Stratifying On-Shell Cluster Varieties

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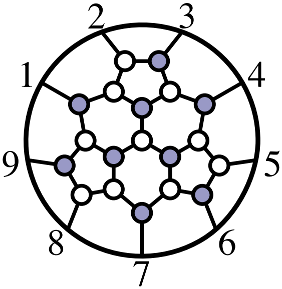
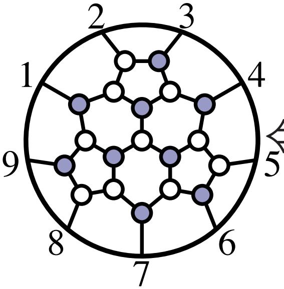
The Combinatorics and Geometry of On-Shell Physics

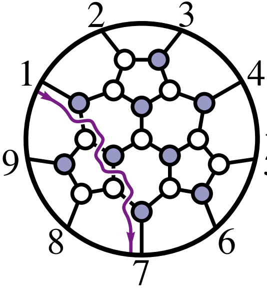


The Combinatorics and Geometry of On-Shell Physics



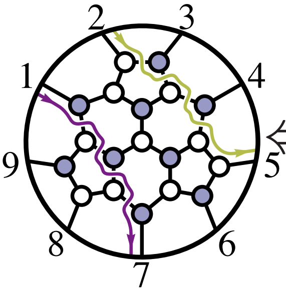
$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ & & & & & & & & \\ & & & & & & & & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



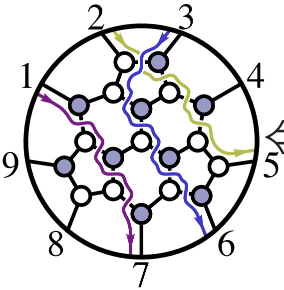
$$\Leftrightarrow \left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & & & & & & & & \\ 7 & & & & & & & & \end{array}\right)$$

The Combinatorics and Geometry of On-Shell Physics



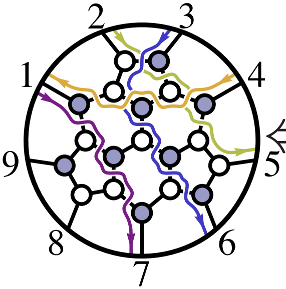
$$\Leftrightarrow\begin{pmatrix}1&2&3&4&5&6&7&8&9\\ \downarrow&\downarrow&&&&&&&\\ 7&5&&&&&&&\end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



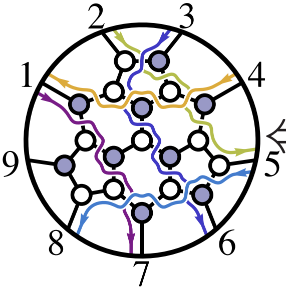
$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & & & & & & \\ 7 & 5 & 6 & & & & & & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



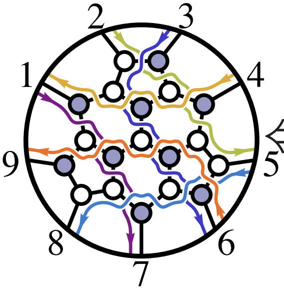
$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & & & & & \\ 7 & 5 & 6 & 10 & & & & & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



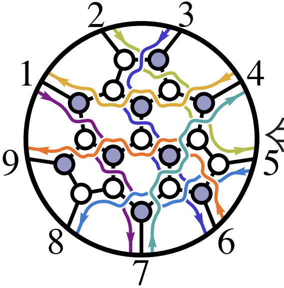
$$\Leftrightarrow\begin{pmatrix}1&2&3&4&5&6&7&8&9\\ \downarrow&\downarrow&\downarrow&\downarrow&\downarrow&&&&\\ 7&5&6&10&8&&&&\end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



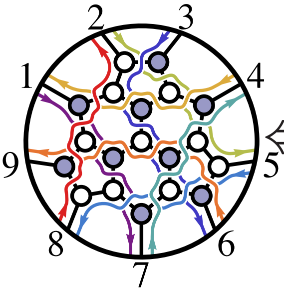
$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \\ 7 & 5 & 6 & 10 & 8 & 9 & & & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



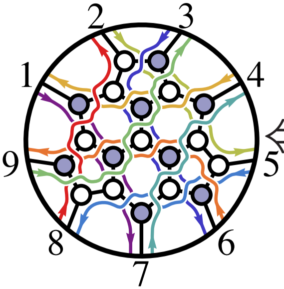
$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics



$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

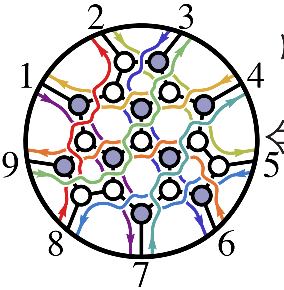


$$\Leftrightarrow \left(\begin{array}{ccccccccc}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12\end{array}\right)$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

$$\Updownarrow$$

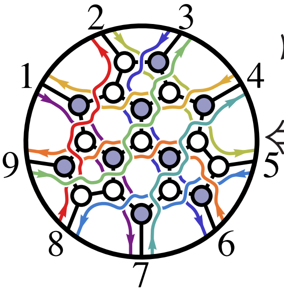


$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha) \equiv \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \in G_+(4,9)$$

$$\Nearrow \qquad \Updownarrow$$

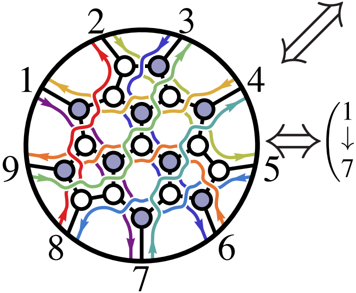


$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha) \equiv \left(\begin{array}{ccccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & 0 & 0 & 1 \end{array}\right) \in G_+(4,9)$$

$$\Updownarrow$$

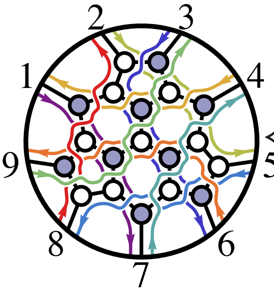


$$\Leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix}1&0&0&0&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&0&0&1&0&\alpha_3\\0&0&\alpha_1&0&0&-\alpha_2\,\alpha_1&0&0&1\end{pmatrix}\in G_+(4,9)$$

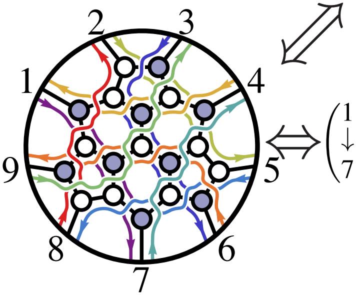
$$\Updownarrow$$



$$\Leftrightarrow\begin{pmatrix}1&2&3&4&5&6&7&8&9\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\7&5&6&10&8&9&13&11&12\end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & 0 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$



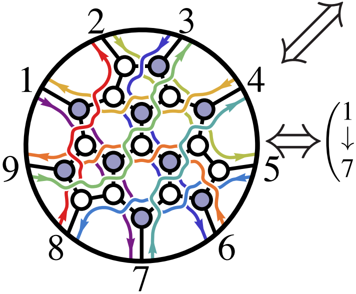
$$\Updownarrow$$

$$\Longleftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_3 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & 0 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

$$\Updownarrow$$

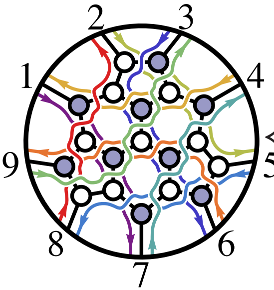


$$\Longleftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\left(\begin{array}{ccccccccc} 1 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & 0 & 0 & 1 \end{array}\right)\in G_+(4,9)$$

$\Updownarrow$

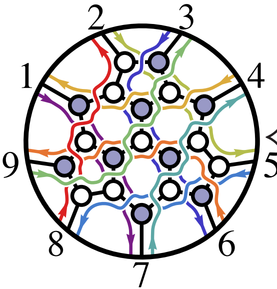


$$\Leftrightarrow\left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{array}\right)$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_4 & \alpha_7\,\alpha_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 \\ 0 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

$$\swarrow\!\!\!\nearrow \qquad \Updownarrow$$

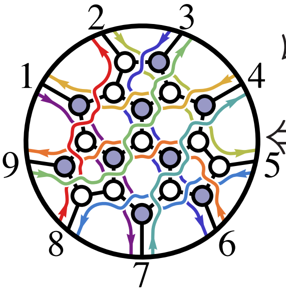


$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix}1&\alpha_8&\alpha_5&0&0&0&0&0&0\\0&0&0&1&0&\alpha_4&\alpha_7\,\alpha_4&0&0\\0&0&0&0&0&0&1&\alpha_6&\alpha_3\\0&0&\alpha_1&0&0&-\alpha_2\,\alpha_1&-\alpha_7\,\alpha_2\,\alpha_1&0&1\end{pmatrix}\in G_+(4,9)$$

⤢ $\Updownarrow$

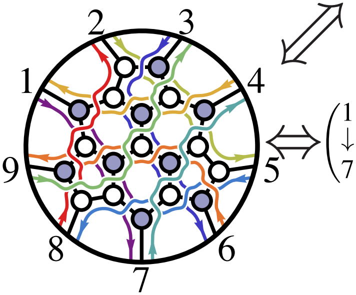


$$\Longleftrightarrow\begin{pmatrix}1&2&3&4&5&6&7&8&9\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\7&5&6&10&8&9&13&11&12\end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix}1&\alpha_8&\alpha_5&0&0&0&0&0&0\\0&0&0&1&0&\alpha_4&\alpha_7\,\alpha_4&0&0\\-\alpha_9\,\alpha_3&0&0&0&0&0&1&\alpha_6&\alpha_3\\-\alpha_9&0&\alpha_1&0&0&-\alpha_2\,\alpha_1&-\alpha_7\,\alpha_2\,\alpha_1&0&1\end{pmatrix}\in G_+(4,9)$$

$$\Updownarrow$$

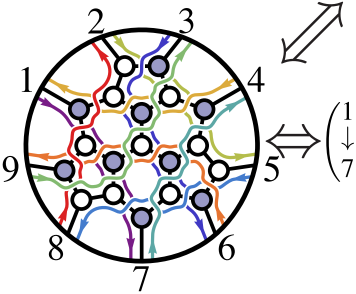


$$\Longleftrightarrow\begin{pmatrix}1&2&3&4&5&6&7&8&9\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\7&5&6&10&8&9&13&11&12\end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & \alpha_8 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4 & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 \\ -\alpha_9 & 0 & \alpha_1 & 0 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

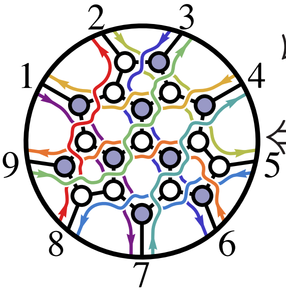
$$\Updownarrow$$



$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & \alpha_8 & \alpha_5 & \alpha_{11}\,\alpha_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4 & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 \\ -\alpha_9 & 0 & \alpha_1 & \alpha_{11}\,\alpha_1 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

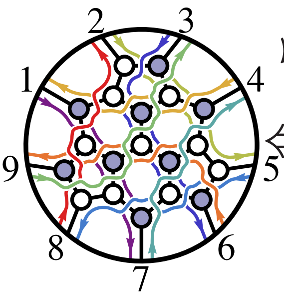


$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & \alpha_8 & \alpha_5 & \alpha_{11}\,\alpha_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4 & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & (\alpha_3+\alpha_{12}\,\alpha_6) \\ -\alpha_9 & 0 & \alpha_1 & \alpha_{11}\,\alpha_1 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

$$\Updownarrow$$

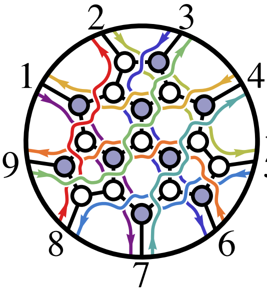


$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & \alpha_8 & \alpha_5 & \alpha_{11}\,\alpha_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & (\alpha_4+\alpha_{13}\,\alpha_{10}) & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & (\alpha_3+\alpha_{12}\,\alpha_6) \\ -\alpha_9 & 0 & \alpha_1 & \alpha_{11}\,\alpha_1 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

$$\Updownarrow$$

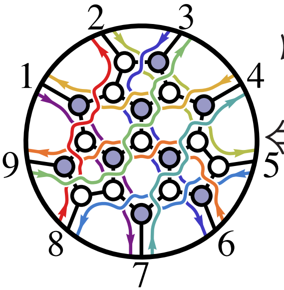


$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & \alpha_8 & (\alpha_5+\alpha_{14}\,\alpha_8) & \alpha_{11}\,\alpha_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & (\alpha_4+\alpha_{13}\,\alpha_{10}) & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & (\alpha_3+\alpha_{12}\,\alpha_6) \\ -\alpha_9 & 0 & \alpha_1 & \alpha_{11}\,\alpha_1 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

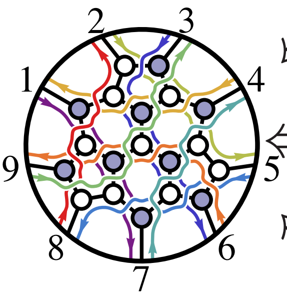
$$\Updownarrow$$



$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & \alpha_8 & (\alpha_5+\alpha_{14}\,\alpha_8) & \alpha_{11}\,\alpha_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & (\alpha_4+\alpha_{13}\,\alpha_{10}) & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & (\alpha_3+\alpha_{12}\,\alpha_6) \\ -\alpha_9 & 0 & \alpha_1 & \alpha_{11}\,\alpha_1 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

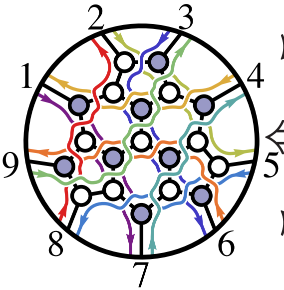


$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

$$f_\sigma\equiv\int\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\;\;\delta^{k\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\delta^{k\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\delta^{2\times(n-k)}\big(\lambda\cdot C(\alpha)^{\perp}\big)$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\begin{pmatrix} 1 & \alpha_8 & (\alpha_5+\alpha_{14}\,\alpha_8) & \alpha_{11}\,\alpha_5 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & \alpha_{10} & (\alpha_4+\alpha_{13}\,\alpha_{10}) & \alpha_7\,\alpha_4 & 0 & 0\\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & (\alpha_3+\alpha_{12}\,\alpha_6)\\ -\alpha_9 & 0 & \alpha_1 & \alpha_{11}\,\alpha_1 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{pmatrix}\in G_+(4,9)$$

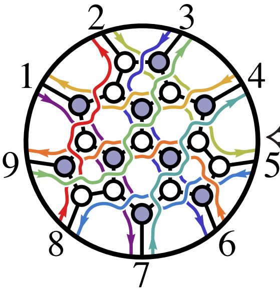


$$\Leftrightarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9\\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow\\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{pmatrix}$$

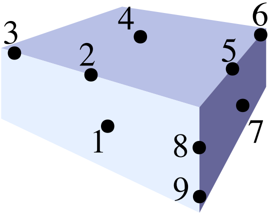
$$f_\sigma\equiv\int\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\;\;\delta^{k\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\delta^{k\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\delta^{2\times(n-k)}\big(\lambda\cdot C(\alpha)^{\perp}\big)$$

The Combinatorics and Geometry of On-Shell Physics

$$C(\alpha)\equiv\left(\begin{array}{ccccccccc} 1 & \alpha_8 & (\alpha_5+\alpha_{14}\,\alpha_8) & \alpha_{11}\,\alpha_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & (\alpha_4+\alpha_{13}\,\alpha_{10}) & \alpha_7\,\alpha_4 & 0 & 0 \\ -\alpha_9\,\alpha_3 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & (\alpha_3+\alpha_{12}\,\alpha_6) \\ -\alpha_9 & 0 & \alpha_1 & \alpha_{11}\,\alpha_1 & 0 & -\alpha_2\,\alpha_1 & -\alpha_7\,\alpha_2\,\alpha_1 & 0 & 1 \end{array}\right)\in G_+(4,9)$$



$$\Leftrightarrow\left(\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 7 & 5 & 6 & 10 & 8 & 9 & 13 & 11 & 12 \end{array}\right)$$



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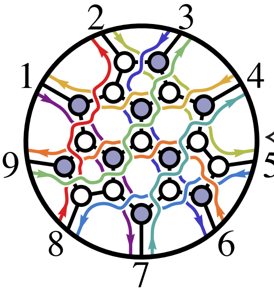

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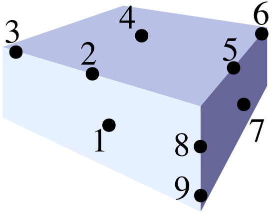

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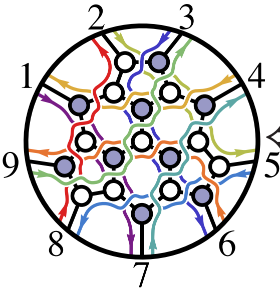

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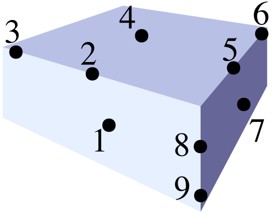

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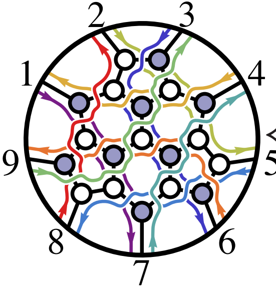
$$\Updownarrow$$



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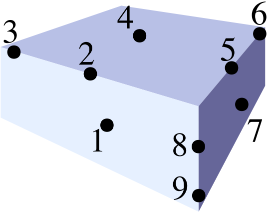

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$$f_\sigma\equiv\oint\limits_{\sigma}\frac{d^{4\times9}C}{GL(4)}\frac{\delta^{k\times4}\big(C\cdot\widetilde{\eta}\big)\,\delta^{k\times2}\big(C\cdot\widetilde{\lambda}\big)\,\delta^{2\times(n-k)}\big(\lambda\cdot C^{\perp}\big)}{\langle1234\rangle\langle2345\rangle\langle3456\rangle\langle4567\rangle\langle5678\rangle\langle6789\rangle\langle7891\rangle\langle8912\rangle\langle9123\rangle}$$

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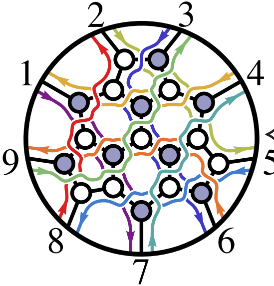

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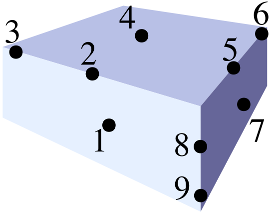

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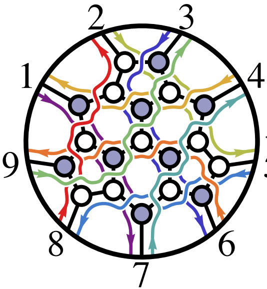

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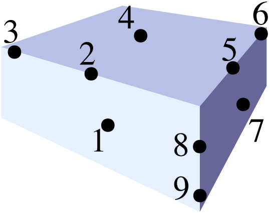

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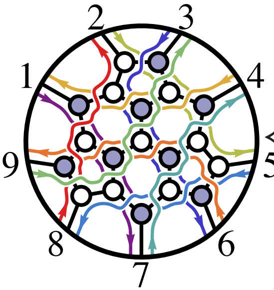

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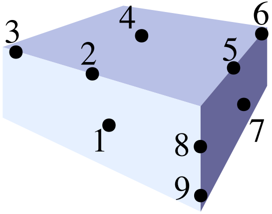

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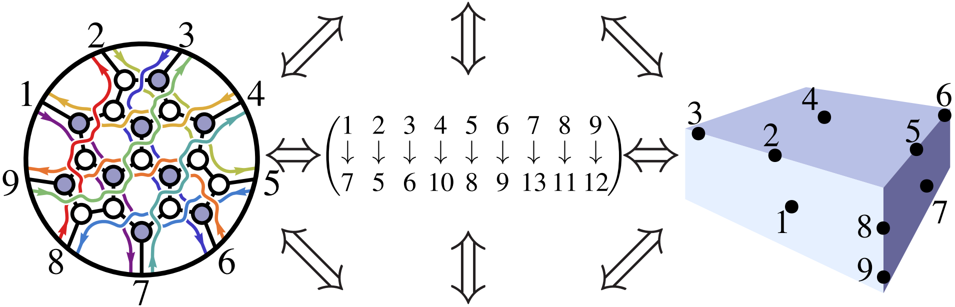

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Organization and Outline

1. From On-Shell Physics to the (Positive) Grassmannian (II)
   - Geometry of Parke-Taylor Amplitudes
   - *Generalized* Parke-Taylor 'Amplitudes' & Grassmannian Residues
   - The *Positroid* Stratification of the (Positive) Grassmannian
2. Building-Up the Grassmannian Correspondence: On-Shell Varieties
   - *Grassmannian* Representations of On-Shell Functions
   - Iterative Construction of Grassmannian 'On-Shell' Varieties
   - Characteristics of Grassmannian Representations
3. The Classification of On-Shell (Cluster) Varieties
   - Warm-Up: Classifying On-Shell Functions of $G(2,n)$
   - Definitions, Stratifications, and Conjectures
   - Application: the Stratification of On-Shell Varieties in $G(3,6)$
4. Conclusions and Future Directions

The Grassmannian Geometry of Scattering Amplitudes

Consider all the *iterated factorizations* of the Parke-Taylor Amplitude:

$$A_4^{(2)}(1,2,3,4)$$

$$\frac{\langle 1\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\langle 4\,1\rangle}$$

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$\lambda \in G(2,4)$

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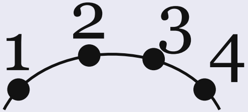

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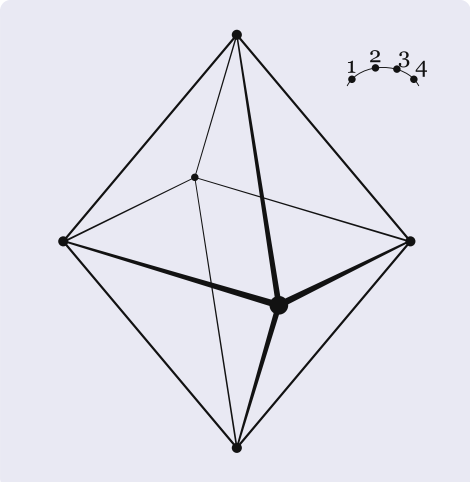

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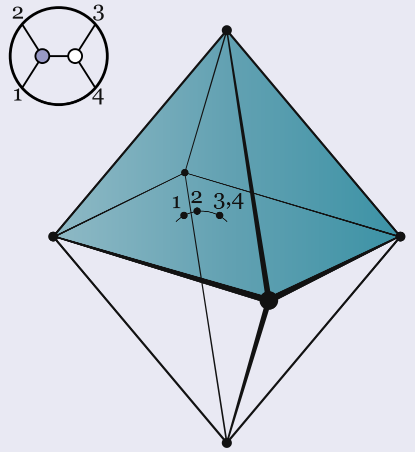
$$\lambda_3\longrightarrow\alpha_{3\,4}\lambda_4$$



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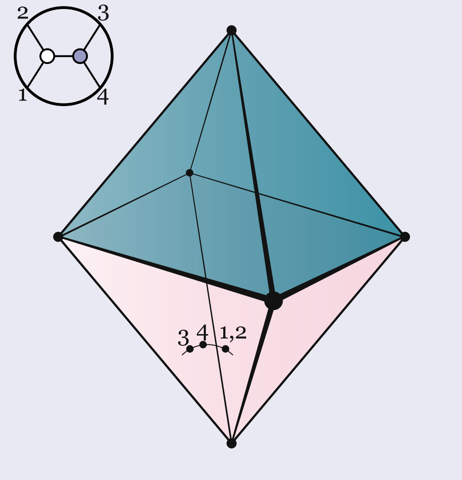
$\lambda\in G(2,4)\qquad\lambda_a\in\mathbb{P}^1$

$$\begin{pmatrix}\lambda_1^1&\lambda_2^1&\lambda_3^1&\lambda_4^1\\0&0&\lambda_3^2&\lambda_4^2\end{pmatrix}$$

$$\lambda_1\longrightarrow\alpha_{1\,2}\lambda_2$$



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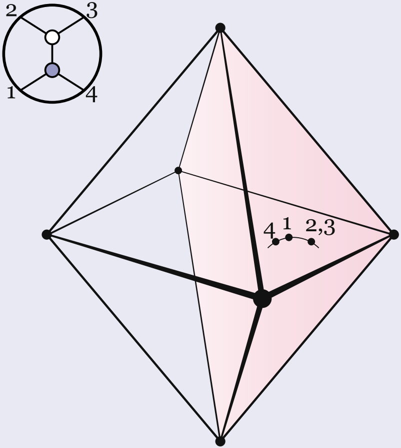

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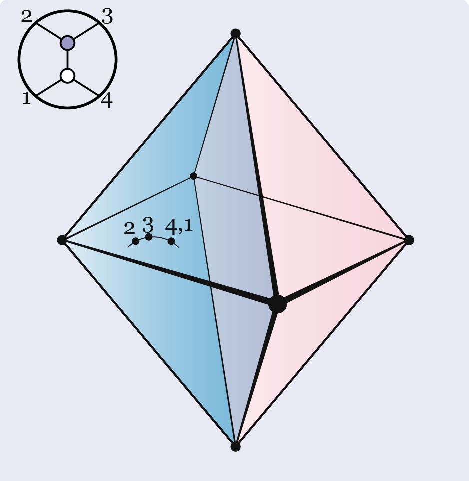
$$\lambda_4\longrightarrow\alpha_{4\,1}\lambda_1$$



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$$\lambda_1\longrightarrow\alpha_{1\,2}\lambda_2$$
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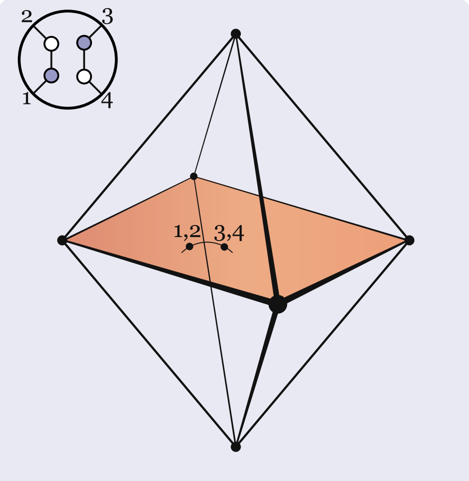

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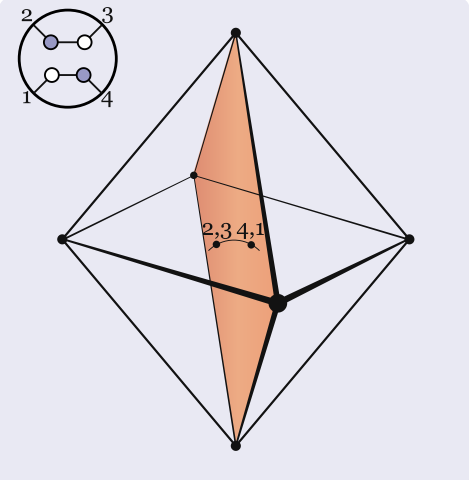
$\lambda \in G(2,4) \qquad \lambda_a \in \mathbb{P}^1$

$$\left(\begin{array}{cccc}\lambda_1^1 & 0 & 0 & \lambda_4^1\\ 0 & \lambda_2^2 & \lambda_3^2 & 0\end{array}\right)$$

$$\lambda_2 \longrightarrow \alpha_{2\,3}\lambda_3$$
$$\lambda_4 \longrightarrow \alpha_{4\,1}\lambda_1$$



The Grassmannian Geometry of Scattering Amplitudes

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$\lambda\in G(2,4)\qquad\lambda_a\in\mathbb{P}^1$

$$\begin{pmatrix}\lambda_1^1&\lambda_2^1&\lambda_3^1&\lambda_4^1\\\lambda_1^2&0&0&0\end{pmatrix}$$

$$\lambda_2\longrightarrow\alpha_{2\,3}\lambda_3$$
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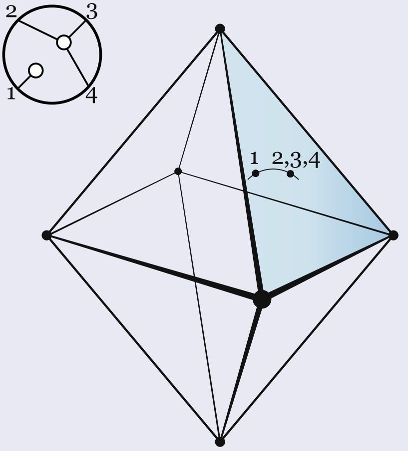

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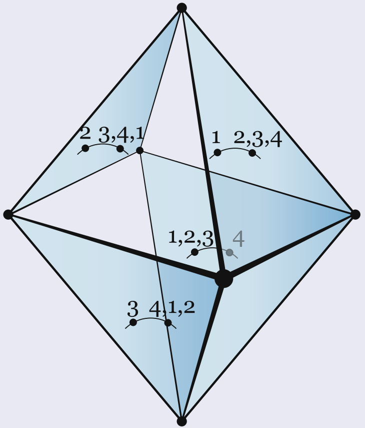

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$$\lambda_4\longrightarrow\alpha_{4\,1}\lambda_1$$
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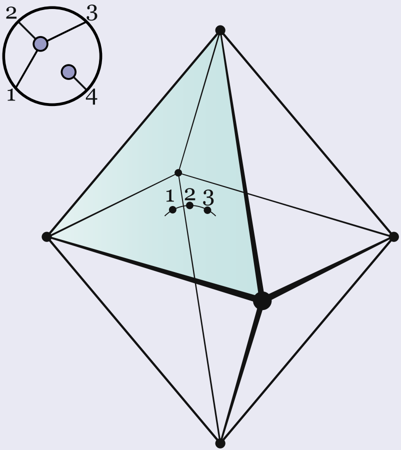
$$(\therefore\,\lambda_4=0)$$



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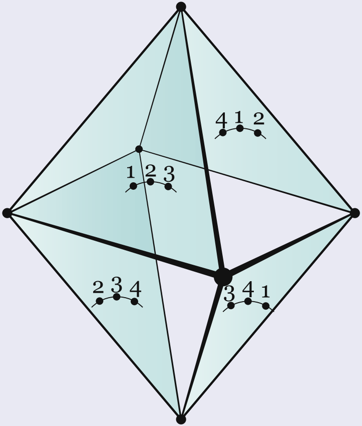

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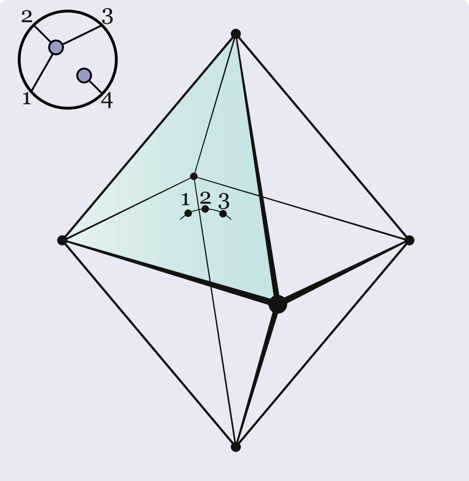

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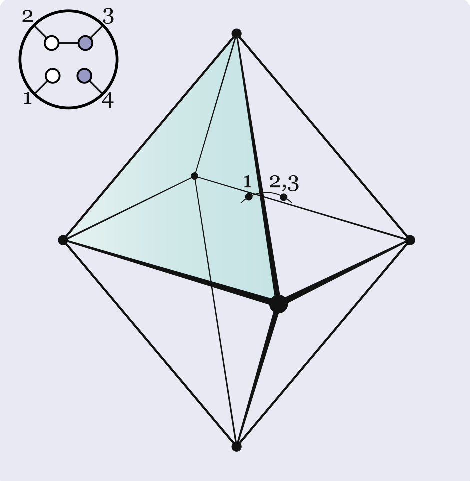

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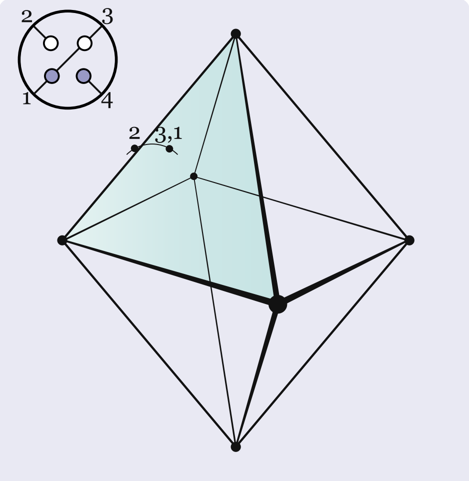

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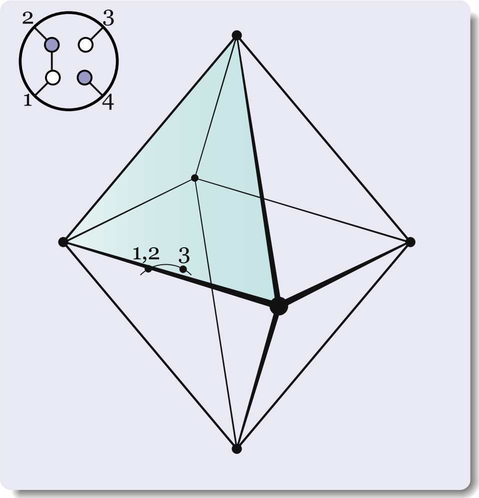

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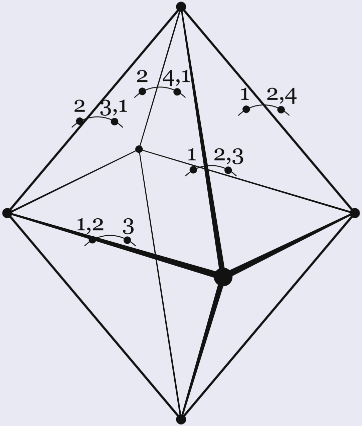

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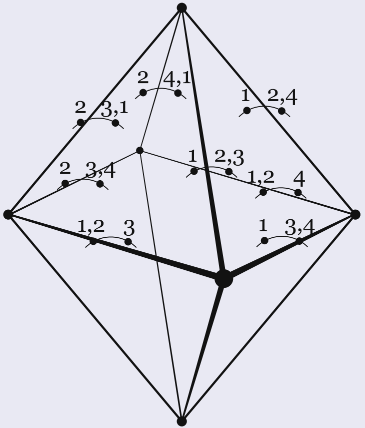

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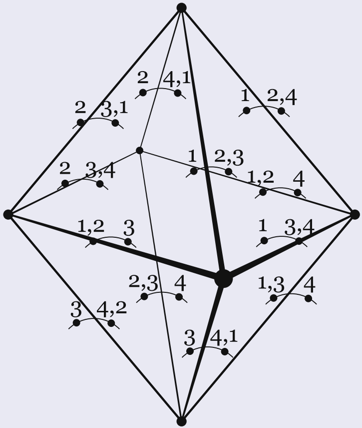

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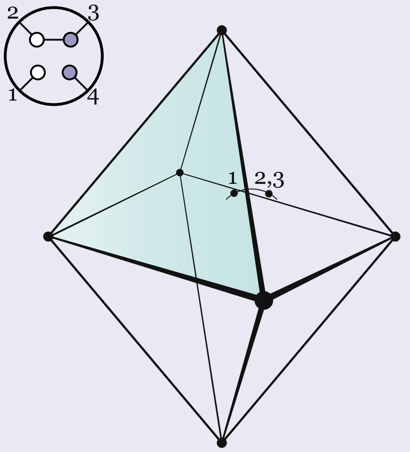

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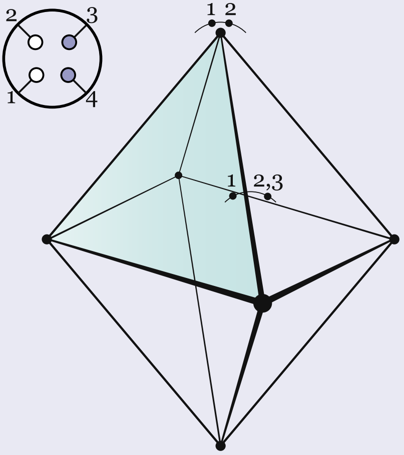
$$\begin{aligned}\lambda_4 &\longrightarrow \alpha_{4\,1}\lambda_1\\ \lambda_4 &\longrightarrow \alpha_{4\,3}\lambda_3\\ \lambda_3 &\longrightarrow \alpha_{3\,2}\lambda_2\\ \lambda_3 &\longrightarrow \alpha_{3\,1}\lambda_1\end{aligned}$$



