_{13} X_{23})^{1/2}} \right)^S \langle O_1 O_2 \Sigma_3 \rangle .$$

Spin-Raising Operator

Consider

$$\langle \varphi \varphi \Sigma \rangle = (X_{12}^{4-\Delta} X_{23}^{\Delta} X_{31}^{\Delta})^{-1/2} ,$$

$$\langle \varphi \tilde{\varphi} \Sigma \rangle = (X_{12}^{3-\Delta} X_{23}^{\Delta-1} X_{31}^{\Delta+1})^{-1/2} = \left(\frac{X_{12} X_{23}}{X_{31}} \right)^{1/2} \langle \varphi \varphi \Sigma \rangle ,$$

$$\langle \varphi \tilde{\varphi} \Sigma^{(1)} \rangle = \frac{(Z_3 \cdot X_1)(X_2 \cdot X_3) - (Z_3 \cdot X_2)(X_1 \cdot X_3)}{(X_{12} X_{23} X_{31})^{1/2}} \langle \varphi \tilde{\varphi} \Sigma \rangle .$$

Ex: Show that

$$\langle \varphi \tilde{\varphi} \Sigma^{(1)} \rangle = -\frac{2}{\Delta} \mathcal{S}_{32} \langle \varphi \varphi \Sigma \rangle ,$$

where $\quad \mathcal{S}_{32} = (X_3 \cdot X_2) Z_3 \cdot \dfrac{\partial}{\partial X_3} - (Z_3 \cdot X_2) X_3 \cdot \dfrac{\partial}{\partial X_3}.$

We see that $\mathcal{S}_{32}$ raises the spin at 3 and lowers the weight at 2.

In Fourier space, we get

$$\mathcal{S}_{32} = z_3^i \left[K_{32}^i + \frac{1}{2} k_3^i K_{32}^j K_{32}^j \right] , \quad K_{32}^i \equiv \partial_{k_3^i} - \partial_{k_2^i} .$$

Finally, we preform a shadow transform to get

$$\langle \varphi \varphi \Sigma^i \rangle = k_2 \langle \varphi \tilde{\varphi} \Sigma^i \rangle = k_2 \mathcal{S}_{32}^i \langle \varphi \varphi \Sigma \rangle \equiv \mathcal{S}_L^i \langle \varphi \varphi \Sigma \rangle .$$

Repeated application of $\mathcal{S}_L^i$ would raise the spin further.

Raising Internal Spin

Using the spin-raising operator, we can write

$$\underbrace{\hat{F}_S}_{\text{spin-S exchange}} = \sum_{\lambda=0}^{S} \underbrace{P^{(\lambda)}_{i_1 \dots i_S j_1 \dots j_S}}_{\text{polarization tensor}} \underbrace{(\mathcal{S}_L^{i_1} \cdots \mathcal{S}_L^{i_S})(\mathcal{S}_R^{j_1} \cdots \mathcal{S}_R^{j_S})}_{\text{spin raising}} \underbrace{\hat{F}_0}_{\text{spin-0 exchange}} .$$

Writing this in terms of u and v, we get

$$\hat{F}_S = \sum_{\lambda=0}^{S} \Pi_{S,\lambda}(\text{angles})\, \mathcal{D}_{uv}^{(S,\lambda)} \hat{F}_0 .$$

For spin-1 and spin-2 exchange, we find

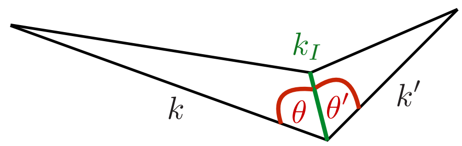
$$\begin{aligned} \hat{F}_1 &= (\Pi_{1,1}\, D_{uv} + \Pi_{1,0}\, \Delta_u)\, \hat{F}_0 , \\ \hat{F}_2 &= \big(\Pi_{2,2}\, D_{uv}^2 + \Pi_{2,1}\, D_{uv}(\Delta_u - 2) + \Pi_{2,0}\, \Delta_u(\Delta_u - 2)\big)\, \hat{F}_0 , \end{aligned}$$

where $D_{uv} \equiv (uv)^2 \partial_u \partial_v$.

