Lectures on

The Cosmological Bootstrap

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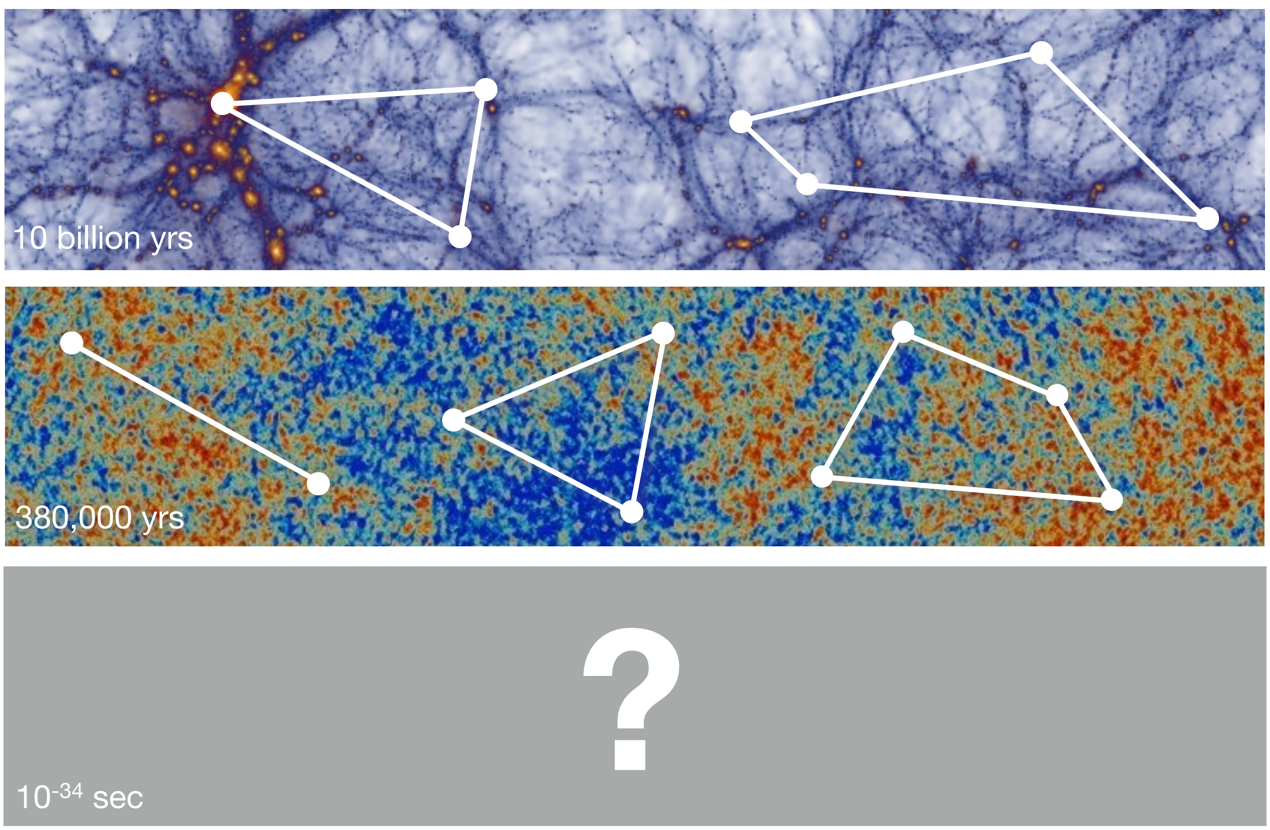
Motivation

The physics of the early universe is encoded in the spatial correlations between cosmological structures at late times:



A central challenge of modern cosmology is to construct a consistent history of the universe that explains these correlations.

In these lectures, I will describe a new approach to determine cosmological correlation functions from consistency conditions alone = **bootstrap**.

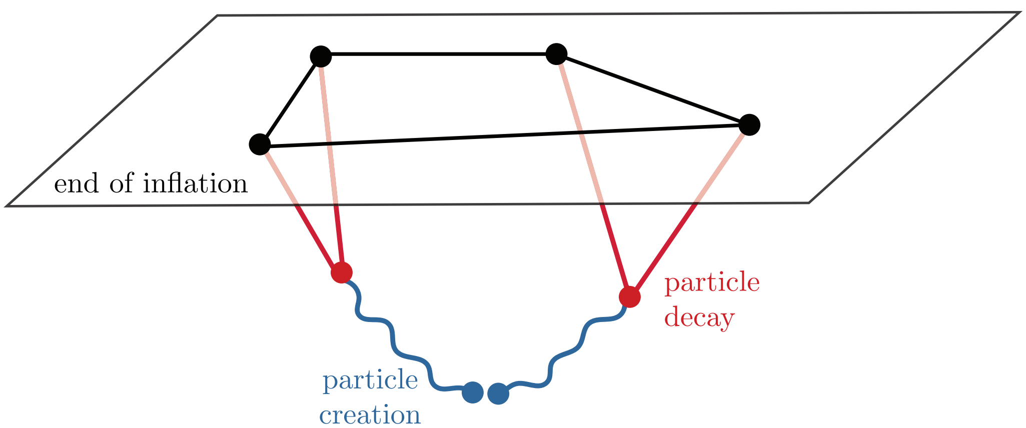
We will take inspiration from the **S-matrix bootstrap**, where the structure of scattering amplitudes is fixed by Lorentz invariance, locality and unitarity:

$$A(s,t) = \sum a_{nm} s^n t^m \; + \; \frac{g^2}{s - M^2} P_S \left(1 + \frac{2t}{M^2}\right)$$

- No Lagrangian and Feynman diagrams are needed to derive this.
- Basic principles allow only a small menu of possibilities.

Can we obtain a similar understanding of cosmological correlators?

The connection to scattering amplitudes is also relevant because the early universe was like a giant particle collider = **cosmological collider physics**



The cosmological bootstrap is a systematic way to study this physics.

Outline for the rest of the lectures:

I. Review of Cosmological Correlations

II. Bootstrapping Inflationary Correlators

III. Summary and Future Directions

Contents

Useful References:

Background in Cosmology:

Baumann, *TASI Lectures on Primordial Cosmology* [arXiv:1807.03098]
Baumann, *TASI Lectures on Inflation* [arXiv:0907.5424]
Baumann and McAllister, *Inflation and String Theory* [arXiv:1404.2601]
Maldacena, *Non-Gaussian Features of Primordial Fluctuations* [arXiv:astro-ph/0210603]
Chen, *Primordial Non-Gaussianities from Inflation Models* [arXiv:1002.1416]

Background in Conformal Field Theory:

Di Francesco, Mathieu and Senechal, *Conformal Field Theory* [Springer, 1997]
Qualls, *Lectures on Conformal Field Theory* [arXiv:1511.04074]
Rychkov, *Lectures on Conformal FIeld Theory in* $D \geq 3$ *Dimensions* [arXiv:1601.05000]
Simmons-Duffin, *TASI Lectures on the Conformal Bootstrap* [arXiv:1602.07982]
Penedones, *TASI Lectures on AdS/CFT* [arXiv:1608.04948]
Ginsparg, *Applied Conformal Field Theory* [arXiv:hep-th/9108028]

Relevant Research Literature:

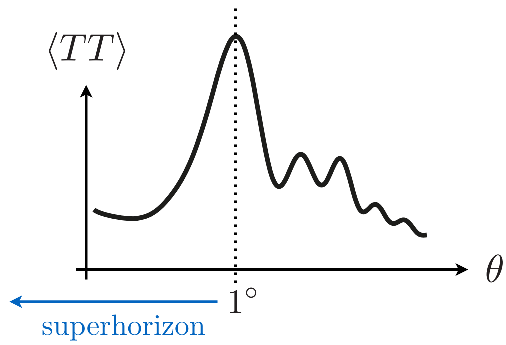
Arkani-Hamed, Baumann, Lee and Pimentel, *The Cosmological Bootstrap* [arXiv:1811.00024]
Arkani-Hamed and Maldacena, *Cosmological Collider Physics* [arXiv:1503.08043]
Lee, Baumann and Pimentel, *Non-Gaussianity as a Particle Detector* [arXiv:1607.03735]
Bzowski, McFadden and Skenderis, *Conformal Invariance in Momentum Space* [arXiv:1304.7760]
Mata, Raju and Trivedi, *CMB from CFT*, [arXiv:1211.5482]
Maldacena and Pimentel, *On Graviton Non-Gaussianities During Inflation*, [arXiv:1104.2846]
Costa, Penedones, Poland and Rychkov, *Spinning Conformal Correlators*, [arXiv:1107.3554]

1 Review of Cosmological Correlations

1.1 Observed Correlations

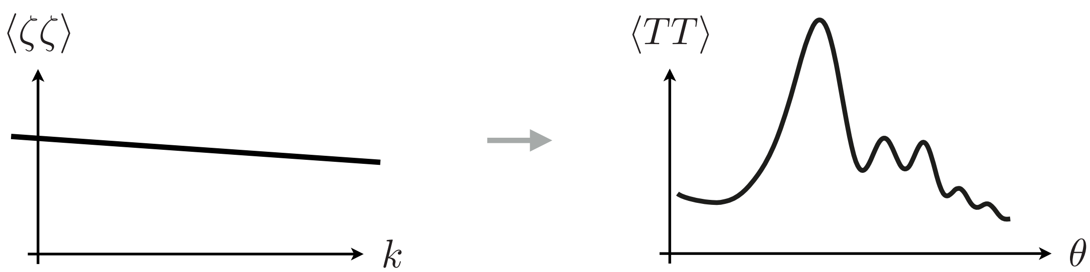
Observations have revealed two important facts:

- The CMB fluctuations are correlated over superhorizon scales:



They were created *before the hot Big Bang*!

- The primordial fluctuations were approximately scale invariant:



In Fourier space, this means

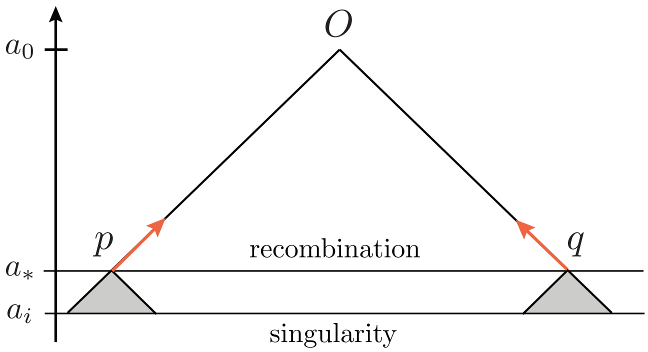
$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \rangle = \frac{2\pi^2}{k_1^3} P(k_1)\, \delta_D(\mathbf{k}_1 + \mathbf{k}_2)\,, \quad \text{with} \quad P(k_1) \approx const\,,$$

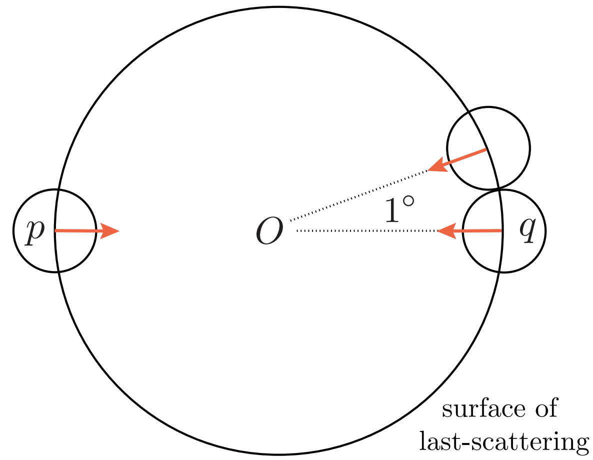
where $\nabla^2 \zeta = R^{(3)}$ is the comoving curvature perturbation.