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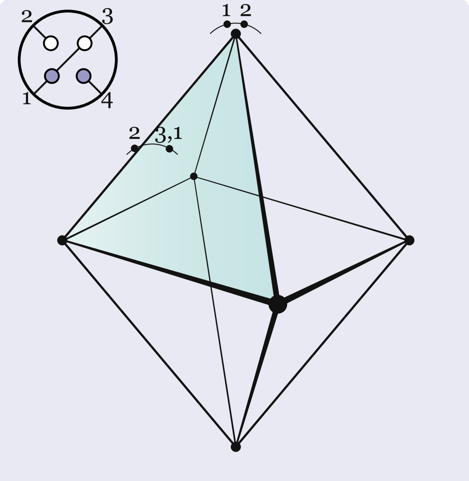

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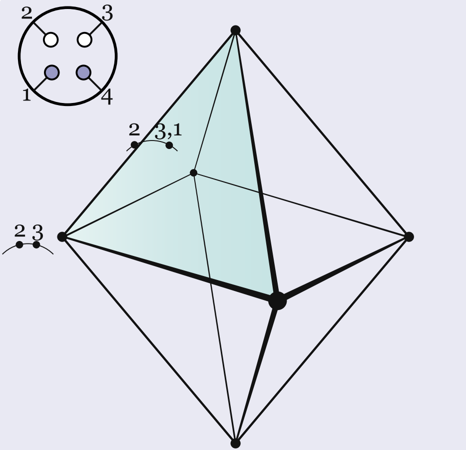

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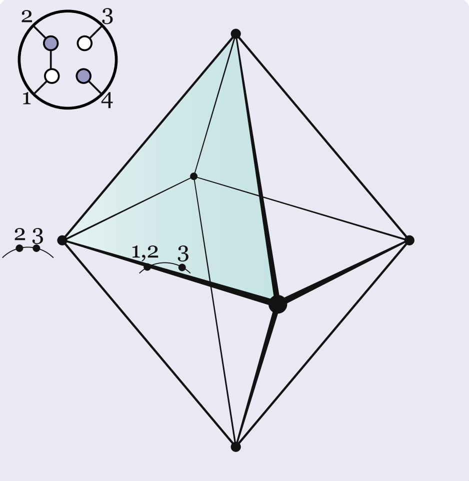

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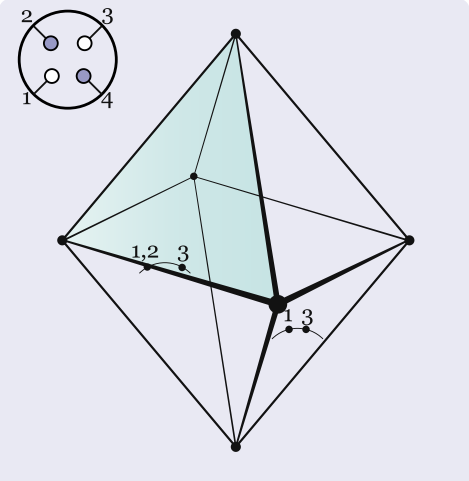

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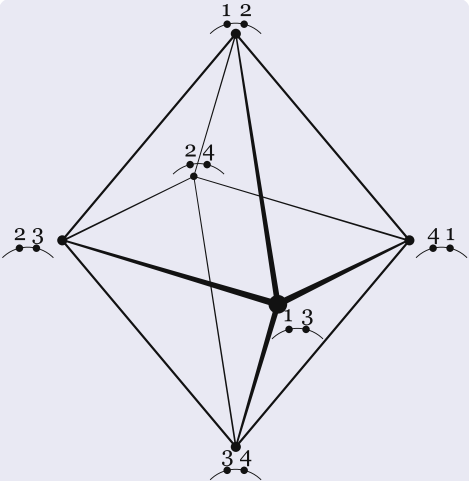

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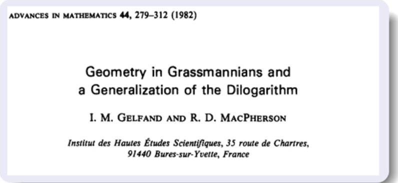

ADVANCES IN MATHEMATICS **44**, 279–312 (1982)

Geometry in Grassmannians and
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Institut des Hautes Études Scientifiques, 35 route de Chartres,
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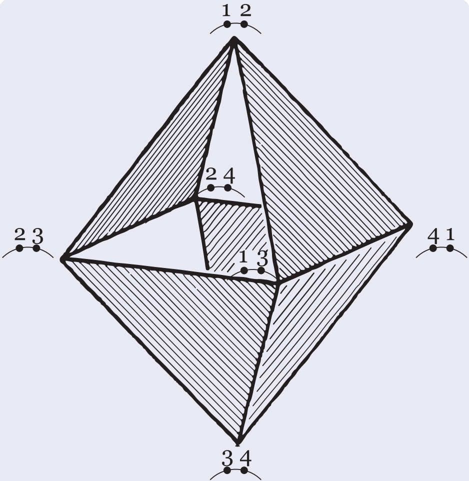

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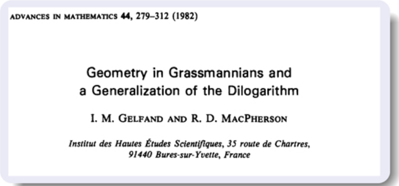

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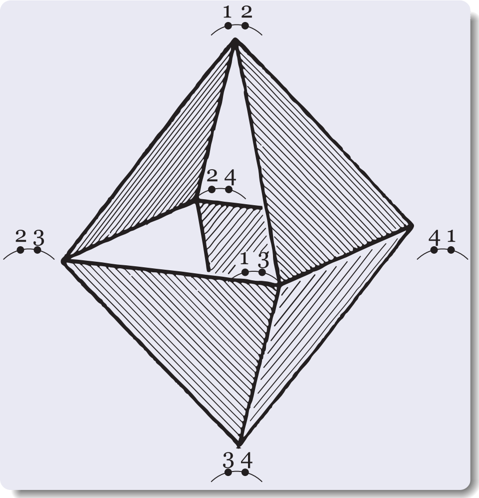

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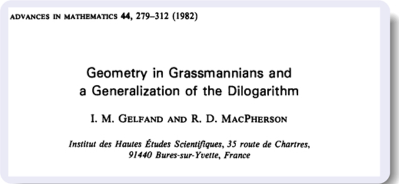

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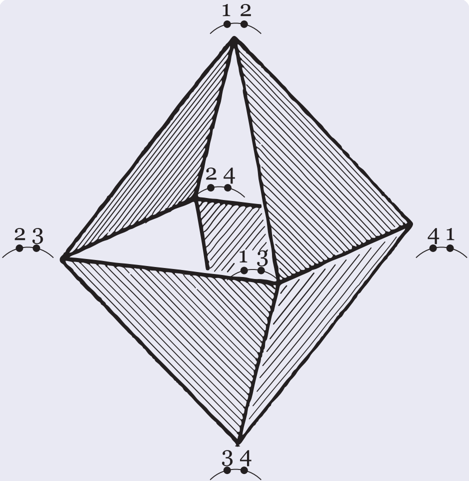

Parke-Taylor 'Amplitudes' and Grassmannian Residues

Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 2 negative-helicity gluons

$$\mathcal{A}_n^{(2)} = \frac{\delta^{2\times4}(\lambda\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle1\,2\rangle\langle2\,3\rangle\langle3\,4\rangle\cdots\langle n\,1\rangle}$$

Parke-Taylor 'Amplitudes' and Grassmannian Residues

Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 2 negative-helicity gluons

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$$\lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \cdots & \lambda_n^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \cdots & \lambda_n^2 \end{pmatrix}$$

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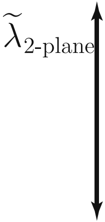

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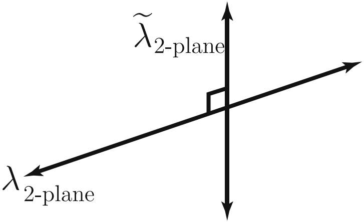


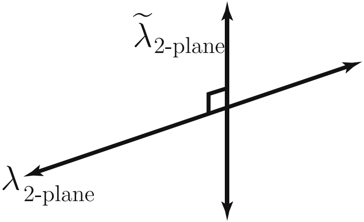
Parke-Taylor 'Amplitudes' and Grassmannian Residues

Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with m negative-helicity gluons:

$$\mathcal{A}_n^{(m)} \overset{?}{=} \frac{\delta^{m\times 4}\big(C\cdot\widetilde{\eta}\big)\,\delta^{m\times 2}\big(C\cdot\widetilde{\lambda}\big)}{\langle 1\cdots m\rangle\langle 2\cdots m{+}1\rangle\cdots\langle n\cdots m{-}1\rangle}$$

$$C \equiv \begin{pmatrix} c_1^1 & c_2^1 & c_3^1 & \cdots & c_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1^m & c_2^m & c_3^m & \cdots & c_n^m \end{pmatrix}$$



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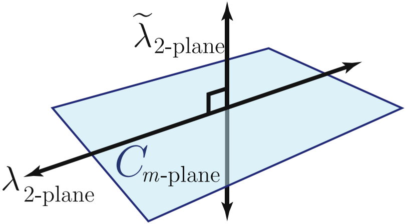

In order for momentum conservation, $\delta^{2\times 2}\big(\lambda\cdot\widetilde{\lambda}\big)$, to be part of the constraints, we must have that $C \supset \lambda$

Parke-Taylor 'Amplitudes' and Grassmannian Residues

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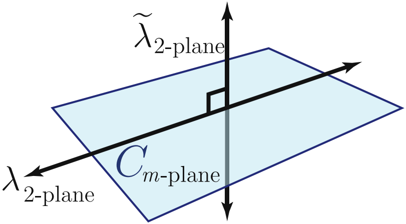

In order for momentum conservation, $\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)$, to be part of the constraints, we must have that $C\supset\lambda$, imposed via $\delta^{2\times(n-m)}\big(\lambda\cdot C^{\perp}\big)$

Parke-Taylor 'Amplitudes' and Grassmannian Residues

Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 3 negative-helicity gluons—e.g.,

$$\mathcal{A}_6^{(3)} \overset{?}{=} \frac{\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)}{\langle1\,2\,3\rangle\langle2\,3\,4\rangle\langle3\,4\,5\rangle\langle4\,5\,6\rangle\langle5\,6\,1\rangle\langle6\,1\,2\rangle}$$

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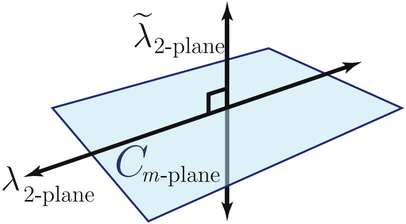

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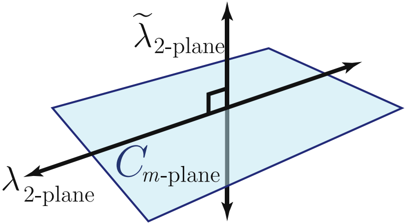

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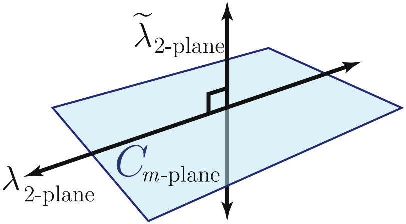
$$\dim(C) = 3\times6 - 3\times3 = 9$$



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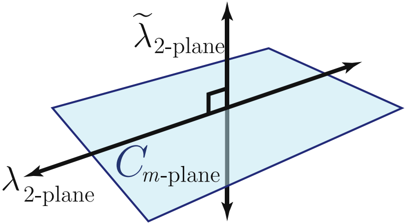
$$C \equiv \begin{pmatrix} c_1^1 & c_2^1 & c_3^1 & c_4^1 & c_5^1 & c_6^1 \\ c_1^2 & c_2^2 & c_3^2 & c_4^2 & c_5^2 & c_6^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 & c_5^3 & c_6^3 \end{pmatrix}$$

$$\dim(C) = 3\times6 - 3\times3 = 9 = 8 + 1$$



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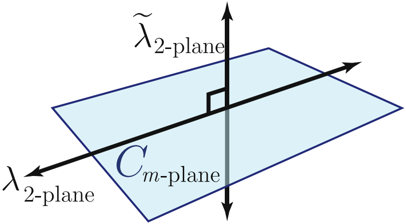

Parke-Taylor 'Amplitudes' and Grassmannian Residues

Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 3 negative-helicity gluons—e.g.,

$$\oint \frac{d\tau \qquad\qquad \delta^{3\times4}(C\cdot\widetilde{\eta}) \qquad \delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 123\rangle(\tau)\cdot\langle 234\rangle(\tau)\cdot\langle 345\rangle(\tau)\cdot\langle 456\rangle(\tau)\cdot\langle 561\rangle(\tau)\cdot\langle 612\rangle(\tau)}$$

$$C \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ c_1^3 & c_2^3 & c_3^3 & c_4^3 & c_5^3 & c_6^3 \end{pmatrix}$$

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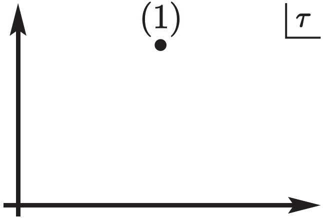
Parke-Taylor 'Amplitudes' and Grassmannian Residues

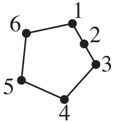
Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 3 negative-helicity gluons—e.g.,

$$\oint_{\langle 123\rangle=0}\frac{d\tau}{\langle 123\rangle(\tau)\cdot\langle 234\rangle(\tau)\cdot\langle 345\rangle(\tau)\cdot\langle 456\rangle(\tau)\cdot\langle 561\rangle(\tau)\cdot\langle 612\rangle(\tau)}\,\delta^{3\times 4}(C\cdot\widetilde{\eta})\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$

$$C\equiv\begin{pmatrix}\lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1\\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2\\ 0 & 0 & 0 & c_4^3 & c_5^3 & c_6^3\end{pmatrix}$$





Parke-Taylor 'Amplitudes' and Grassmannian Residues

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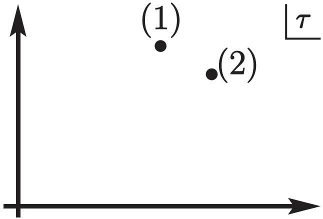
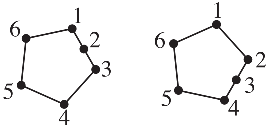
$$\oint_{\langle 234\rangle=0} \frac{d\tau \qquad \delta^{3\times4}(C\cdot\widetilde{\eta}) \qquad \delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 123\rangle(\tau)\cdot\langle 234\rangle(\tau)\cdot\langle 345\rangle(\tau)\cdot\langle 456\rangle(\tau)\cdot\langle 561\rangle(\tau)\cdot\langle 612\rangle(\tau)}$$

$$C \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ c_1^3 & 0 & 0 & 0 & c_5^3 & c_6^3 \end{pmatrix}$$





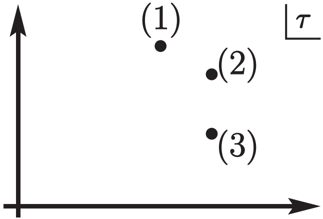
Parke-Taylor 'Amplitudes' and Grassmannian Residues

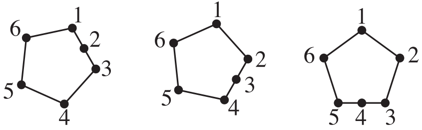
Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 3 negative-helicity gluons—e.g.,

$$\oint_{\langle 345\rangle=0} \frac{d\tau \qquad \delta^{3\times4}(C\cdot\widetilde{\eta}) \qquad \delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 123\rangle(\tau)\cdot\langle 234\rangle(\tau)\cdot\langle 345\rangle(\tau)\cdot\langle 456\rangle(\tau)\cdot\langle 561\rangle(\tau)\cdot\langle 612\rangle(\tau)}$$

$$C\equiv\begin{pmatrix}\lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1\\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2\\ c_1^3 & c_2^3 & 0 & 0 & 0 & c_6^3\end{pmatrix}$$





Parke-Taylor 'Amplitudes' and Grassmannian Residues

Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 3 negative-helicity gluons—e.g.,

$$\oint_{\langle 456\rangle=0}\frac{d\tau\qquad\delta^{3\times4}(C\cdot\widetilde{\eta})\qquad\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle123\rangle(\tau)\cdot\langle234\rangle(\tau)\cdot\langle345\rangle(\tau)\cdot\langle456\rangle(\tau)\cdot\langle561\rangle(\tau)\cdot\langle612\rangle(\tau)}$$

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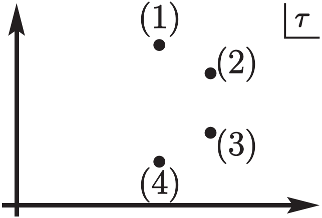
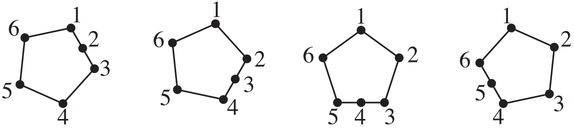




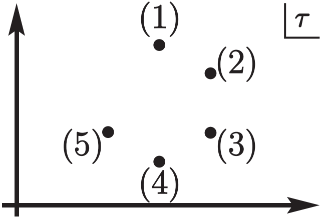
Parke-Taylor 'Amplitudes' and Grassmannian Residues

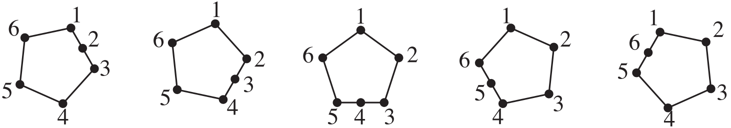
Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 3 negative-helicity gluons—e.g.,

$$\oint_{\langle 561\rangle=0}\frac{d\tau\qquad\delta^{3\times4}(C\cdot\widetilde{\eta})\qquad\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle123\rangle(\tau)\cdot\langle234\rangle(\tau)\cdot\langle345\rangle(\tau)\cdot\langle456\rangle(\tau)\cdot\langle561\rangle(\tau)\cdot\langle612\rangle(\tau)}$$

$$C\equiv\begin{pmatrix}\lambda_1^1&\lambda_2^1&\lambda_3^1&\lambda_4^1&\lambda_5^1&\lambda_6^1\\\lambda_1^2&\lambda_2^2&\lambda_3^2&\lambda_4^2&\lambda_5^2&\lambda_6^2\\0&c_2^3&c_3^3&c_4^3&0&0\end{pmatrix}$$





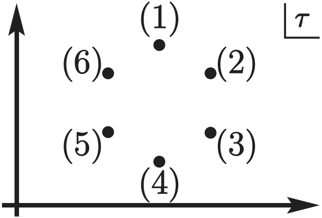
Parke-Taylor 'Amplitudes' and Grassmannian Residues

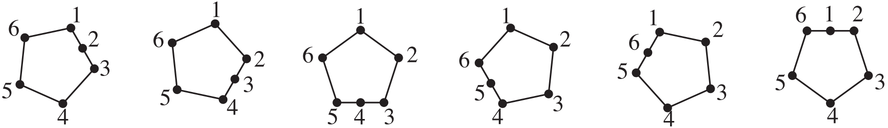
Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with 3 negative-helicity gluons—e.g.,

$$\oint_{\langle 612\rangle=0} \frac{d\tau \qquad \delta^{3\times4}(C\cdot\widetilde{\eta}) \qquad \delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 123\rangle(\tau)\cdot\langle 234\rangle(\tau)\cdot\langle 345\rangle(\tau)\cdot\langle 456\rangle(\tau)\cdot\langle 561\rangle(\tau)\cdot\langle 612\rangle(\tau)}$$

$$C \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & c_3^3 & c_4^3 & c_5^3 & 0 \end{pmatrix}$$





Parke-Taylor 'Amplitudes' and Grassmannian Residues

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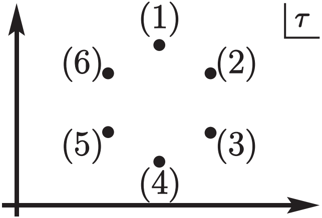

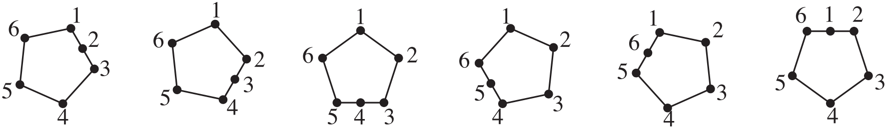

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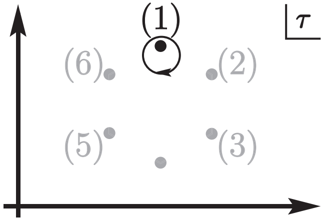

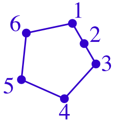

Parke-Taylor 'Amplitudes' and Grassmannian Residues

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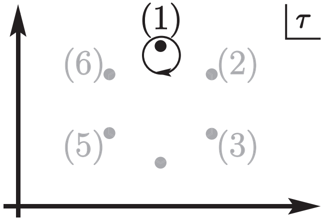

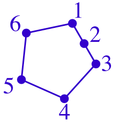

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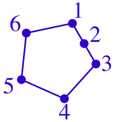
$$\frac{1}{\qquad\qquad\qquad\qquad}$$



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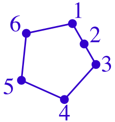
$$\frac{1}{\langle 2\,3\rangle[5\,6]}$$



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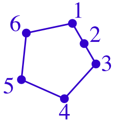

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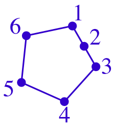

$$\frac{1}{\langle 2\,3\rangle[5\,6][6|(5+4)|3\rangle s_{4\,5\,6}}$$

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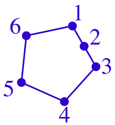

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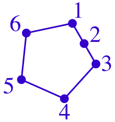

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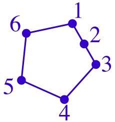

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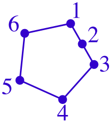
$$\frac{\langle 2|(4+6)|5]^4}{\langle 2\,3\rangle[5\,6][6|(5+4)|3\rangle s_{4\,5\,6}\langle 1|(6+5)|4][4\,5]\langle 1\,2\rangle}$$



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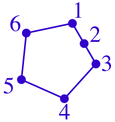
$$\mathcal{A}_6^{(3)}(+,-,+,-,+,-)=(1+r^2+r^4)\frac{\langle2|(4+6)|5]^4}{\langle2\,3\rangle[5\,6][6|(5+4)|3\rangle s_{4\,5\,6}\langle1|(6+5)|4][4\,5]\langle1\,2\rangle}$$



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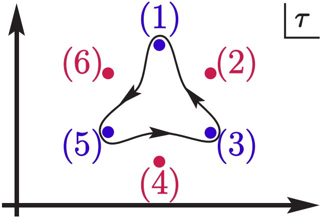
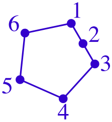
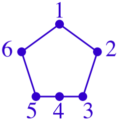
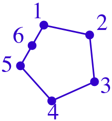


$$\mathcal{A}_6^{(3)}(+,-,+,-,+,-) = (1)+(3)+(5)$$







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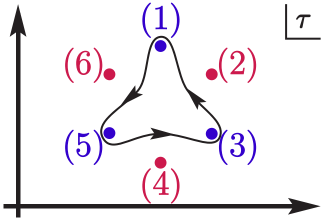

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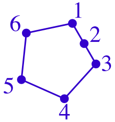

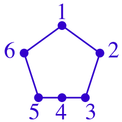

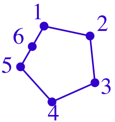

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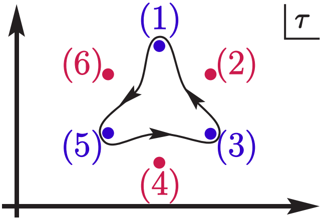
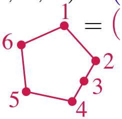
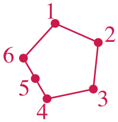
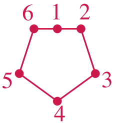


$$\begin{aligned}\mathcal{A}_6^{(3)}(+,-,+,-,+,-) &= (1)+(3)+(5)\\ &= (2)+(4)+(6)\end{aligned}$$







Parke-Taylor 'Amplitudes' and Grassmannian Residues

Recall the natural desire to generalize the Parke-Taylor formula according to:

Amplitudes with m negative-helicity gluons:

$$\mathcal{A}_n^{(m)} = \oint \frac{d^{m\times n}C}{\mathrm{vol}(GL(m))} \frac{\delta^{m\times 4}\big(C\cdot\widetilde{\eta}\big)\,\delta^{m\times 2}\big(C\cdot\widetilde{\lambda}\big)\,\delta^{2\times(n-m)}\big(\lambda\cdot C^{\perp}\big)}{\langle 1\,2\,\cdots\,m\rangle\langle 2\,3\,\cdots\,m{+}1\rangle\cdots\langle n\;1\cdots\,m{-}1\rangle}$$

$$C \equiv \begin{pmatrix} c_1^1 & c_2^1 & c_3^1 & \cdots & c_n^1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_1^m & c_2^m & c_3^m & \cdots & c_n^m \end{pmatrix}$$

Parke-Taylor 'Amplitudes' and Grassmannian Residues

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Grassmannian Correspondence:

The residues of $\mathcal{L}_{n,m}$ are in one-to-one correspondence with on-shell functions of $\mathcal{N}=4$

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A complete, $GL(k)$-invariant description of any contour of $\mathcal{L}_{n,k}$ would be a list of all the ranks of spaces spanned by all consecutive chains of columns.

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$|1\ \cdots\ 1|$

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$|1 \;\cdots\; 2|$

$|1 \;\cdots\; 1|$

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$|1 \cdots n{-}2|$

$\vdots$

$|1 \;\cdots\; 2|$

$|1 \;\cdots\; 1|$

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Ranks of Consecutive Chains of Columns

$$\begin{array}{c}
|1\cdots n{-}1| \\
|1\cdots n{-}2| \\
\vdots \\
|1\;\cdots\;2| \\
|1\;\cdots\;1|
\end{array}$$

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$|1\ \cdots\ n|$

$|1\cdots n{-}1|$

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Ranks of Consecutive Chains of Columns

$$\begin{array}{cc}
 & |2\cdots n{+}1| \\
|1\ \cdots\ n\ | & |2\ \cdots\ n\ | \\
|1\cdots n{-}1| & |2\ \cdots n{-}1| \\
|1\cdots n{-}2| & |2\ \cdots n{-}2| \\
\vdots & \vdots \\
|1\ \cdots\ 2| & |2\ \cdots\ 2| \\
|1\ \cdots\ 1| &
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$$\begin{array}{ccccc}
 & & & & |n\ \cdots\ 2n| \\
 & |2\cdots n{+}1| & & \iddots & \vdots \\
|1\ \cdots\ n| & |2\ \cdots\ n| & \iddots & |n{-}1\ \cdots\ n| & |n\ \cdots\ n| \\
|1\cdots n{-}1| & |2\ \cdots n{-}1| & \iddots & |n{-}1\cdots n{-}1| & \\
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\vdots & \iddots & \iddots & \vdots & \vdots \\
|1\cdots n| & |2\cdots n| & \iddots & |n{-}1\cdots n| & |n\cdots n| \\
|1\cdots n{-}1| & |2\cdots n{-}1| & \iddots & |n{-}1\cdots n{-}1| & \\
|1\cdots n{-}2| & |2\cdots n{-}2| & \iddots & & \\
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$\lvert 1\cdots 2n\rvert$	$\lvert 2\cdots 2n\rvert$	$\cdots$	$\lvert n{-}1\cdots 2n\rvert$	$\lvert n\cdots 2n\rvert$
$\vdots$	$\iddots$	$\iddots$	$\vdots$	$\vdots$
$\lvert 1\cdots n\rvert$	$\lvert 2\cdots n\rvert$	$\iddots$	$\lvert n{-}1\cdots n\rvert$	$\lvert n\cdots n\rvert$
$\lvert 1\cdots n{-}1\rvert$	$\lvert 2\cdots n{-}1\rvert$	$\iddots$	$\lvert n{-}1\cdots n{-}1\rvert$	
$\lvert 1\cdots n{-}2\rvert$	$\lvert 2\cdots n{-}2\rvert$	$\iddots$		
$\vdots$	$\vdots$	$\iddots$		
$\lvert 1\cdots 2\rvert$	$\lvert 2\cdots 2\rvert$			
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$$\begin{array}{ccccc}
k & k & \cdots & k & k\\
\vdots & \iddots & \iddots & \vdots & \vdots\\
k & |2\;\cdots\;n\,| & \iddots & |n{-}1\;\cdots\;n\,| & |\,n\;\cdots\;n\,|\\
|1\cdots n{-}1| & |2\cdots n{-}1| & \iddots & |n{-}1\cdots n{-}1| & \\
|1\cdots n{-}2| & |2\cdots n{-}2| & \iddots & & \\
\vdots & \vdots & \iddots & & \\
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k & k & \cdots & k & k \\
\vdots & \iddots & \iddots & \vdots & \vdots \\
k & |2\ \cdots\ n\ | & \iddots & |n{-}1\ \cdots\ n\ | & |n\ \cdots\ n\ | \\
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\vdots & \iddots & \iddots & \vdots & \vdots \\
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|1 \cdots n{-}1| & |2 \cdots n{-}1| & \iddots & |n{-}1 \cdots n{-}1| & |n \cdots n{-}1| \\
|1 \cdots n{-}2| & |2 \cdots n{-}2| & \iddots & |n{-}1 \cdots n{-}2| & \\
\vdots & \vdots & \iddots & & \\
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|1 \cdots 0| & & & &
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\vdots & \iddots & \iddots & \vdots & \vdots \\
k & |2\ \cdots\ n\ | & \iddots & |n{-}1\ \cdots\ n\ | & |\,n\ \cdots\ n\ | \\
|\,1\cdots n{-}1| & |\,2\cdots n{-}1| & \iddots & |n{-}1\cdots n{-}1| & 0 \\
|\,1\cdots n{-}2| & |\,2\cdots n{-}2| & \iddots & 0 & \\
\vdots & \vdots & \iddots & & \\
|\,1\ \cdots\ 2| & |\,2\ \cdots\ 2| & \iddots & & \\
|\,1\ \cdots\ 1| & 0 & & & \\
0 & & & &
\end{array}$$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$

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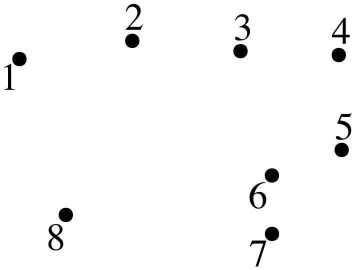

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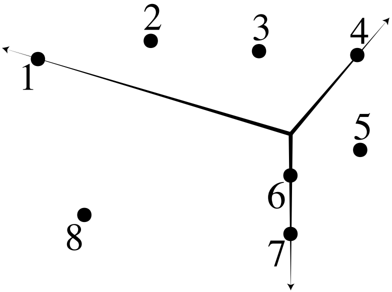

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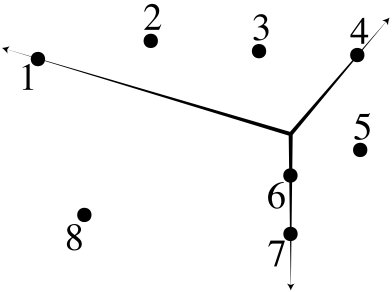

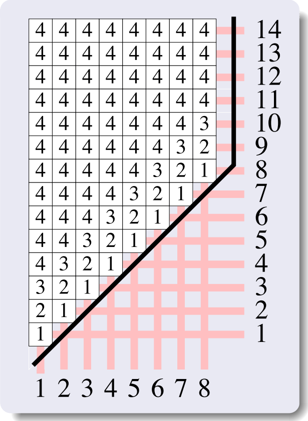

4	4	4	4	4	4	4	4	14
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
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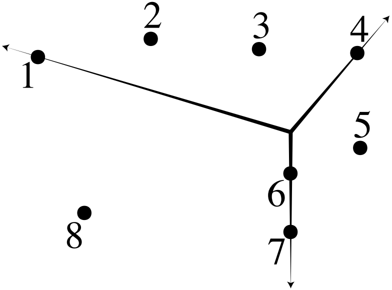
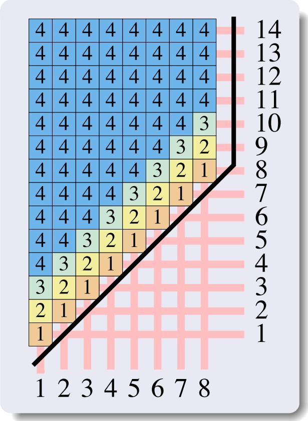
4	4	4	3	2	1			6
4	4	3	2	1				5
4	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$





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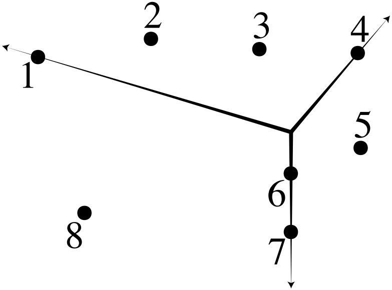
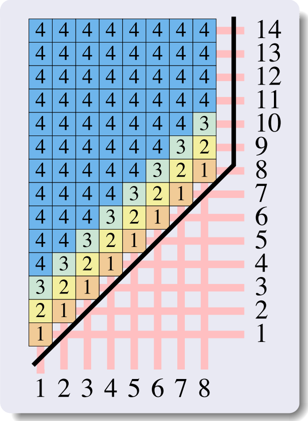
4	4	4	3	2	1			6
4	4	3	2	1				5
4	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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4	4	3	2	1				5
4	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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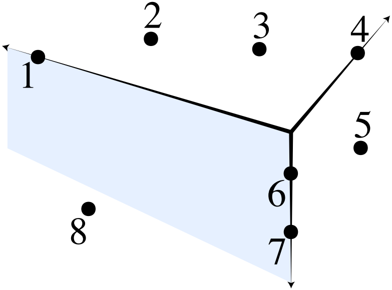

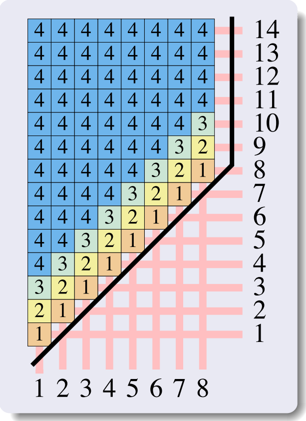

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4	4	4	4	3	2	1		7
4	4	4	3	2	1			6
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4	3	2	1					4
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2	1							2
1								1
1	2	3	4	5	6	7	8	

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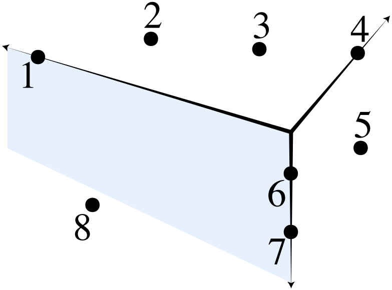

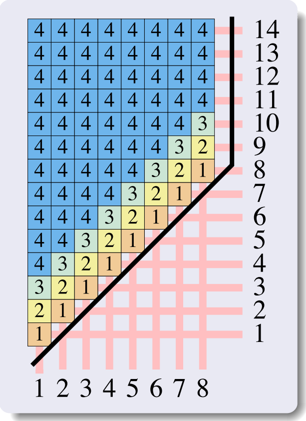

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4	3	2	1					4
3	2	1						3
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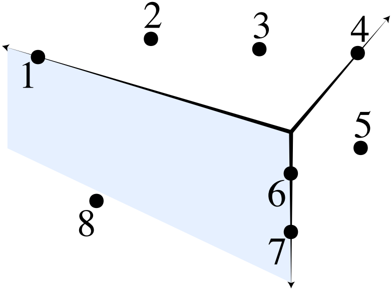

