Exercises

Lecce Lectures on the Cosmological Bootstrap

1. Consider a massless scalar field in four-dimensional de Sitter space, with metric

$$\mathrm{d}s^2 = \frac{-\mathrm{d}\eta^2 + \mathrm{d}\mathbf{x}^2}{(H\eta)^2}\,,$$

where η is conformal time and H is the Hubble parameter.

a. Determine the wave equation satisfied by the field. Show that for each Fourier mode you get solutions of the form

$$f_k(\eta) = \frac{H}{\sqrt{2k^3}}(1+ik\eta)e^{-ik\eta}\,,$$

and its complex conjugate.

b. Promote $\phi_{\mathbf{k}}$ to a quantum operator and write

$$\hat{\phi}_{\mathbf{k}}(\eta) = f_k(\eta)\hat{a}_{\mathbf{k}} + f_k^*(\eta)\hat{a}^\dagger_{-\mathbf{k}}\,.$$

Using the canonical commutation relation for $\hat{\phi}$ and its conjugate momentum $\hat{\pi}$, find the commutator $[\hat{a}_{\mathbf{k}}, \hat{a}^\dagger_{\mathbf{k}'}]$. What is the physical meaning $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$.

c. Define the Bunch Davies vacuum through $\hat{a}_{\mathbf{k}}|0\rangle = 0$. Then compute

$$\langle 0|\hat{\phi}_{\mathbf{k}}(\eta)\hat{\phi}_{\mathbf{k}'}(\eta)|0\rangle = \frac{2\pi^2}{k^3}\,P(k,\eta)\,\delta_D(\mathbf{k}+\mathbf{k}')\,.$$

Determine the superhorizon limit of $P(k,\eta)$.

d. By summing over all Fourier modes, calculate the equal time two-point function of the massless scalar in real space. What is the physical meaning of the IR divergence?

e. Repeat the exercise for a conformally coupled scalar field, with $m^2 = 2H^2$.

f. Photons are massless, but they are not produced during inflation. Why?

2. Add a ϕ^3 interaction to the above problem and compute the three-point function,

$$\langle 0|\,\hat{\phi}_{\mathbf{k}_1}\hat{\phi}_{\mathbf{k}_2}\hat{\phi}_{\mathbf{k}_3}\,|0\rangle = \frac{(2\pi^2)^2}{(k_1k_2k_3)^2}\,B(k_1,k_2,k_3)\,\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)\,.$$

Understand the integration contour necessary for the computation. How do you kill off the oscillatory early time pieces? Understand that you need to rotate the contour into the imaginary direction.

Warning: This is not an easy problem. If you want a simpler warm-up problem, do the next problem first.

3. Consider a conformally coupled scalar field in a fixed de Sitter background.

a. Add a ϕ^4 interaction to the above problem and compute the four-point function,

$$\langle 0|\,\hat{\phi}_{\mathbf{k}_1}\hat{\phi}_{\mathbf{k}_2}\hat{\phi}_{\mathbf{k}_3}\hat{\phi}_{\mathbf{k}_4}\,|0\rangle = F(k_1,k_2,k_3,k_4,k_I,k_I')\,(2\pi)^4\delta_D(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)\,,$$

where $k_I \equiv |\mathbf{k}_1+\mathbf{k}_2|$ and $k_I' \equiv |\mathbf{k}_1+\mathbf{k}_3|$.

Consider only the "s-channel" permutations of the external momenta and write the result in terms of

$$u^{-1} \equiv \frac{k_1+k_2}{k_I}\,, \qquad v^{-1} \equiv \frac{k_3+k_4}{k_I}\,.$$

Check that the result is conformally invariant, i.e. that the four-point function $\hat{F}(u,v) \equiv k_I F$ satisfies

$$(\Delta_u - \Delta_v)\hat{F} = 0\,,$$

where $\Delta_u \equiv u^2(1-u^2)\partial_u^2 - 2u^3\partial_u$.

b. Now, consider the interaction $(\partial_\mu\phi)^4$ and compute the corresponding four-point function for "s-channel" permutations of the external momenta. Write the result in terms of u and v, and check conformal invariance. Show that the result can be written in the form

$$\hat{F} = \left(a_0 + a_1\Delta_u + a_2\Delta_u^2\right)\hat{F}_0\,,$$

where $\hat{F}_0$ is the solution found in part **a.** and a_n are constants to be determined.

4. Consider a massless scalar field in an inflationary background. Add a $(\partial_\mu\phi)^4$ interaction.

a. Evaluating one leg on the background leads to a cubic interaction for the inflationary fluctuations

$$\mathcal{L} \subset \frac{(\partial_\mu\phi)^4}{8\Lambda^4} \quad \Rightarrow \quad \mathcal{L}_{\rm int} = \frac{\dot{\bar{\phi}}}{4\Lambda^4}\,\dot{\varphi}(\partial_\mu\varphi)^2\,,$$

where $\varphi(x) \equiv \phi(x) - \bar{\phi}(t)$. Compute the three-point function associated with this interaction.

b. Re-derive this result from the cosmological bootstrap.