Quantum fluctuations during inflation explain these facts.

1.2 Horizon Problem

In the standard Big Bang, light travels a finite distance before recombination:





The CMB (naively) consists of 10^4 disconnected regions.

- Why is it so uniform?
- Why is it correlated?

= **horizon problem**

Related to the horizon problem are the flatness problem, the monopole problem, the entropy problem, etc.

1.3 Inflation

All of these problems are solved if the early universe went through an extended period of **quasi-de Sitter expansion**:

$$H \equiv \frac{1}{a}\frac{da}{dt} \approx const. \tag{1.1}$$

The comoving horizon then becomes

$$\eta = \int \frac{\mathrm{d}t}{a(t)} \approx -\frac{1}{aH}\,, \tag{1.2}$$

which receives large contributions from early times.

1.4 Quantum Fluctuations

Consider a massless scalar field during inflation:

$$S = \frac{1}{2}\int \mathrm{d}\eta \mathrm{d}^3x\, a^2\left(\dot{\phi}^2 - (\partial_i\phi)^2\right), \tag{1.3}$$

where $a(\eta) = -(H\eta)^{-1}$.

Write the corresponding quantum operator as

$$\hat{\phi}_{\mathbf{k}}(\eta) = \phi_k(\eta)\hat{a}_{\mathbf{k}} + c.c. \quad \text{where} \qquad [\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}] = (2\pi)^3\delta_D(\mathbf{k}+\mathbf{k}'), \\ a(\eta)\phi_k(\eta) = \frac{1}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)e^{-ik\eta}. \tag{1.4}$$

Bunch-Davies mode function

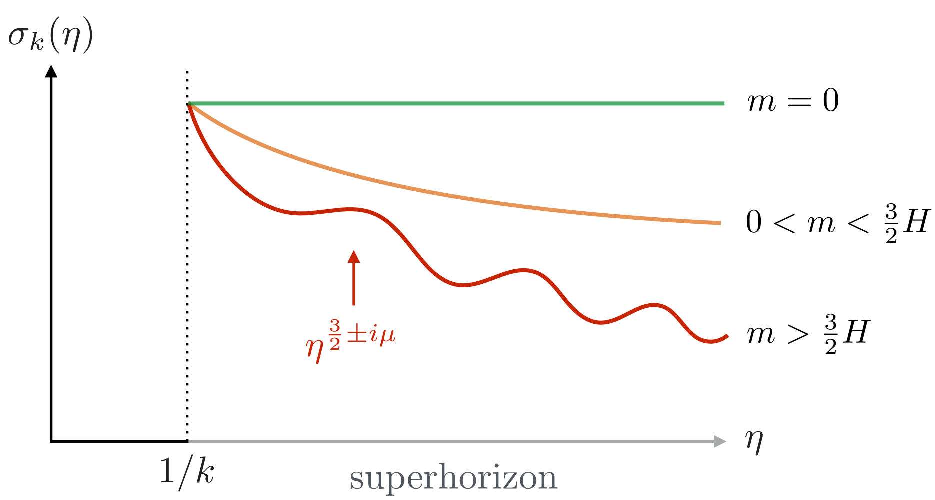
The quantum variance of the operator is $\langle 0|\,\hat{\phi}_{\mathbf{k}}\hat{\phi}_{-\mathbf{k}}\,|0\rangle = |\phi_k(\eta)|^2$
and the corresponding power spectrum is

$$P_\phi(k,\eta) \equiv \frac{k^3}{2\pi^2}|\phi_k(\eta)|^2 \xrightarrow{k\eta\to 0} \left(\frac{H}{2\pi}\right)^2. \tag{1.5}$$

If ϕ is the inflaton, then $\zeta = (H/\dot{\phi})\delta\phi$, and we get

$$P_\zeta(k,\eta) = \left(\frac{H}{\dot{\phi}}\right)^2 P_\phi(k,\eta) \xrightarrow{k\eta\to -1} \frac{1}{4\pi^2}\left(\frac{H^2}{\dot{\phi}}\right)^2\Bigg|_{-k\eta=1} \equiv A_s k^{n_s-1}. \tag{1.6}$$

Massive fields are also produced, but don't survive until late times:



Their imprints can still be found in the correlations of the light fields.

1.5 Non-Gaussianity

So far, we have only measured the two-point function of scalar fluctuations. In the future, we will look for higher-point correlations (= non-Gaussianity).

1.5.1 Bispectrum

The main diagnostic of primordial non-Gaussianity is the *bispectrum*:

$$\langle \zeta_{\mathbf{k}_1} \zeta_{\mathbf{k}_2} \zeta_{\mathbf{k}_3} \rangle = \frac{(2\pi^2)^2}{(k_1 k_2 k_3)^2} B(k_1, k_2, k_3)\, \delta_D(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3)\,. \tag{1.7}$$

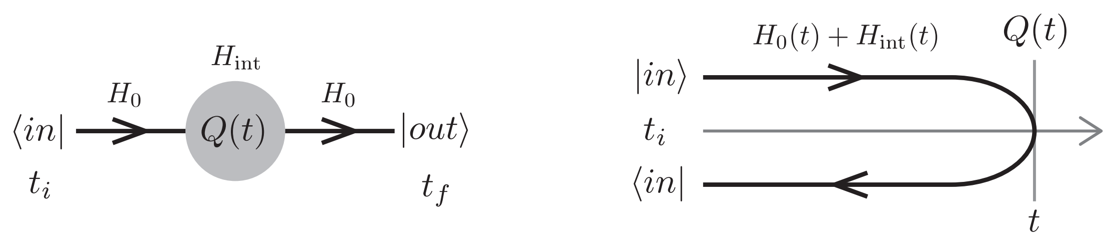
The amplitude of the bispectrum is defined as

$$f_{\rm NL} \equiv \frac{5}{18} \frac{B(k,k,k)}{P^2(k)}\,. \tag{1.8}$$

Observational constraints on $f_{\rm NL}$ depend on the momentum dependence (or 'shape') of the bispectrum.

1.5.2 In-In Formalism

particle physics vs. cosmology



The in-in master formula is:

$$\begin{aligned}
\langle \hat{Q}(\eta) \rangle &\equiv \langle in|\, \hat{Q}(\eta)\, |in\rangle \\
&= \langle 0| \left[\bar{T} e^{i \int_{-\infty}^{\eta} {\rm d}\eta'\, H^I_{\rm int}(\eta')} \right] \hat{Q}^I(\eta) \left[T e^{i \int_{-\infty}^{\eta} {\rm d}\eta''\, H^I_{\rm int}(\eta'')} \right] |0\rangle \\
&= \boxed{\underbrace{-i \int_{-\infty}^{\eta} {\rm d}\eta' \langle 0| \left[\hat{Q}^I(\eta), H^I_{\rm int}(\eta') \right] |0\rangle}_{\text{tree level}} \ + \underbrace{\ \cdots\ }_{\text{loops}}}
\end{aligned} \tag{1.9}$$

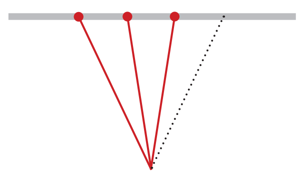
where $\hat{Q} \equiv \hat{\zeta}_{\mathbf{k}_1} \hat{\zeta}_{\mathbf{k}_2} \cdots \hat{\zeta}_{\mathbf{k}_n}$.

1.6 Examples

1.6.1 Contact Interactions

Consider higher-derivative interactions during inflation.
For example:

$$\frac{(\partial_\mu \phi)^4}{8\Lambda^4} \quad \Rightarrow$$



$$\Leftrightarrow \quad \mathcal{L}_{\rm int} = -\frac{\dot{\phi}}{4\Lambda^4}\,\dot{\pi}(\partial_\mu \pi)^2 + \cdots\,,$$

where $\pi(x) \equiv \phi(x) - \bar{\phi}(\eta)$.

Ex: Show that corresponding bispectrum is of the form

$$\frac{B(k_1,k_2,k_3)}{P^2} = \frac{8}{k_1k_2k_3}\frac{\dot{\phi}^2}{\Lambda^4}\frac{\text{Poly}[k^5]}{K^2}\,, \tag{1.10}$$

where $K \equiv k_1 + k_2 + k_3$.

Note that:

- The signal peaks in the equilateral configuration, $k_1 = k_2 = k_3$.
- The squeezed limit, $\lim_{k_3\to 0}\langle \zeta_{\mathbf{k}_1}\zeta_{\mathbf{k}_2}\zeta_{\mathbf{k}_3}\rangle$, is an analytic function of k_3/k_1.

Solution.—Feeding $\mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int}$ into the in-in master formula, we find

$$\frac{B}{P^2} = -\frac{4i}{k_1k_2k_3}\frac{\dot{\phi}^2}{\Lambda^4}\int_{-\infty}^{0} \mathrm{d}\eta\,\Big(\underset{\substack{\uparrow\\ \dot{\pi}^3}}{-k_1^2k_2^2k_3^2\,\eta^2} - \underset{\substack{\uparrow\\ \dot{\pi}(\partial_i\pi)^2}}{(\mathbf{k}_1\cdot\mathbf{k}_2)k_3^2(1-ik_1\eta)(1-ik_2\eta)}\Big)\,e^{iK\eta} + \text{perms.} + c.c.\,, \tag{1.11}$$

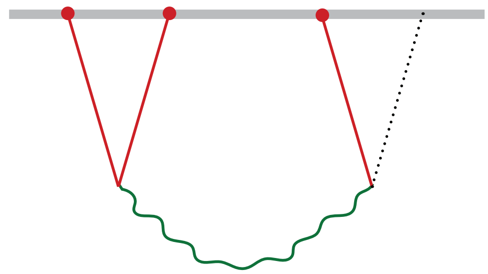
where we have used the Bunch-Davies mode function. Evaluating the integral, we get

$$\frac{B}{P^2} = \frac{8}{k_1k_2k_3}\frac{\dot{\phi}^2}{\Lambda^4}\frac{1}{K^2}\left(\sum_n k_n^5 + \sum_{n\neq m}(2k_n^4k_m - 3k_n^3k_m^2) + \sum_{n\neq m\neq l}(k_n^3k_mk_l - 4k_n^2k_m^2k_l)\right), \tag{1.12}$$

which is of the form shown in (1.10).

1.6.2 Graviton Exchange

A universal amount of non-Gaussianity in slow-roll inflation comes from graviton exchange:



This was computed by Maldacena in 2002. The result is

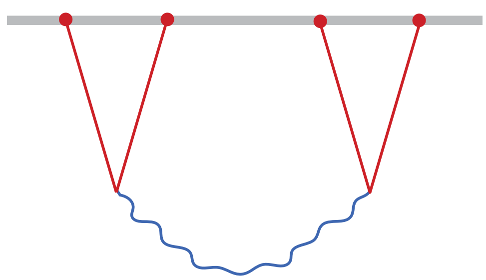
$$\frac{B(k_1,k_2,k_3)}{P^2} = \frac{\varepsilon}{k_1k_2k_3}\left[\sum_{n\neq m} k_n k_m^2 + \frac{8}{K}\sum_{n>m} k_n^2 k_m^2\right] + \frac{n_s-1}{k_1k_2k_3}\sum_n k_n^3\,, \qquad (1.13)$$

where $\varepsilon = -\dot{H}/H^2$. Note that the signal is still analytic in the squeezed limit.