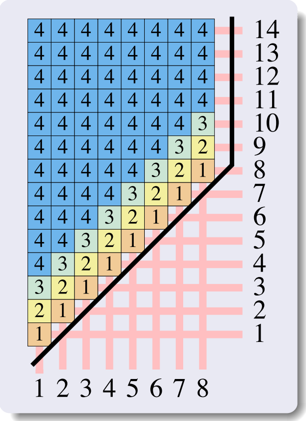

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4	3	2	1					4
3	2	1						3
2	1							2
1								1
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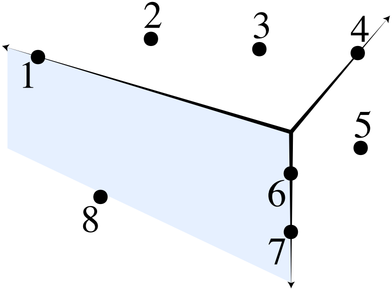

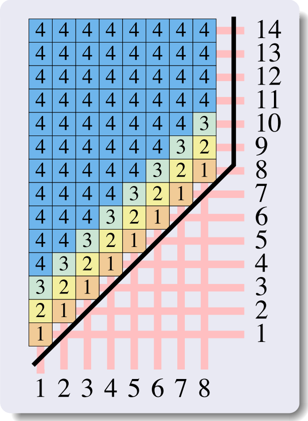

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4	4	3	2	1				5
4	3	2	1					4
3	2	1						3
2	1							2
1								1
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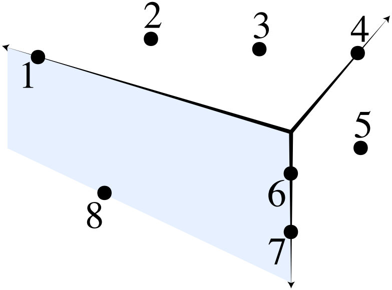

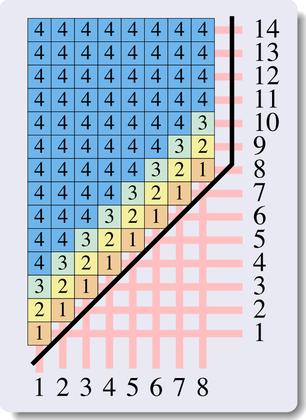

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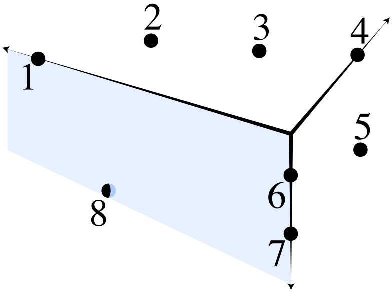

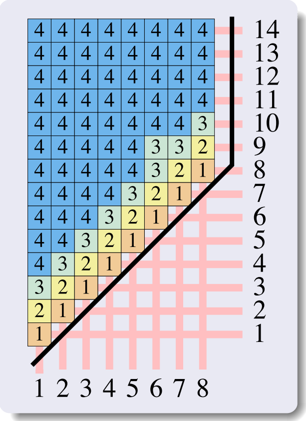

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3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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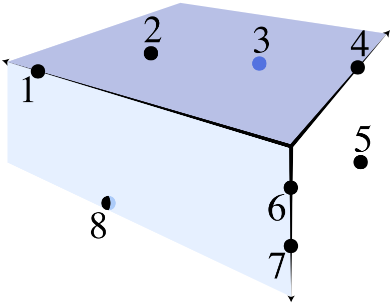

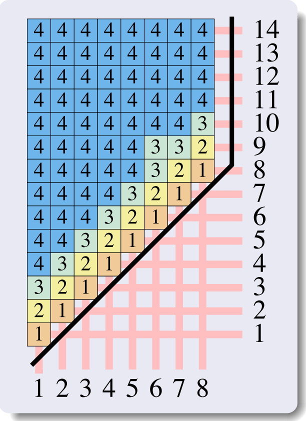

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4	4	4	3	2	1			6
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4	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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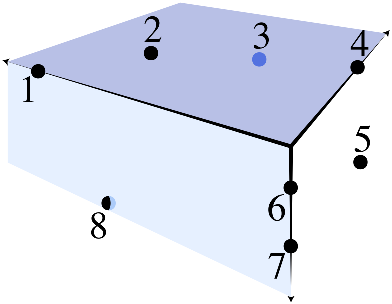

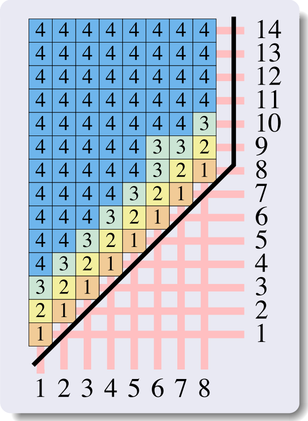

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4	4	3	2	1				5
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3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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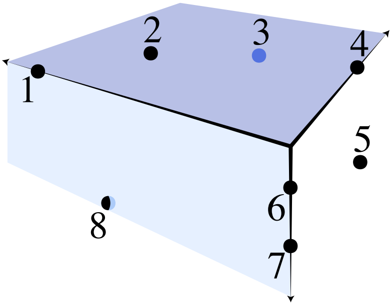

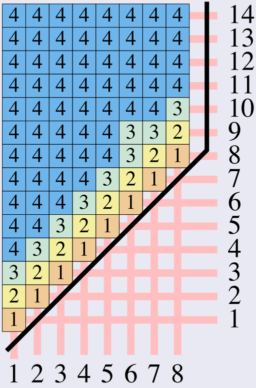

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2	2	1						
1	1							
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Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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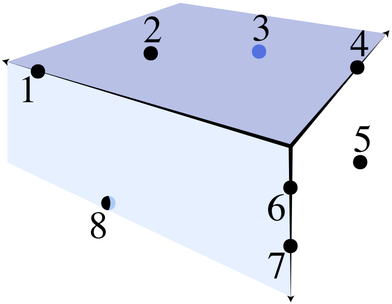

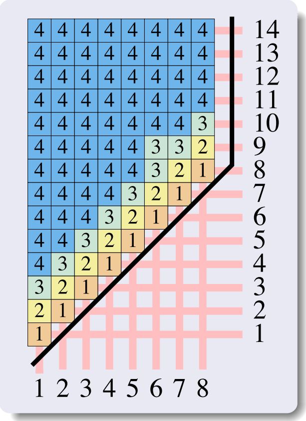

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4	4	3	2	1				5
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3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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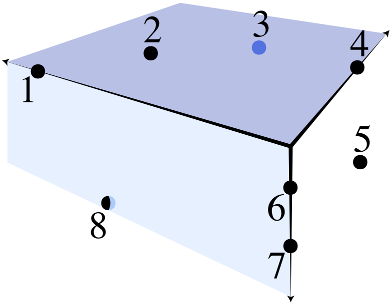

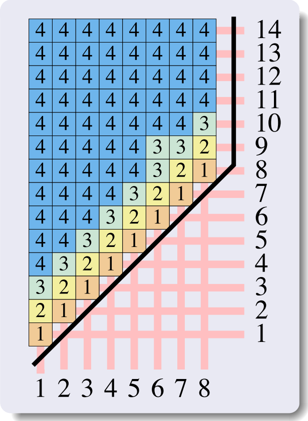

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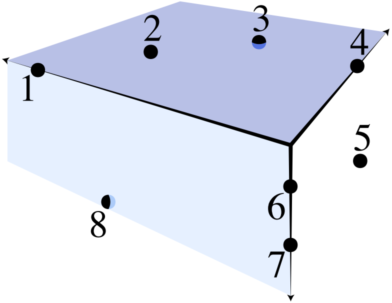

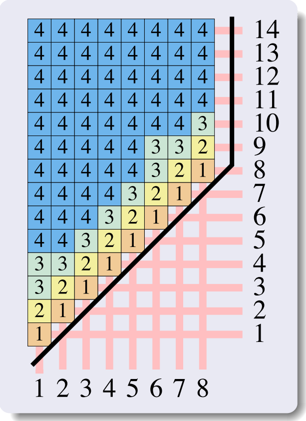

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4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
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1								1
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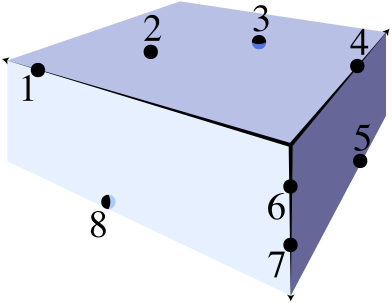

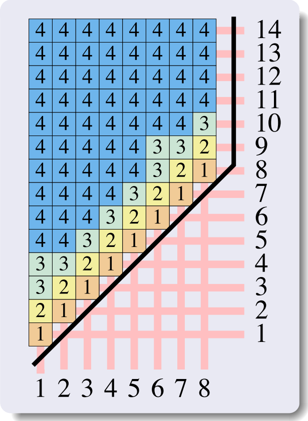

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1								1
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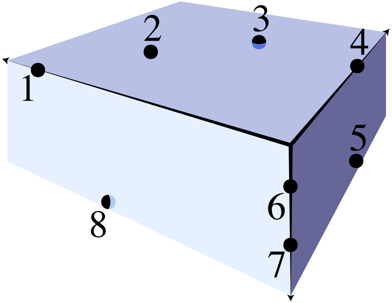

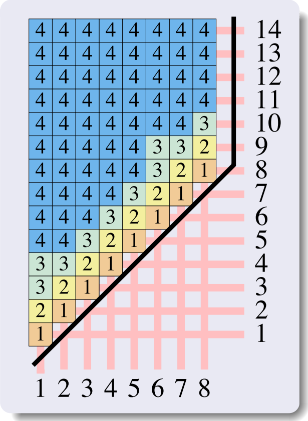

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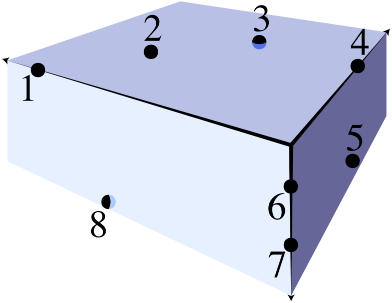
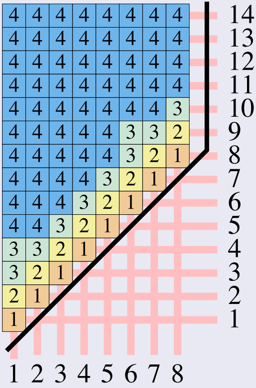
4	4	4	4	4	4	4	3	10
4	4	4	4	4	3	3	2	9
4	4	4	4	4	3	2	1	8
4	4	4	4	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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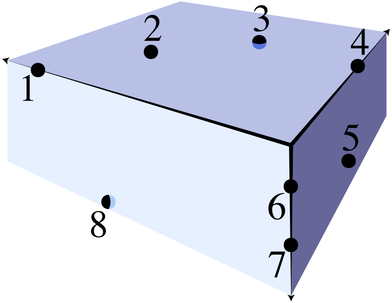
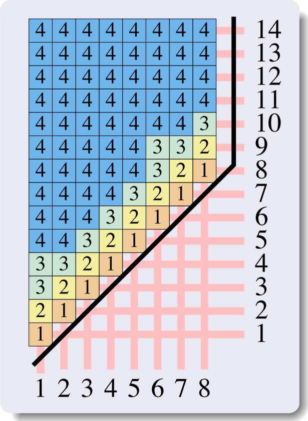
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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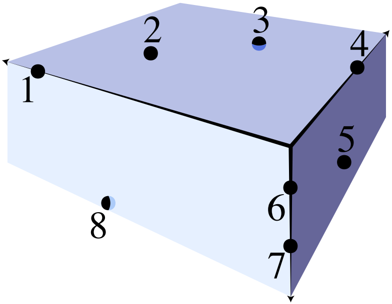
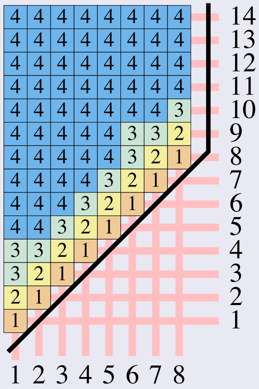
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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4	4	4	4	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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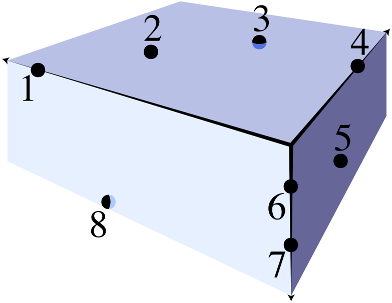

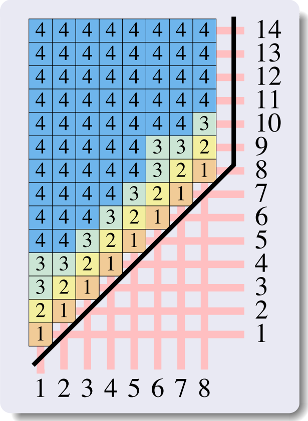

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4	4	4	4	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
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1								1
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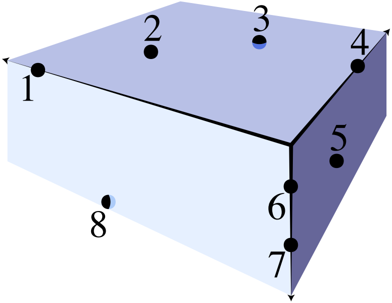

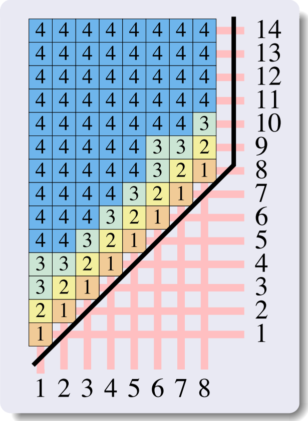

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4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
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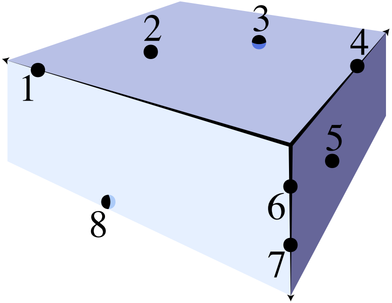

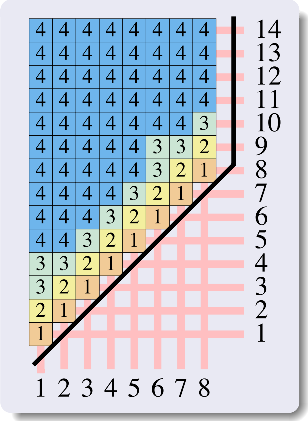

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4	4	4	4	4	3	2	1	8
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4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
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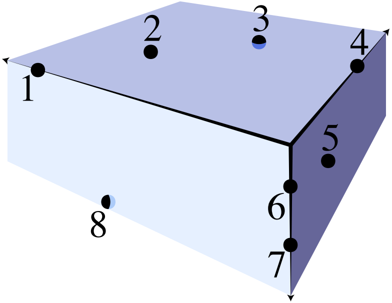

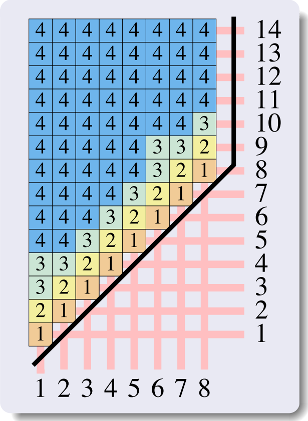

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4	4	4	4	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
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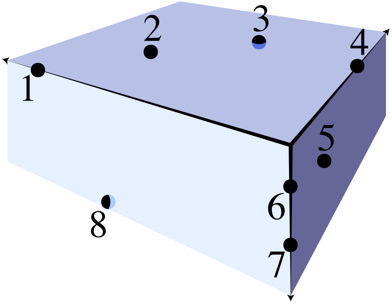

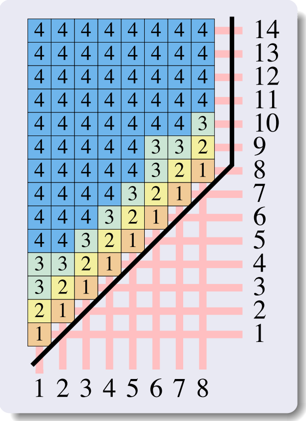

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4	4	4	4	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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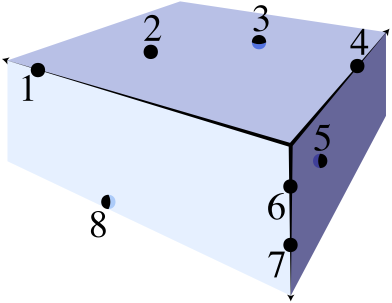

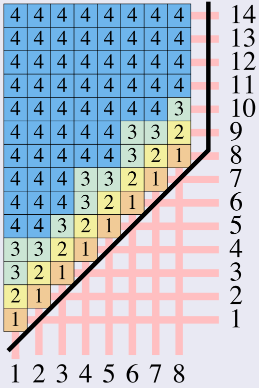

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4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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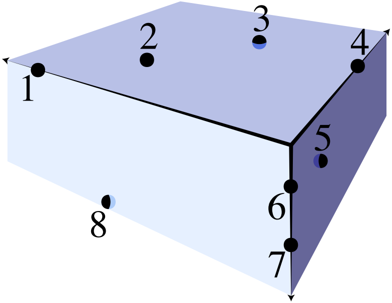

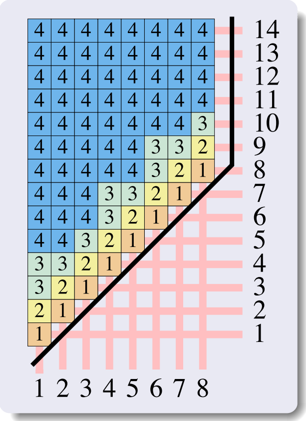

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4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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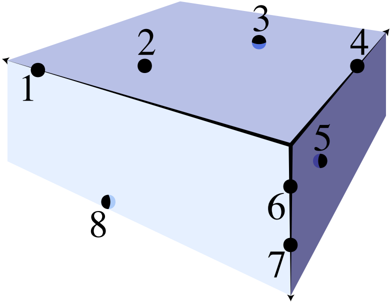

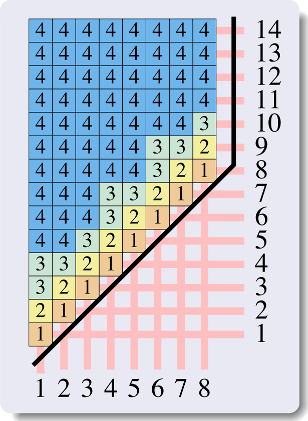

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4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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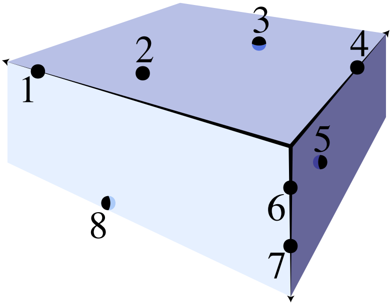

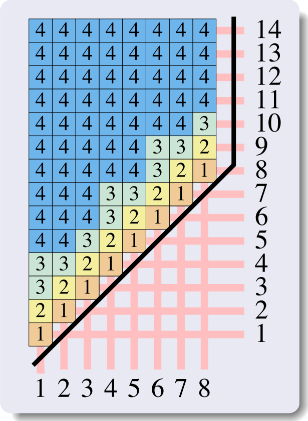

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5	4	4	3	2	1			
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3	3	2	1					
2	2	1						
1	1							
	1	2	3	4	5	6	7	8

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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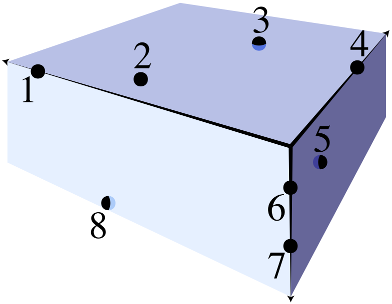

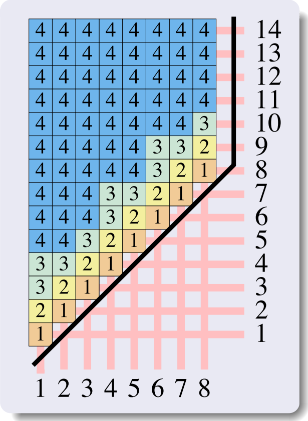

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4	4	3	2	1				5
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2	1							2
1								1
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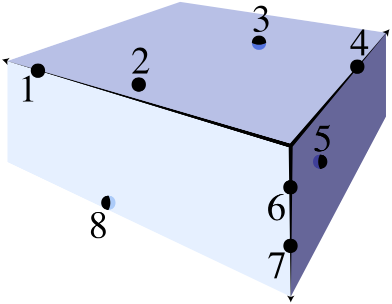

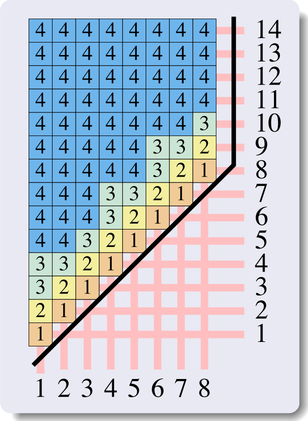

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4	4	3	2	1				5
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3	2	1						3
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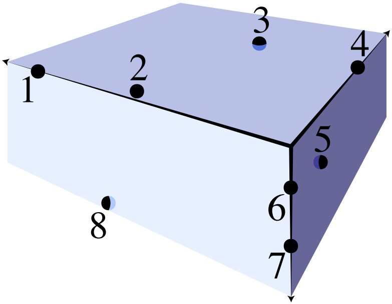

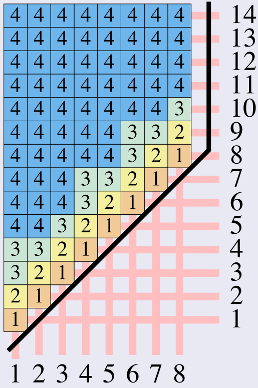

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4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
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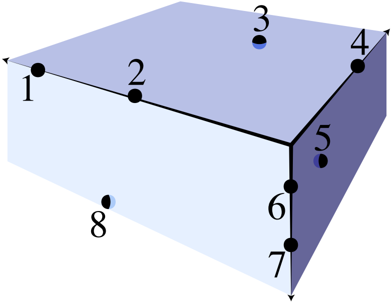

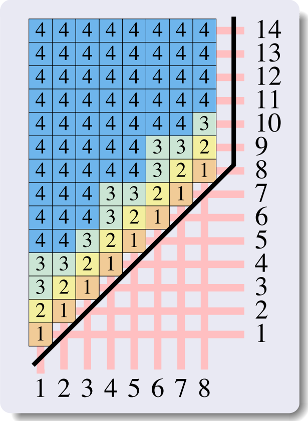

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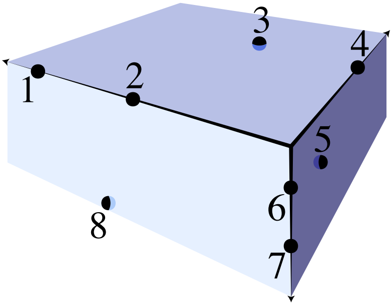
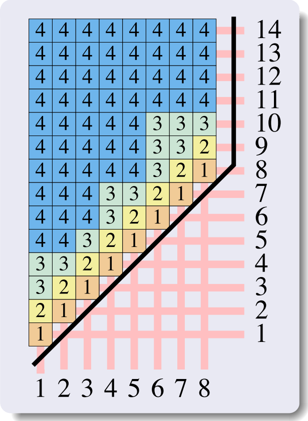
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
4	4	4	4	4	4	4	3	10
4	4	4	4	4	3	3	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





4	4	4	4	4	4	4	4	14
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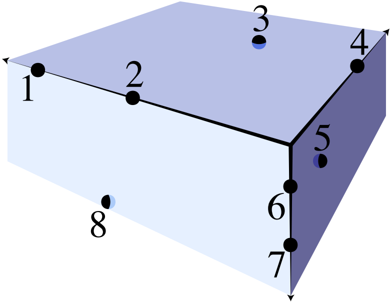
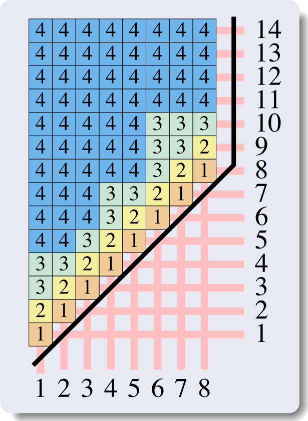
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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4	4	4	4	4	3	2	1	8
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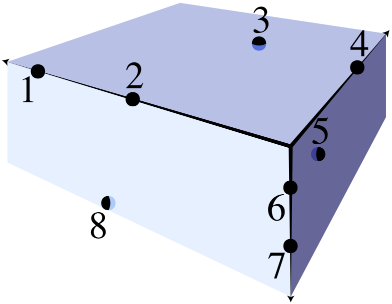
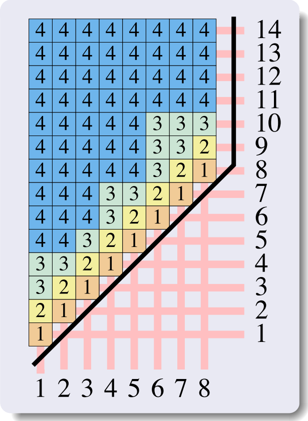
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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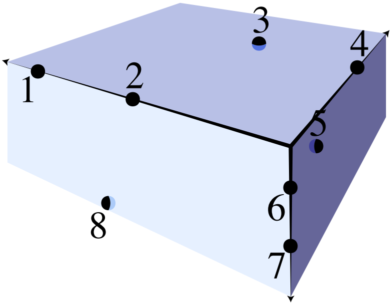
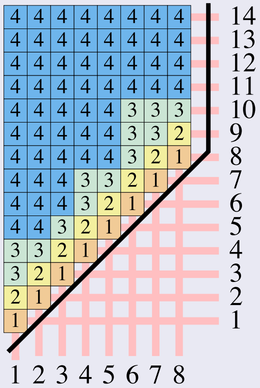
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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6	4	4	4	3	2	1		
5	4	4	3	2	1			
4	3	3	2	1				
3	3	2	1					
2	2	1						
1	1							
	1	2	3	4	5	6	7	8

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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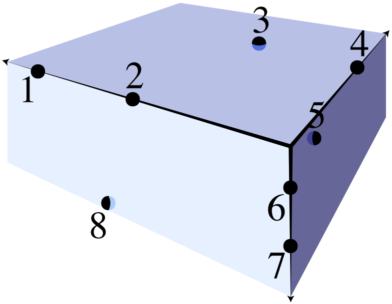

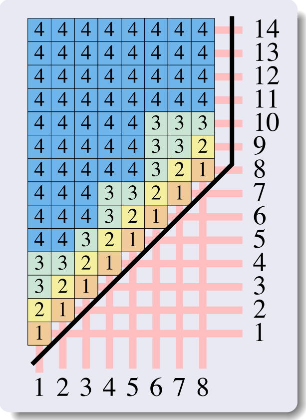

	1	2	3	4	5	6	7	8
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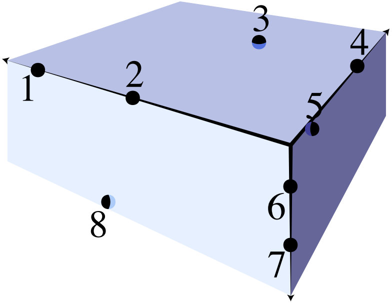
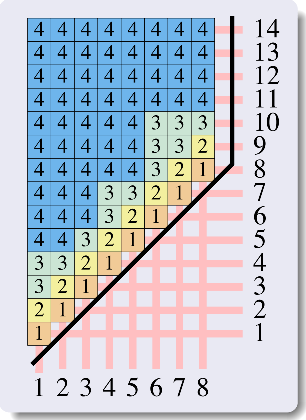
7	4	4	4	3	3	2	1	
6	4	4	4	3	2	1		
5	4	4	3	2	1			
4	3	3	2	1				
3	3	2	1					
2	2	1						
1	1							

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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4	4	4	4	4	4	4	4	12
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4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	3	2	9
4	4	4	4	4	3	2	1	8
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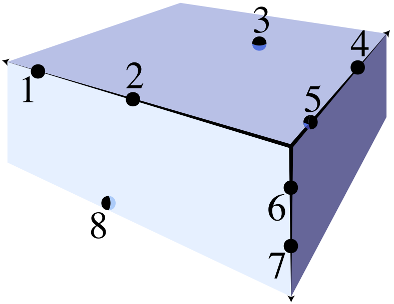
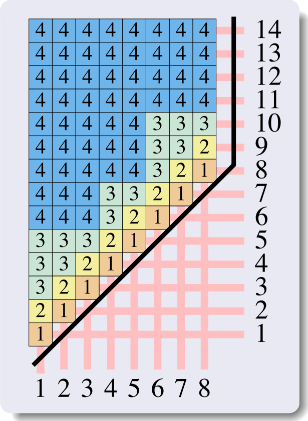
4	4	4	3	2	1			6
4	4	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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4	4	4	4	4	3	2	1	8
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3	3	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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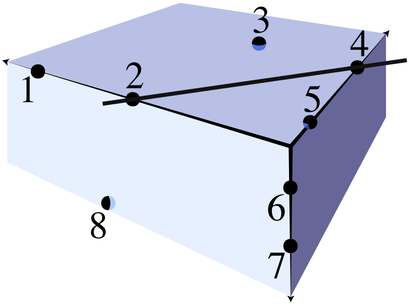
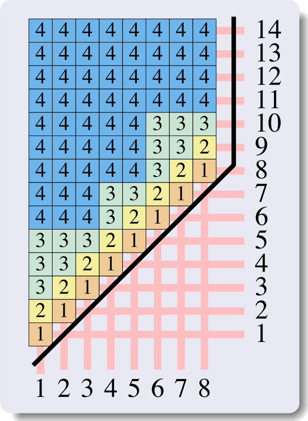
Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	3	2	9
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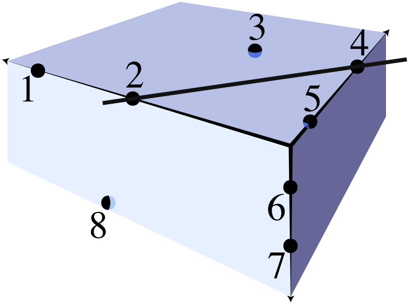
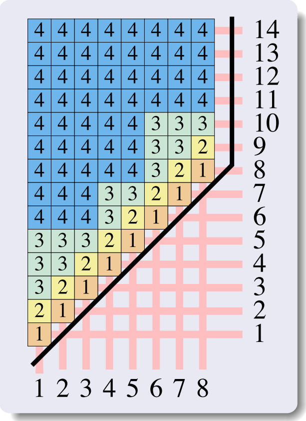
4	4	4	3	2	1			6
3	3	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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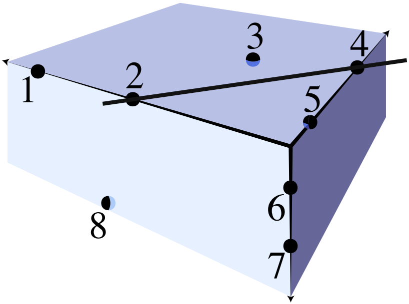
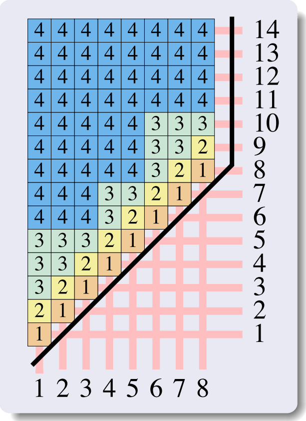
4	4	4	4	4	3	3	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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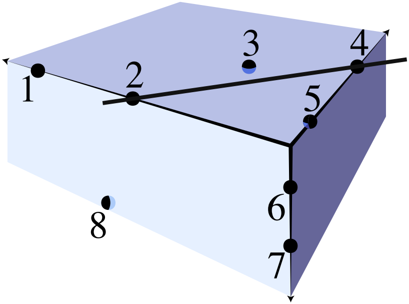
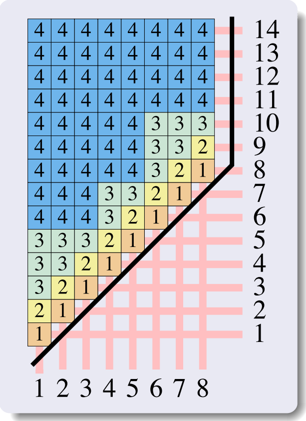
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





1	2	3	4	5	6	7	8	
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4	4	4	4	4	4	4	4	11
4	4	4	4	4	3	3	3	10
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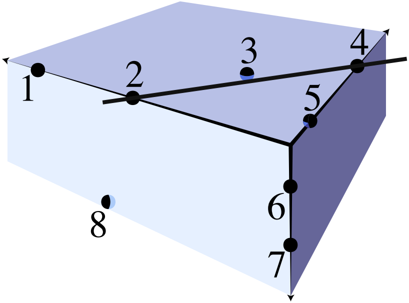
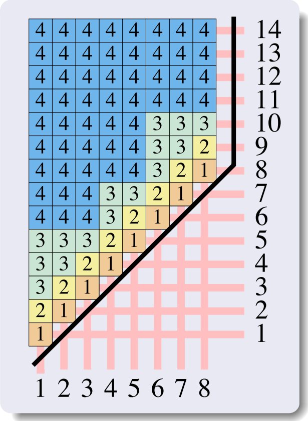
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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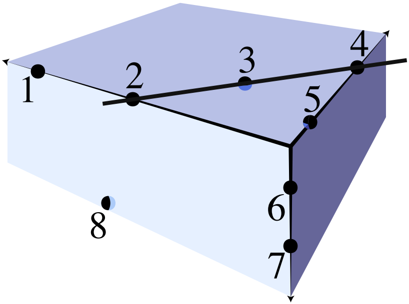
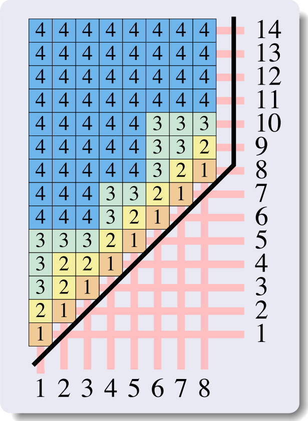
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	3	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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7	4	4	4	3	3	2	1	
6	4	4	4	3	2	1		
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3	3	2	1					
2	2	1						
1	1							

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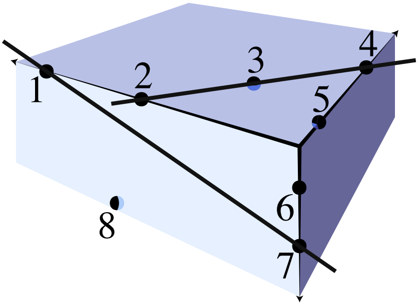
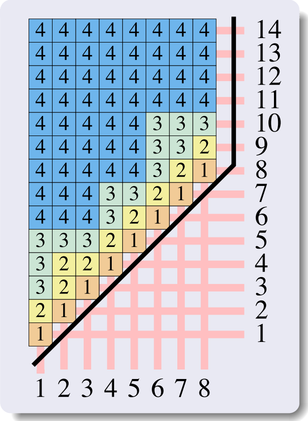
Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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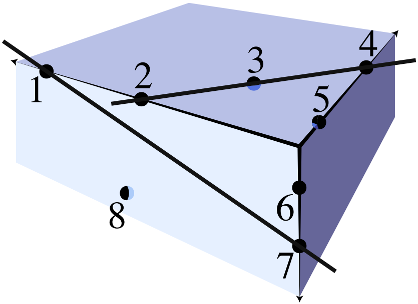
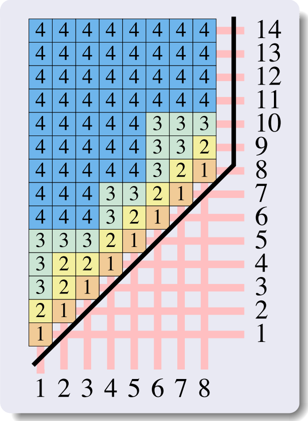
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

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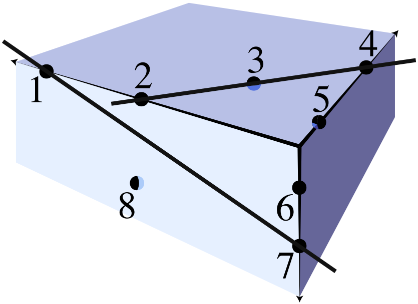
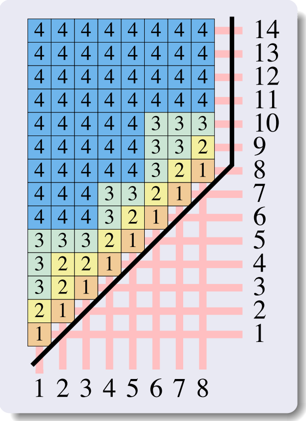
Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:

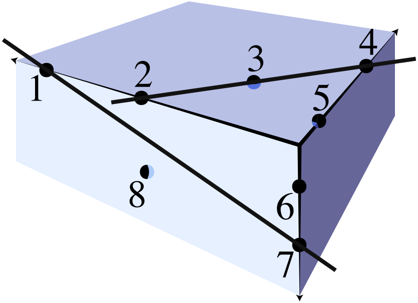


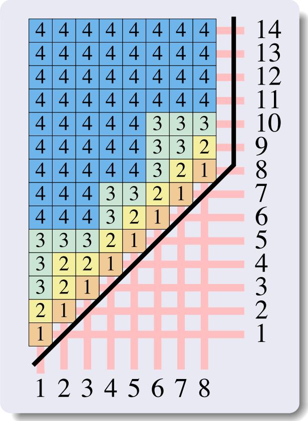


4	4	4	4	4	4	4	4	14
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	3	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:

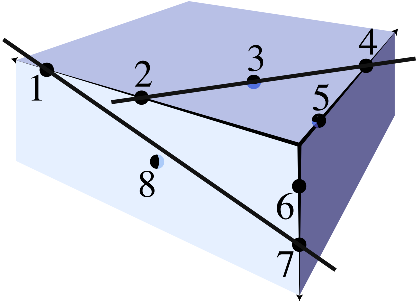


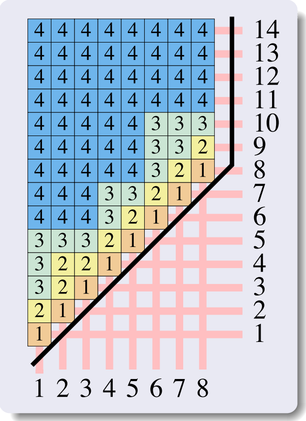


4	4	4	4	4	4	4	4	14
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	3	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:

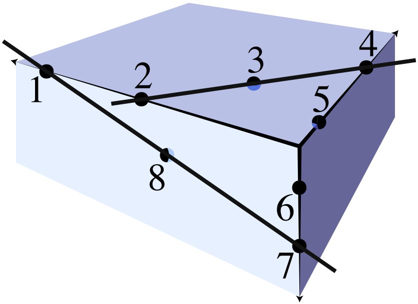


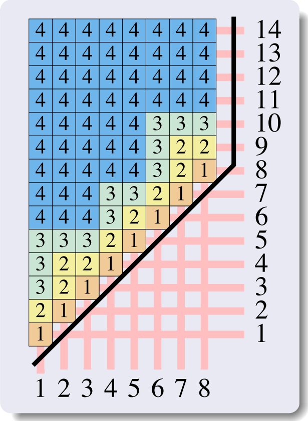


4	4	4	4	4	4	4	4	14
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	3	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:





4	4	4	4	4	4	4	4	14
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	2	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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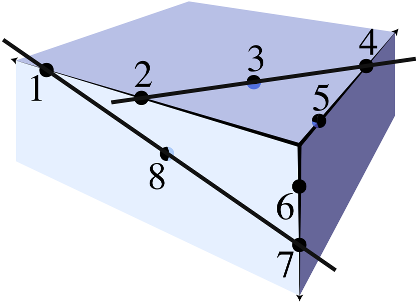

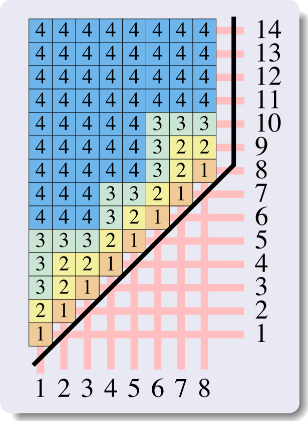

4	4	4	4	4	4	4	4	14
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	2	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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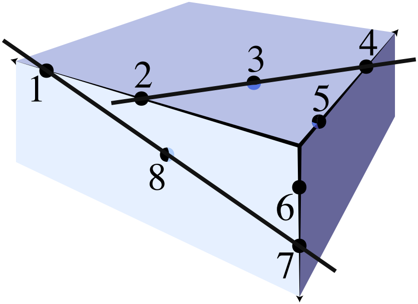

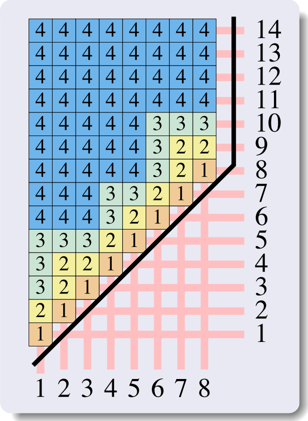

4	4	4	4	4	4	4	4	14
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	2	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1
1	2	3	4	5	6	7	8	

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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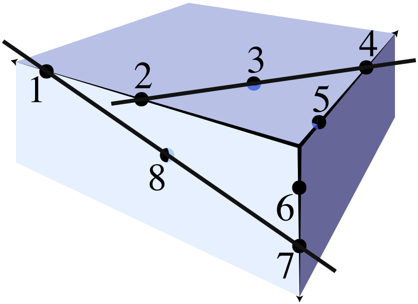

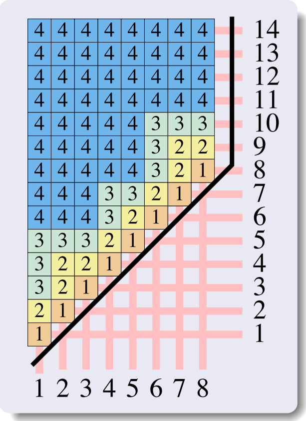

1	2	3	4	5	6	7	8	
4	4	4	4	4	4	4	4	14
4	4	4	4	4	4	4	4	13
4	4	4	4	4	4	4	4	12
4	4	4	4	4	4	4	4	11
4	4	4	4	4	3	3	3	10
4	4	4	4	4	3	2	2	9
4	4	4	4	4	3	2	1	8
4	4	4	3	3	2	1		7
4	4	4	3	2	1			6
3	3	3	2	1				5
3	2	2	1					4
3	2	1						3
2	1							2
1								1

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

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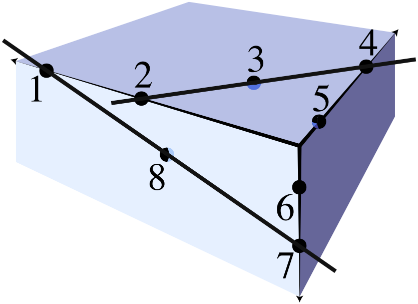
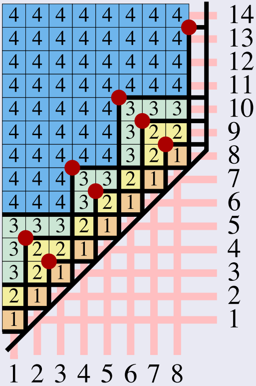
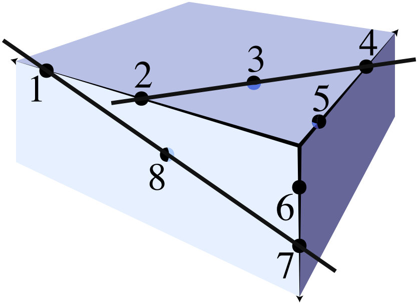


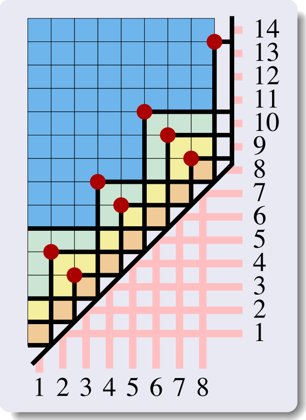


	1	2	3	4	5	6	7	8
14	4	4	4	4	4	4	4	4
13	4	4	4	4	4	4	4	4
12	4	4	4	4	4	4	4	4
11	4	4	4	4	4	4	4	4
10	4	4	4	4	4	3	3	3
9	4	4	4	4	4	3	2	2
8	4	4	4	4	4	3	2	1
7	4	4	4	3	3	2	1	
6	4	4	4	3	2	1		
5	3	3	3	2	1			
4	3	2	2	1				
3	3	2	1					
2	2	1						
1	1							

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

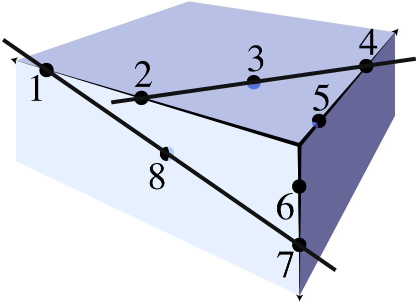
Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:

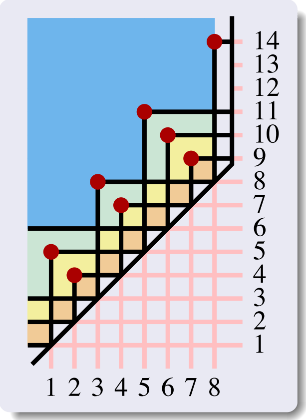




Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

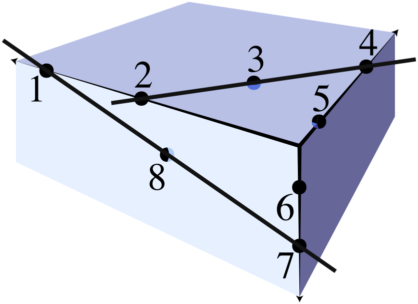
Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:

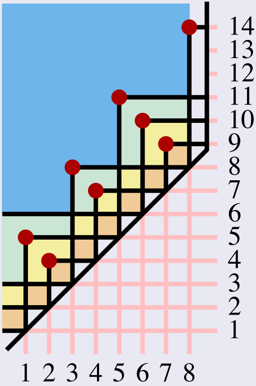




Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:

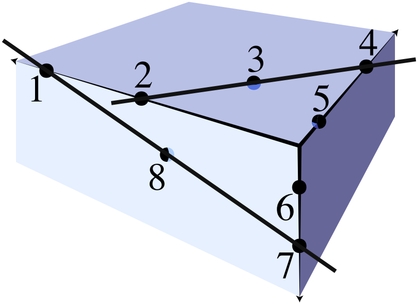


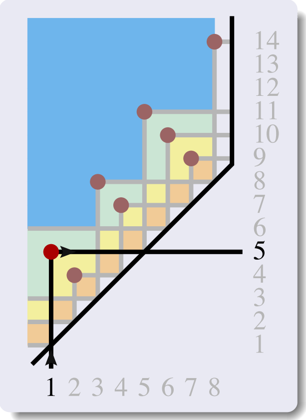


Permutation Label

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:



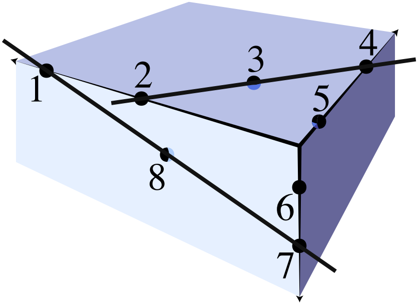


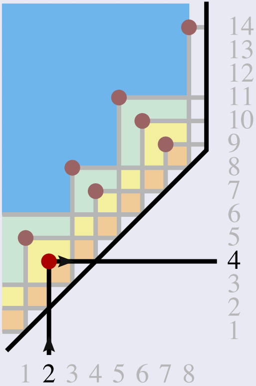
Permutation Label

$\sigma(1) = 5$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:



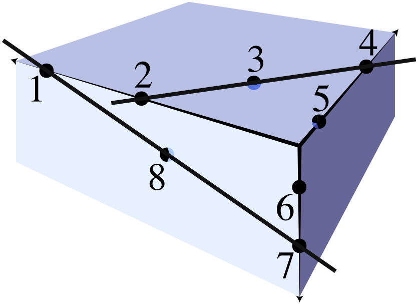


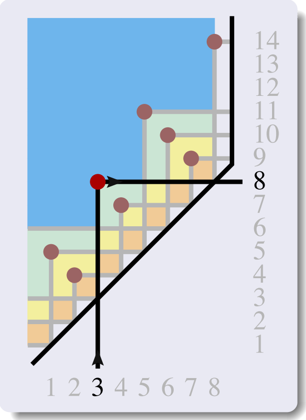
Permutation Label

$\sigma(2) = 4$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:



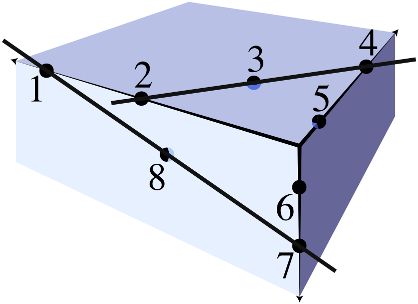


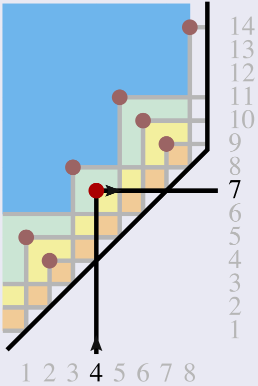
Permutation Label

$\sigma(3) = 8$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:



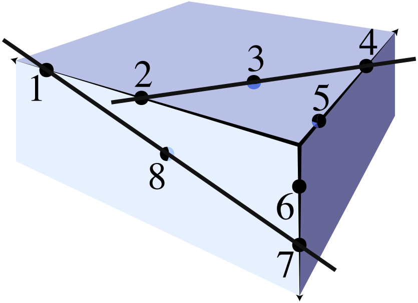


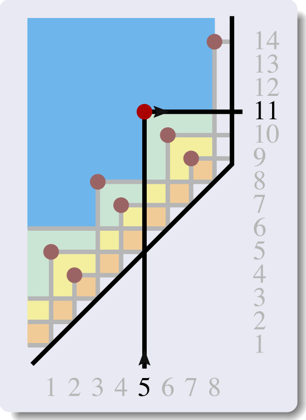
Permutation Label

$\sigma(4)=7$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:



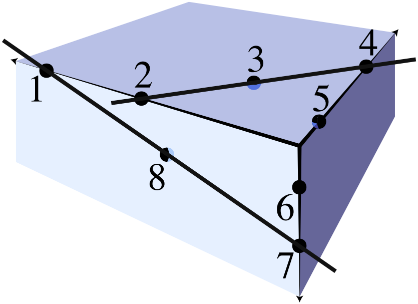


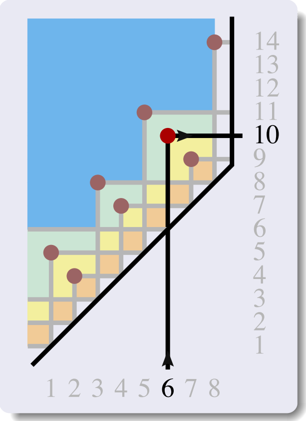
Permutation Label

$$\sigma(5) = 3 + 8 = 11$$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:



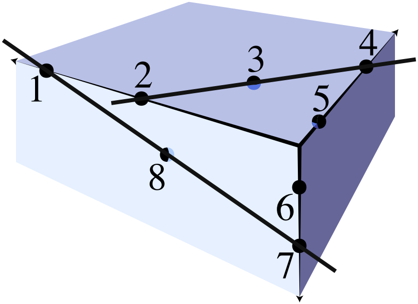


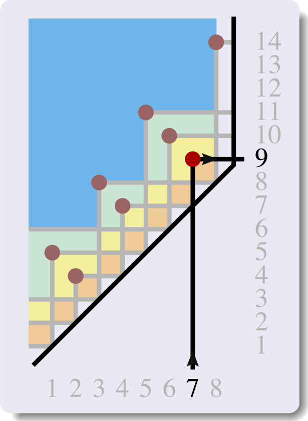
Permutation Label

$$\sigma(6) = 2 + 8 = 10$$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:



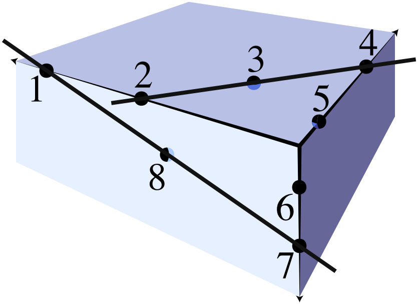


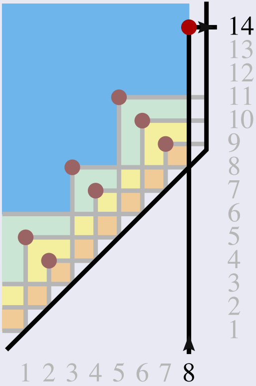
Permutation Label

$$\sigma(7)=1+8=9$$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:



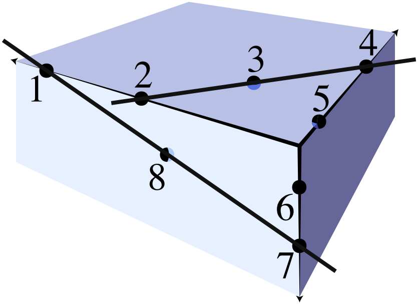


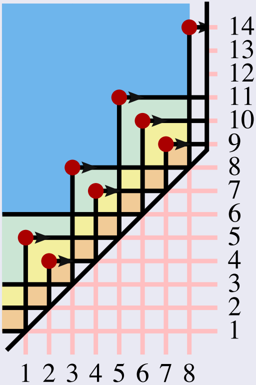
Permutation Label

$$\sigma(8)=6+8=14$$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C \in G(4,8)$, $c_a \in \mathbb{P}^3$, and impose the following sequence of consecutive constraints:



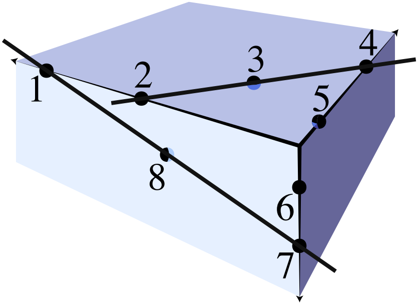


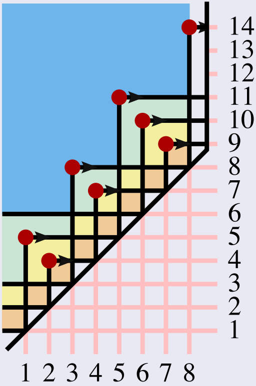
Permutation Label

$\{5, 4, 8, 7, 11, 10, 9, 14\}$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:





Geometric meaning of the permutation $\sigma(a)=b$

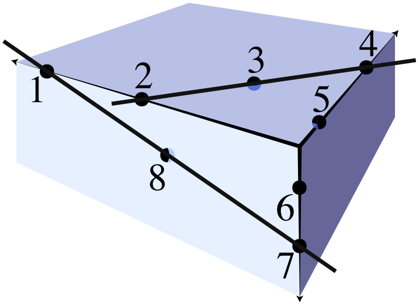
For each column a, there is a unique, **nearest** column $b\geq a$ such that $a\in\text{span}\{a{+}1,\ldots,b{-}1,b\}$

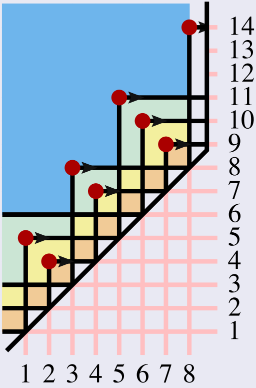
Permutation Label

$\{5,4,8,7,11,10,9,14\}$

Exempli Gratia: Iterated Residues of $\mathcal{L}_{8,4}$

Take a *generic* $C\in G(4,8)$, $c_a\in\mathbb{P}^3$, and impose the following sequence of consecutive constraints:





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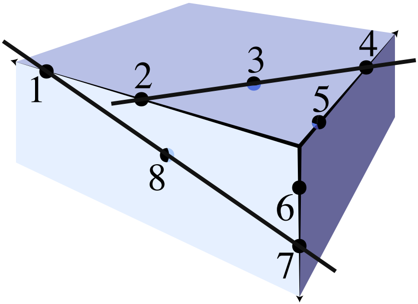
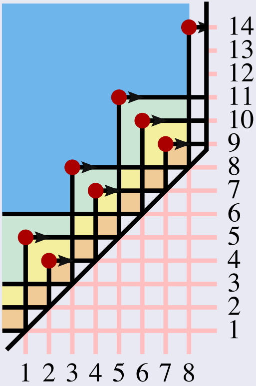
Permutation Label

$\{5,4,8,7,11,10,9,14\}$

The *Positroid* Stratification of the Grassmannian

The dimensionality of the configuration is encoded by its permutation label as follows:





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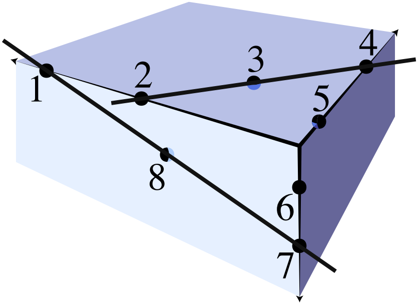

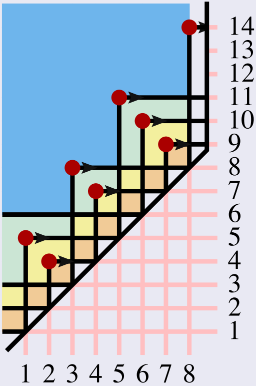

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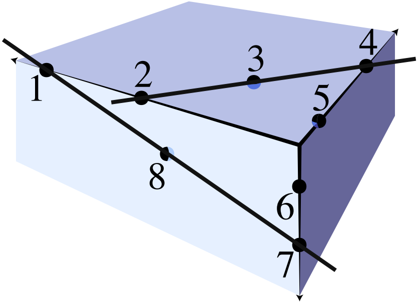

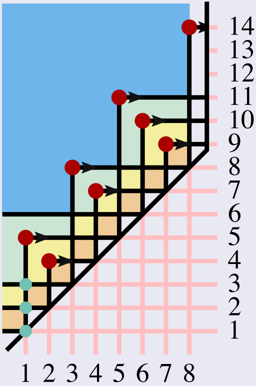

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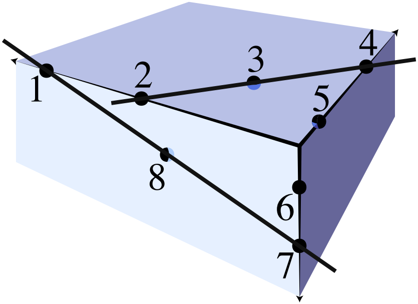

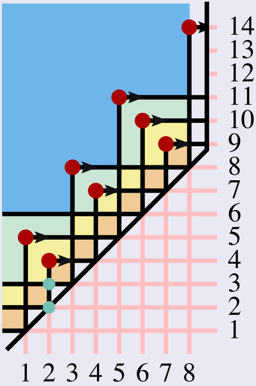

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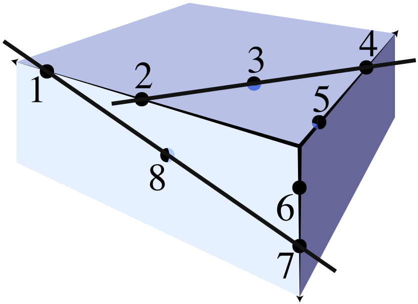

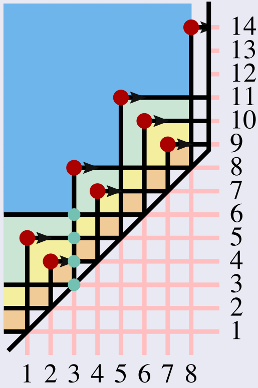

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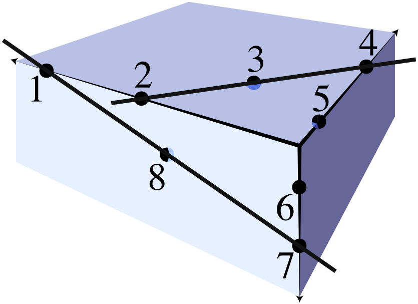

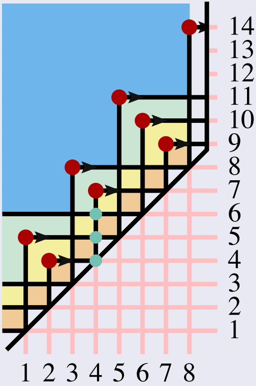

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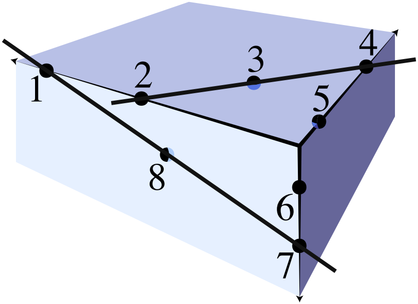

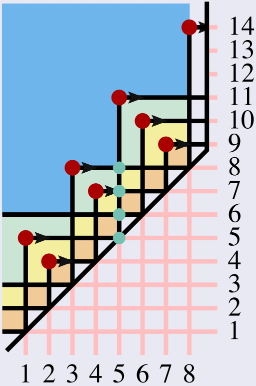

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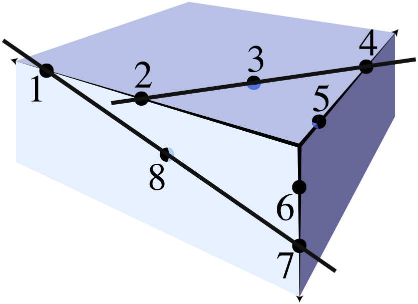

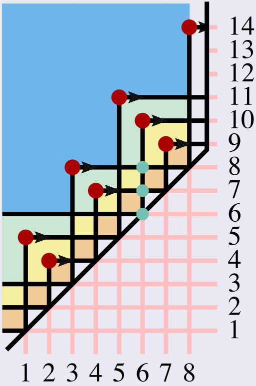

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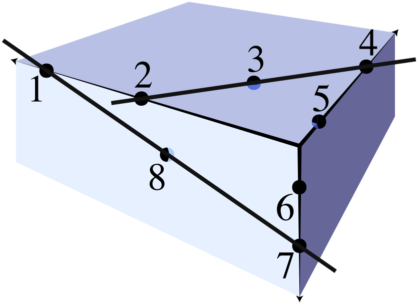

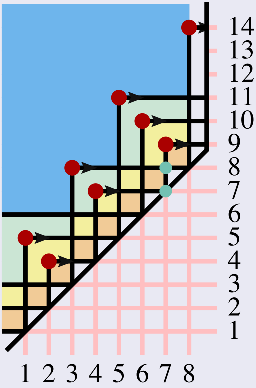

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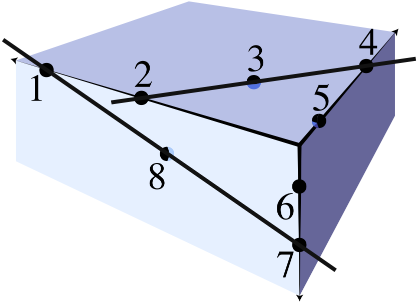

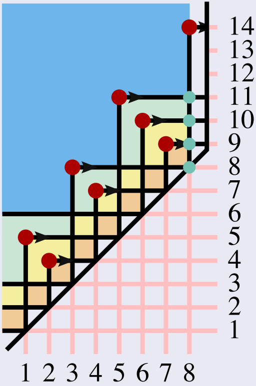

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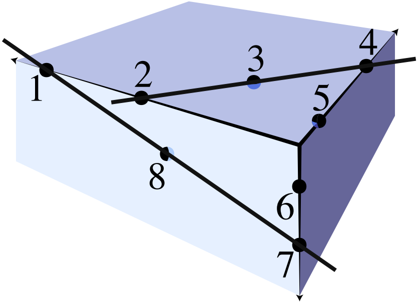

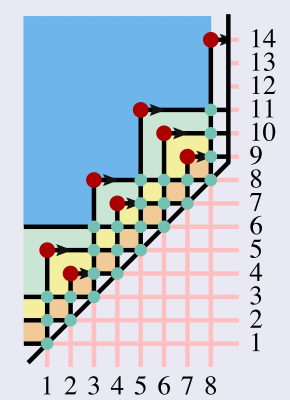

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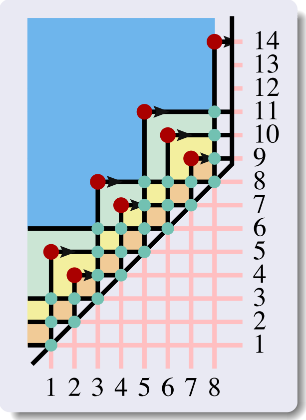

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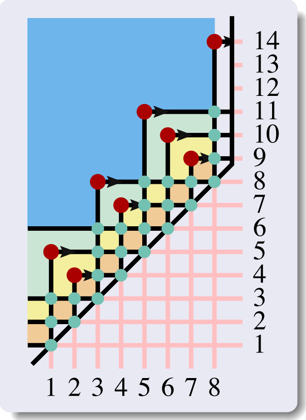

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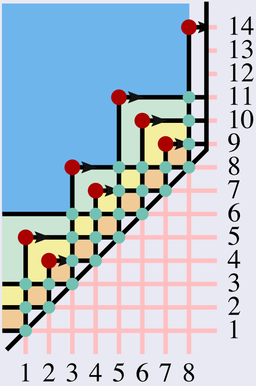

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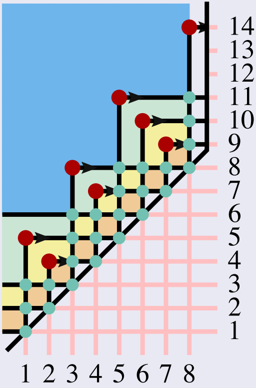

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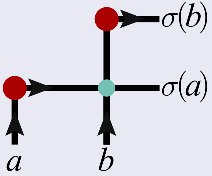

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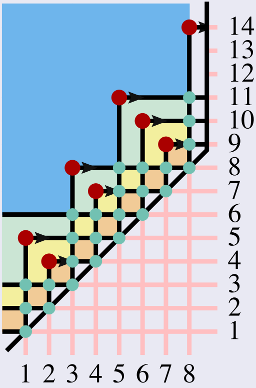

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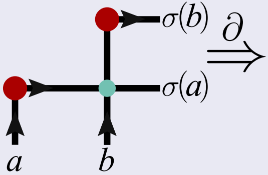

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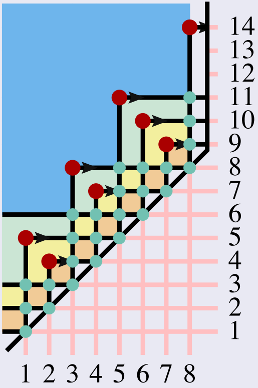

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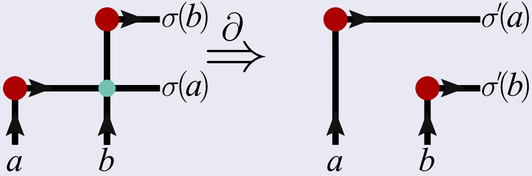

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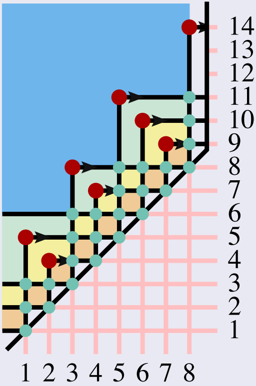

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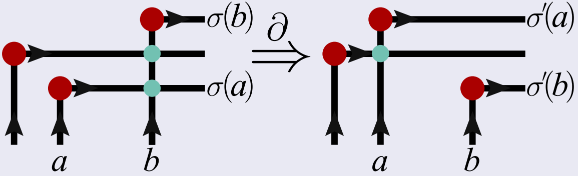

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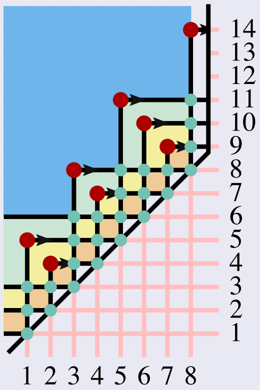

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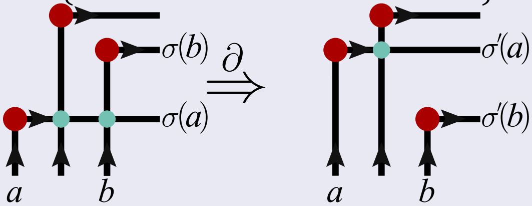

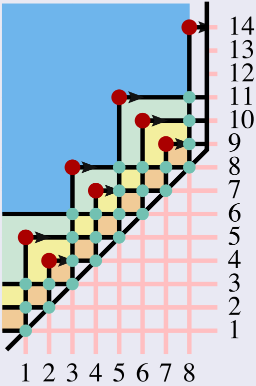

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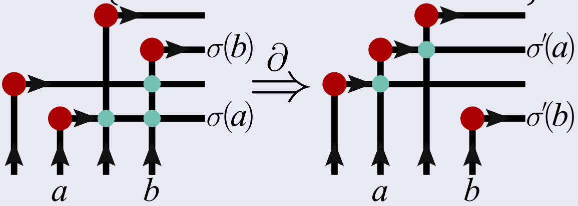

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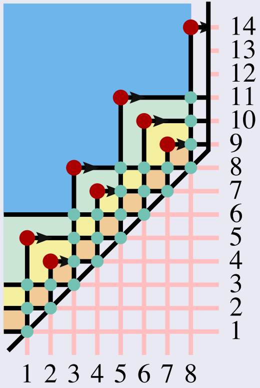

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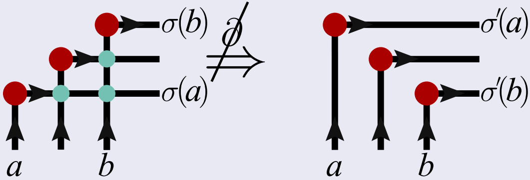

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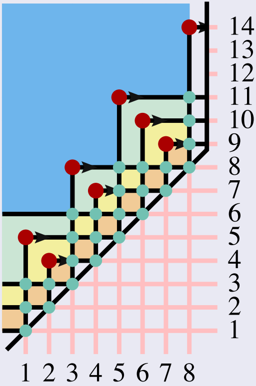

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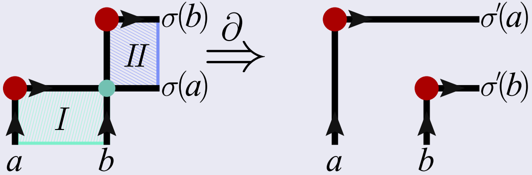

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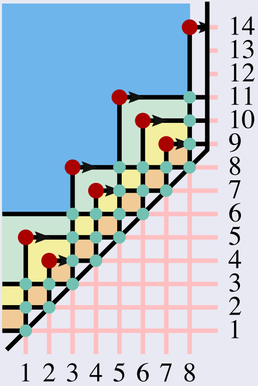

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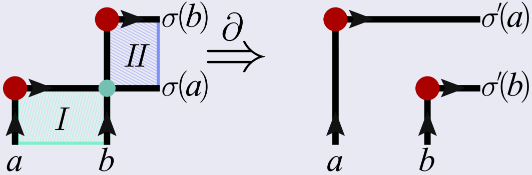

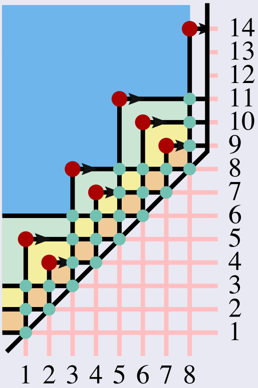

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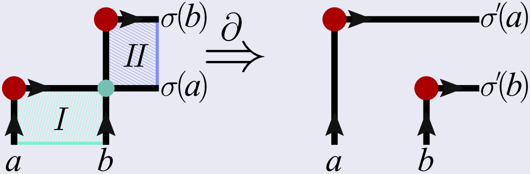
The boundaries of a configuration labelled by σ:

$$\partial\sigma \equiv \left\{\sigma' = (a\,b)\circ\sigma \,\middle|\, \dim(\sigma') = \dim(\sigma) - 1\right\}$$



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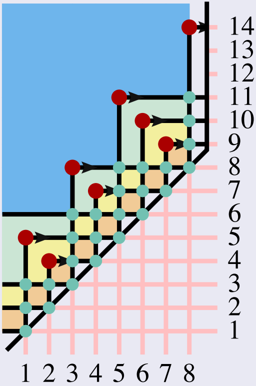