Soft Limit and Spectroscopy

In the collapsed limit $u \to 0$, this gives

$$\lim_{u \to 0} \hat{F}_S \; =$$

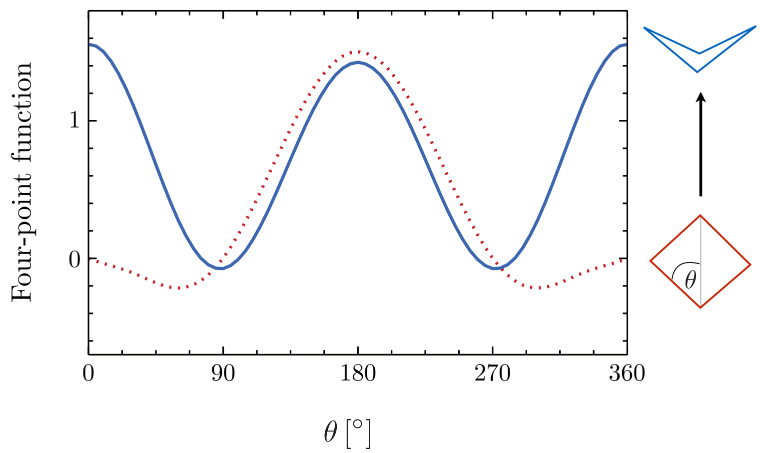


$$\propto \left(\frac{k_I^2}{kk'}\right)^{\Delta} \sum_{\lambda} I_{S,\lambda}(\Delta)\, P_S^{\lambda}(\cos\theta) P_S^{-\lambda}(\cos\theta')\,.$$

$\uparrow$

fixed by
conformal symmetry

The spin of the new particles is encoded in the angular dependence:

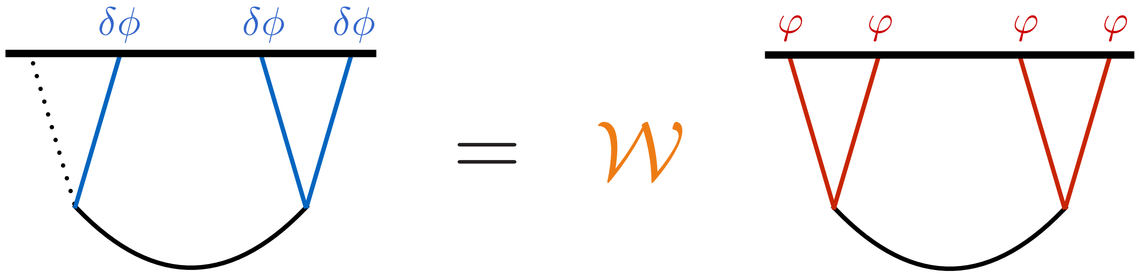


This is the analog of the angular dependence of the final state particles in collider physics.

2.6 Inflationary Three-Point Functions

Strategy

Find a differential operator that relates the four-point function of conformally coupled scalars to that of massless scalars:



Evaluate one leg on the time-dependent background to obtain inflationary three-point functions.

Massless External Fields

Recall that

$$\begin{aligned}\langle\varphi\varphi\varphi\varphi\rangle &= \frac{1}{X_{12}^2 X_{34}^2}\, f(u,v)\,,\\ \langle\phi\phi\phi\phi\rangle &= \frac{1}{X_{12}^3 X_{34}^3}\, h(u,v)\,.\end{aligned}$$

Ex: Show that

$$\langle\phi\phi\phi\phi\rangle = \mathcal{W}_L\mathcal{W}_R\langle\varphi\varphi\varphi\varphi\rangle\,,$$

$$\text{where}\quad \mathcal{W}_L \equiv \left(\frac{\partial}{\partial X_{1,M}} + \frac{X_1^M}{3}\frac{\partial^2}{X_1^2}\right)\left(\frac{\partial}{\partial X_2^M} + \frac{X_{2,M}}{\underset{\uparrow \atop 2\Delta-1}{3}}\frac{\partial^2}{X_2^2}\right)\qquad \text{weight-raising operator}$$