1.6.3 Massive Particles

Non-analyticity in the squeezed limit arises from massive particles:

$$F \;=$$



$$\sim\; -g^2 \int \frac{\mathrm{d}\eta}{\eta^2}\frac{\mathrm{d}\eta'}{\eta'^2}\, e^{i(k_1+k_2)\eta} e^{i(k_3+k_4)\eta'}\, G(|\mathbf{k}_1+\mathbf{k}_2|,\eta,\eta')\,. \qquad (1.14)$$

↑ written for conformally coupled scalars

↑ complicated function of Hankel functions

Instead of trying to compute the integral, we note that G satisfies

$$\left(\eta^2\partial_\eta^2 - 2\eta\partial_\eta + k_I^2\eta^2 + m^2\right) G(k_I,\eta,\eta') = -i\eta^2\eta'^2\,\delta(\eta-\eta')\,. \qquad (1.15)$$

Since $k_I^3 G$ depends only on $k_I\eta$ and $k_I\eta'$, we can trade η-derivatives for k_I-derivatives. This gives

$$\frac{1}{k_I}\left(k_I^2\partial_{k_I}^2 - 2k_I\partial_{k_I} - k_I^2\partial_{k_1+k_2}^2 + m^2 - 2\right)(k_I^2 F) = g^2\frac{k_I}{E}\,. \qquad (1.16)$$

It is useful to write this as

$$\boxed{\big(u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + m^2 - 2\big)\,\hat{F} = g^2\,\frac{uv}{u+v}}\,, \tag{1.17}$$

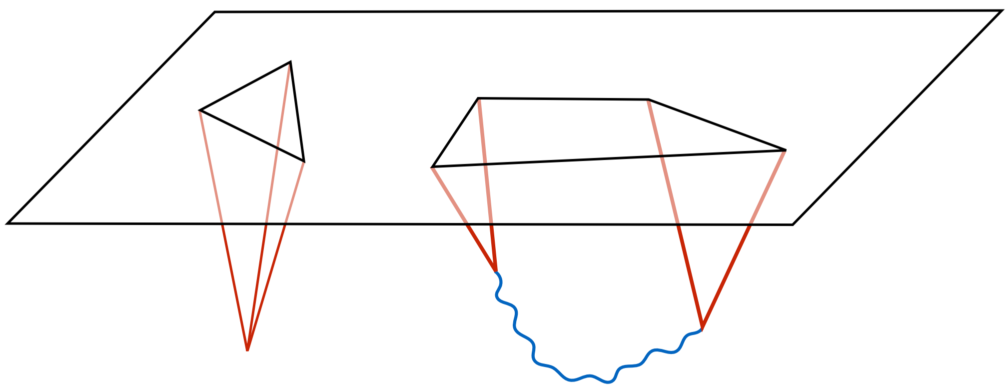
where $\hat{F}(u,v) \equiv k_I F$, with $u^{-1} \equiv (k_1 + k_2)/k_I$ and $v^{-1} \equiv (k_3 + k_4)/k_I$. Permutation symmetry implies a second equation with $u \leftrightarrow v$.

In the next section, I will first show that this differential equation for F can be derived without refering to the bulk, and then present its solutions.

2 Bootstrapping Inflationary Correlators

2.1 Time Without Time

All cosmological correlations can be traced back to the spacelike boundary of the inflationary quasi-de Sitter spacetime:



The time dependence of bulk interactions is encoded in the momentum dependence of these boundary correlators.

Is there a purely boundary way to derive these correlators?

2.2 De Sitter Space

The metric of de Sitter space (in conformal coordinates) is

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2}{(H\eta)^2}\,. \tag{2.1}$$

Besides ordinary spatial rotations and translations, the metric is invariant under spacetime dilatations and special conformal transformations:

$$\text{D:} \quad \begin{aligned} \eta &\to \lambda\eta \\ \mathbf{x} &\to \lambda\mathbf{x} \end{aligned} \tag{2.2}$$

$$\text{SCT:} \quad \begin{aligned} \eta &\to \frac{\eta}{1+2(\mathbf{b}\cdot\mathbf{x})+b^2(x^2-\eta^2)} \\ \mathbf{x} &\to \frac{\mathbf{x}+(x^2-\eta^2)\mathbf{b}}{1+2(\mathbf{b}\cdot\mathbf{x})+b^2(x^2-\eta^2)} \end{aligned} \tag{2.3}$$

In the limit $\eta \to 0$, these symmetries act as conformal transformations on $\mathbb{R}^3$.

Consider a massive scalar field in de Sitter space:

$$\phi'' - \frac{2}{\eta}\phi' - \nabla^2\phi + \frac{m^2}{H^2}\frac{\phi}{\eta^2} = 0\,. \tag{2.4}$$

At late times, the solution is

$$\phi(\eta, \mathbf{x}) \approx C_+ \eta^{\Delta_+} O_+(\mathbf{x}) + C_- \eta^{\Delta_-} O_-(\mathbf{x})\,, \tag{2.5}$$

where

$$\Delta_\pm = \frac{3}{2} \pm \underbrace{\sqrt{\frac{9}{4} - \frac{m^2}{H^2}}}_{i\mu}\,. \tag{2.6}$$

In the limit $\eta \to 0$, the second part of the solution dominates.

Bunch-Davies mode functions.— For general m, the solution of (2.4), with Bunch-Davies initial conditions, is

$$\phi_k(\eta) = \frac{H}{k^{3/2}} \times \frac{\sqrt{\pi}}{2} e^{-\frac{\pi}{2}\mu + i\frac{\pi}{4}} (-k\eta)^{3/2} H^{(1)}_{i\mu}(-k\eta)\,, \tag{2.7}$$

where $H^{(1)}_{i\mu}$ is a Hankel function of the first kind.

Two important special cases are

$$m^2 = 0: \qquad \phi_k(\eta) = \frac{H}{\sqrt{2k^3}}(1 + ik\eta)e^{-ik\eta}\,, \tag{2.8}$$

$$m^2 = 2H^2: \qquad \phi_k(\eta) = \frac{H}{\sqrt{2k^3}}\, ik\eta\, e^{-ik\eta}\,. \tag{2.9}$$

The case of conformally coupled scalars ($m^2 = 2H^2$) is particular useful for analytic computations. It will play an important role in these lectures.

We define the operator $O \equiv O_+$ (with $\Delta \equiv \Delta_+$) and its shadow $\tilde{O} \equiv O_-$ (with $\tilde{\Delta} \equiv \Delta_- = 3 - \Delta$). Correlators of O and $\tilde{O}$ are related by

$$\langle \tilde{O}(\mathbf{k}_1)\tilde{O}(\mathbf{k}_2)\cdots\tilde{O}(\mathbf{k}_N)\rangle' = \frac{\langle O(\mathbf{k}_1)O(\mathbf{k}_2)\cdots O(\mathbf{k}_N)\rangle'}{(k_1k_2\cdots k_N)^{2\Delta-3}}\,. \tag{2.10}$$

The form of the boundary correlators is constrained by conformal symmetry.

2.3 Conformal Field Theory

Consider $\mathbb{R}^d$, with $\mathrm{d}s^2 = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$.

A conformal transformation is a coordinate transformation $x^i \to \tilde{x}^i$ that leaves the metric invariant up to a scale change,

$$g_{ij}(x) \to \tilde{g}_{ij}(\tilde{x}) = \Omega^2(x) g_{ij}(x)\,. \tag{2.11}$$

For $d > 2$, the infinitesimal transformation $x^i \to x^i + \epsilon^i(x)$ is a conformal transformation if

$$\epsilon^i(x) \;=\; \underbrace{a^i}_{\mathrm{T}} \;+\; \underbrace{r^{ij}x_j}_{\mathrm{R}} \;+\; \underbrace{\alpha x^i}_{\mathrm{D}} \;+\; \underbrace{x^2 b^i - 2(b\cdot x)x^i}_{\mathrm{SCT}}\,, \tag{2.12}$$

where $r_{ij} = -r_{ji}$.

Derivation.—The metric transforms as

$$\begin{aligned} g_{ij} \to \tilde{g}_{ij} = \frac{dx^a}{d\tilde{x}^i}\frac{dx^a}{d\tilde{x}^j}\, g_{ab} &= (\delta_i^a - \partial_i\epsilon^a)(\delta_j^b - \partial_j\epsilon^b) g_{ab} \\ &= g_{ij} - (\partial_i\epsilon_j + \partial_j\epsilon_i)\,. \end{aligned}$$

For a conformal transformation, we require $\partial_i\epsilon_j + \partial_j\epsilon_i$ to be proportional to the metric,

$$\partial_i\epsilon_j + \partial_j\epsilon_i = f(x)\, g_{ij}\,. \tag{$\star$}$$

The factor $f(x)$ is found by taking the trace on both sides:

$$f(x) = \frac{2}{d}\,\partial\cdot\epsilon\,.$$

Acting with ∂_k on ($\star$), permuting the indices and taking a linear combination, we find

$$2\partial_i\partial_j\epsilon_k = (-g_{ij}\partial_k + g_{ki}\partial_j + g_{jk}\partial_i) f\,. \tag{$\star\star$}$$

Contracting with g^{ij}, this becomes

$$2\Box\epsilon_i = (2-d)\partial_i f\,.$$

Applying ∂_j to this expression, and using ($\star$), we find

$$g_{ij}\Box f = (2-d)\partial_i\partial_j f\,.$$

Contracting this with g^{ij}, we obtain $(d-1)\Box f = 0$, so that the previous expression becomes

$$\partial_i\partial_j f = 0\,.$$

The function $f(x)$ is therefore at most linear in x. Substituting $f(x) = A + B_i x^i$ into ($\star\star$), we see that $\partial_i \partial_j \epsilon_k$ is constant, which means that ϵ_i is at most quadratic in x:

$$\epsilon_i = a_i + b_{ij} x^j + c_{ijk} x^j x^k \,.$$

Since the constraints ($\star$) and ($\star\star$) hold for all x, we can treat each power of the transformation separately. Substituing the linear term into ($\star$), we find

$$b_{ij} = \alpha\, g_{ij} + r_{ij} \,,$$

where $r_{ij} = -r_{ji}$. Similarly, substituting the quadratic term into ($\star\star$), we get

$$c_{ijk} = g_{ij} b_k + g_{ik} b_j - g_{jk} b_i \,, \quad b_i \equiv \frac{1}{d}\, g^{lm} c_{lmi} \,.$$

The special conformal transformation then takes the form

$$\begin{aligned} \epsilon_i &= c_{ijk} x^j x^k \\ &= (g_{ij} b_k + g_{ik} b_j - g_{jk} b_i) x^j x^k \\ &= 2(b \cdot x) x_i - x^2 b_i \,, \end{aligned}$$

which establishes the result (2.12).

The corresponding finite transformations are

$$\text{T:} \qquad \tilde{x}^i = a^i \qquad \Omega(x) = 1 \tag{2.13}$$

$$\text{R:} \qquad \tilde{x}^i = R^{ij} x_j \qquad \Omega(x) = 1 \tag{2.14}$$

$$\text{D:} \qquad \tilde{x}^i = \lambda x^i \qquad \Omega(x) = \lambda^{-1} \tag{2.15}$$

$$\text{SCT:} \qquad \tilde{x}^i = \frac{x^i - b^i x^2}{1 - 2\, b \cdot x + b^2 x^2} \quad \Omega(x) = 1 - 2\, b \cdot x + b^2 x^2 \,. \tag{2.16}$$

The special conformal transformation can also be written as

$$\frac{\tilde{x}^i}{\tilde{x}^2} = \frac{x^i}{x^2} - b^i \,, \tag{2.17}$$

i.e. it can be thought of as an translation, preceded and followed by an inversion. The distance between two points transforms as

$$|x_n - x_m| \to |\tilde{x}_n - \tilde{x}_m| = \frac{|x_n - x_m|}{(\Omega_n \Omega_m)^{1/2}} \,, \tag{2.18}$$

where $\Omega_n = \Omega(x_n)$.

Next, consider

$$\langle O(x_1)\cdots O(x_N)\rangle = \frac{1}{Z}\int [\mathrm{d}O]\, O(x_1)\cdots O(x_N)\exp(-S[O])\,. \tag{2.19}$$

We wish to understand how these correlators are constrained by conformal symmetry.