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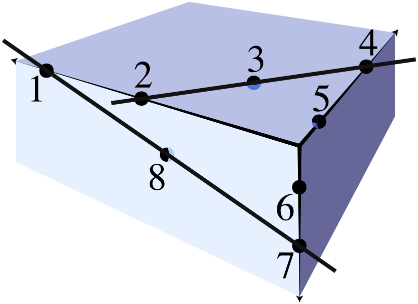

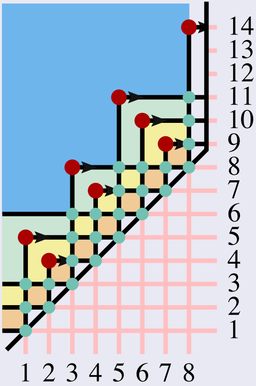

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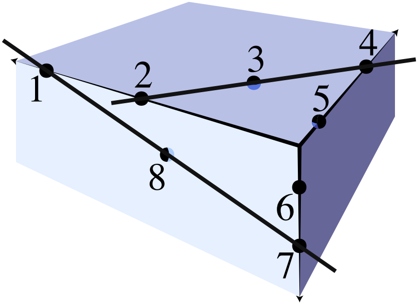

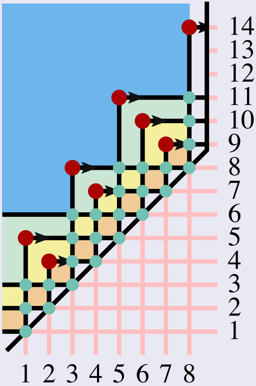

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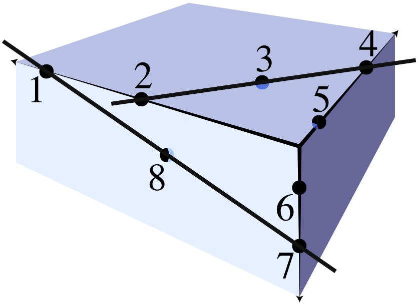

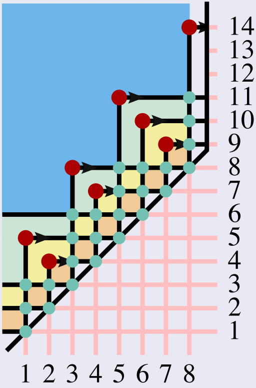

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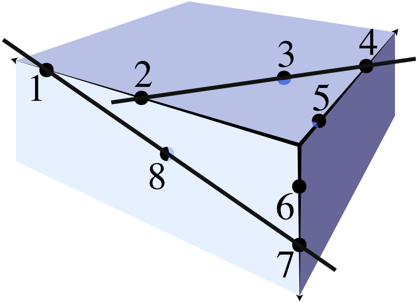

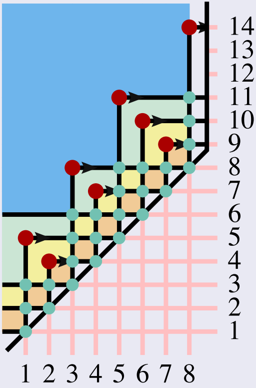

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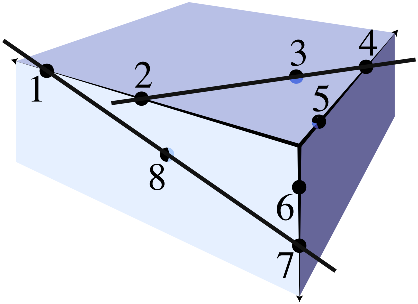

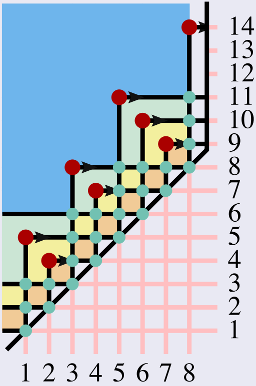

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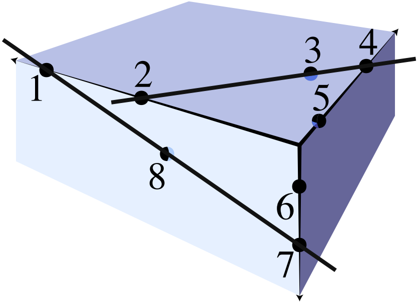

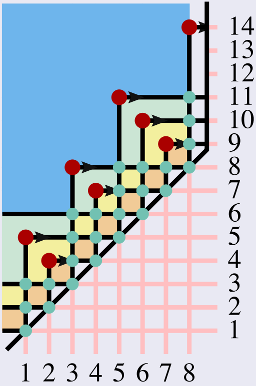

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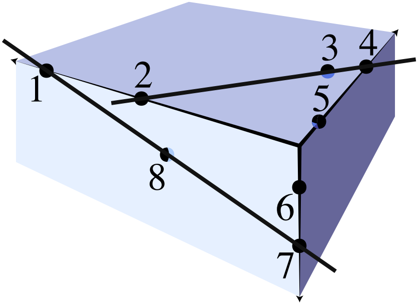

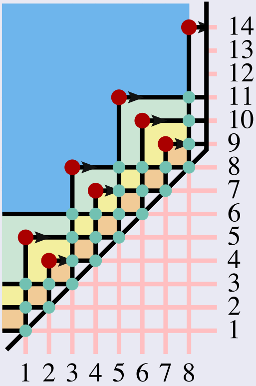

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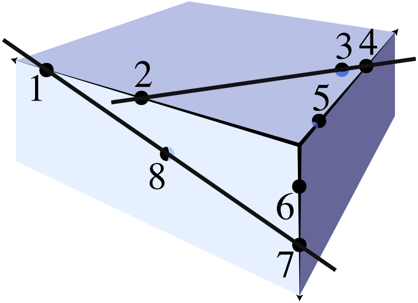

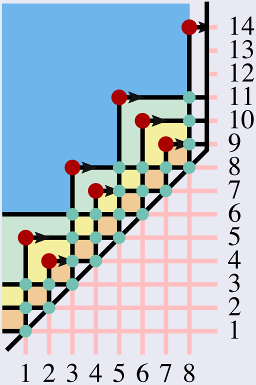

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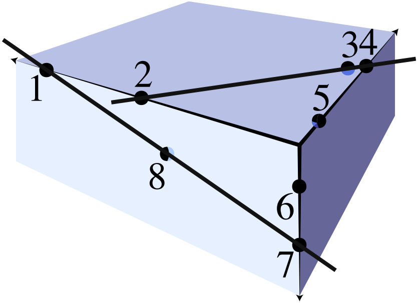

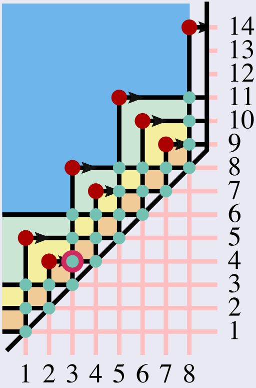

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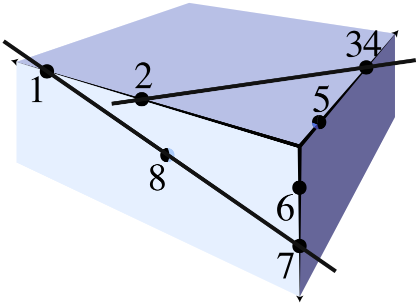

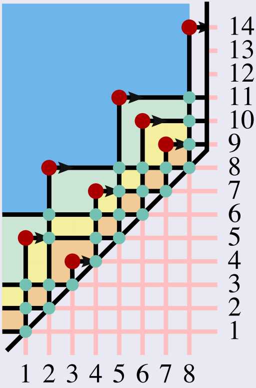

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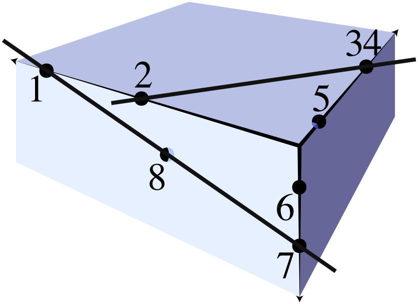

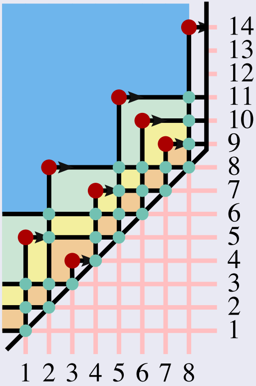

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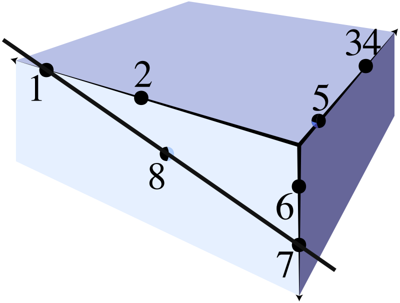

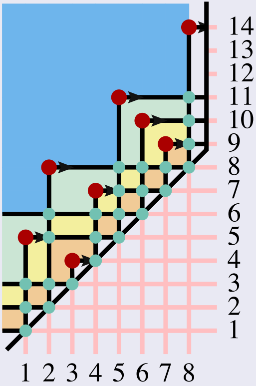

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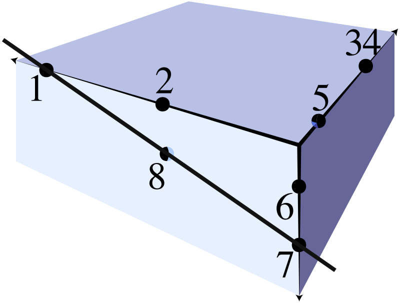

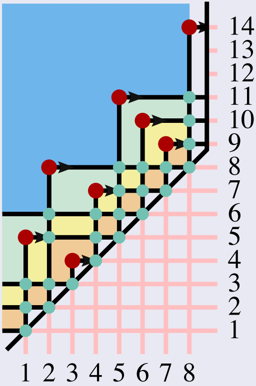

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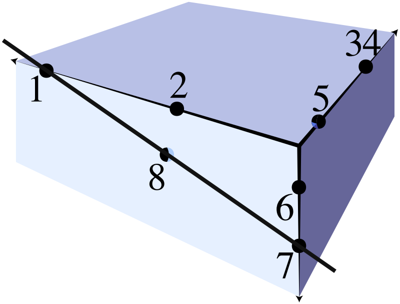

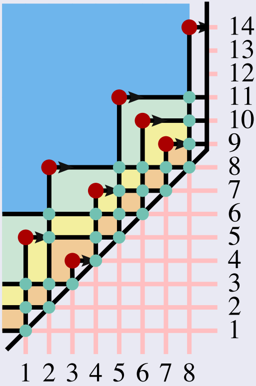

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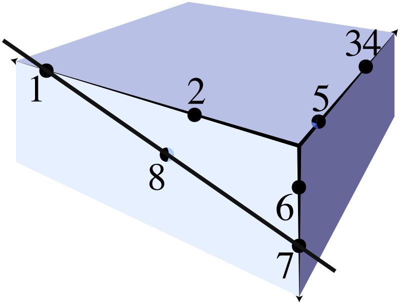

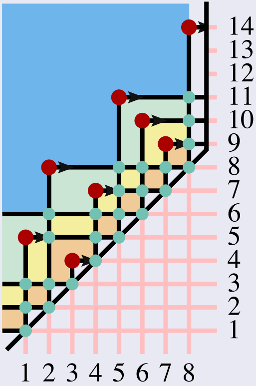

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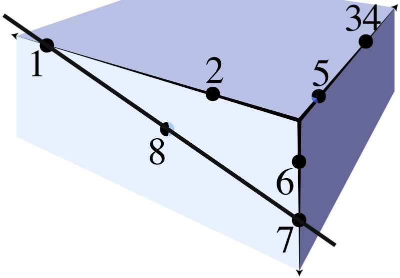

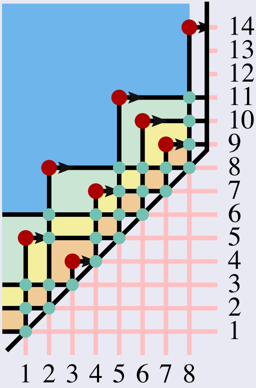

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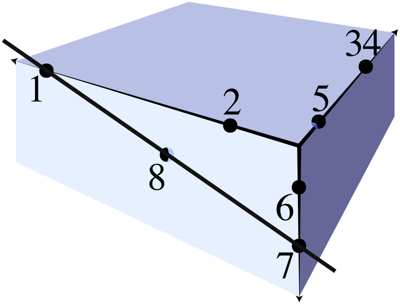

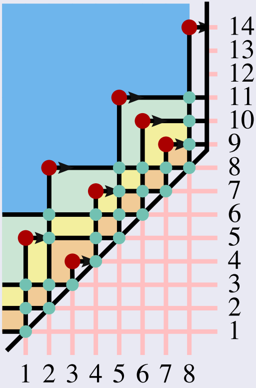

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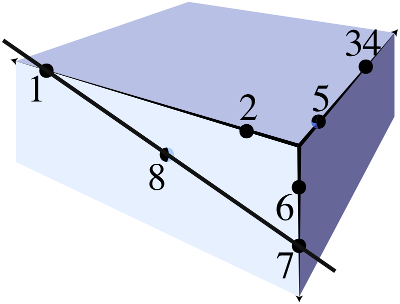

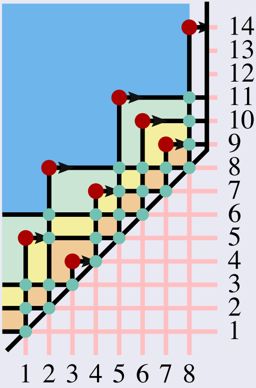

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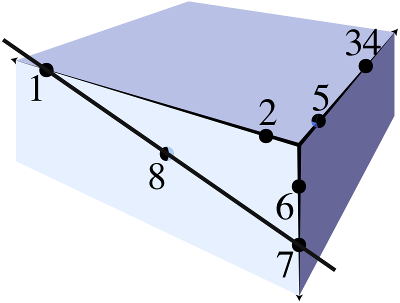

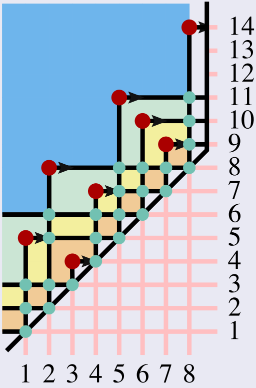

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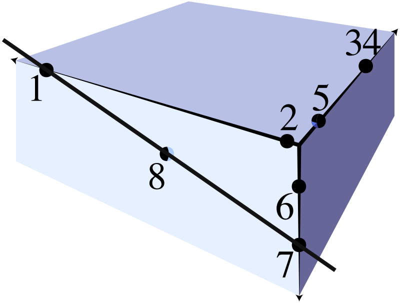

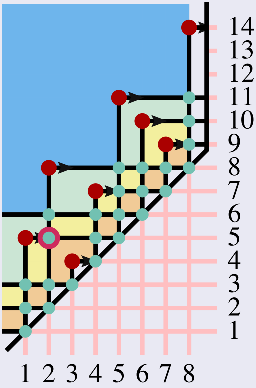

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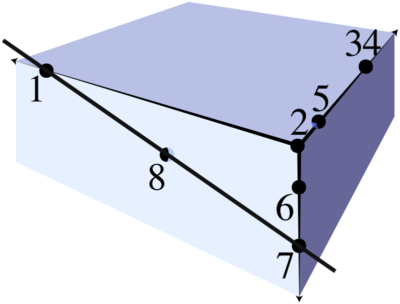

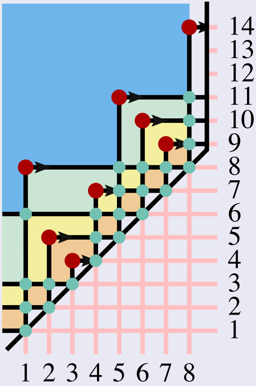

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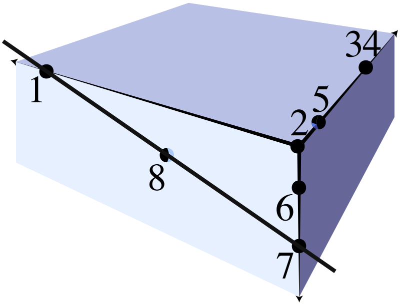

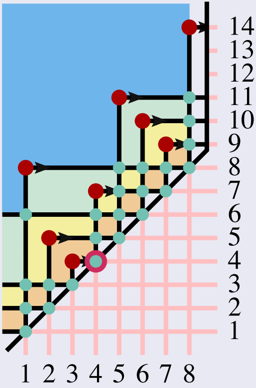

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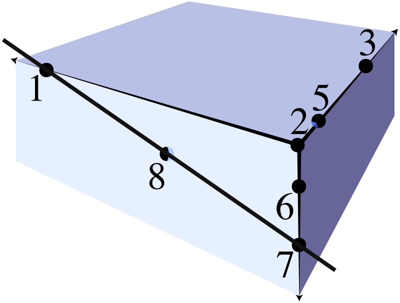

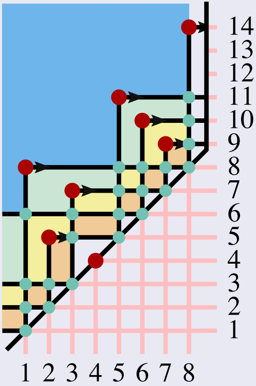

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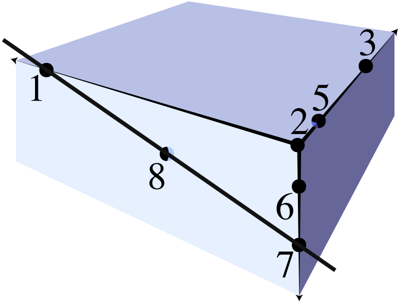

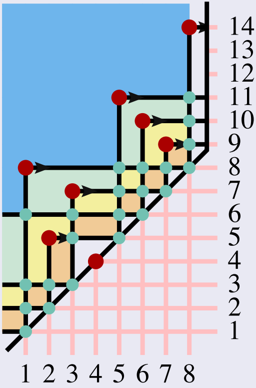

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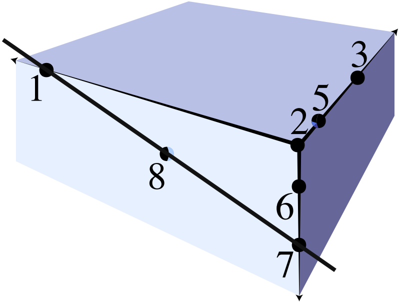

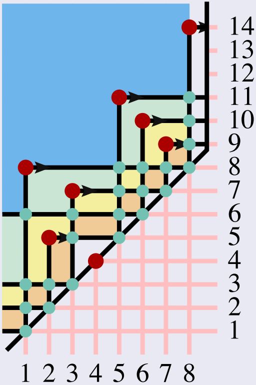

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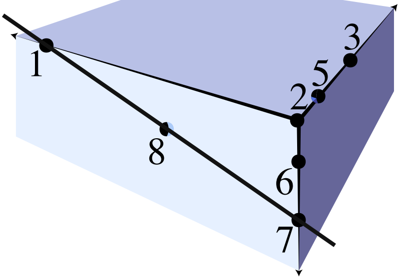

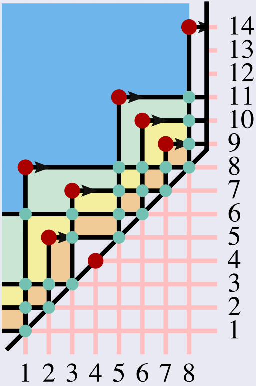

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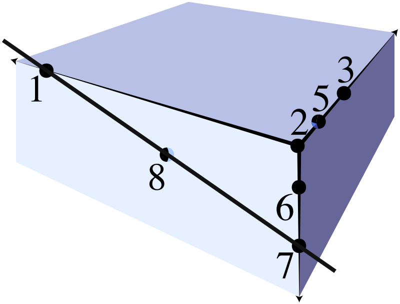

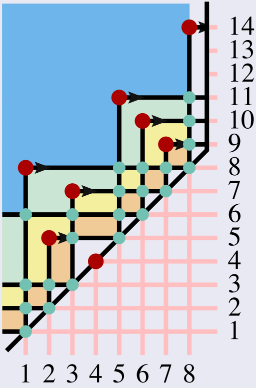

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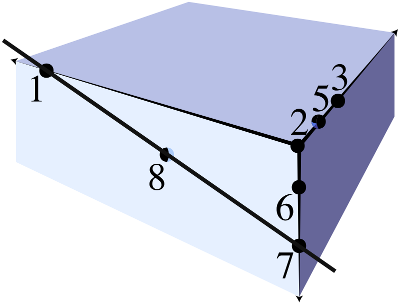

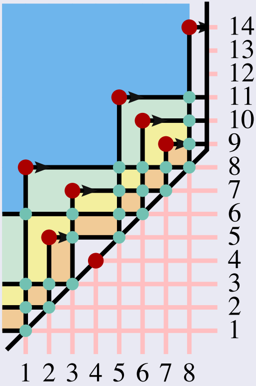

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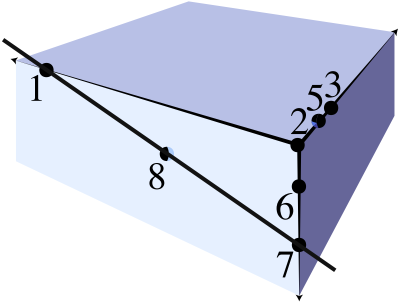

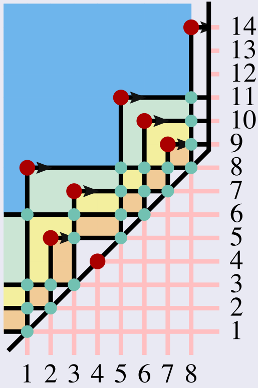

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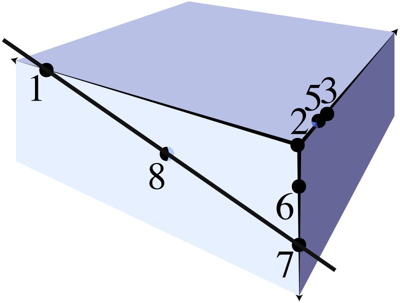

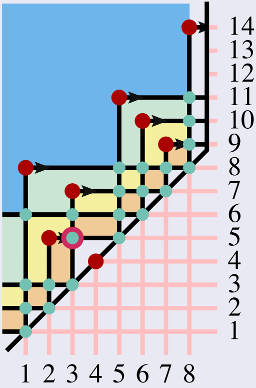

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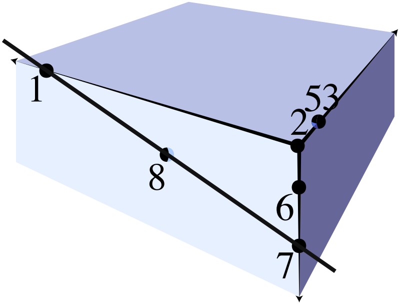

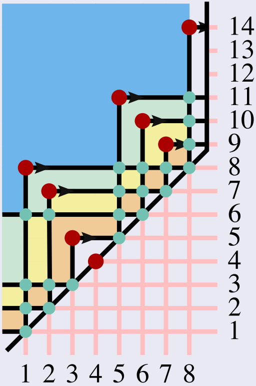

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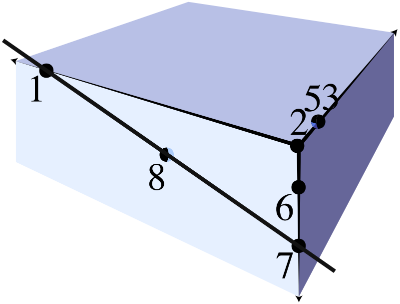

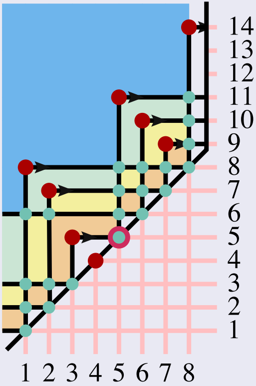

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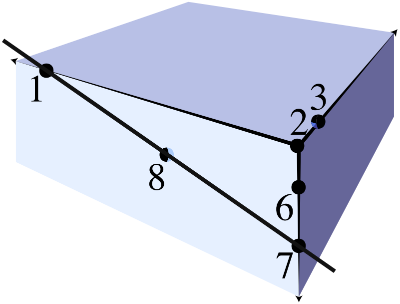

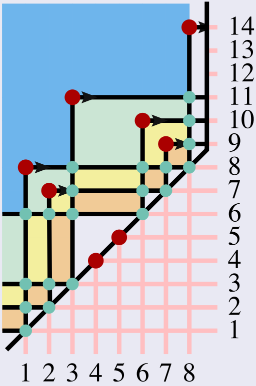

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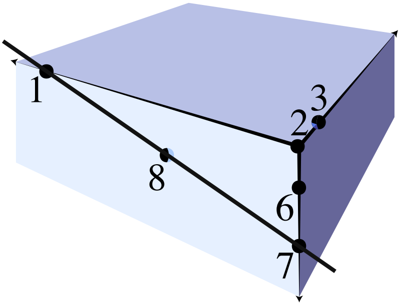

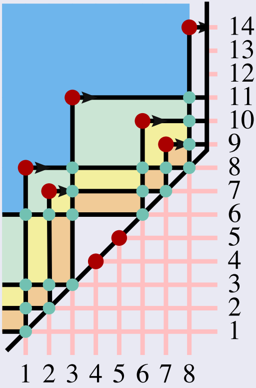

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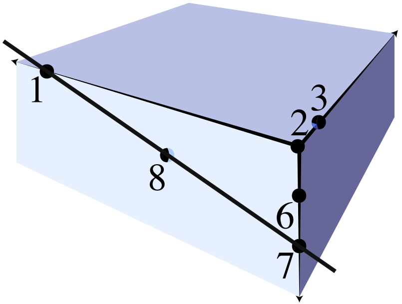

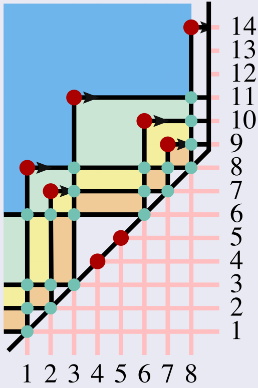

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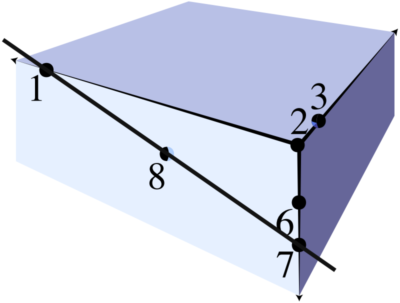

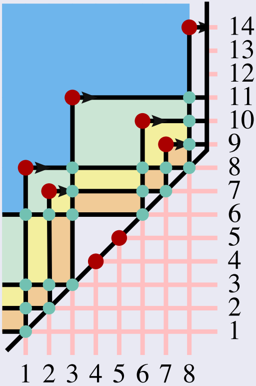

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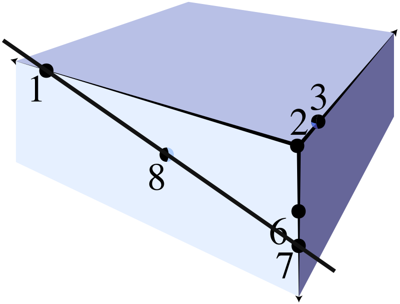

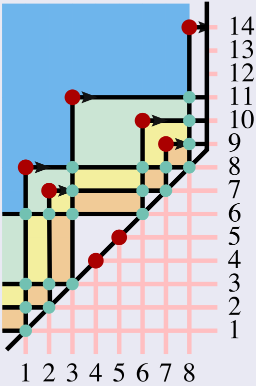

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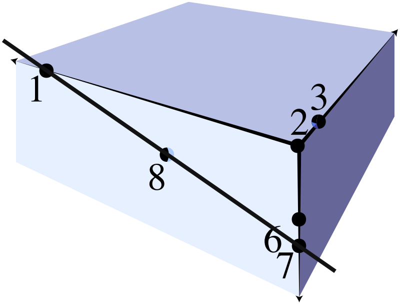

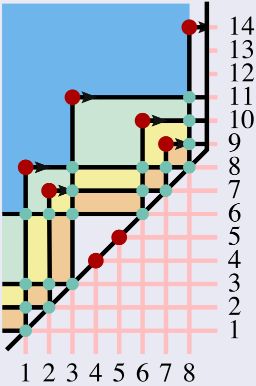

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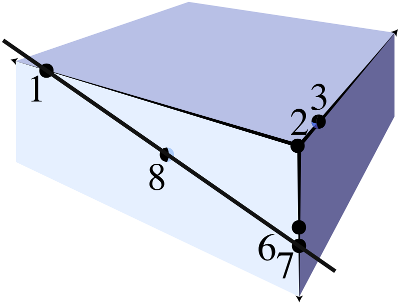

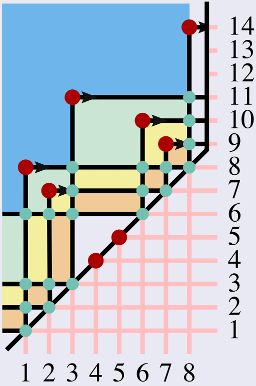

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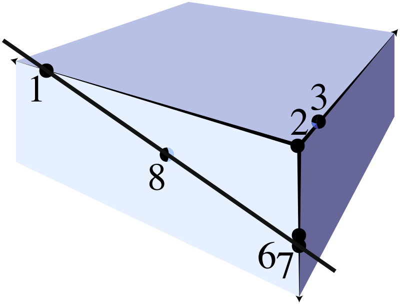

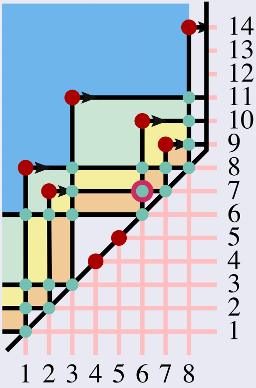

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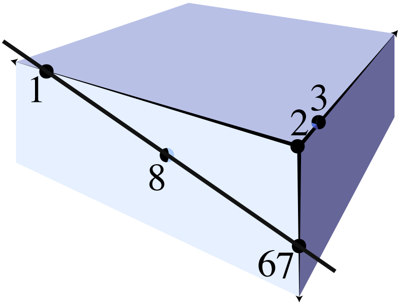

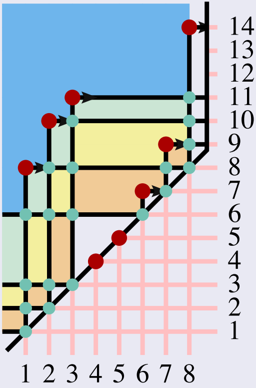

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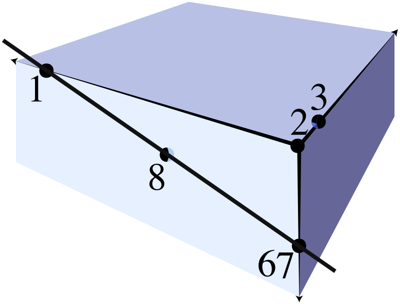

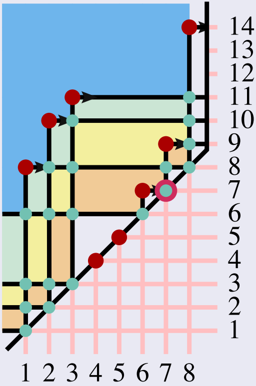

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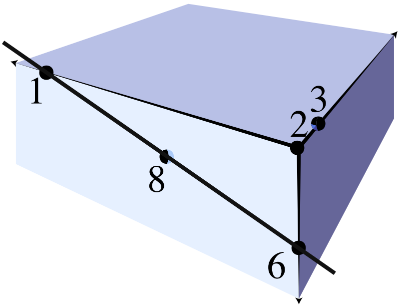

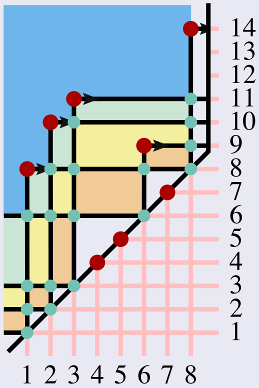

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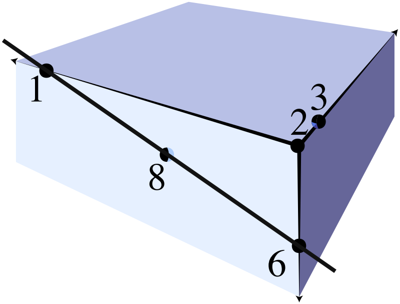

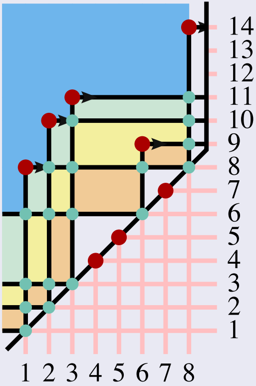

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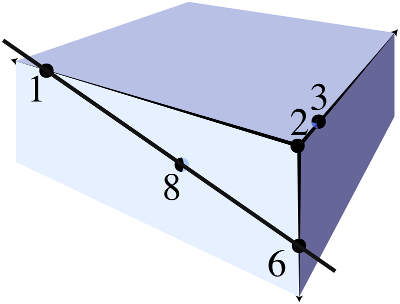

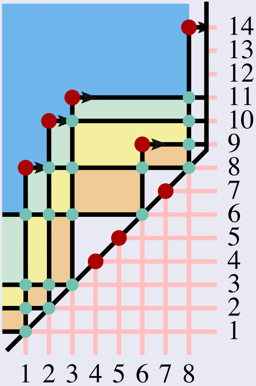

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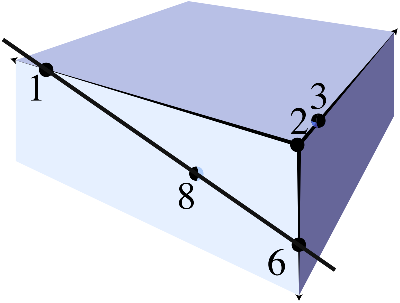

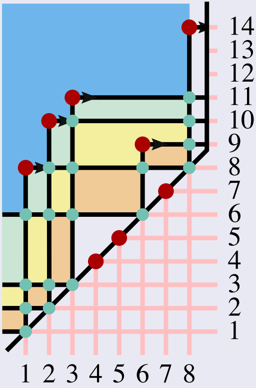

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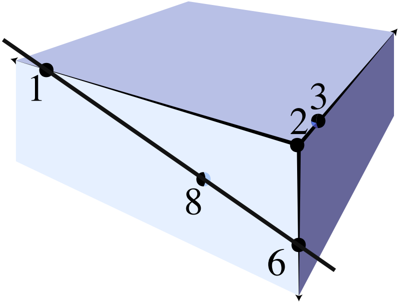

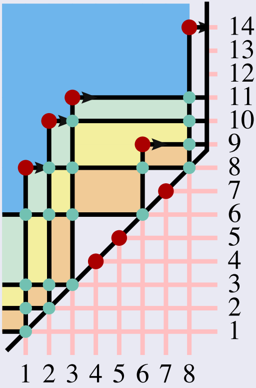

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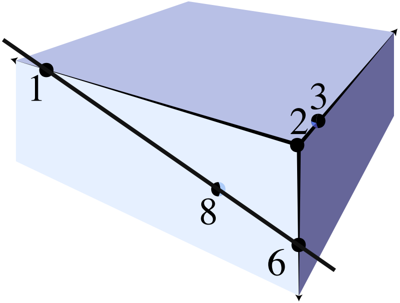

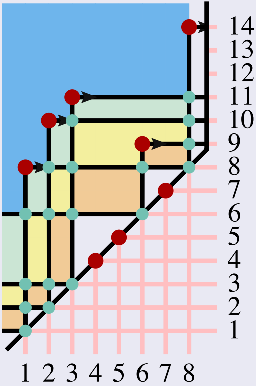

Geometric meaning of the permutation $\sigma(a)=b$

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Equivalence with On-Shell Diagram Permutation Labels

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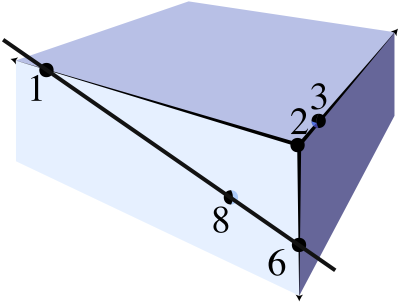