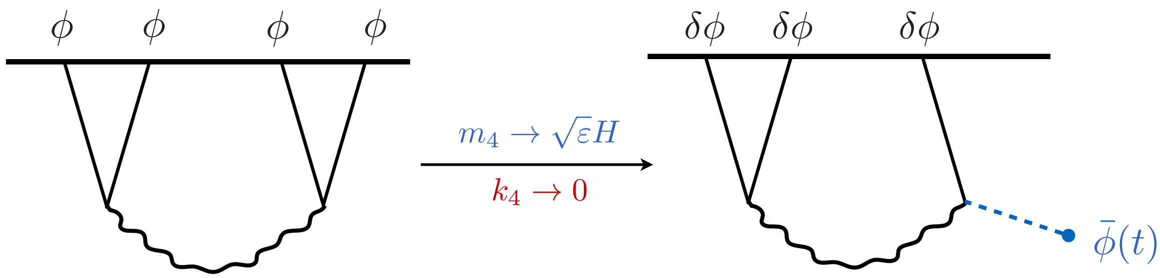
For scalar exchange, we find

$$F_{\Delta=3} = \mathcal{W}_L\mathcal{W}_R\hat{F}_{\Delta=2}\,,$$

$$\text{where}\qquad \mathcal{W}_L(\cdot) \equiv \frac{1}{2}\left(1-\frac{k_1k_2}{k_1+k_2}\partial_{k_1+k_2}\right)\left[\frac{1-u^2}{u^2}\partial_u(u\,\cdot)\right].$$

For spin exchange, $\mathcal{W}_{L,R}$ is more complicated (in Fourier space).

Perturbed de Sitter



The inflationary bispectrum is

$$B = \lim_{k_4 \to 0} F_{\Delta=3-\varepsilon} + \text{perms}\,,$$

where

$$\begin{aligned} F_{\Delta=3-\varepsilon} &= \mathcal{W}_L \mathcal{W}_R\, F_{\Delta=2-\varepsilon} \\ &= \mathcal{W}_L \left(\underset{\substack{\uparrow \\ 0}}{\bar{\mathcal{W}}_R} + \underset{\substack{\uparrow \\ 1}}{\varepsilon}\, \delta \mathcal{W}_R + \cdots \right) \left(F_{\Delta=2} + \varepsilon\, F_{\Delta=2} + \cdots \right) \\ & \qquad\qquad \text{for } \; k_4 \to 0. \end{aligned}$$

We hence find

$$\boxed{B(k_1,k_2,k_3) \;=\; \varepsilon\, \mathcal{W}_L \lim_{v \to 1} F_{\Delta=2} \;+\; \text{perms}}\,. \qquad (\star)$$

For spin exchange, only the longitudinal mode contributes:

$$F^S_{\Delta=2} \to \Pi_{S,0}\, \mathcal{D}^{(S,0)}_{uv}\, \hat{F}^{S=0}_{\Delta=2}\,.$$

Contact Interactions

For the simplest contact solution, we have

$$\lim_{v \to 1} \hat{F}^{(0)}_c = \frac{u}{u+1}\,.$$

Substituting this into ($\star$), we get

$$\begin{aligned} B(k_1,k_2,k_3) = \frac{\varepsilon}{4K^2} \Bigg[& \sum_n k_n^5 + \sum_{n \neq m} (2k_n^4 k_m - 3k_n^3 k_m^2) \\ & + \sum_{n \neq m \neq l} (k_n^3 k_m k_l - 4k_n^2 k_m^2 k_l) \Bigg]\,, \end{aligned}$$

which (up to a shadow transform) is the bispectrum arising from $(\partial_\mu \phi)^4$.