Acting on scalar primary operators O, a conformal transformation implies

$$O(x) \to \tilde{O}(\tilde{x}) = \Omega(x)^{\Delta} O(x)\,, \tag{2.20}$$

where Δ is the scaling dimension of the operator.

The above correlators must then satisfy

$$\langle O_1(\tilde{x}_1)\dots O_N(\tilde{x}_N)\rangle = \Omega(x_1)^{\Delta_1}\cdots\Omega(x_N)^{\Delta_N}\langle O_1(x_1)\dots O_N(x_N)\rangle\,. \tag{2.21}$$

Derivation.—To verify the stated identity, we write

$$\begin{aligned}
\langle O1(\tilde{x}_1)\cdots O_N(\tilde{x}_N)\rangle &= \frac{1}{Z}\int [\mathrm{d}O]\, O_1(\tilde{x}_1)\cdots O_N(\tilde{x}_N)\ \exp(-S[O]) \\
&= \frac{1}{Z}\int [\mathrm{d}\tilde{O}]\, \tilde{O}_1(\tilde{x}_1)\cdots \tilde{O}_N(\tilde{x}_N)\ \exp(-S[\tilde{O}]) \\
&= \frac{1}{Z}\int [\mathrm{d}O]\, \Omega(x_1)^{-\Delta_1}O_1(\tilde{x}_1)\cdots \Omega(x_N)^{-\Delta_N}O_N(\tilde{x}_N)\ \exp(-S[O]) \\
&= \Omega(x_1)^{-\Delta_1}\cdots\Omega(x_N)^{-\Delta_N}\langle O_1(x_1)\dots O_N(x_N)\rangle\,,
\end{aligned}$$

where we have assumed that both the action S and the integral measure $[\mathrm{d}O]$ are invariant under the transformation.

Ex: Show that (2.21) implies

$$\langle O_1 O_2\rangle = \frac{1}{x_{12}^{2\Delta_1}}\,\delta_{\Delta_1,\Delta_2}\,, \tag{2.22}$$

$$\langle O_1 O_2 O_3\rangle = \frac{c_{123}}{x_{12}^{\Delta_t-2\Delta_3}x_{23}^{\Delta_t-2\Delta_1}x_{31}^{\Delta_t-2\Delta_2}}\,, \tag{2.23}$$

where $O_n \equiv O_n(x_n)$, $x_{nm} \equiv |x_n - x_m|$ and $\Delta_t \equiv \sum \Delta_n$.

Solution.—Specializing to dilatations, the identity (2.21) implies

$$\langle O_1(\lambda x_1)O_2(\lambda x_2)\rangle = \lambda^{-\Delta_1}\lambda^{-\Delta_2}\langle O_1(x_1)O_2(x_2)\rangle\,.$$

Invariance under rotations and translations implies that the two-point function can only be a function of $|x_1 - x_2|$, i.e.

$$\langle O_1(x_1)O_2(x_2)\rangle = f(|x_1 - x_2|)\,,$$

with $f(\lambda x) = \lambda^{-\Delta_1-\Delta_2} f(x)$, and hence

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{c_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,,$$

where c_{12} is a constant coefficient, which can be set to $c_{12} \equiv 1$ by a constant rescaling of the operators. For a SCT, the constraint (2.21) implies

$$\frac{1}{|\tilde{x}_1 - \tilde{x}_2|^{\Delta_1+\Delta_2}} = \frac{(\Omega_1\Omega_2)^{(\Delta_1+\Delta_2)/2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{\Omega_1^{\Delta_1}\Omega_2^{\Delta_2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}\,,$$

where we have used (2.18) in the first equality. We see that this is only satisfied if $\Delta_1 = \Delta_2$, which proves (2.22).

Solution.—Invariance under rotations, translations, and dilatations force the three-point function to have the form

$$\langle O_1 O_2 O_3\rangle = \frac{c_{123}^{(abc)}}{x_{12}^a x_{23}^b x_{31}^c}\,,$$

where $x_{nm} \equiv |x_n - x_m|$ and

$$a + b + c = \Delta_1 + \Delta_2 + \Delta_3 \equiv \Delta_t\,.$$

Invariance under SCTs demands

$$\begin{aligned} a &= \Delta_1 + \Delta_2 - \Delta_3 = \Delta_t - 2\Delta_3\,,\\ b &= \Delta_2 + \Delta_3 - \Delta_1 = \Delta_t - 2\Delta_1\,,\\ c &= \Delta_3 + \Delta_1 - \Delta_2 = \Delta_t - 2\Delta_2\,, \end{aligned}$$

which leads to (2.23).

The four-point function of scalar operators is

$$\langle O_1 O_2 O_3 O_4\rangle = f(u,v)\prod_{n<m}^{4} x_{nm}^{\Delta_t/3-\Delta_n-\Delta_m}\,, \tag{2.24}$$

where we have introduced the conformally invariant cross-ratios

$$u \equiv \left(\frac{x_{12}x_{34}}{x_{13}x_{24}}\right)^2 , \quad v \equiv \left(\frac{x_{12}x_{34}}{x_{23}x_{14}}\right)^2 . \tag{2.25}$$

Comment.—The reason that there are two cross-ratios can be understood as follows:

- Using SCTs, we can move x_4 to infinity.
- Using T, we can move x_1 to zero.
- Using R and D, we can move x_3 to $(1, 0, \ldots, 0)$.
- Using R, with x_3 fixed, we can move x_2 to $(x, y, 0, \ldots, 0)$.

This leaves two undetermined quantities x and y, giving two independent conformal invariants:

$$u = z\bar{z} , \quad v = (1-z)(1-\bar{z}) ,$$

where $z \equiv x + iy$.

The above constraints can also be expressed as Ward identities.

Consider $x^i \to \tilde{x}^i = x^i + \epsilon^i(x)$ and define $\delta O(x) \equiv \tilde{O}(x) - O(x)$.

For a dilatation, we get

$$\begin{aligned}
\delta O(x) &= \tilde{O}(x) - O(x) \\
&= \tilde{O}(\tilde{x} - \epsilon) - O(x) \\
&= (1+\alpha)^{-\Delta} O(x) - \alpha x^j \partial_j O(x) - O(x) \\
&= -\alpha \big[\Delta + x^j \partial_j\big] O(x) \\
&\equiv \alpha \, D \, O(x) .
\end{aligned} \tag{2.26}$$

Similarly, for a special conformal transformation, we find

$$\begin{aligned}
\delta O(x) &= \tilde{O}(x) - O(x) \\
&= \tilde{O}(\tilde{x} - \epsilon) - O(x) \\
&= (1 - 2\, b \cdot x)^{-\Delta} O(x) - (x^2 b^j - 2(b \cdot x) x^j) \partial_j O(x) - O(x) \\
&= b^i \big[2\Delta\, x_i + 2 x_i x^j \partial_j - x^2 \partial_i\big] O(x) \\
&\equiv b^i \, K_i \, O(x) .
\end{aligned} \tag{2.27}$$

The invariance of the correlators then implies

$$\sum_{n=1}^{N} \langle O_1 \cdots \delta O_n \cdots O_N \rangle = 0 , \tag{2.28}$$

or, explicitly,

$$\mathrm{D}: \quad 0 = \sum_{n=1}^{N} \left(\Delta_n + x_n^j \frac{\partial}{\partial x_n^j} \right) \langle O_1 \cdots O_N \rangle \,, \tag{2.29}$$

$$\mathrm{SCT}: \quad 0 = \sum_{n=1}^{N} \left(\Delta_n x_n^i + x_n^i x_n^j \frac{\partial}{\partial x_n^j} - \frac{x_n^2}{2} \frac{\partial}{\partial x_{n,i}} \right) \langle O_1 \cdots O_N \rangle \,. \tag{2.30}$$

Ex: Show that (2.22) and (2.23) are solutions of these Ward identities.

In cosmology, we are interested in these constraints in Fourier space:

$$\mathrm{D}: \quad 0 = \sum_{n=1}^{N} \left((\Delta_n - 3) - k_n^j \frac{\partial}{\partial k_n^j} \right) \langle O_1 \cdots O_N \rangle' \,, \tag{2.31}$$

$$\mathrm{SCT}: \quad 0 = \sum_{n=1}^{N} \left((\Delta_n - 3) \frac{\partial}{\partial k_n^i} - k_n^j \frac{\partial^2}{\partial k_n^j k_n^i} + \frac{k_n^i}{2} \frac{\partial^2}{\partial k_n^j k_n^i} \right) \langle O_1 \cdots O_N \rangle' \,. \tag{2.32}$$

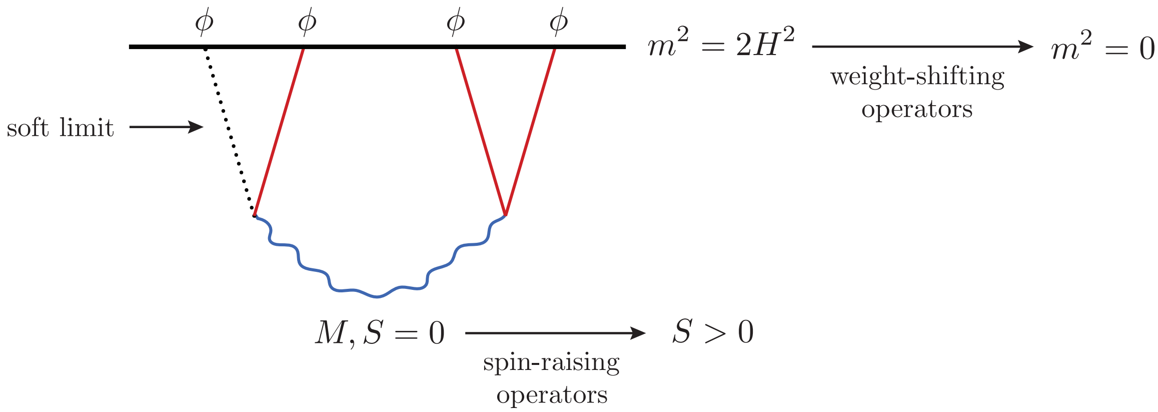
In the following, we will study the solutions to these equations.

2.4 De Sitter Four-Point Functions

2.4.1 A Road Map

The fundamental object will be the four-point function of conformally coupled scalars, $F = \langle \varphi\varphi\varphi\varphi \rangle$, mediated by the exchange of a massive scalar.

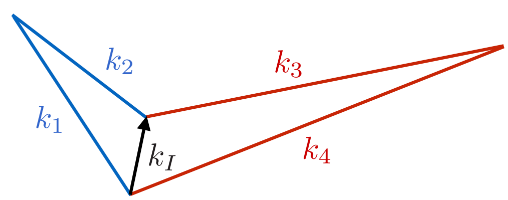
Everything else can be derive from this building block:



2.4.2 Kinematics

The four-point function of conformally coupled scalars can be written as

$$\langle \varphi_{\mathbf{k}_1}\varphi_{\mathbf{k}_2}\varphi_{\mathbf{k}_3}\varphi_{\mathbf{k}_4} \rangle' =$$



$$= \frac{1}{k_I}\hat{F}(u, v)\,, \tag{2.33}$$

where we have introduced the dimensionless variables

$$u^{-1} = \frac{k_1 + k_2}{k_I}\,, \qquad v^{-1} = \frac{k_3 + k_4}{k_I}\,. \tag{2.34}$$

Ex: Show that this ansatz solves the dilatation Ward identity (2.31).

2.4.3 Conformal Ward Identities

After some work, the conformal Ward identity (2.32) can be written as

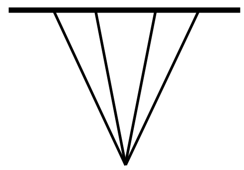
$$\boxed{(\nabla_u - \nabla_v)\hat{F} = 0}\,, \tag{2.35}$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$ (hypergeometric).