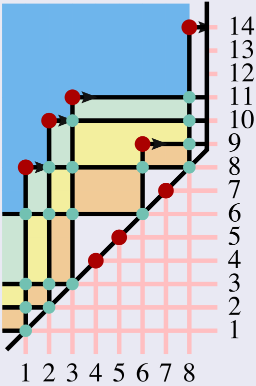

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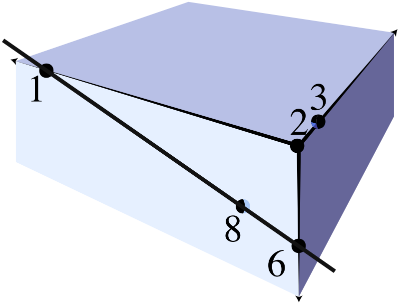

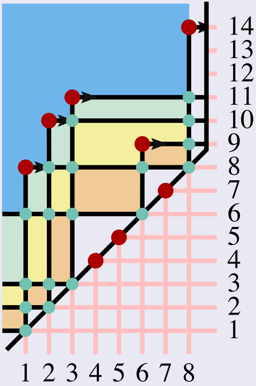

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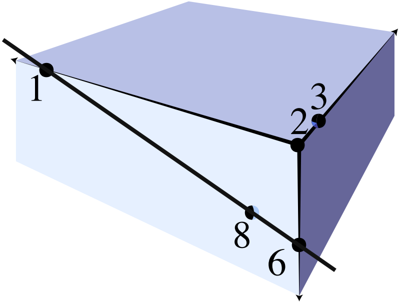

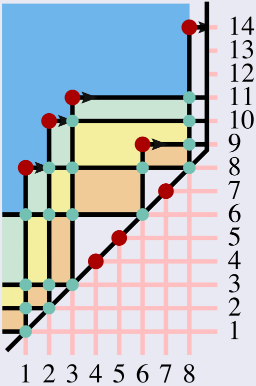

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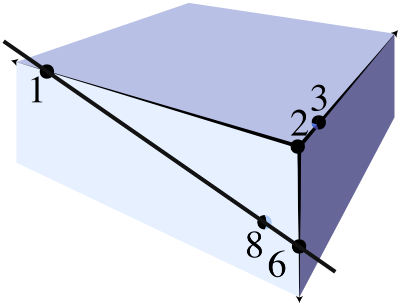

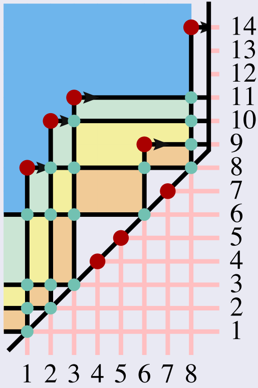

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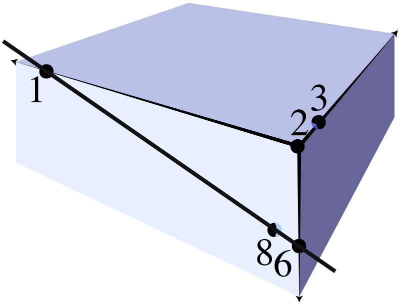

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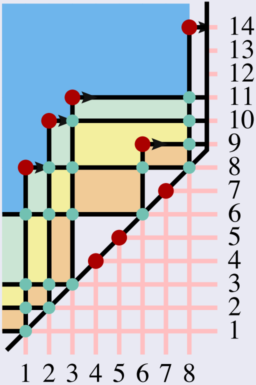

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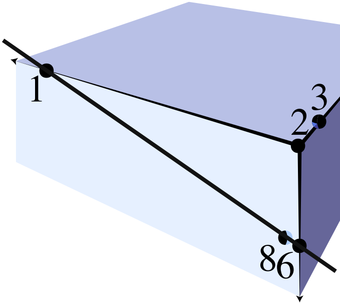

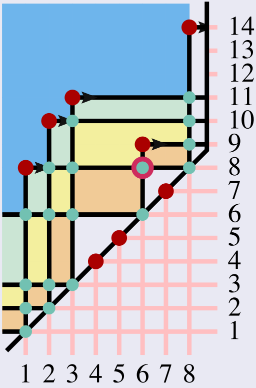

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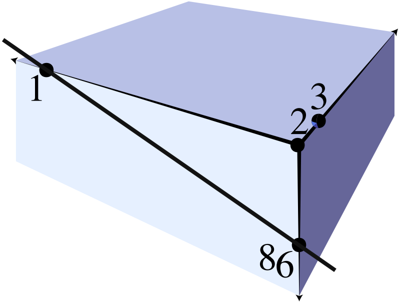

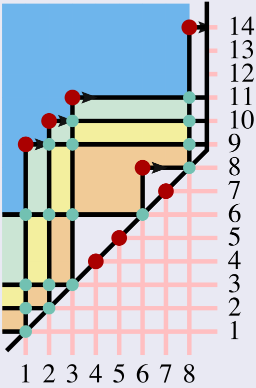

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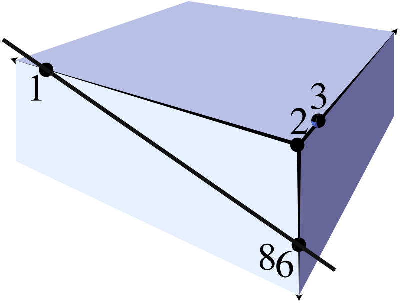

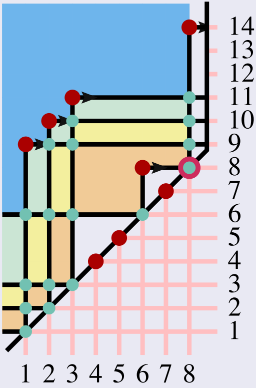

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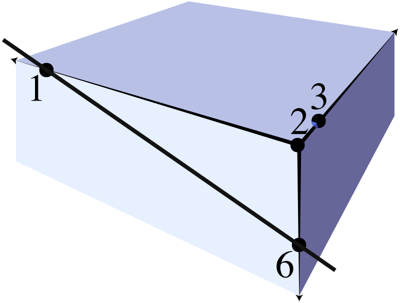

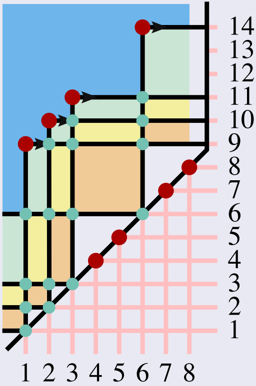

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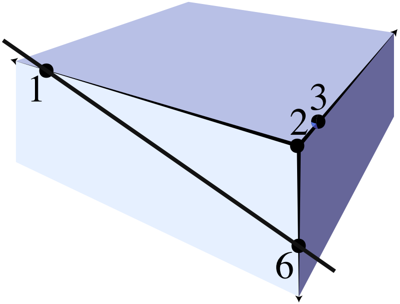

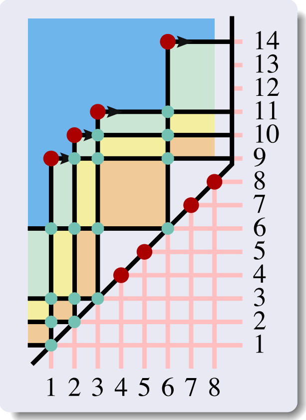

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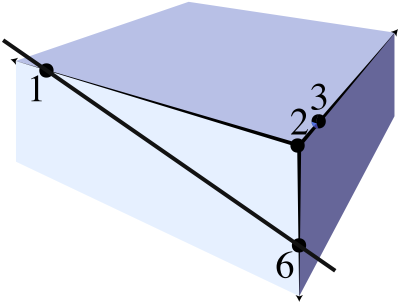

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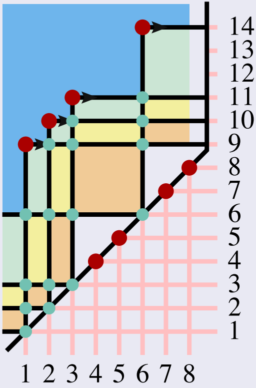

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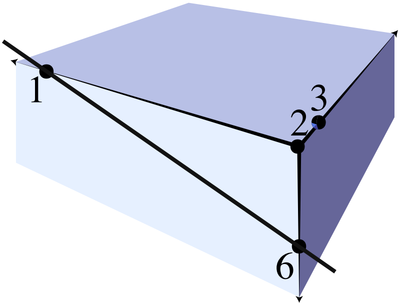

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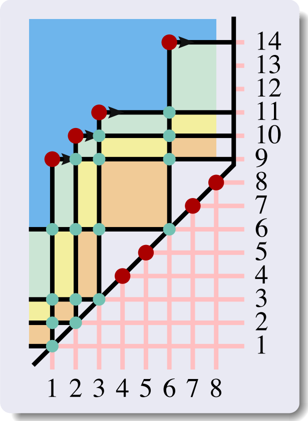

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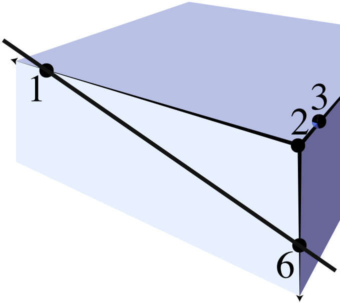

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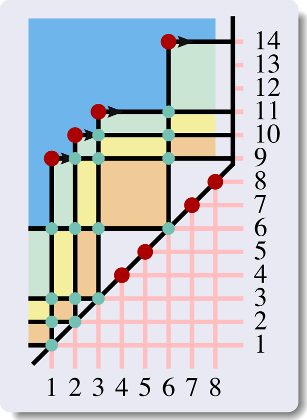

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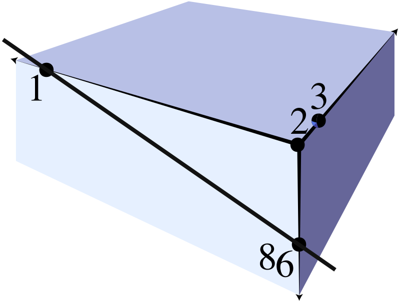

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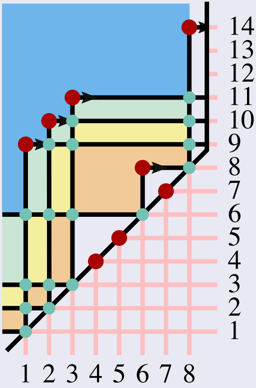

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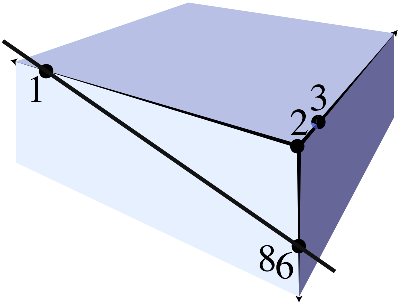

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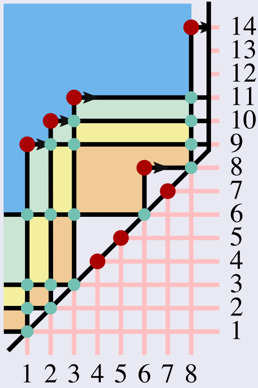

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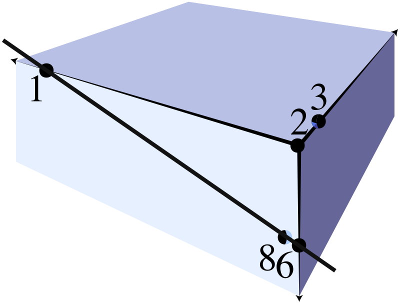

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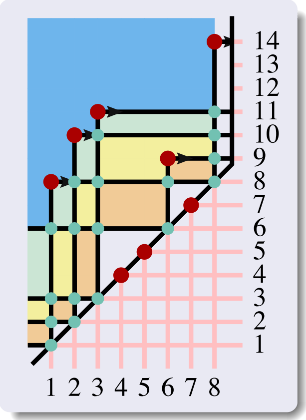

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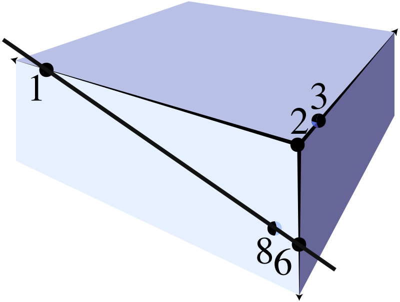

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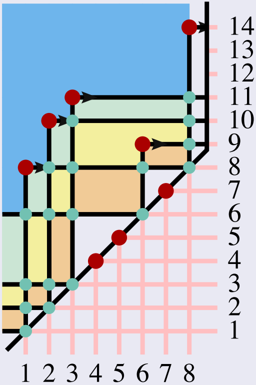

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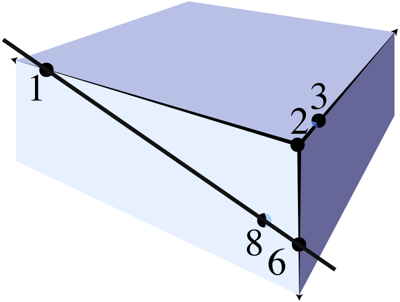

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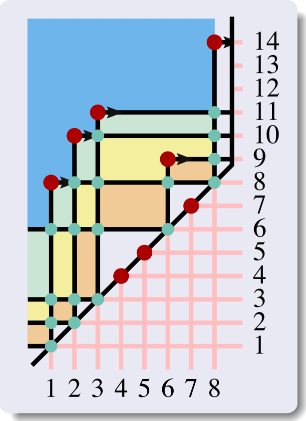

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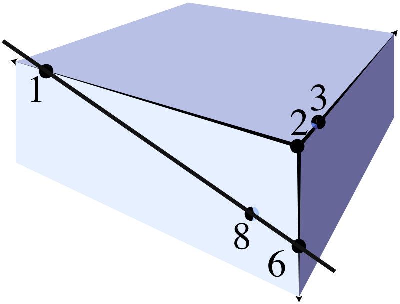

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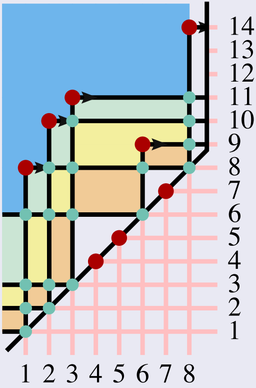

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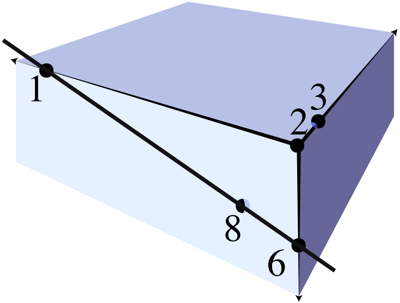

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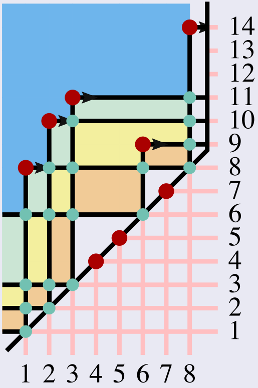

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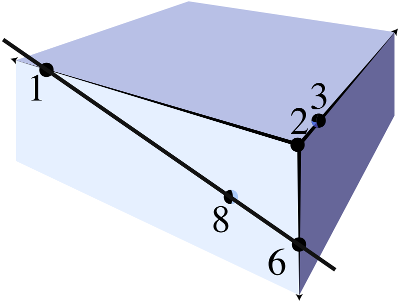
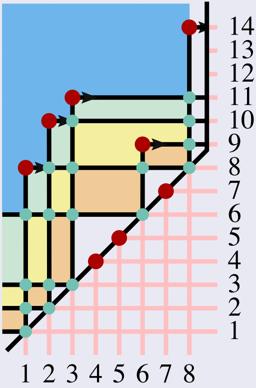


$$C \equiv \begin{array}{c} \begin{array}{cccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \end{array} \\ \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & \alpha_8 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_9 \end{array}\right) \end{array}$$

$$(1\,6)\!: \quad c_6 \mapsto c_6 + \alpha_8\, c_1$$



Permutation Label

$\{8, 10, 11, 4, 5, 9, 7, 14\}$

Equivalence with On-Shell Diagram Permutation Labels

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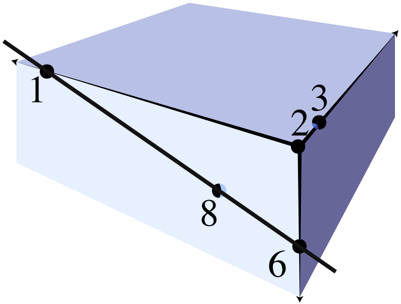

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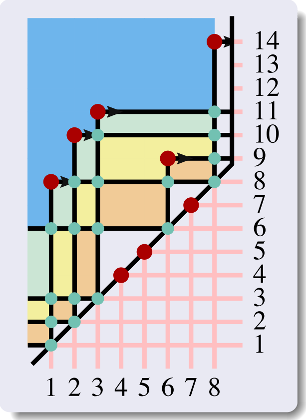

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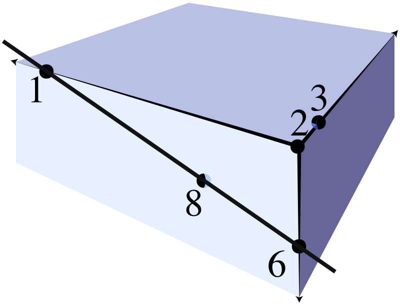

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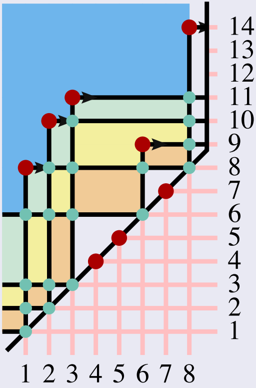

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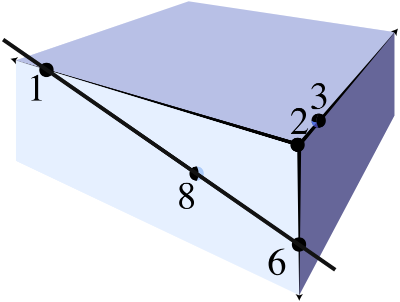

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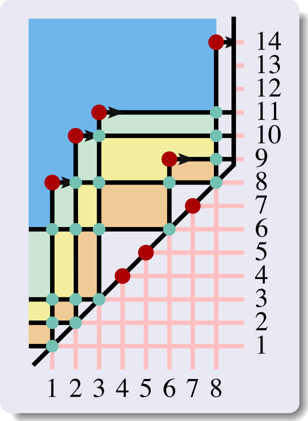

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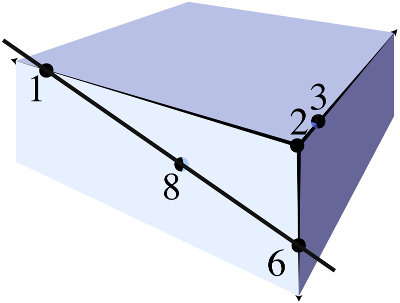

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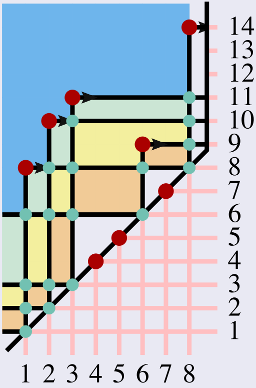

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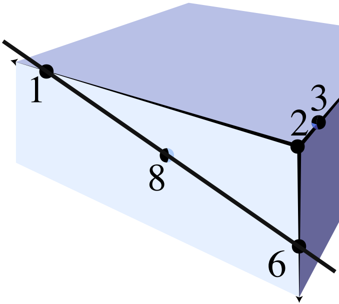

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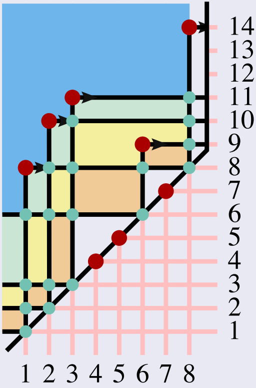

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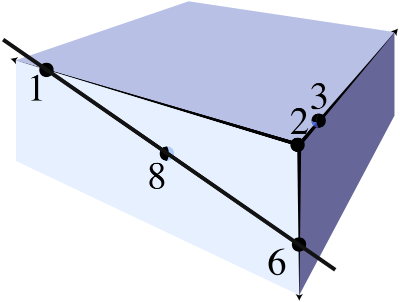

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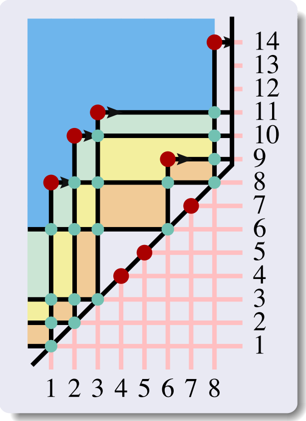

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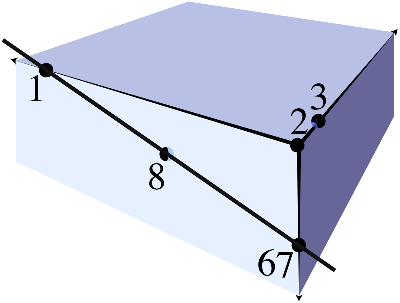
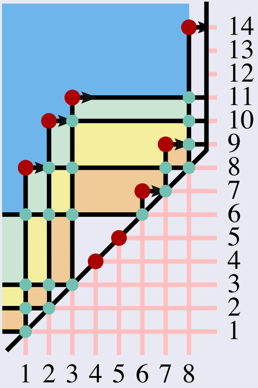


$$C\equiv\left(\begin{array}{cccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \\ \hline 1 & 0 & 0 & 0 & 0 & \alpha_8 & \alpha_7\alpha_8 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha_7 & \alpha_9 \end{array}\right)$$

$(6\,7)$: $c_7 \mapsto c_7 + \alpha_7\, c_6$



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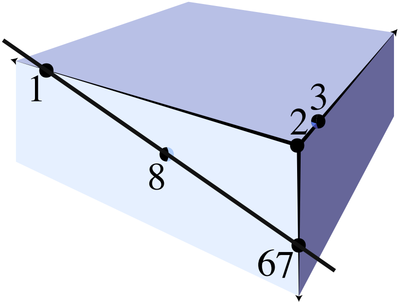

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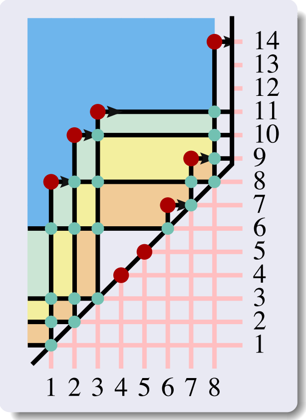

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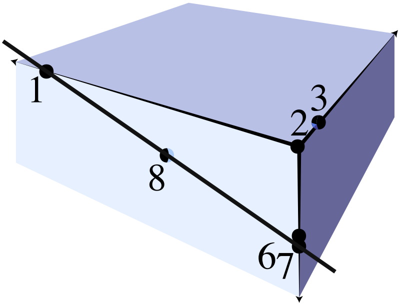

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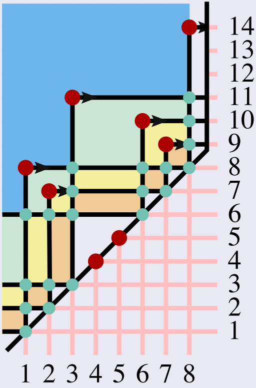

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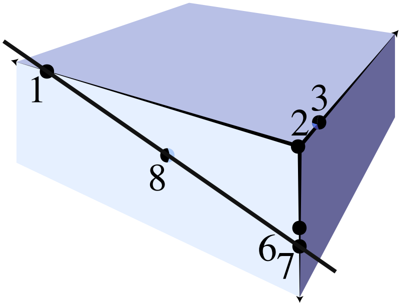

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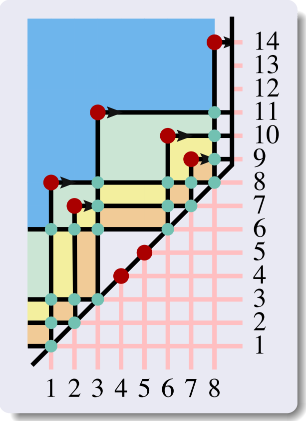

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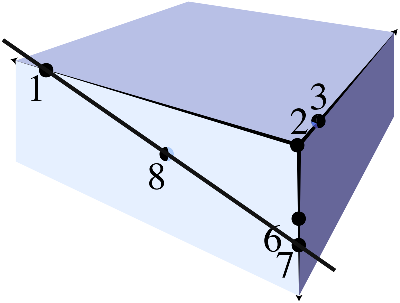

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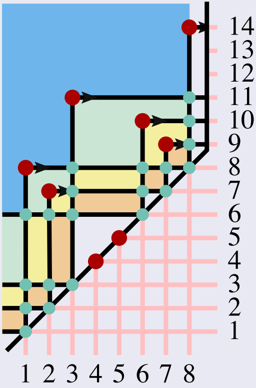

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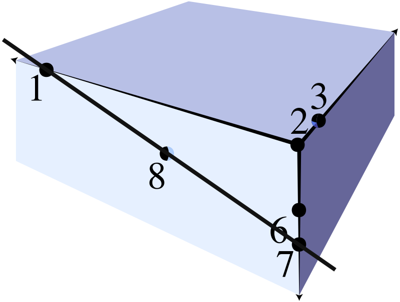

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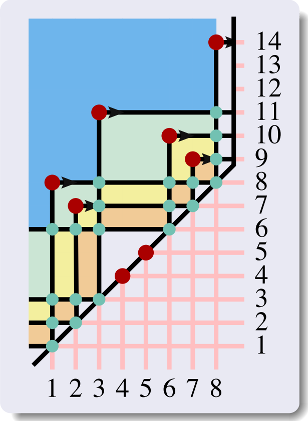

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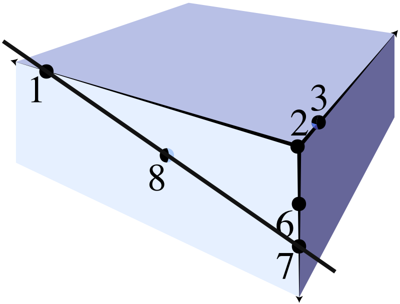

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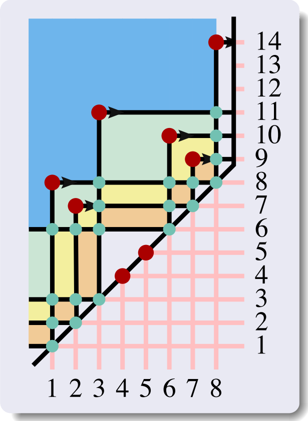

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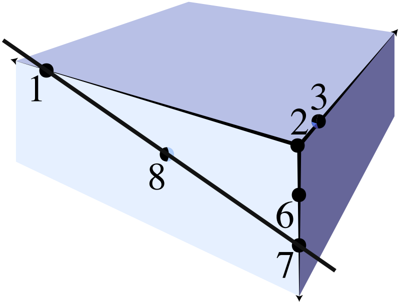

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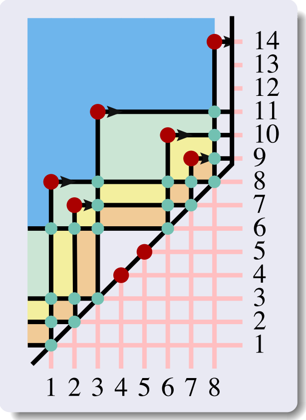

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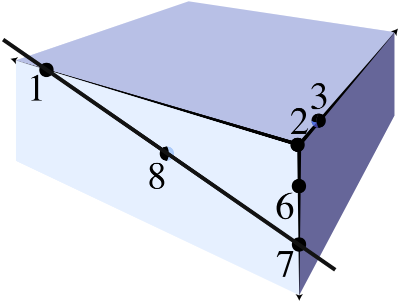

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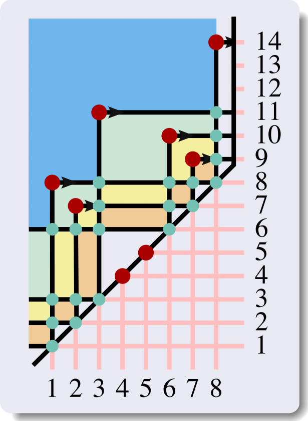

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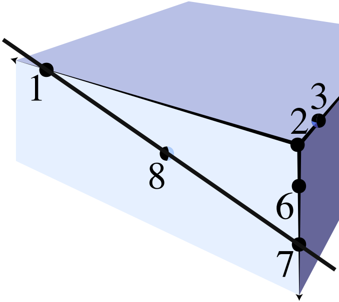

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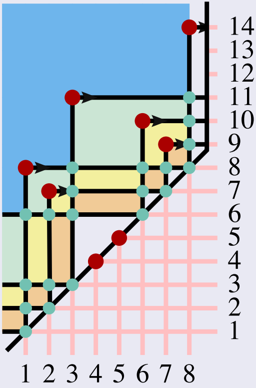

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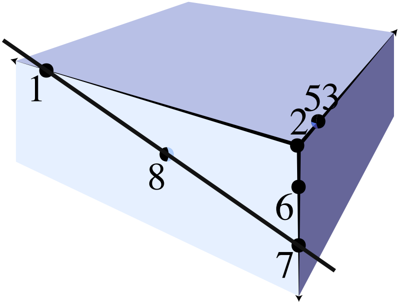

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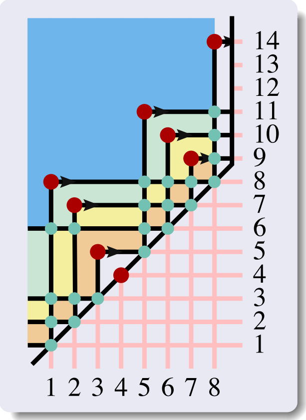
$$(3\,5): \quad c_5 \mapsto c_5 + \alpha_5\, c_3$$

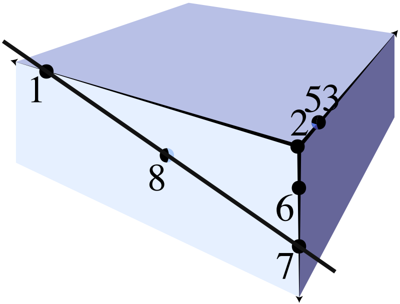


Permutation Label

$\{8, 7, 5, 4, 11, 10, 9, 14\}$

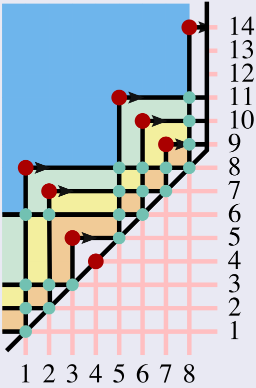
Equivalence with On-Shell Diagram Permutation Labels

As for permutations labeling on-shell diagrams, we can decompose σ into adjacent transpositions:



$$C \equiv \begin{array}{c} \begin{array}{cccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \end{array} \\ \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & \alpha_8 & \alpha_7\alpha_8 & 0 \\ 0 & 1 & 0 & 0 & 0 & \alpha_6 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha_7 & \alpha_9 \end{array}\right) \end{array}$$

$(3\,5)$: $c_5 \mapsto c_5 + \alpha_5\, c_3$



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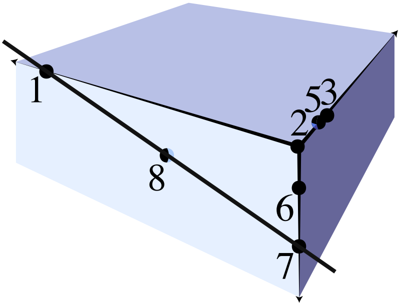

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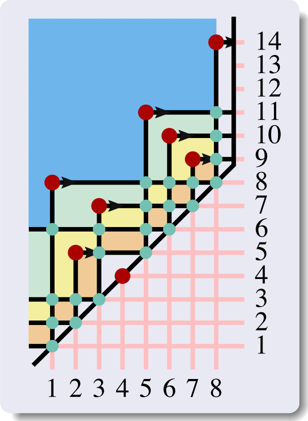
$(2\,3)$: $c_3 \mapsto c_3 + \alpha_4\, c_2$



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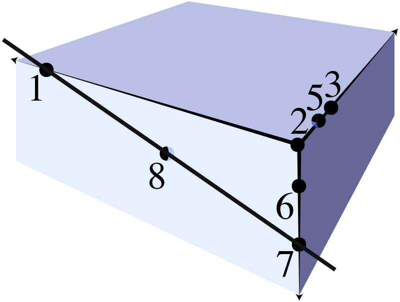

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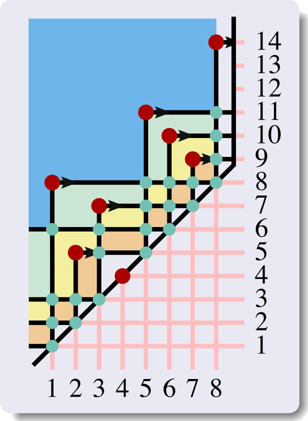

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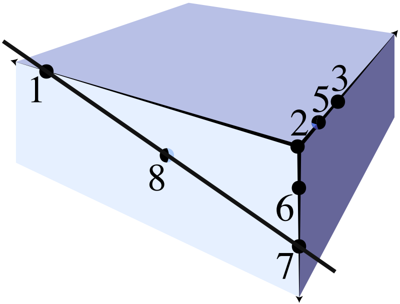

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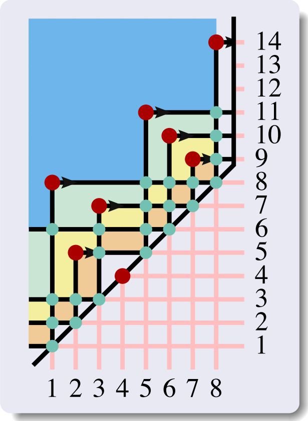
$$(2\,3)\!: \quad c_3 \mapsto c_3 + \alpha_4\, c_2$$



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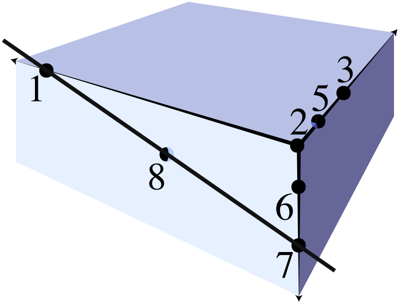

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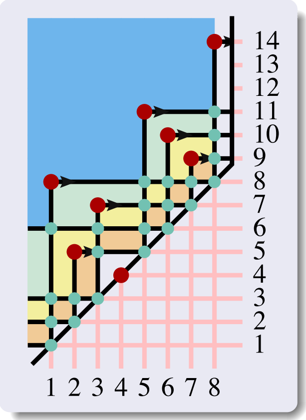
$(2\,3)$: $c_3 \mapsto c_3 + \alpha_4\, c_2$



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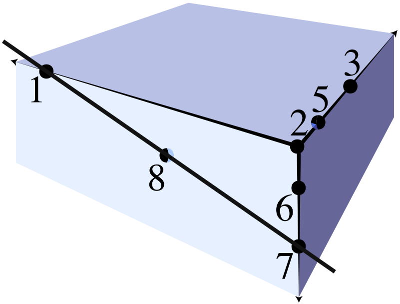

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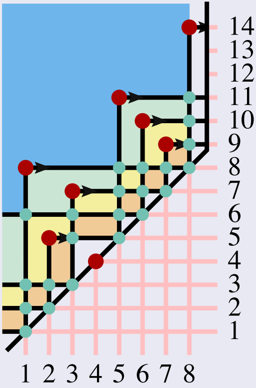

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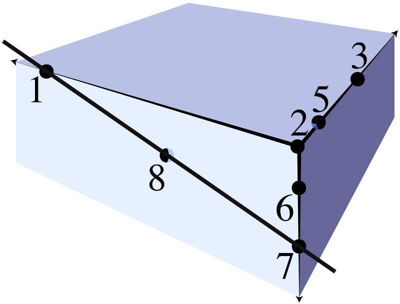
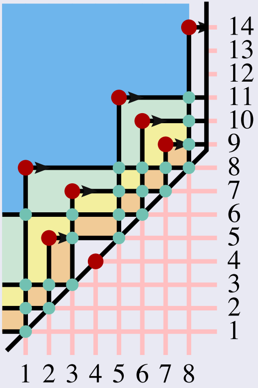


$$C \equiv \begin{array}{c}\begin{array}{cccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \end{array}\\ \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & \alpha_8 & \alpha_7\alpha_8 & 0 \\ 0 & 1 & \alpha_4 & 0 & 0 & \alpha_6 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha_7 & \alpha_9 \end{array}\right)\end{array}$$