Graviton Exchange

For massless spin-2 exchange, we have

$$\begin{aligned} \lim_{v\to 1} \Delta_u(\Delta_u - 2)\hat{F}_{\Delta=2} \;=\; \lim_{v\to 1} \Delta_u \hat{F}_c^{(-1)} \;&=\; \lim_{v\to 1} \hat{F}_c^{(0)} \\ &=\; \frac{u}{u+1}\,. \end{aligned}$$

Substituting this into ($\star$), we get

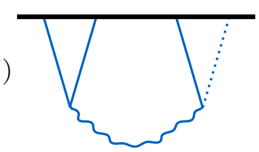
$$B(k_1,k_2,k_3) \;=\; \varepsilon\left[\sum_{n\neq m} k_n k_m^2 + \frac{8}{K}\sum_{n>m} k_n^2 k_m^2\right] + (n_s - 1)\sum_n k_n^3\,,$$

which (up to a shadow transform) is the standard three-point function of slow-roll inflation.

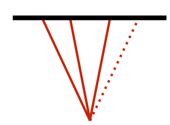
Massive Particles

The effects of massive particles during inflation are characterized in terms of just two basis functions:

$$B(k_1,k_2,k_3) \;=\; \mathcal{W}_L \Bigg[\sum_S a_S\, \mathcal{S}^{(S)}$$



$$+ \sum_n b_n\, \Delta_u^n$$



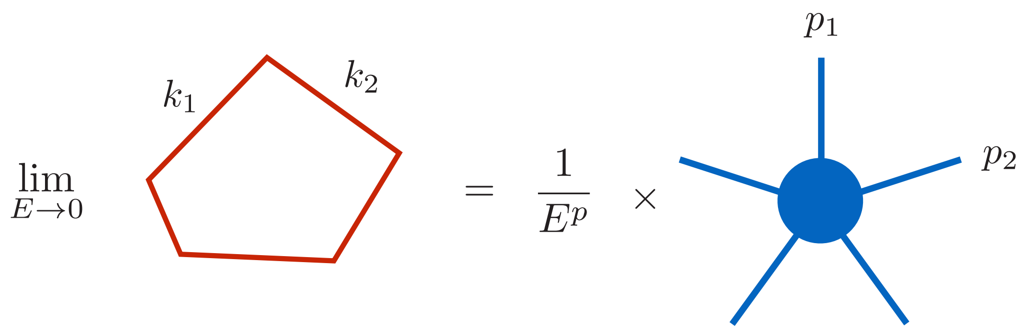
$$\Bigg] + \text{perms}$$

This result is valid for all momenta, not just soft limits.

3 Future Directions

3.1 Amplitudes Meet Cosmology

Remarkably, correlation functions contain scattering amplitudes:

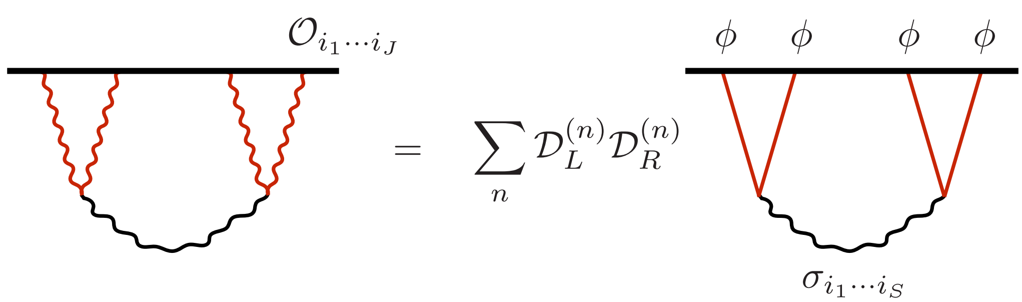


where $E \equiv \sum |\mathbf{k}_n|$.

Insights from the physics of scattering amplitudes should therefore translate to cosmology.