2.4.4 Contact Interactions

The simplest solutions to (2.40) correspond to contact interactions:

$$\hat{F}_c \equiv$$



$$= \sum_n \underset{\substack{\uparrow \\ \varphi^4,\,(\partial_\mu\varphi)^4,\,\cdots}}{\frac{c_n(u,v)}{E^{2n+1}}}\,, \tag{2.36}$$

which have poles at vanishing total energy

$$E \equiv \sum_n k_n = \frac{u+v}{uv}\, k_I\,. \tag{2.37}$$

Note that $F_c^{(n)} = \Delta_u^n F_c^{(0)}$, where $F_c^{(0)} \equiv uv/(u+v)$.

2.4.5 Exchange Interactions

For tree exchange, we try

$$(\Delta_u + M^2)\hat{F} = \hat{F}_c\,, \tag{2.38}$$

$$(\Delta_v + M^2)\hat{F} = \hat{F}_c\,, \tag{2.39}$$

where $\hat{F}_c$ is a contact solution.

Ex: Show that $\hat{F}_c$ must satisfy $(\nabla_u - \nabla_v)\hat{F}_c = 0$.

Solution.—Consider ∇_v (2.38) $-$ ∇_u (2.39). This leads to

$$0 = (\nabla_v - \nabla_u)\hat{F}_c\,,$$

which is the claimed result.

Using the simplest contact interaction as a source, we have

$$\left[u^2(1-u^2)\partial_u^2 - 2u^3\partial_u + M^2\right]\hat{F} = g^2\,\frac{uv}{u+v}\,, \tag{2.40}$$

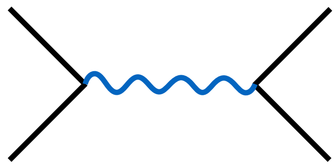
which is the same as (1.17) if $M^2 \equiv m^2 - 2 = \mu^2 + \dfrac{1}{4}$.

Let's study this!

2.4.6 Singularities

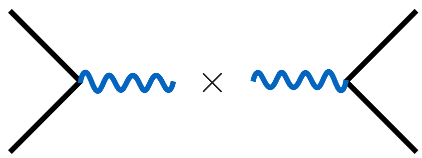
The equation has a number of interesting singularities:

- Flat-space limit: $\lim_{u \to -v} \hat{F} = A_4 \, (u+v) \log(u+v)$

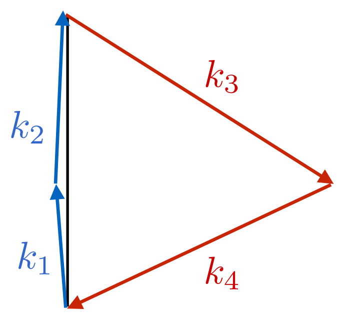


The correlator contains the scattering amplitude.

- Factorization limit: $\lim_{u,v \to -1} \hat{F} = A_3 \log(1+u) \times A_3 \log(1+v)$

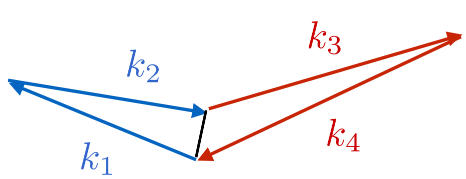


- Folded limit: $\lim_{u \to +1} \hat{F} \propto \log(1-u)$



This singularity should be absent in the standard vacuum.

- Collapsed limit: $\lim_{u \to 0} \hat{F} \propto u^{\frac{1}{2}+i\mu}$



This non-analyticity corresponds to spontaneous particle production.

Imposing regularity in the folded limit and the correct normalization in the factorization limit uniquely fixes the solution.

2.4.7 EFT Expansion

A formal solution of (2.40) is

$$\hat{F} = \frac{\hat{F}_c^{(0)}}{\Delta_u + M^2} = \sum_n \frac{1}{n!}\left(\frac{\Delta_u}{M^2}\right)^n \frac{\hat{F}_c^{(0)}}{M^2} \tag{2.41}$$

$$= \underset{\underset{\varphi^4}{\uparrow}}{\frac{\hat{F}_c^{(0)}}{M^2}} + \underset{\underset{\varphi^2(\partial_\mu\varphi)^2}{\uparrow}}{\frac{\hat{F}_c^{(1)}}{M^4}} + \underset{\underset{(\partial_\mu\varphi)^4}{\uparrow}}{\frac{1}{2}\frac{\hat{F}_c^{(2)}}{M^6}} + \cdots \tag{2.42}$$

However, this EFT expansion misses the important physics of particle production in the expanding spacetime.

2.4.8 Particle Production

Consider first the limit $v \to 0$, where the source is small.

Writing $e^t \equiv u/v \equiv \xi$ and $f \equiv (uv)^{-1/2}\hat{F}$, equation (2.40) becomes

$$\left[\frac{d^2}{dt^2} + \mu^2\right] f = \frac{1}{2\cosh(\frac{1}{2}t)}\,, \tag{2.43}$$

which is the equation of a forced harmonic oscillator.

The homogeneous solutions are

$$f_\pm = e^{\pm i\mu t} = \xi^{\pm i\mu}\,. \tag{2.44}$$

Around $\xi = 0$, the inhomogeneous solution is

$$f_<(\xi) = \sqrt{\xi}\sum_{n=0}^{\infty}(-1)^n \frac{\xi^n}{(n+\frac{1}{2})^2 + \mu^2}\,, \tag{2.45}$$

which is convergent for $\xi \le 1$ and divergent for $\xi > 1$.

Around $\xi = \infty$, we have

$$f_>(\xi) = \frac{1}{\sqrt{\xi}}\sum_{n=0}^{\infty}(-1)^n \frac{\xi^{-n}}{(n+\frac{1}{2})^2 + \mu^2}\,, \tag{2.46}$$

which is convergent for $\xi \ge 1$ and divergent for $\xi < 1$.

Matching the solutions at $\xi = 1$, we find

$$\tilde{F}_<(\xi) = \begin{cases} \displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{\xi^{n+1}}{(n+\frac{1}{2})^2+\mu^2} & \xi \leq 1\,, \\ \underbrace{\displaystyle\sum_{n=0}^{\infty}(-1)^n \frac{\xi^{-n}}{(n+\frac{1}{2})^2+\mu^2}}_{\text{EFT expansion}} + \underbrace{\frac{\pi}{\cosh\pi\mu}\frac{\xi^{\frac{1}{2}-i\mu}-\xi^{\frac{1}{2}+i\mu}}{2i\mu}}_{\text{non-perturbative correction}} & \xi \geq 1\,. \end{cases} \tag{2.47}$$

The general solution is derived in a similar way.

The homogeneous solutions are

$$\hat{F}_\pm(u) = \left(\frac{iu}{2\mu}\right)^{\frac{1}{2}\pm i\mu} {}_2F_1\left[\left.\begin{matrix}\frac{1}{4}\pm\frac{i\mu}{2}, \frac{3}{4}\pm\frac{i\mu}{2} \\ 1\pm i\mu\end{matrix}\,\right|\, u^2\right]. \tag{2.48}$$

Around $u = 0$, the inhomogeneous solution is

$$\hat{F}_<(u,v) \sum_{m,n=0}^{\infty} c_{mn}(\mu) u^{2m+1}(u/v)^n\,, \tag{2.49}$$

where $c_{mn}(\mu)$ are known coefficients [arXiv:1811.00024].

Around $u = \infty$, we have

$$F_>(u,v) = F_<(v,u)\,. \tag{2.50}$$

Matching at $u = v$, we find

$$\hat{F}_<(u,v) = \begin{cases} \displaystyle\sum_{m,n=0}^{\infty} c_{mn} u^{2m+1}(u/v)^n & u \leq v\,, \\ \displaystyle\sum_{m,n=0}^{\infty} c_{mn} v^{2m+1}(v/u)^n + \frac{\pi}{\cosh\pi\mu}\hat{F}_h(u,v) & u \geq v\,, \end{cases} \tag{2.51}$$

where $\hat{F}_h(u,v) \equiv \hat{F}_+(v)\hat{F}_-(u) - \hat{F}_-(v)\hat{F}_+(u)$.

We still have the freedom to add homogeneous solutions. In fact, we *must* add them to satisfy the boundary conditions

$$\lim_{u\to+1} \hat{F} = \text{regular} \tag{2.52}$$

$$\lim_{u,v\to-1} \hat{F} = \frac{1}{2}\log(1+u)\log(1+v)\,. \tag{2.53}$$

The final result is

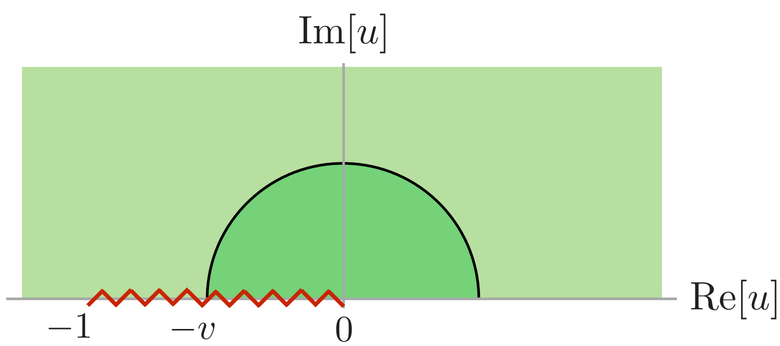
$$\hat{F}(u,v) = \begin{cases} \displaystyle\sum_{m,n=0}^{\infty} c_{mn} u^{2m+1} (u/v)^n + \frac{\pi}{2\cosh \pi\mu}\, \hat{g}(u,v) & u \le v\,, \\ \displaystyle\sum_{m,n=0}^{\infty} c_{mn} v^{2m+1} (v/u)^n + \frac{\pi}{2\cosh \pi\mu}\, \hat{g}(v,u) & u \ge v\,, \end{cases} \tag{2.54}$$

where $\hat{g}(u,v)$ is a known function [arXiv:1811.00024].

2.4.9 Flat-Space Limit

An interesting limit is $u \to -v$ (or $E = \sum k_n \to 0$).

In this limit, the solution has a branch cut singularity:



The discontinuity across the cut is

$$\lim_{u \to -v} \frac{\mathrm{Disc}[\hat{F}']}{2\pi i} = \frac{1}{(k_1+k_2)^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2} = A_4\,. \tag{2.55}$$

This relates curved-space particle production to flat-space scattering.

2.4.10 Soft Limit

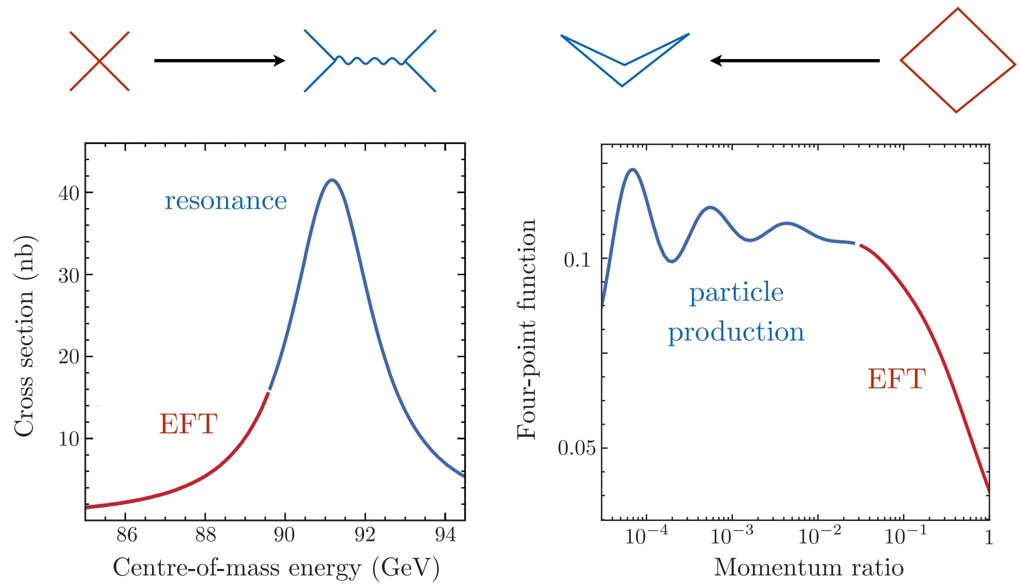
Another important limit is $u, v \to 0$.