$(2\,3)$: $\quad c_3 \mapsto c_3 + \alpha_4\, c_2$



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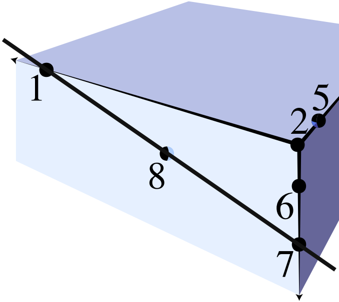

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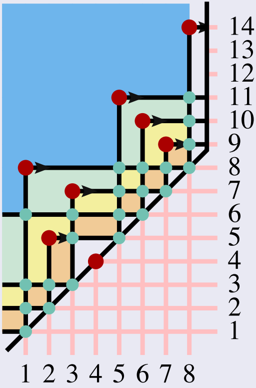

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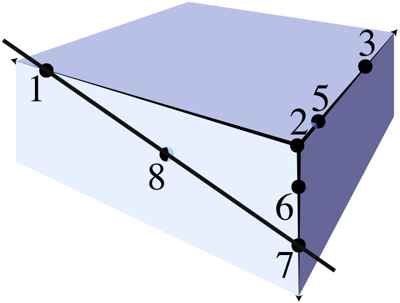

$$C\equiv\begin{array}{c}\begin{array}{cccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \end{array}\\ \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & \alpha_8 & \alpha_7\alpha_8 & 0 \\ 0 & 1 & \alpha_4 & 0 & 0 & \alpha_6 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha_7 & \alpha_9 \end{array}\right)\end{array}$$

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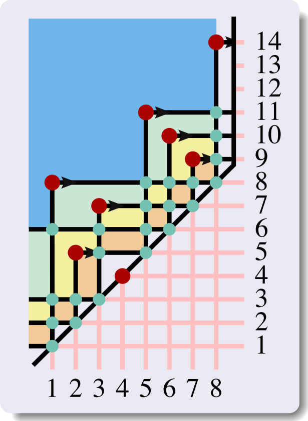

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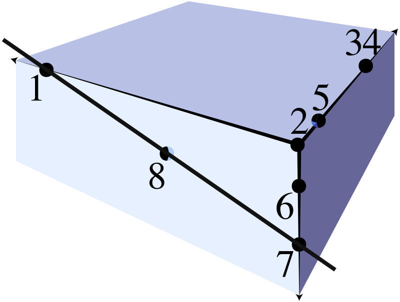

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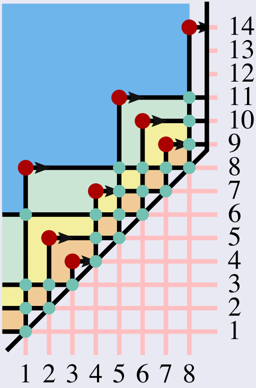
$(3\,4)$: $c_4 \mapsto c_4 + \alpha_3\, c_3$



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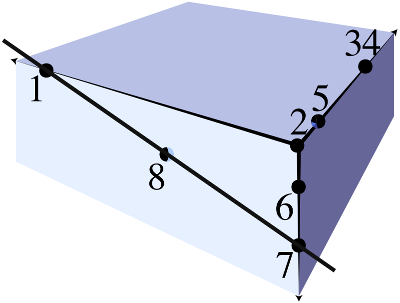

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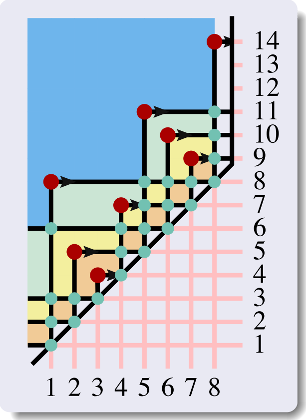

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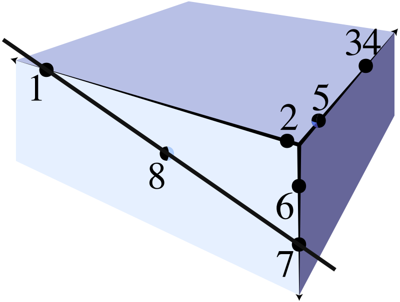

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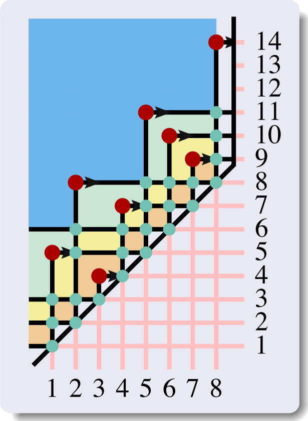
$(1\,2)$: $\quad c_2\mapsto c_2+\alpha_2\,c_1$



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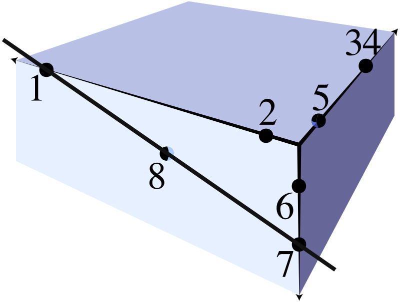

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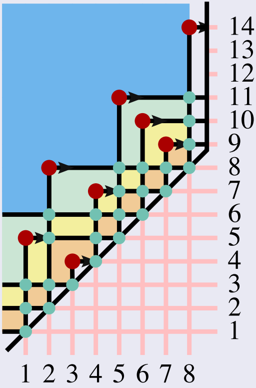

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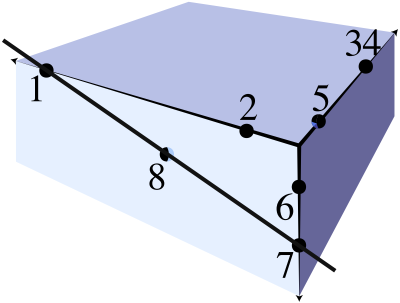

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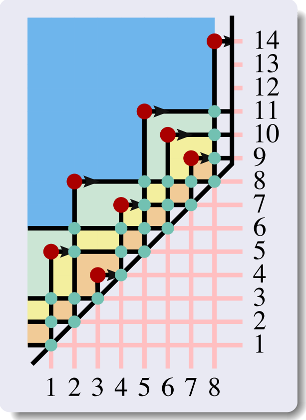

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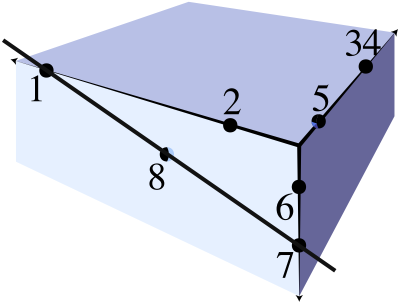

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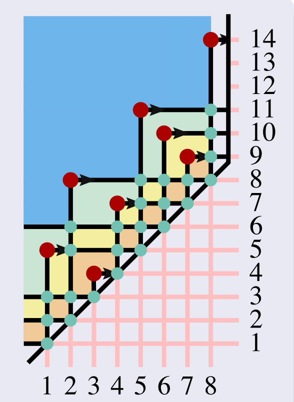

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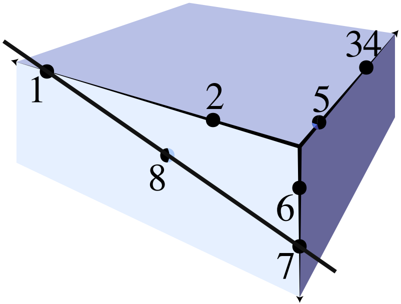

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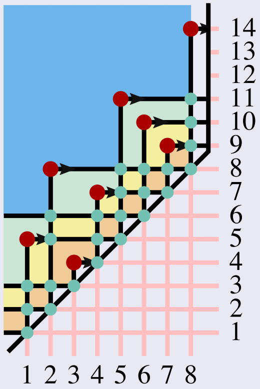

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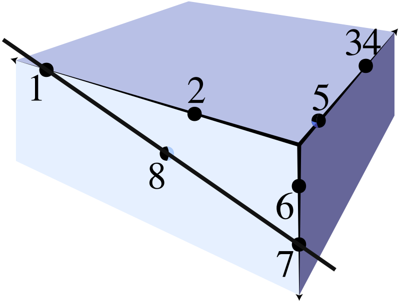

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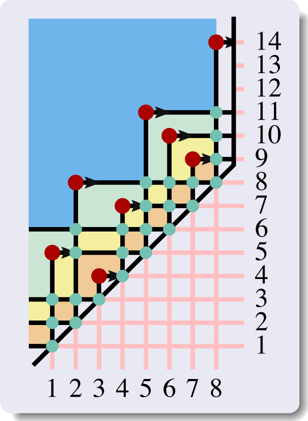

Permutation Label

$\{5, 8, 4, 7, 11, 10, 9, 14\}$

Equivalence with On-Shell Diagram Permutation Labels

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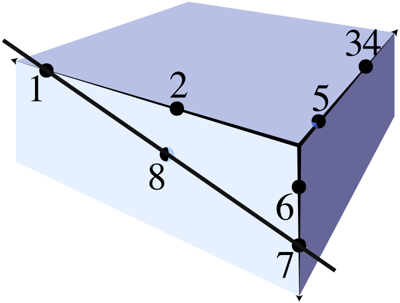

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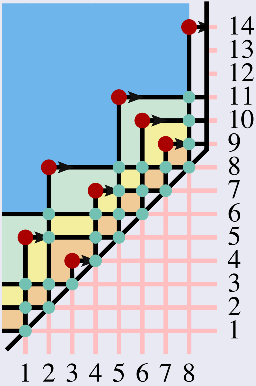

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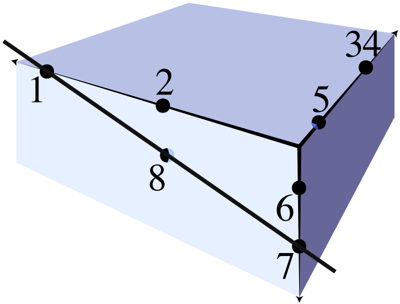

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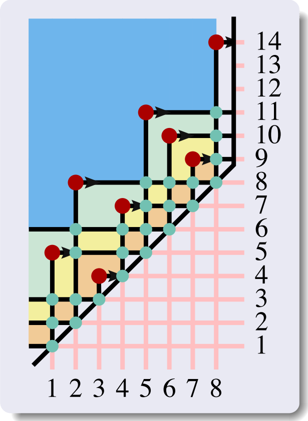

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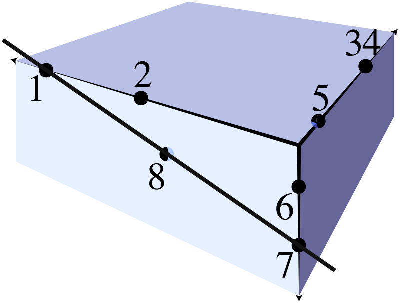

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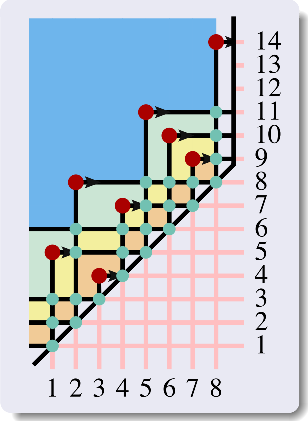

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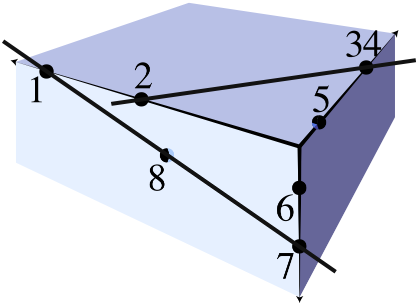

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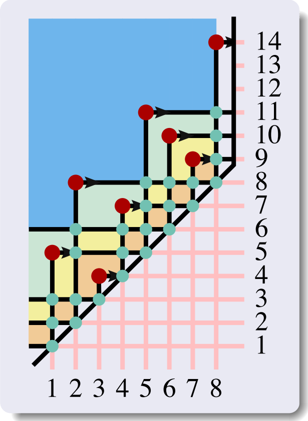

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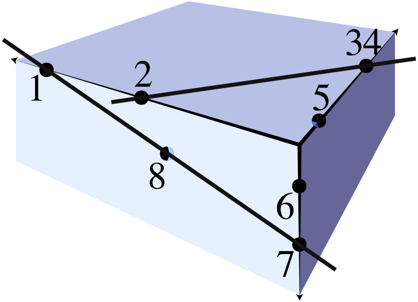

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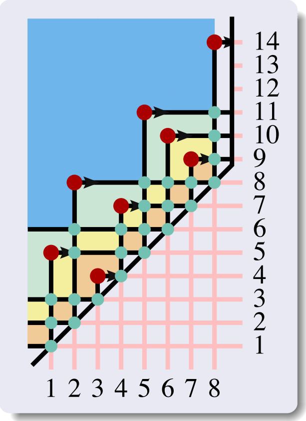

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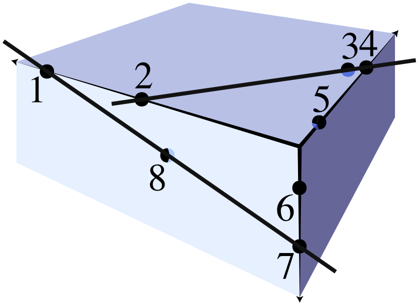

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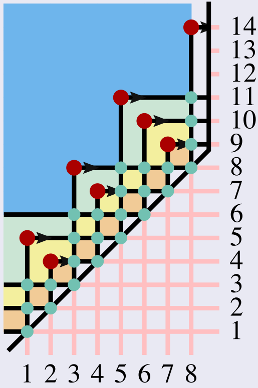

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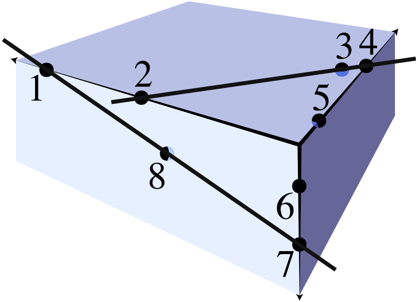

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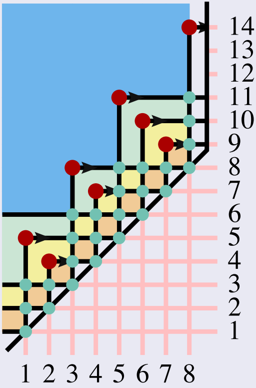

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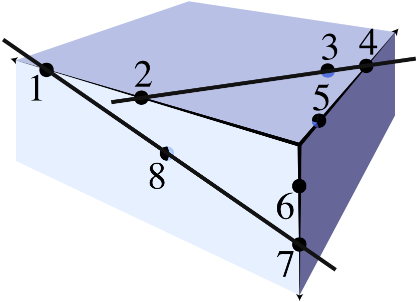

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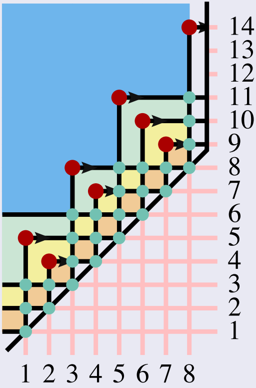

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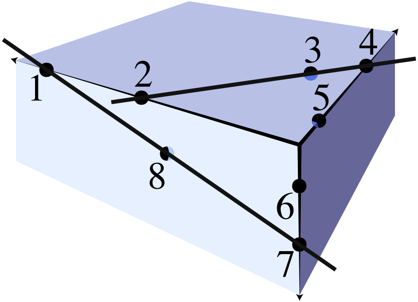

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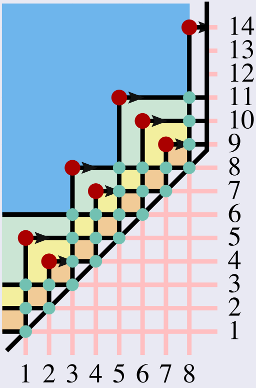

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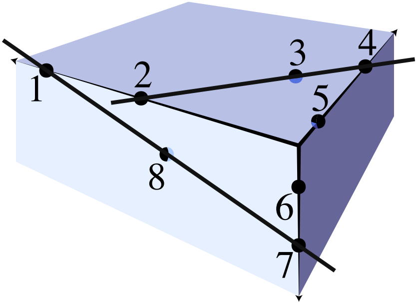

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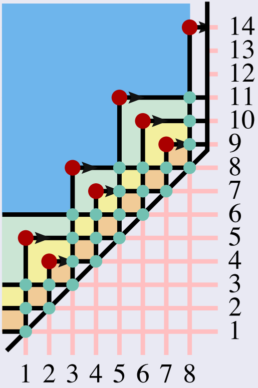

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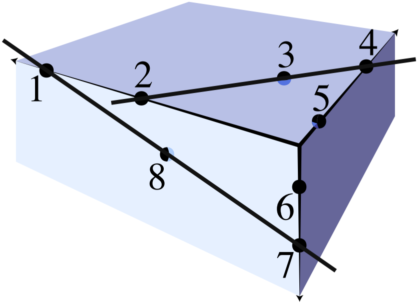

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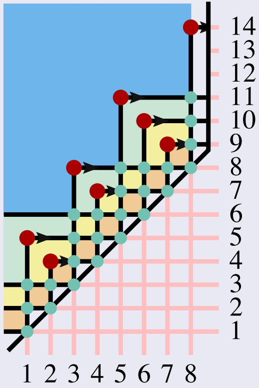

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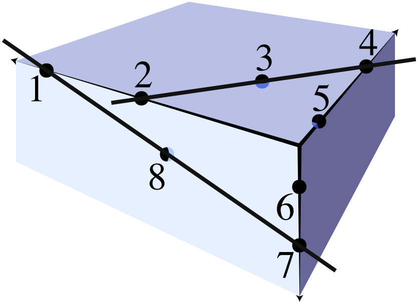

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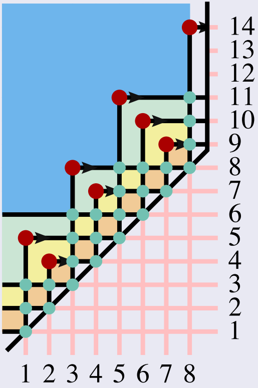

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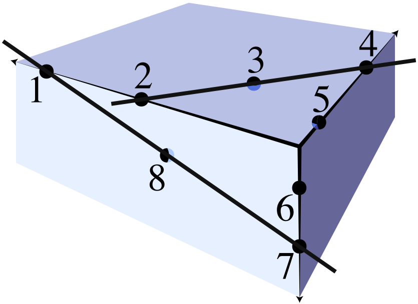

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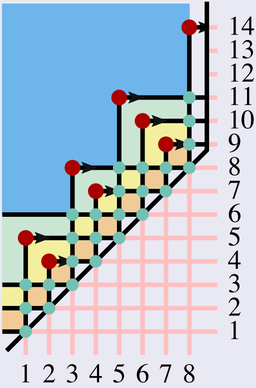

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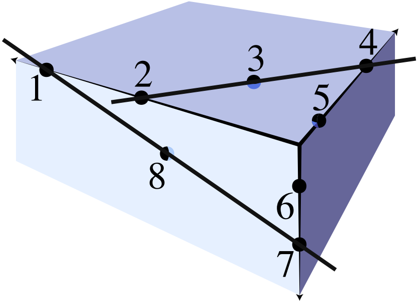

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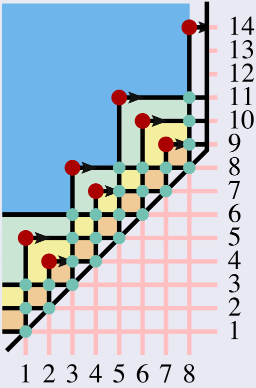

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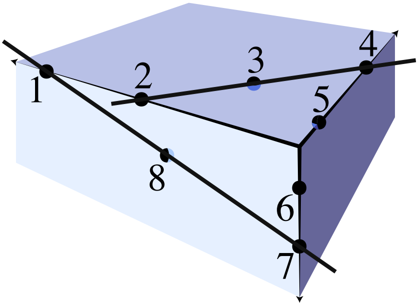

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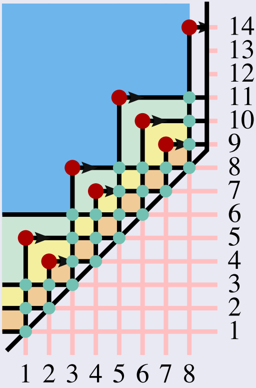

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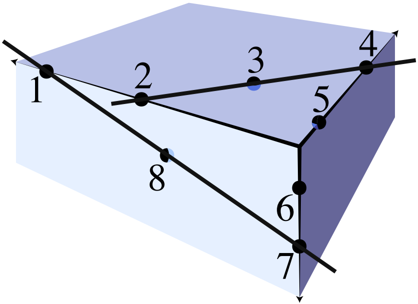

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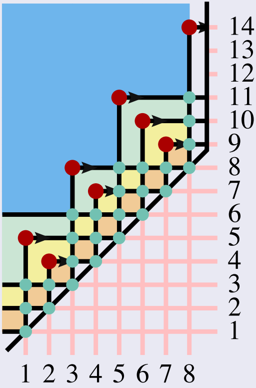

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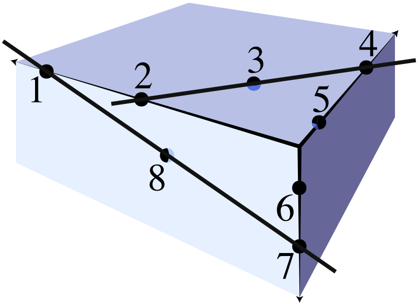
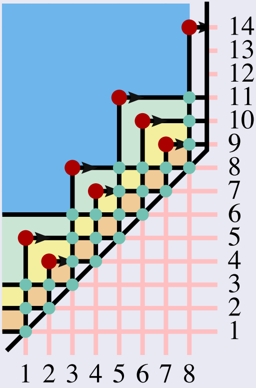


$$C \equiv \begin{array}{c} \begin{array}{cccccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \end{array} \\ \left(\begin{array}{cccccccc} 1 & \alpha_2 & \alpha_1\alpha_2 & 0 & 0 & \alpha_8 & \alpha_7\alpha_8 & 0 \\ 0 & 1 & \alpha_1+\alpha_4 & \alpha_3\alpha_4 & 0 & \alpha_6 & 0 & 0 \\ 0 & 0 & 1 & \alpha_3 & \alpha_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \alpha_7 & \alpha_9 \end{array}\right) \end{array}$$



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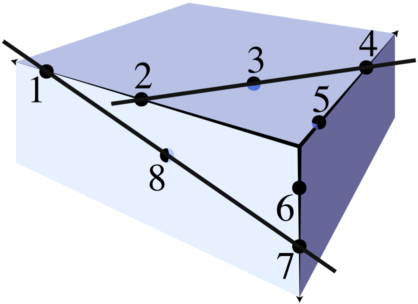

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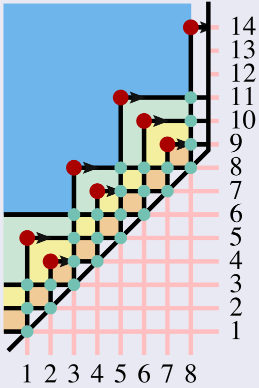

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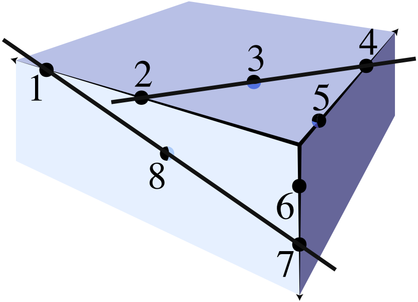

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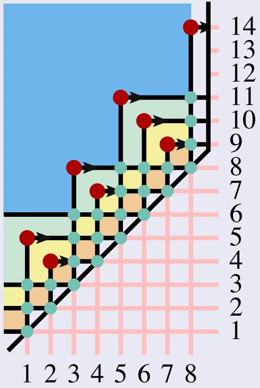

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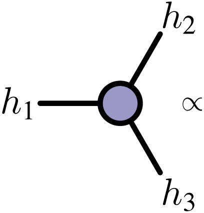
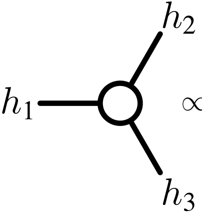
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).



$$\propto \quad \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}$$

$$h_1+h_2+h_3\leq 0$$



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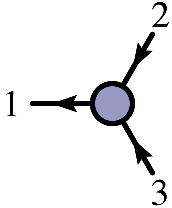
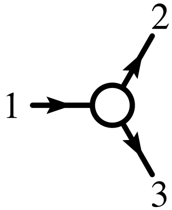


$$= \frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$



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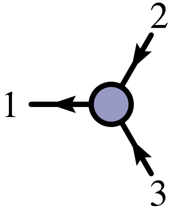

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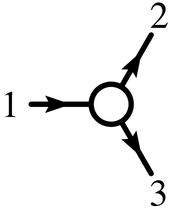

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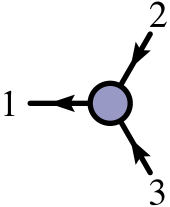

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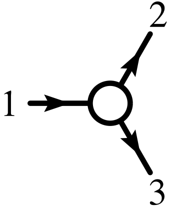

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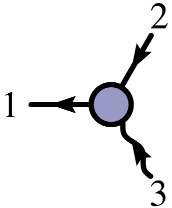

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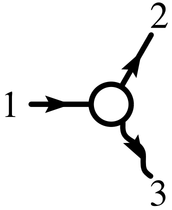

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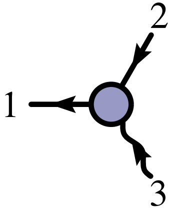

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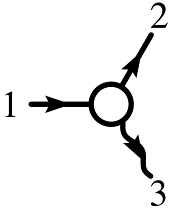

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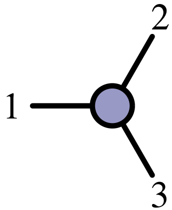
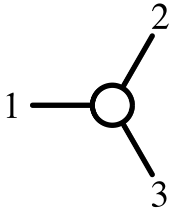


$$= \quad \frac{\delta^{2\times4}\big(\lambda\cdot\tilde{\eta}\big)}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,\delta^{2\times2}\big(\lambda\cdot\tilde{\lambda}\big)\equiv\mathcal{A}_3^{(2)}$$



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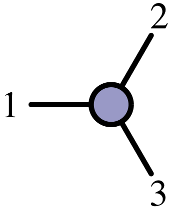

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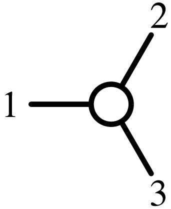

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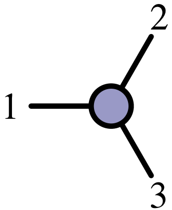

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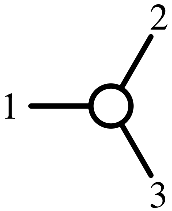

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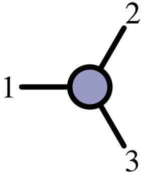

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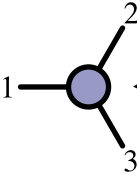

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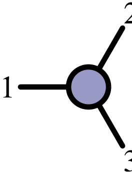

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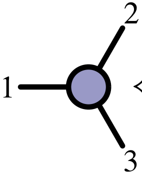

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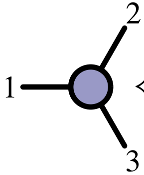

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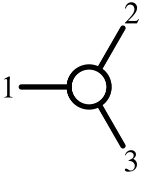

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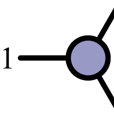

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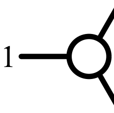

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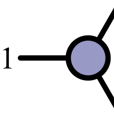

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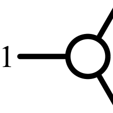

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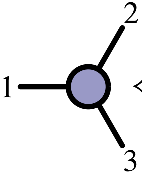
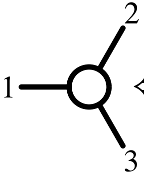


$$\Leftrightarrow B\equiv\begin{pmatrix} b_1^1 & b_2^1 & b_3^1\\ b_1^2 & b_2^2 & b_3^2\end{pmatrix}$$



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$$\mathcal{A}_3^{(2)}=\frac{\delta^{2\times4}\left(\lambda\cdot\widetilde{\eta}\right)}{\langle12\rangle\langle23\rangle\langle31\rangle}\delta^{2\times2}\left(\lambda\cdot\widetilde{\lambda}\right)\equiv\int\frac{d^{2\times3}B}{\mathrm{vol}(GL_2)}\frac{\delta^{2\times4}\left(B\cdot\widetilde{\eta}\right)}{(12)(23)(31)}\;\delta^{2\times2}\left(B\cdot\widetilde{\lambda}\right)\;\delta^{1\times2}\left(\lambda\cdot B^{\perp}\right)$$

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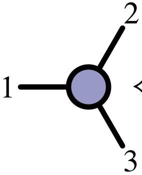

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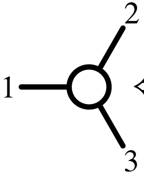

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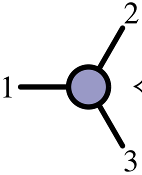

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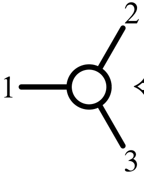

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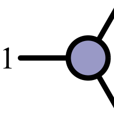

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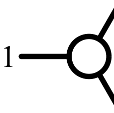

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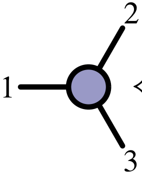

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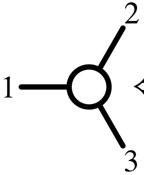

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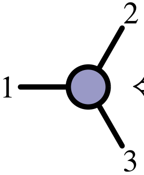

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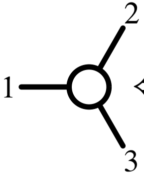

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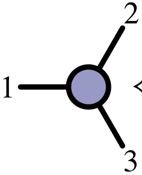

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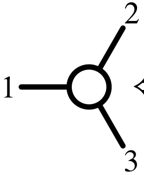

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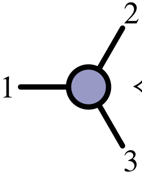
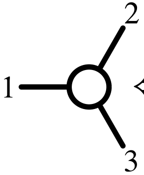


$$\Leftrightarrow B \equiv \begin{pmatrix} b_1^1 & 1 & 0 \\ b_1^2 & 0 & 1 \end{pmatrix}$$



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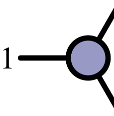

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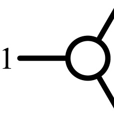

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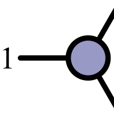

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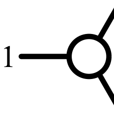

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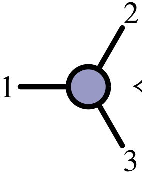

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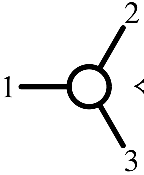

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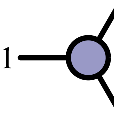

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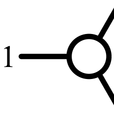

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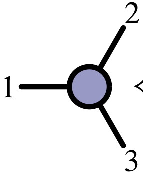

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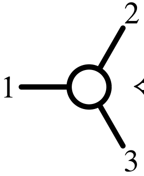

$$\Leftrightarrow W\equiv\begin{pmatrix}w_1^1 & w_2^1 & w_3^1\end{pmatrix}$$

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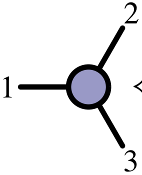

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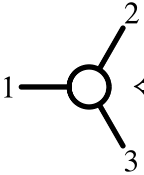

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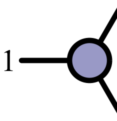

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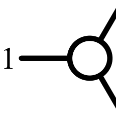

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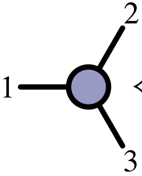

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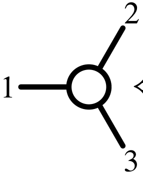

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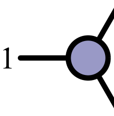

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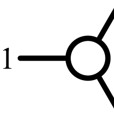

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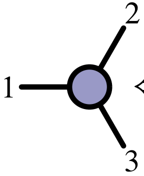

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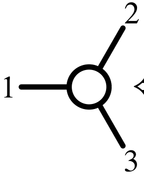

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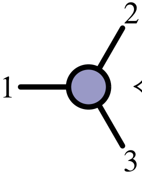

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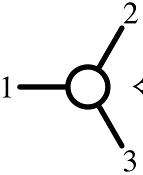

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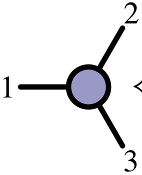

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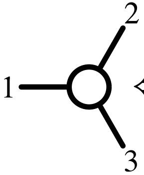

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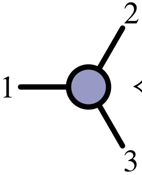

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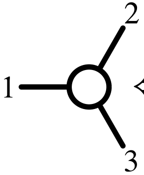

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Constructing the Correspondence: Amalgamations & Bridges

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Direct/Outer Products

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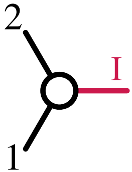

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$$\begin{array}{ccc} 1 & 2 & \mathrm{I} \\ \hline \end{array}$$
$$\left(\begin{array}{ccc} 1 & w_2 & w_{\mathrm{I}} \end{array}\right)$$

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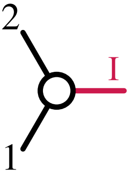

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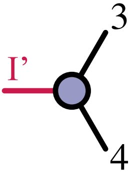

$$\left(\begin{array}{ccc} \mathrm{I'} & 3 & 4 \\ \hline 1 & 0 & b_4^1 \\ 0 & 1 & b_4^2 \end{array}\right)$$

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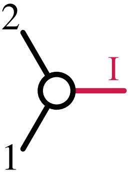

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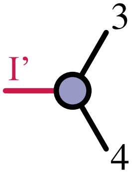

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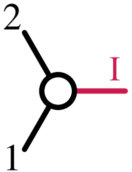

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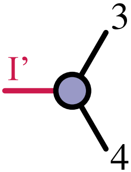

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$$(f_1,f_2)\mapsto\ f_1\times f_2$$
$$(C_1,C_2)\mapsto C_1\oplus C_2\subset G(k_1{+}k_2,n_1{+}n_2)$$
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$$\begin{array}{ccc} 1 & 2 & \mathrm{I} \\ \hline 1 & w_2 & w_{\mathrm{I}} \end{array}$$

$$\begin{array}{ccc} \mathrm{I'} & 3 & 4 \\ \hline 1 & 0 & b_4^1 \\ 0 & 1 & b_4^2 \end{array}$$

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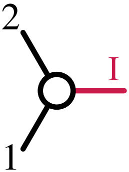

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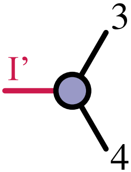

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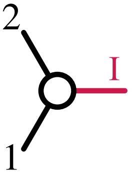

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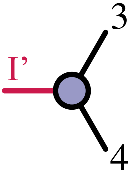

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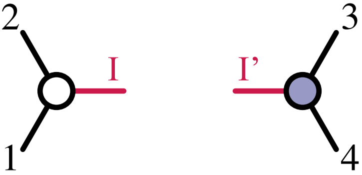

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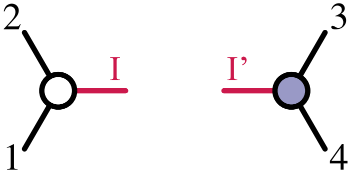

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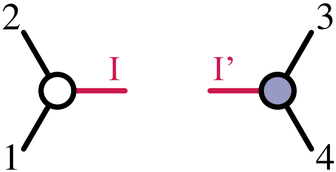

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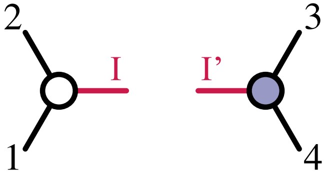

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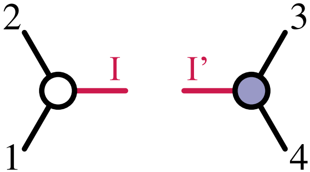