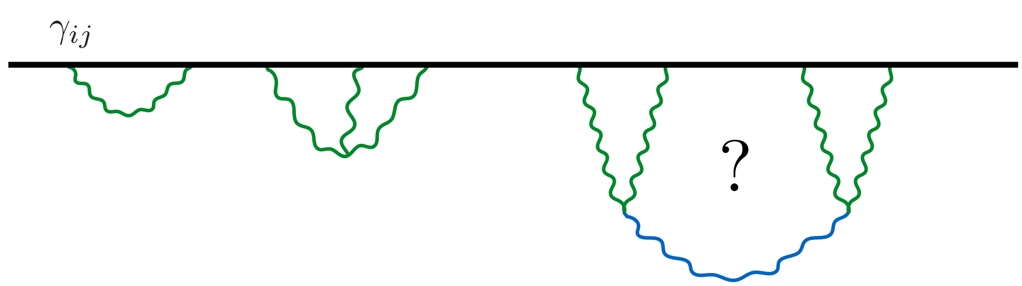
3.2 Spinning Correlators

Spinning correlators can also be bootstrapped from our scalar building blocks:



3.3 Graviton Correlators

An important special case are graviton correlators:

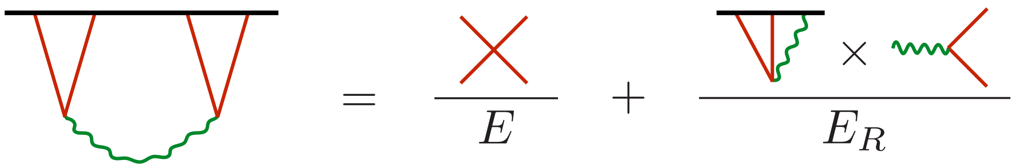


In de Sitter space, very little is known beyond three-point functions.
In flat space, a consistent S-matrix of gravitons is very constrained.

What is the cosmological analog of these results?

3.4 Factorization

For massless spin exchange, we find



Does consistent factorization allow for an efficient construction of graviton correlators?

3.5 Double Copy

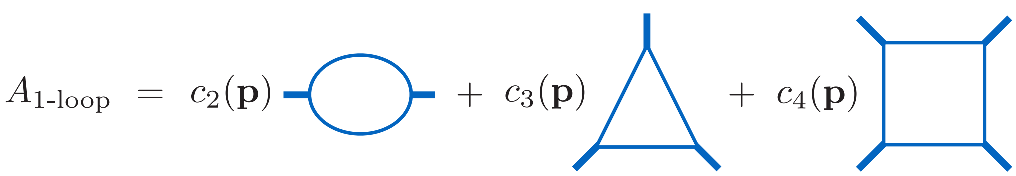
Gravity amplitudes can be written as the square of gauge theory amplitudes:

$$\text{Gravity} = \text{YM}^2$$

Is there an analog of this for cosmological correlators?

3.6 Loop Corrections

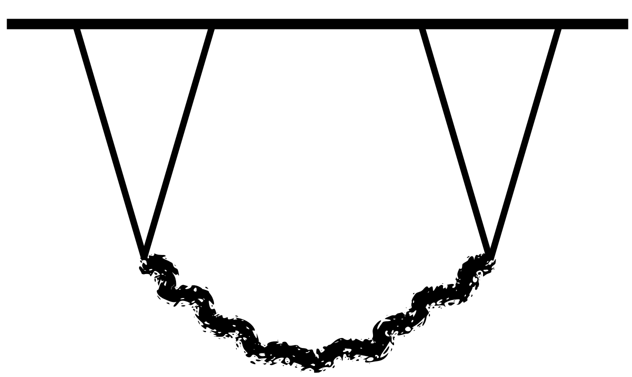
How does the bootstrapping of de Sitter correlators generalize to loops? One-loop amplitudes can be written as



Is there a cosmological analog of this?

3.7 Ultraviolet Completion

What is the space of consistent UV completions of inflationary correlators?



- What is the cosmological analog of positivity bounds?
- What is the Veneziano correlator in de Sitter space?