In this limit, the particle production piece dominates:

$$\lim_{u,v\to 0} \hat{F}(u,v) = \underset{\substack{\uparrow \\ \text{positivity}}}{g^2} \underset{\substack{\uparrow \\ \text{oscillations}}}{\left(\frac{uv}{4}\right)^{\frac{1}{2}+i\mu}} \underset{\substack{\uparrow \\ e^{-\pi\mu},\ \text{for large } \mu}}{(1 + i \sinh \pi\mu)} \frac{\Gamma(\frac{1}{2}+i\mu)^2 \Gamma(-i\mu)^2}{2\pi} + c.c. \tag{2.56}$$

2.4.11 Spectroscopy

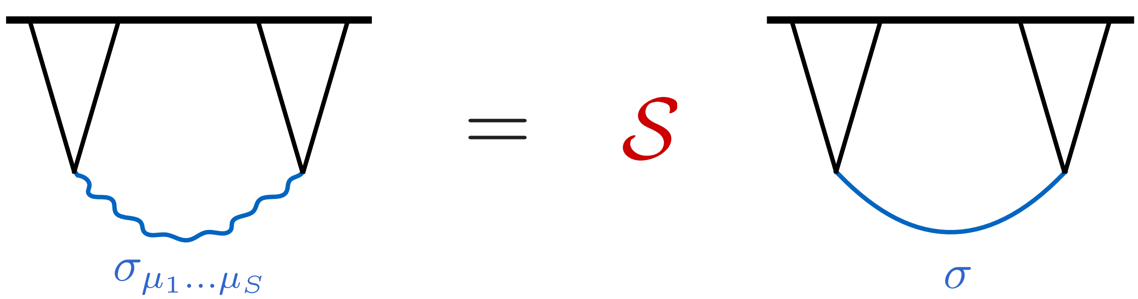
In the collapsed limit, the signal oscillates with a frequency given by the mass of the new particles. This is the analog of resonances in collider physics.



2.5 Exchange of Spinning Particles

2.5.1 Strategy

Find differential operators that relate scalar exchange to spin exchange:



It turns out that the spin raising is best implemented in embedding space and then Fourier transformed.

2.5.2 CFTs in Embedding Space

Conformal transformations are complicated, especially for spinning operators. By embedding $\mathbb{R}^d$ into $\mathbb{R}^{1,d+1}$, they become as simple as Lorentz transformations.

Projective Null Cone

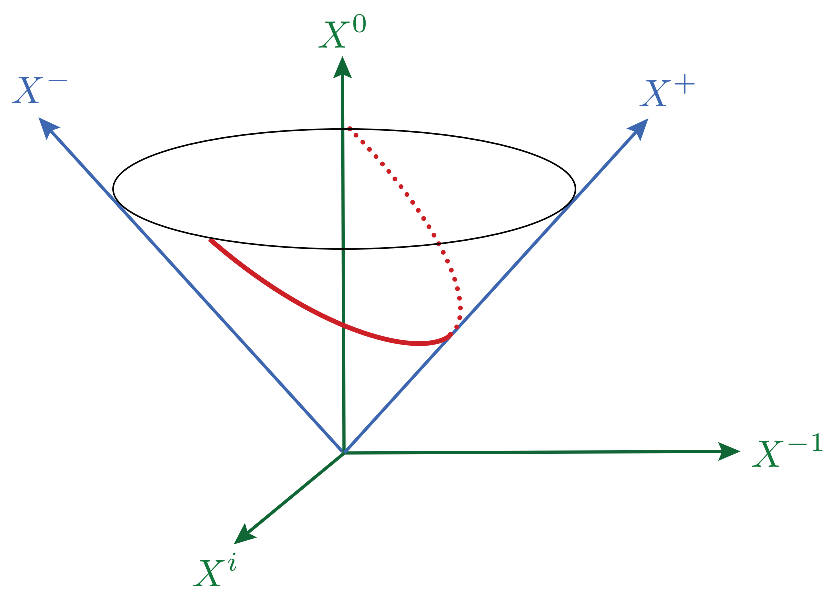
Consider $d+2$ dimensional Minkowski space, with coordinates

$$X^M,\ M = -1, 0, 1, \ldots, d\,.$$
$$X^\pm \equiv X^0 \pm X^{-1}$$

The embedding of $\mathbb{R}^d$ into $\mathbb{R}^{1,d+1}$ is defined by

- $X^2 = 0$ (null cone)
- $X^+ = 1$ (Euclidean section)

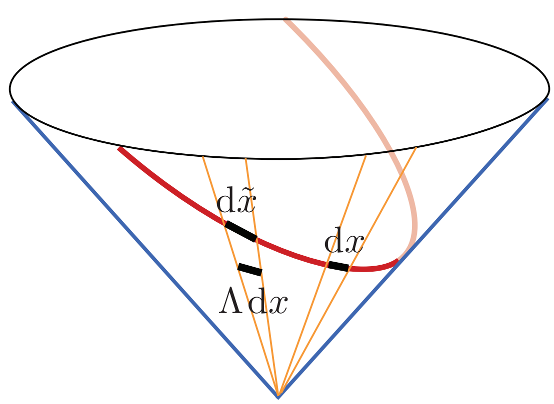
$$\Rightarrow \quad X^M = (X^+, X^-, X^i) = (1, x^2, x^i)$$



Lorentz transformations on $\mathbb{R}^{1,d+1}$ become conformal transformations on $\mathbb{R}^d$:

- $X^M \to \Lambda^M{}_N X^N$
- $X^M \to \lambda\, X^M$

$$\Rightarrow \quad g_{ij} \to \tilde{g}_{ij} = \Omega^2(x) g_{ij}, \ \text{with} \ \ \Omega(x) = \lambda(X).$$



Conformal transformations of fields on $\mathbb{R}^d$ are scaling transformations on $\mathbb{R}^{1,d+1}$:

$$O(\lambda X) = \lambda^{-\Delta} O(X) \quad \Rightarrow \quad O(\tilde{x}) = \Omega(x)^{\Delta} O(x)\,. \tag{2.57}$$

To determine conformal correlators, we simply write the most general Lorentz-invariant expression that is consistent with the scaling in (2.57).

Examples

- The two- and three-point functions of scalar operators are:

$$\langle O_1 O_2 \rangle = \frac{1}{X_{12}^{\Delta_1}}\,, \tag{2.58}$$

$$\langle O_1 O_2 O_3 \rangle = \frac{c_{123}}{X_{12}^{(\Delta_1+\Delta_2-\Delta_3)/2} X_{23}^{(\Delta_2+\Delta_3-\Delta_1)/2} X_{31}^{(\Delta_3+\Delta_1-\Delta_2)/2}}\,, \tag{2.59}$$

  where $X_{nm} \equiv X_n \cdot X_m$.

  Using

$$\begin{aligned} X_{12} &= \delta_{ij} X_1^i X_2^j - \frac{1}{2}(X_1^+ X_2^- + X_1^- X_2^+) \\ &= \delta_{ij} x_1^i x_2^j - \frac{1}{2}(x_1^2 + x_2^2) \\ &= -\frac{1}{2}(x_1 - x_2)^2 \\ &= -\frac{1}{2} x_{12}^2\,, \end{aligned} \tag{2.60}$$

  we see that this is equivalent to our previous results.

- The scalar four-point function, for identical fields ($\Delta_i = \Delta$), is

$$\langle O_1 O_2 O_3 O_4 \rangle = \frac{1}{X_{12}^{\Delta} X_{34}^{\Delta}}\, f(u,v)\,, \tag{2.61}$$

  with

$$u \equiv \frac{X_{12} X_{34}}{X_{13} X_{24}}\,, \quad v = u(2 \leftrightarrow 4)\,. \tag{2.62}$$

  Projecting to the Euclidean section, this reproduces our previous result.

Fields with Spin

Consider symmetric traceless tensors on $\mathbb{R}^{1,d+1}$. They are projected to tensors on the Euclidean section via

$$O_{i_1 i_2 \dots}(x) = O_{M_1 M_2 \dots}(X)\, \frac{\partial X^{M_1}}{\partial x^{i_1}} \frac{\partial X^{M_2}}{\partial x^{i_2}} \cdots \,, \tag{2.63}$$

where

$$\frac{\partial X^M}{\partial x^i} = (0, 2x_j, \delta_i^j)\,. \tag{2.64}$$

Extra components are removed by

- $X^M O_{M\cdots}(X) = 0$ (transversality)
- $O_{M\cdots} + X_M(\cdots) \sim O_{M\cdots}$ ("gauge invariance")

Lorentz transformations on $\mathbb{R}^{1,d+1}$ become conformal transformations on $\mathbb{R}^d$:

- $\tilde{O}_{M_1\cdots}(\tilde{X}) = \Lambda_{M_1}{}^{N_1} \cdots O_{N_1\cdots}(X)$
- $O_{M_1\cdots}(\lambda X) = \lambda^{-\Delta} O_{M_1\cdots}(X)$

$$\Rightarrow \quad \tilde{O}_{i_1\cdots}(\tilde{x}) = \Omega(x)^{\Delta}\, M_{i_1}^{j_1} \cdots O_{j_1\cdots}(x)$$

Examples

Let

$$\Sigma_n^{(S)} \equiv Z_n^{M_1} \cdots Z_n^{M_S} \Sigma_{M_1 \dots M_S}(X_n)\,, \tag{2.65}$$

where Z is an arbitrary null vector.

The two-point function of spin-S fields then is

$$\langle \Sigma_1^{(S)} \Sigma_2^{(S)} \rangle = \left(Z_1 \cdot Z_2 - \frac{Z_1 \cdot X_2\, Z_2 \cdot X_1}{X_{12}} \right)^S \langle \Sigma_1 \Sigma_2 \rangle\,, \tag{2.66}$$

where $\langle \Sigma_1 \Sigma_2 \rangle$ is given by (2.58) and the relative coefficient in the tensor structure is fixed by transversality.

The scalar-scalar-spin-S three-point function is

$$\langle O_1 O_2 \Sigma_3^{(S)} \rangle = \left(\frac{(Z_3 \cdot X_1)(X_2 \cdot X_3) - (Z_3 \cdot X_2)(X_1 \cdot X_3)}{(X_{12} X_{13} X_{23})^{1/2}} \right)^S \langle O_1 O_2 \Sigma_3 \rangle\,, \tag{2.67}$$

where $\langle O_1 O_2 \Sigma_3 \rangle$ is given by (2.59).

2.5.3 Spin-Raising Operator

Consider a conformally coupled scalar φ (with $\Delta_\varphi = 2$) and its shadow $\tilde{\varphi}$ (with $\Delta_{\tilde{\varphi}} = 3 - \Delta_\varphi = 1$). Let Σ and $\Sigma^{(1)}$ be operators of generic dimension Δ.
Recall that

$$\langle \varphi\varphi\Sigma\rangle = (X_{12}^{4-\Delta} X_{23}^{\Delta} X_{31}^{\Delta})^{-1/2} \,, \tag{2.68}$$

$$\langle \varphi\tilde{\varphi}\Sigma\rangle = (X_{12}^{3-\Delta} X_{23}^{\Delta-1} X_{31}^{\Delta+1})^{-1/2} = \left(\frac{X_{12}X_{23}}{X_{31}}\right)^{1/2} \langle \varphi\varphi\Sigma\rangle \,, \tag{2.69}$$

$$\langle \varphi\tilde{\varphi}\Sigma^{(1)}\rangle = \frac{(Z_3\cdot X_1)(X_2\cdot X_3) - (Z_3\cdot X_2)(X_1\cdot X_3)}{(X_{12}X_{23}X_{31})^{1/2}} \langle \varphi\tilde{\varphi}\Sigma\rangle \,. \tag{2.70}$$

Ex: Show that

$$\langle \varphi\tilde{\varphi}\Sigma^{(1)}\rangle = -\frac{2}{\Delta} \mathcal{S}_{32} \langle \varphi\varphi\Sigma\rangle \,, \tag{2.71}$$

where

$$\mathcal{S}_{32} = (X_3\cdot X_2) Z_3 \cdot \frac{\partial}{\partial X_3} - (Z_3\cdot X_2) X_3 \cdot \frac{\partial}{\partial X_3} \,. \tag{2.72}$$

Solution.—First, note that

$$\begin{aligned} X_3\cdot \frac{\partial}{\partial X_3}\langle \varphi\varphi\Sigma\rangle &= -\Delta \langle \varphi\varphi\Sigma\rangle \,, \\ Z_3\cdot \frac{\partial}{\partial X_3}\langle \varphi\varphi\Sigma\rangle &= -\frac{\Delta}{2}\left(\frac{Z_3\cdot X_1}{X_{31}} + \frac{Z_3\cdot X_2}{X_{23}}\right)\langle \varphi\varphi\Sigma\rangle \,, \end{aligned}$$

so that

$$\begin{aligned} \mathcal{S}_{32}\langle \varphi\varphi\Sigma\rangle &= -\frac{\Delta}{2}\left(Z_3\cdot X_1 \frac{X_{23}}{X_{31}} - Z_3\cdot X_2\right)\langle \varphi\varphi\Sigma\rangle \\ &= -\frac{\Delta}{2}\left(Z_3\cdot X_1 \frac{X_{23}}{X_{31}} - Z_3\cdot X_2\right)\left(\frac{X_{31}}{X_{12}X_{23}}\right)^{1/2}\langle \varphi\tilde{\varphi}\Sigma\rangle \\ &= -\frac{\Delta}{2}\frac{(Z_3\cdot X_1)(X_2\cdot X_3) - (Z_3\cdot X_2)(X_1\cdot X_3)}{(X_{12}X_{23}X_{31})^{1/2}}\langle \varphi\tilde{\varphi}\Sigma\rangle \\ &= -\frac{\Delta}{2}\langle \varphi\tilde{\varphi}\Sigma^{(1)}\rangle \,. \end{aligned}$$

We see that the operator $\mathcal{S}_{32}$ raises the spin at position 3 and lowers the weight at position 2. In Fourier space, this operator becomes

$$\mathcal{S}_{32} = z_3^i \left[K_{32}^i + \frac{1}{2} k_3^i K_{32}^j K_{32}^j\right] \,, \quad K_{32}^i \equiv \partial_{k_3^i} - \partial_{k_2^i} \,. \tag{2.73}$$

Finally, we can preform a shadow transform to get

$$\langle\varphi\varphi\Sigma^i\rangle = k_2\langle\varphi\tilde{\varphi}\Sigma^i\rangle = k_2\mathcal{S}^i_{32}\,\langle\varphi\varphi\Sigma\rangle \equiv \mathcal{S}^i_L\langle\varphi\varphi\Sigma\rangle\,. \tag{2.74}$$

Repeated application of $\mathcal{S}^i_L$ would raise the spin further.

2.5.4 Raising Internal Spin

Using the spin-raising operator, we can write

$$\underset{\substack{\uparrow\\ \text{spin-S exchange}}}{\hat{F}_S} = \sum_{\lambda=0}^{S} \underset{\substack{\uparrow\\ \text{polarization}\\ \text{tensor}}}{P^{(\lambda)}_{i_1\ldots i_S j_1\ldots j_S}} \underset{\substack{\uparrow\\ \text{spin raising}}}{(\mathcal{S}^{i_1}_L\cdots\mathcal{S}^{i_S}_L)(\mathcal{S}^{j_1}_R\cdots\mathcal{S}^{j_S}_R)}\,\underset{\substack{\uparrow\\ \text{spin-0 exchange}}}{\hat{F}_0}\,. \tag{2.75}$$

Writing this in terms of u and v, we get

$$\hat{F}_S = \sum_{\lambda=0}^{S} \Pi_{S,\lambda}\mathcal{D}^{(S,\lambda)}_{uv}\hat{F}_0\,, \tag{2.76}$$

where the polarization sums $\Pi_{S,\lambda}$ and the differential operators $\mathcal{D}^{(S,\lambda)}_{uv}$ can be found in [arXiv:1811.00024].

For spin-1 and spin-2 exchange, we find

$$\hat{F}_1 = (\Pi_{1,1}\,D_{uv} + \Pi_{1,0}\,\Delta_u)\,\hat{F}_0\,, \tag{2.77}$$

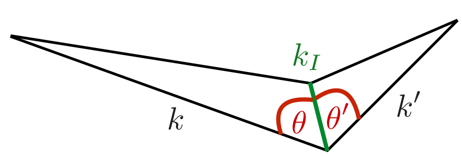
$$\hat{F}_2 = \big(\Pi_{2,2}\,D^2_{uv} + \Pi_{2,1}\,D_{uv}(\Delta_u - 2) + \Pi_{2,0}\,\Delta_u(\Delta_u-2)\big)\,\hat{F}_0\,, \tag{2.78}$$

where $D_{uv} \equiv (uv)^2\partial_u\partial_v$.

2.5.5 Soft Limit

In the collapsed limit $u \to 0$, this gives

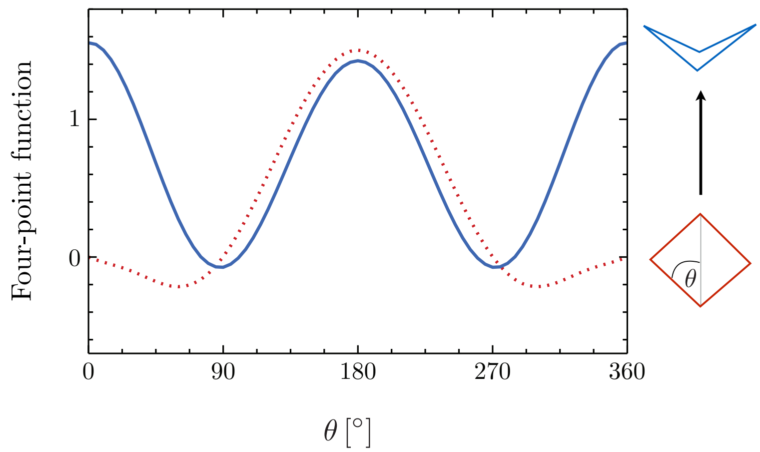
$$\lim_{u\to 0}\hat{F}_S =$$



$$\propto \left(\frac{k_I^2}{kk'}\right)^{\Delta} \sum_{\lambda} I_{S,\lambda}(\Delta)\,P^{\lambda}_S(\cos\theta)P^{-\lambda}_S(\cos\theta')\,. \tag{2.79}$$

2.5.6 Spectroscopy

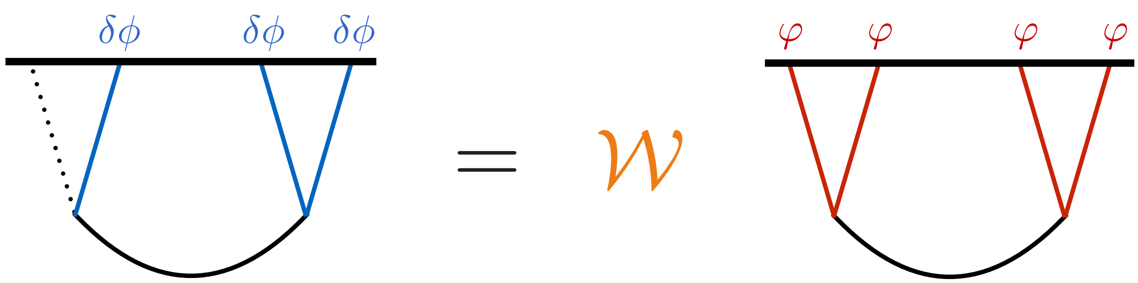
The spin of the new particles is encoded in the angular dependence of the collapsed limit. This is the analog of the angular dependence of the final state particles in collider physics.



2.6 Inflationary Three-Point Functions

2.6.1 Strategy

Find a differential operator that relates the four-point function of conformally coupled scalars to that of massless scalars:



Evaluate one leg on the time-dependent background to obtain inflationary three-point functions.

2.6.2 Massless External Fields

Recall that

$$\langle \varphi\varphi\varphi\varphi \rangle = \frac{1}{X_{12}^2 X_{34}^2}\, f(u,v)\,, \tag{2.80}$$

$$\langle \phi\phi\phi\phi \rangle = \frac{1}{X_{12}^3 X_{34}^3}\, h(u,v)\,. \tag{2.81}$$

Ex: Show that

$$\langle\phi\phi\phi\phi\rangle = \mathcal{W}_L\mathcal{W}_R\langle\varphi\varphi\varphi\varphi\rangle\,, \tag{2.82}$$

where

$$\mathcal{W}_L \equiv \eta^{MN}\left(\frac{\partial}{\partial X_1^M} + \frac{X_{1,M}}{3}\frac{\partial^2}{X_1^2}\right)\left(\frac{\partial}{\partial X_2^N} + \frac{X_{2,N}}{3}\frac{\partial^2}{X_2^2}\right). \tag{2.83}$$

[Hint: Don't do this by hand!]

We see that the operator $\mathcal{W}_L\mathcal{W}_R$ acts as a weight-raising operator. Transforming $\mathcal{W}_{L,R}$ to Fourier space, we can act on our solutions for $\langle\varphi\varphi\varphi\varphi\rangle$ to produce the corresponding $\langle\phi\phi\phi\phi\rangle$.

For scalar exchange, we find

$$F_{\Delta=3} = \mathcal{W}_L\mathcal{W}_R\hat{F}_{\Delta=2}\,, \tag{2.84}$$

where
$$\mathcal{W}_L(\cdot) \equiv \frac{1}{2}\left(1 - \frac{k_1k_2}{k_{12}}\partial_{k_{12}}\right)\left[\frac{1-u^2}{u^2}\partial_u(u\,\cdot)\right]. \tag{2.85}$$

For spin exchange, the form of $\mathcal{W}_L$ and $\mathcal{W}_R$ (in Fourier space) is more complicated and can be found in [arXiv:1811.00024].

As we will see below, only the longitudinal modes of the de Sitter four-point functions contribute to the inflationary three-point functions.

In that case, we find

$$\hat{F}^{(S,\lambda=0)}_{\Delta=3} = \mathcal{W}_L\mathcal{W}_R\left(\Pi_{S,0}\mathcal{D}^{(S,0)}_{uv}\hat{F}^{S=0}_{\Delta=2}\right), \tag{2.86}$$

$$= \frac{1}{2}U_L^{(S,0)}U_R^{(S,0)}\underbrace{\prod_{j=1}^{S}(\Delta_u - (S-j)(S-j+1))\hat{F}^{S=0}_{\Delta=2}}_{\equiv\,\hat{A}^{(S,0)}_{\Delta=2}}\,, \tag{2.87}$$

where

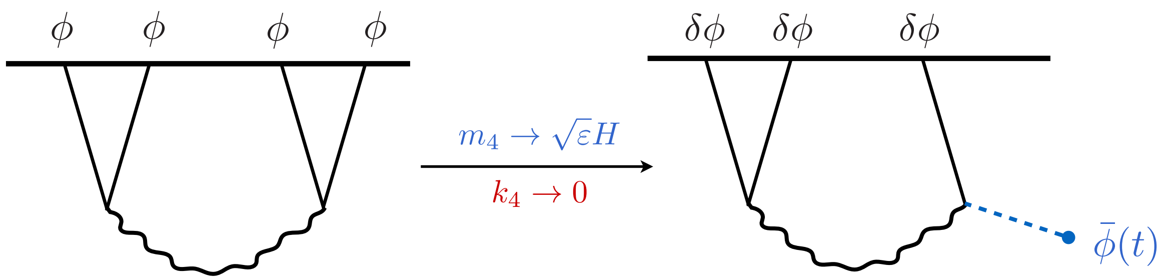
$$U_L^{(S,0)} \equiv \alpha S P_{S-1}(\alpha) + \left(U_L^{(1,0)} - 1 + \frac{(S-1)(S-2)u^2\alpha^2 - (S+2)(S-1)}{4u^2}\right)P_S(\alpha) \tag{2.88}$$

$$U_L^{(1,0)} \equiv \frac{1}{u}\left(1 - \frac{k_1k_2}{k_{12}}\partial_{k_{12}}\right)\left[(1-u^2)\partial_u(u\,\cdot)\right], \tag{2.89}$$

with $\alpha \equiv (k_1-k_2)/k_I$.

2.6.3 Perturbed de Sitter

To obtain inflationary three-point functions, we evaluate one of the external legs on the time-dependent background:



For spin exchange, only the longitudinal mode contributes.

Consider

$$F^{(S,0)}_{\Delta=3-\varepsilon} = U^{(S,0)}_L U^{(S,0)}_R \hat{A}^{(S,0)}_{\Delta=2-\varepsilon} \tag{2.90}$$

$$= U^{(S,0)}_L \left(\bar{U}^{(S,0)}_R - \varepsilon\, \delta U^{(S,0)}_R + \cdots\right)\left(\hat{A}^{(S,0)}_{\Delta=2} + \varepsilon\, \delta\hat{A}^{(S,0)}_{\Delta=2} + \cdots\right), \tag{2.91}$$

where

$$\delta U^{(S,0)}_R \equiv \beta S P_{S-1}(\beta) + \left(\frac{1-v^2}{v}\partial_v + \frac{3-(4+(2S-3)\beta^2)v^2}{2v^2}\right) P_S(\beta)\,, \tag{2.92}$$

with $\beta \equiv (k_3 - k_4)/k_I$.

In the limit $k_4 \to 0$, we find

$$\lim_{k_4\to 0} \bar{U}^{(S,0)}_R = 0\,, \tag{2.93}$$

$$\lim_{k_4\to 0} \delta U^{(S,0)}_R = 1\,. \tag{2.94}$$

The inflationary bispectrum therefore is

$$\boxed{B(k_1,k_2,k_3) = \frac{\varepsilon}{2}\, k_3^3\, U^{(S,0)}_L \hat{b}(u) \ + \ \text{perms}}\,, \tag{2.95}$$

where

$$\hat{b}(u) \equiv \lim_{v\to 1} \hat{A}^{(S,0)}_{\Delta=2}(u,v)\,. \tag{2.96}$$

2.6.4 Contact Interactions

For the simplest contact solution, we have

$$\begin{aligned}\hat{b}(u) &= \lim_{v\to 1} \hat{F}_c^{(0)} \\ &= \frac{u}{u+1}\,. \end{aligned} \tag{2.97}$$

Substituting this into (2.95), we get

$$\begin{aligned} B(k_1,k_2,k_3) = \frac{\varepsilon}{4K^2}\Bigg[\sum_n k_n^5 + \sum_{n\neq m}(2k_n^4 k_m - 3k_n^3 k_m^2) \\ + \sum_{n\neq m\neq l}(k_n^3 k_m k_l - 4k_n^2 k_m^2 k_l)\Bigg]\,, \end{aligned} \tag{2.98}$$

which (up to a shadow transform) is the bispectrum arising from $(\partial_\mu\phi)^4$, cf. (1.12).

2.6.5 Graviton Exchange

For massless spin-2 exchange, we have

$$\begin{aligned}\hat{b}(u) &= \lim_{v\to 1} \Delta_u(\Delta_u - 2)\hat{F}_{\Delta=2} \\ &= \lim_{v\to 1} \Delta_u \hat{F}_c^{(-1)} \\ &= \lim_{v\to 1} \hat{F}_c^{(0)} \\ &= \frac{u}{u+1}\,. \end{aligned} \tag{2.99}$$

Substituting this into (2.95), we get

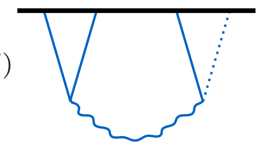
$$B(k_1,k_2,k_3) \;=\; \varepsilon\left[\sum_{n\neq m} k_n k_m^2 + \frac{8}{K}\sum_{n>m} k_n^2 k_m^2\right] + (n_s-1)\sum_n k_n^3\,, \tag{2.100}$$

which (up to a shadow transform) is the standard three-point function of slow-roll inflation.

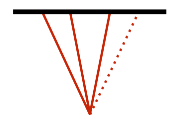
2.6.6 Massive Particles

The effects of massive particles during inflation are characterized in terms of just two basis functions:

$$B(k_1, k_2, k_3) = \mathcal{W}_L \Bigg[\sum_S a_S \, \mathcal{S}^{(S)}$$



$$+ \sum_n b_n \, \Delta_u^n$$



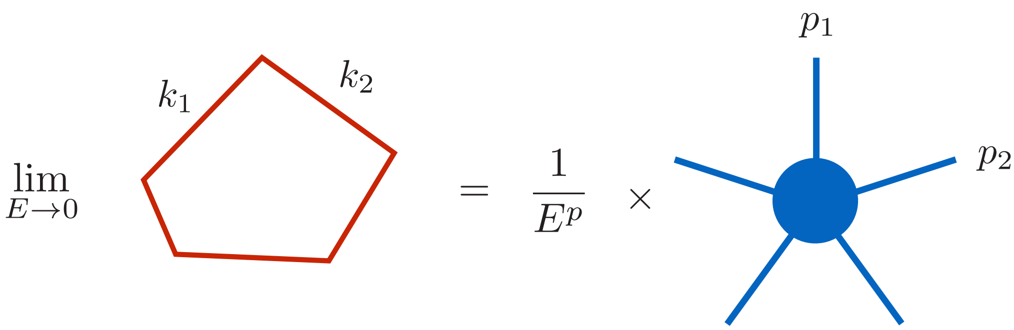
$$\Bigg] + \text{perms} \tag{2.101}$$

This result is valid for all momenta, not just soft limits.

3 Summary and Future Directions

3.1 Amplitudes Meet Cosmology

Remarkably, correlation functions contain scattering amplitudes:

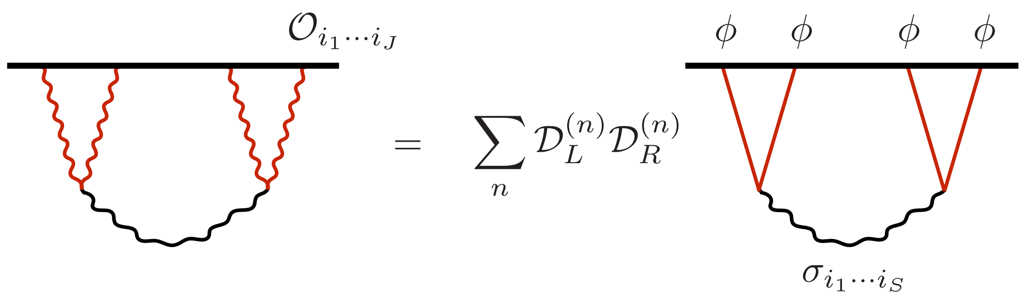


where $E \equiv \sum |\mathbf{k}_n|$.

Insights from the physics of scattering amplitudes should therefore translate to cosmology.

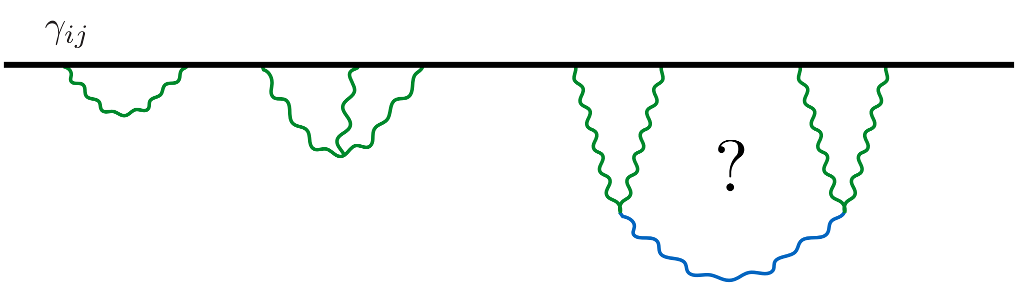
3.2 Spinning Correlators

Spinning correlators can also be bootstrapped from our scalar building blocks:



3.3 Graviton Correlators

An important special case are graviton correlators:

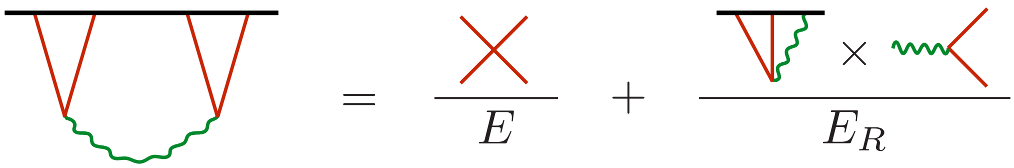


In de Sitter space, very little is known beyond three-point functions.
In flat space, a consistent S-matrix of gravitons is very constrained.

What is the cosmological analog of these results?

3.4 Factorization

For massless spin exchange, we find



Does consistent factorization allow for an efficient construction of graviton correlators?

3.5 Double Copy

Gravity amplitudes can be written as the square of gauge theory amplitudes:

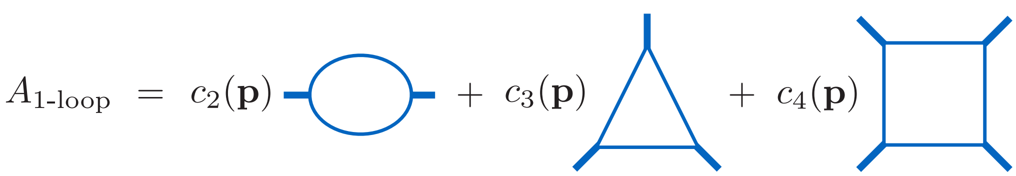
$$\text{Gravity} = \text{YM}^2$$

Is there an analog of this for cosmological correlators?

3.6 Loop Corrections

How does the bootstrapping of de Sitter correlators generalize to loops?

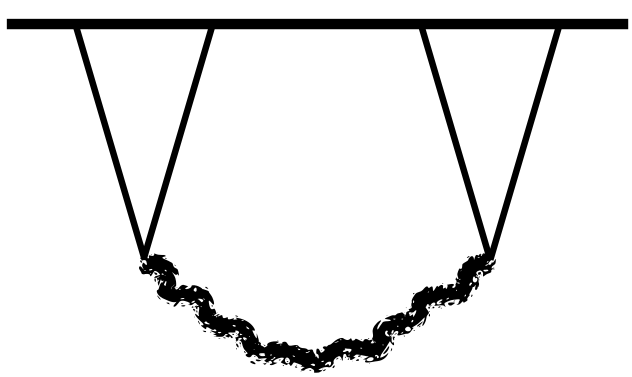
One-loop amplitudes can be written as



Is there a cosmological analog of this?

3.7 Ultraviolet Completion

What is the space of consistent UV completions of inflationary correlators?



- What is the cosmological analog of positivity bounds?
- What is the Veneziano correlator in de Sitter space?