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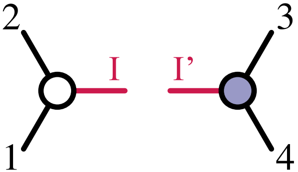

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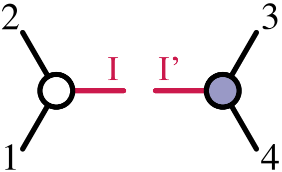

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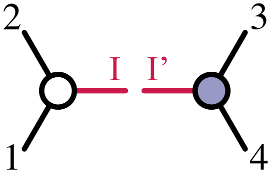

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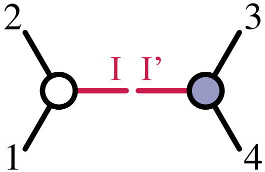

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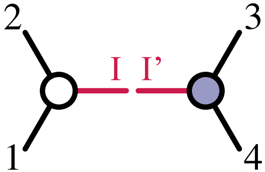

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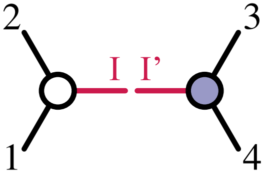

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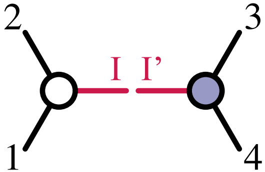

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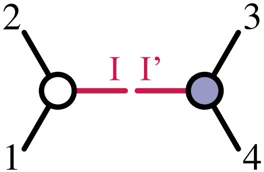

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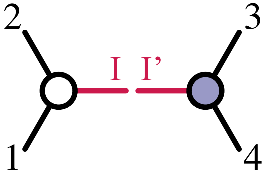

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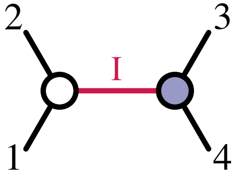

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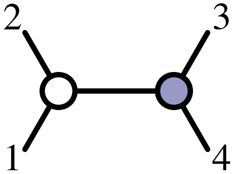

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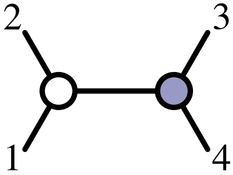

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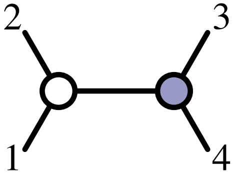

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$$f_{\Gamma} \equiv \int \Omega_C \, \delta^{k\times 2}(C\cdot\widetilde{\lambda})\,\delta^{2\times(n-k)}(\lambda\cdot C^{\perp})$$

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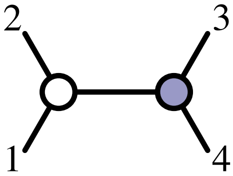

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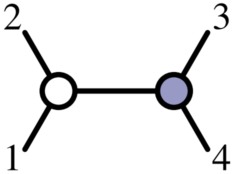

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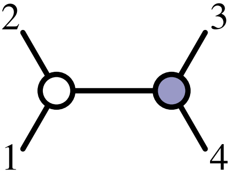

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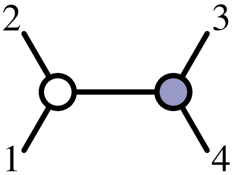

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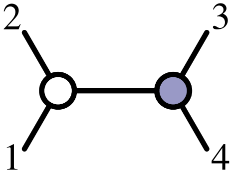

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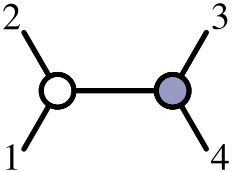

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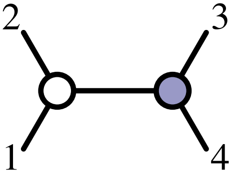

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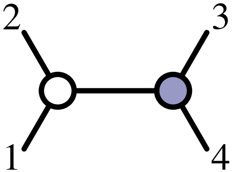

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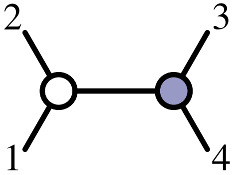

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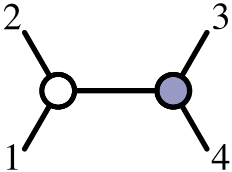

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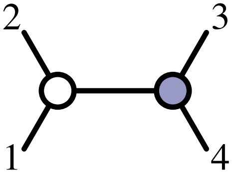

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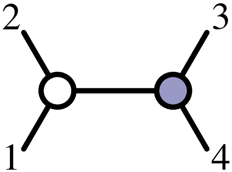

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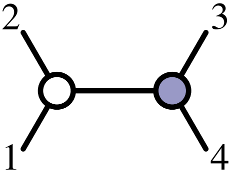

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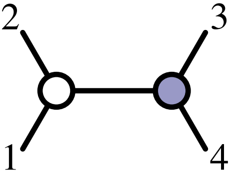

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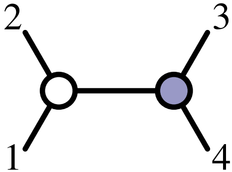

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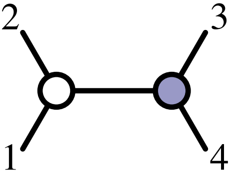

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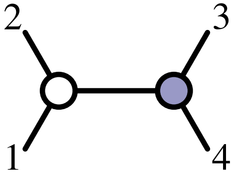

Adding a ‘Bridge’ to Legs (A,B)

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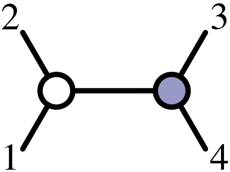

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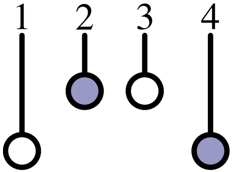

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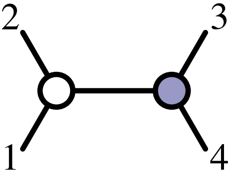

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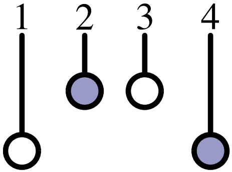

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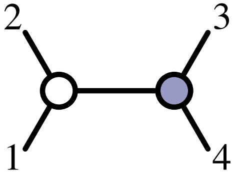

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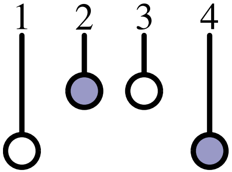

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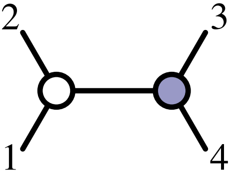

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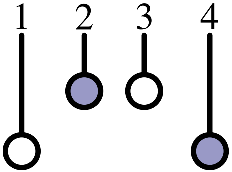

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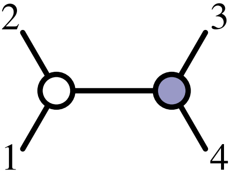

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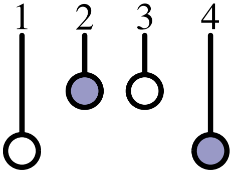

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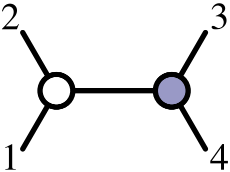

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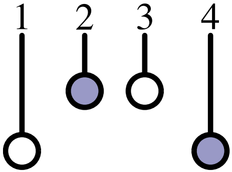

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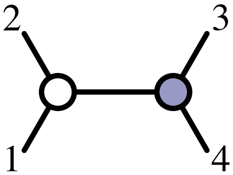

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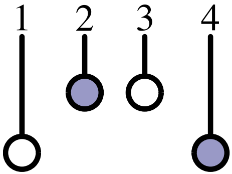

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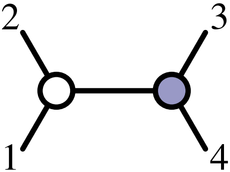

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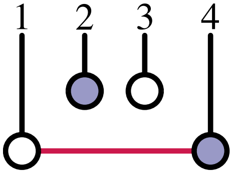

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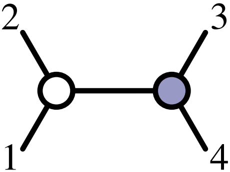

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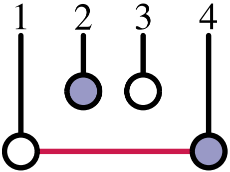

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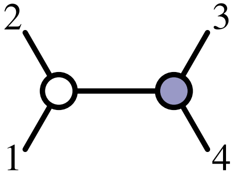

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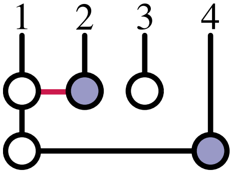

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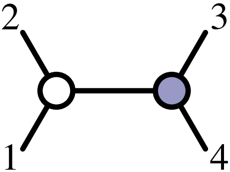

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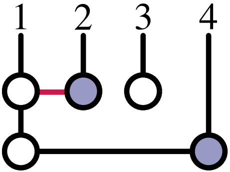

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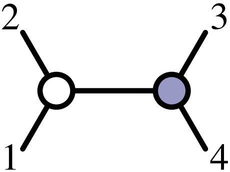

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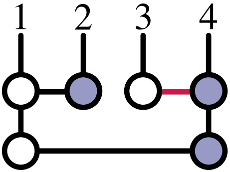

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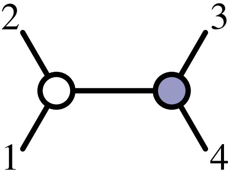

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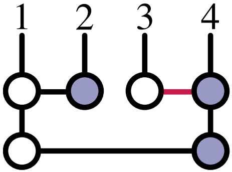

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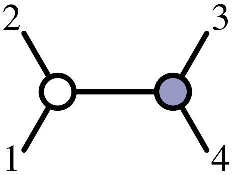

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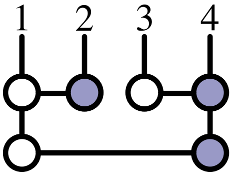

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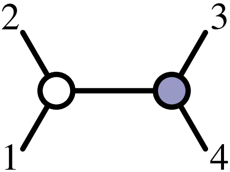

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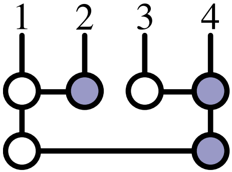

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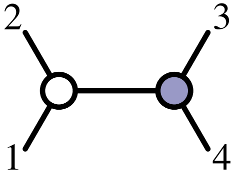

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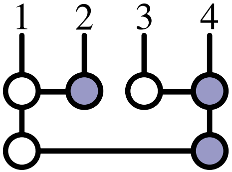

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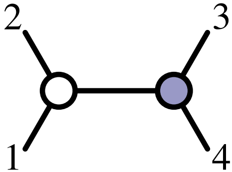

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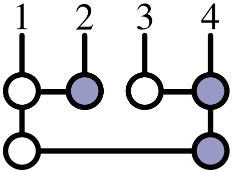

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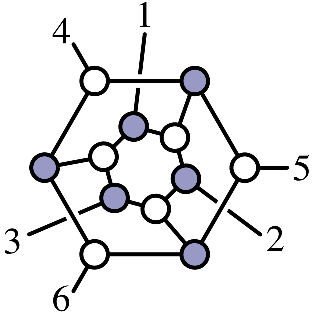

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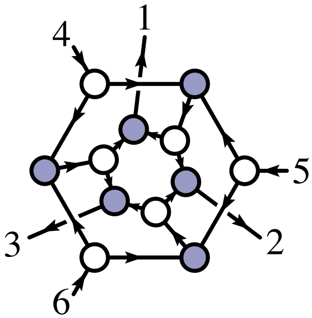

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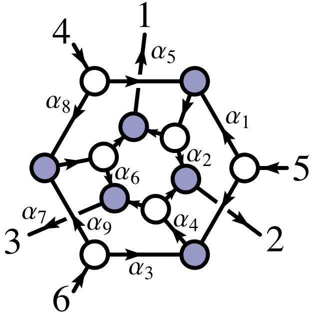

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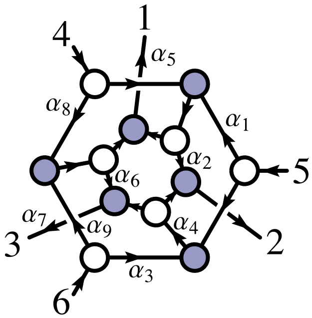

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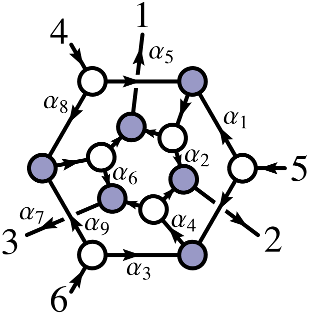

$$C(\alpha)\equiv\left(\begin{array}{cccccc} c_1 & c_2 & c_3 & c_4 & c_5 & c_6\\ \hline \alpha_5(1+\alpha_8) & \alpha_2 & \alpha_6\,\alpha_7\,\alpha_8 & 1 & 0 & 0\\ \alpha_1\,\alpha_5 & \alpha_1\,\alpha_2+\alpha_4 & \alpha_4\,\alpha_7 & 0 & 1 & 0\\ \alpha_5\,\alpha_9 & \alpha_3\,\alpha_4 & \alpha_7(\alpha_3\alpha_4+\alpha_6\alpha_9) & 0 & 0 & 1 \end{array}\right)$$

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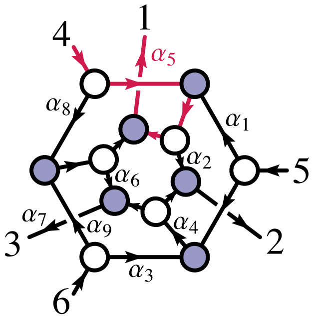

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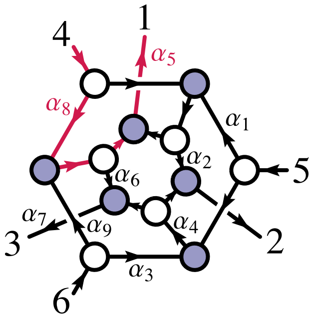

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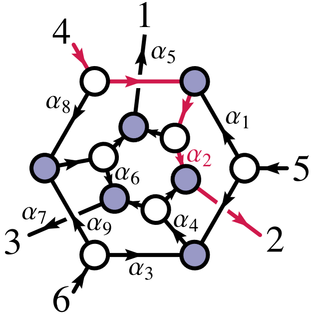

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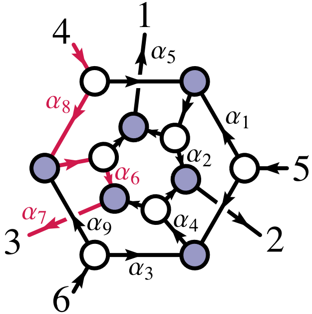

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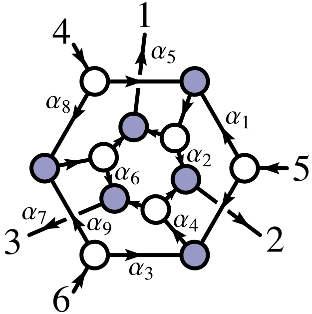

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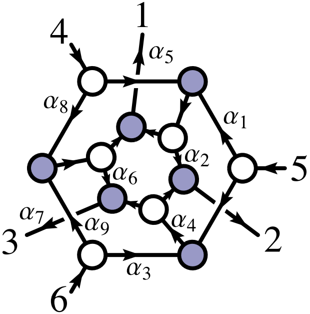

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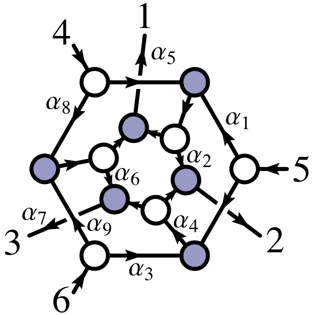

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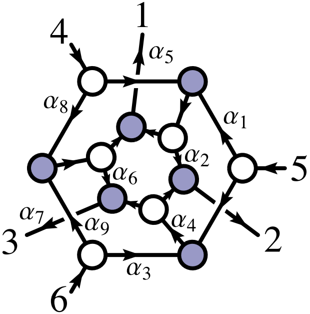

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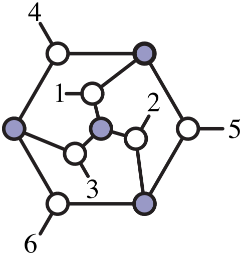

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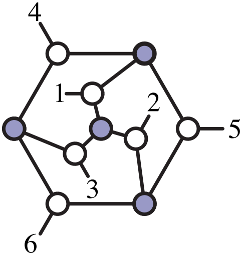

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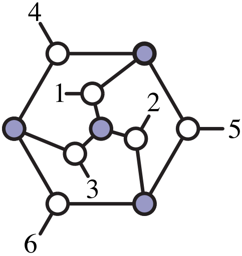

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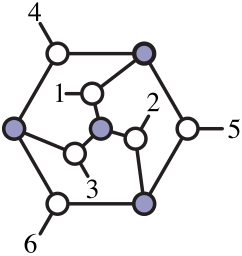

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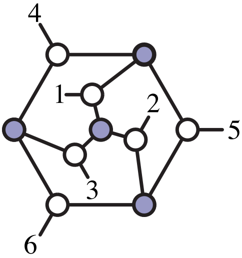

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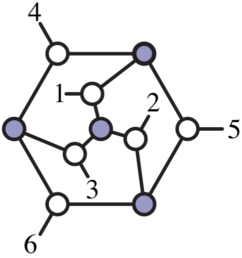

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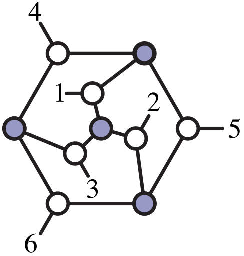

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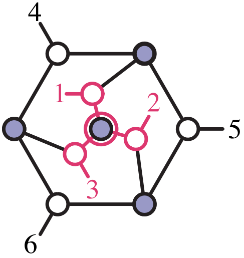

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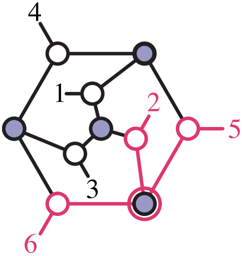

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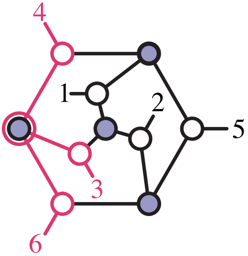

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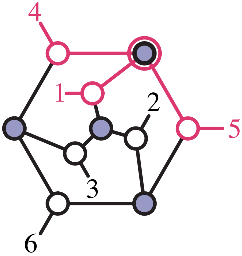

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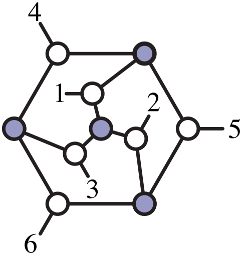

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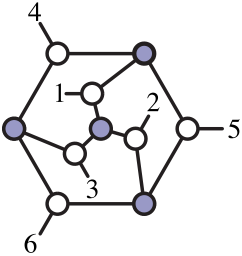

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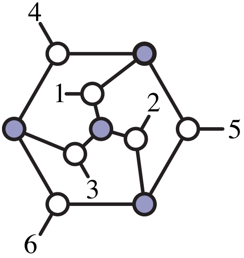

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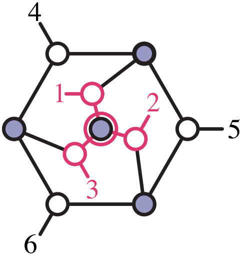

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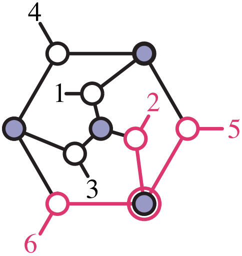

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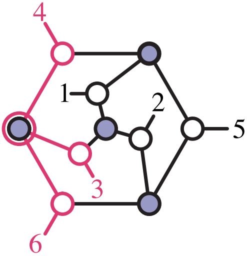

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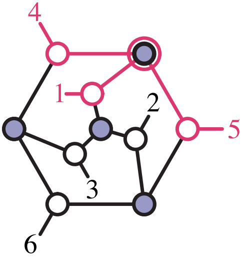

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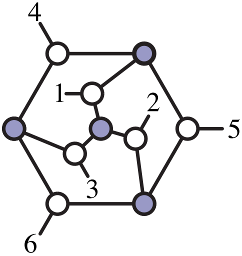

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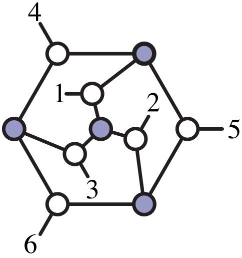

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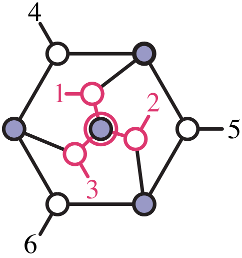

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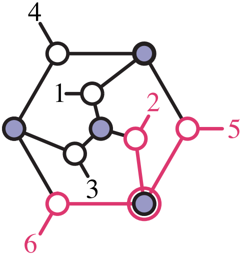

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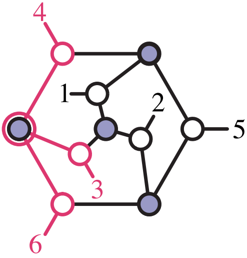

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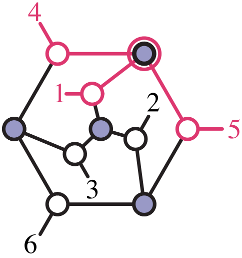

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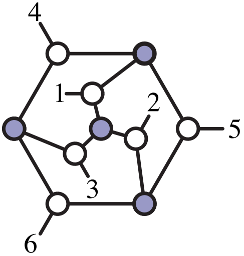

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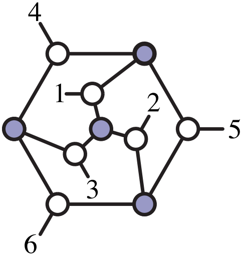

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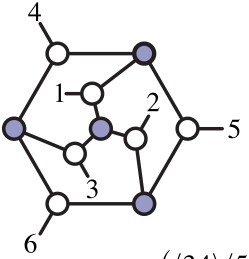

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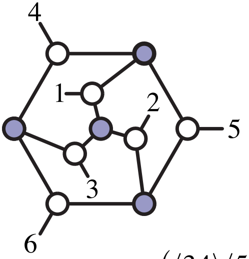

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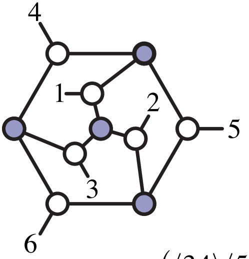

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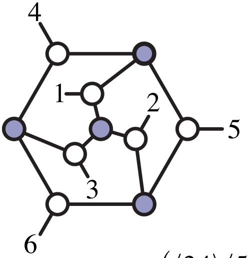

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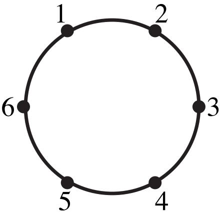

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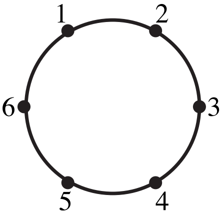

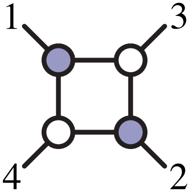

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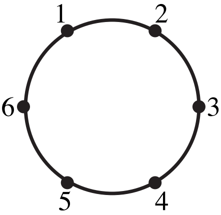

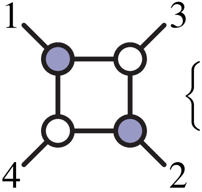

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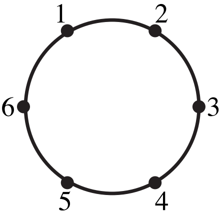

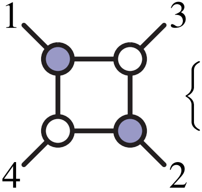

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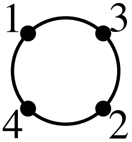

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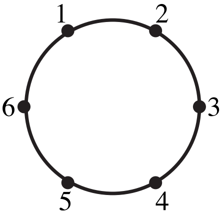

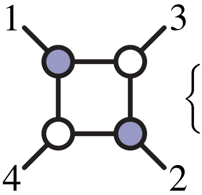

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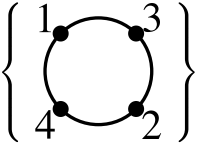

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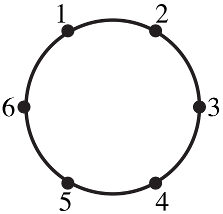

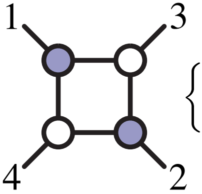

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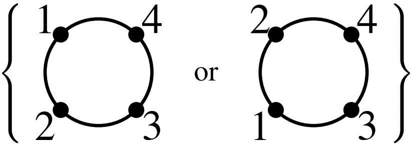

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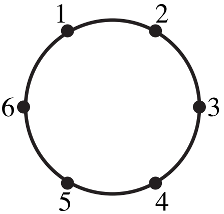

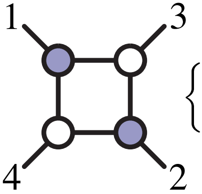

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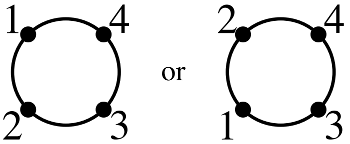

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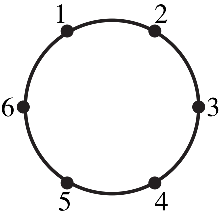
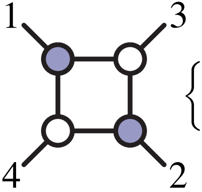
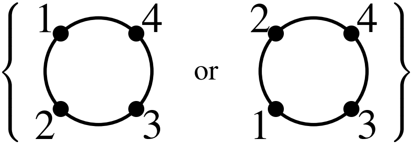




$$\left\{\begin{matrix}(1\,{\color{red}4\,3})\\(3\,2\,4)\end{matrix}\right\} \Leftrightarrow$$



$\mathrm{PT}(1,4,3,2)$, $\mathrm{PT}(1,2,4,3)$

Extended 'Positivity' and Parke-Taylor Completeness

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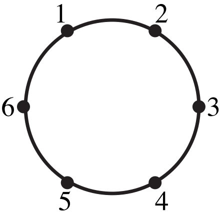

$$\boxed{\widetilde{f}_{\Gamma} = \sum_{\{\sigma\in(\mathfrak{S}_n/\mathbb{Z}_n)|\forall\tau\in T:\,\sigma_{\tau_1}<\sigma_{\tau_2}<\sigma_{\tau_3}\}} \mathrm{PT}(\sigma_1,\ldots,\sigma_n),}$$

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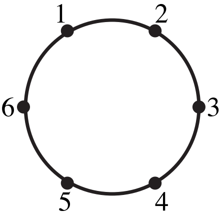

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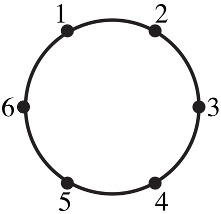

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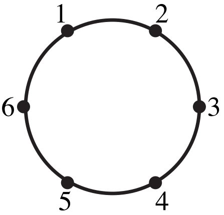

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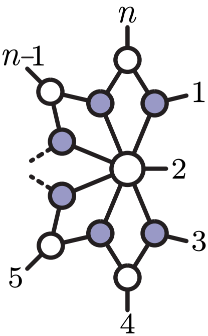

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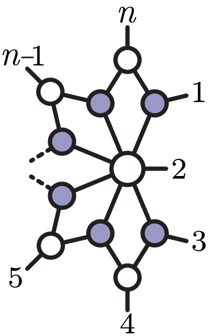

$$\left\{\begin{matrix} (2 & n & 1) \\ (2 & 3 & 4) \\ (2 & 4 & 5) \\ & \vdots & \\ (2 & n-1 & n) \end{matrix}\right\}$$

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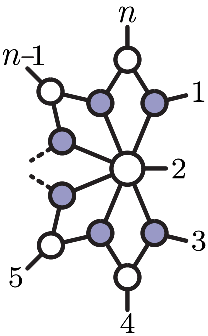

$$\frac{\left\{\begin{matrix}(2 & n & 1)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n{-}1 & n)\end{matrix}\right\}}{\mathrm{PT}(1,2,\ldots,n)}$$

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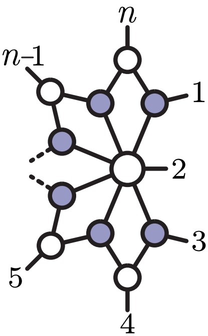

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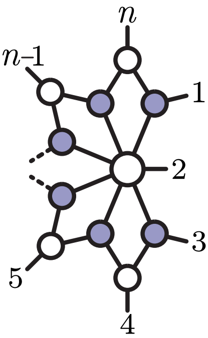

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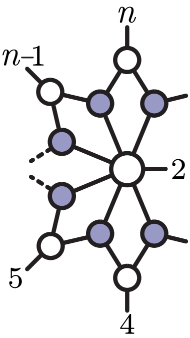

$$(-1)\frac{\left\{\begin{matrix}(2 & n & 1)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n{-}1 & n)\end{matrix}\right\}}{\mathrm{PT}(1,2,\ldots,n)} = \frac{\left\{\begin{matrix}(2 & 1 & n)\\(2 & 3 & 4)\\(2 & 4 & 5)\\ & \vdots & \\(2 & n{-}1 & n)\end{matrix}\right\}}{\mathrm{PT}(1,n,2,\ldots,n{-}1)+\ldots+\mathrm{PT}(1,3,\ldots,n,2)}$$

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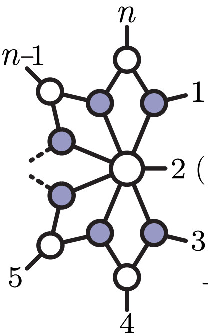

$$\underset{-\,\mathrm{PT}(1,2,\ldots,n)}{\underline{(-1)\left\{\begin{matrix}(2 & n & 1)\\ (2 & 3 & 4)\\ (2 & 4 & 5)\\ & \vdots & \\ (2\,n{-}1\,n) & & \end{matrix}\right\}}} = \underset{\mathrm{PT}(1,n,2,\ldots,n{-}1)+\ldots+\mathrm{PT}(1,3,\ldots,n,2)}{\underline{\left\{\begin{matrix}(2 & 1 & n)\\ (2 & 3 & 4)\\ (2 & 4 & 5)\\ & \vdots & \\ (2\,n{-}1\,n) & & \end{matrix}\right\}}}$$

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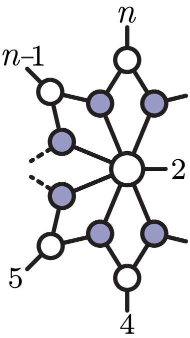

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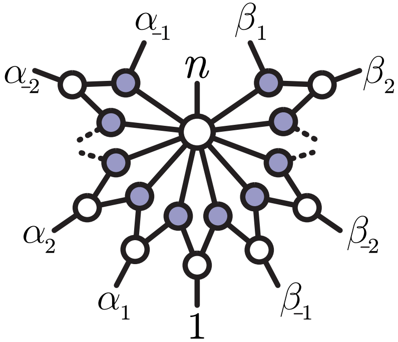

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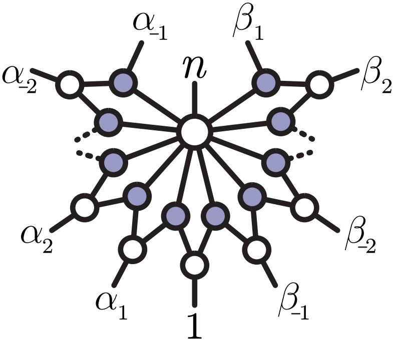

$$\left\{\begin{pmatrix} (\,1 & \alpha_1 & n\,) \\ (\alpha_1 & \alpha_2 & n\,) \\ & \vdots & \\ (\alpha_{-2} & \alpha_{-1} & n\,) \\ (\,n & \beta_1 & \beta_2) \\ & \vdots & \\ (\,n & \beta_{-2} & \beta_{-1}) \\ (\,n & \beta_{-1} & 1\,) \end{pmatrix}\right\}$$

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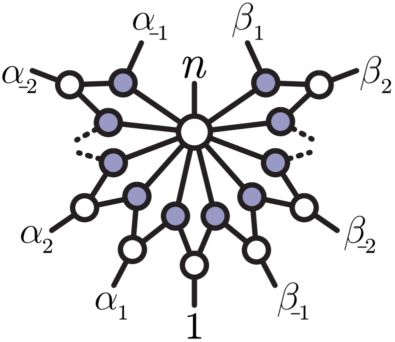

$$\left\{\begin{pmatrix}(1\;\;\alpha_1\;\;n)\\(\alpha_1\;\alpha_2\;\;n)\\\vdots\\(\alpha_{-2}\;\alpha_{-1}\;\;n)\\(n\;\;\beta_1\;\beta_2)\\\vdots\\(n\;\;\beta_{-2}\;\beta_{-1})\\(n\;\;\beta_{-1}\;\;1)\end{pmatrix}\right\}\qquad\left\{\begin{pmatrix}(1\;\;\alpha_1\;\;n)\\(\alpha_1\;\alpha_2\;\;n)\\\vdots\\(\alpha_{-2}\;\alpha_{-1}\;\;n)\\(n\;\;\beta_2\;\beta_1)\\\vdots\\(n\;\;\beta_{-1}\;\beta_{-2})\\(n\;\;1\;\;\beta_{-1})\end{pmatrix}\right\}$$

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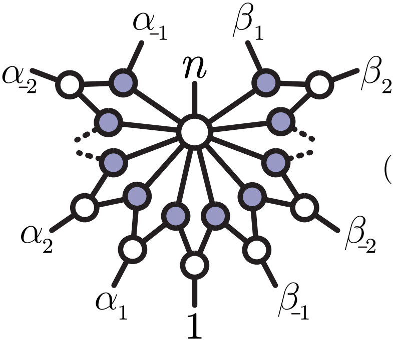

$$(-1)^{n_\beta}\left\{\begin{matrix}(\,1 & \alpha_1 & n\,)\\ (\alpha_1 & \alpha_2 & n\,)\\ & \vdots & \\ (\alpha_{-2} & \alpha_{-1} & n\,)\\ (\,n & \beta_1 & \beta_2)\\ & \vdots & \\ (\,n & \beta_{-2} & \beta_{-1})\\ (\,n & \beta_{-1} & 1\,)\end{matrix}\right\} = \left\{\begin{matrix}(\,1 & \alpha_1 & n\,)\\ (\alpha_1 & \alpha_2 & n\,)\\ & \vdots & \\ (\alpha_{-2} & \alpha_{-1} & n\,)\\ (\,n & \beta_2 & \beta_1)\\ & \vdots & \\ (\,n & \beta_{-1} & \beta_{-2})\\ (\,n & 1 & \beta_{-1})\end{matrix}\right\}$$

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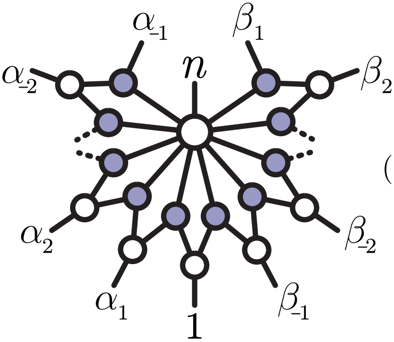

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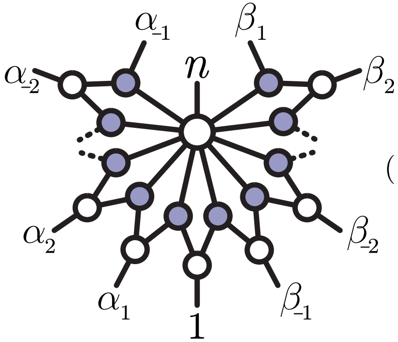

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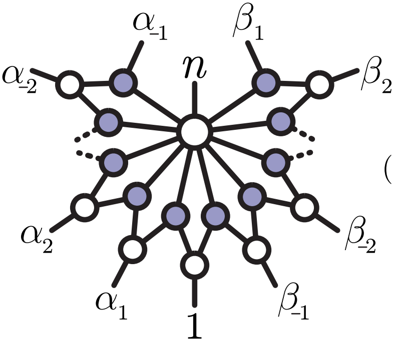

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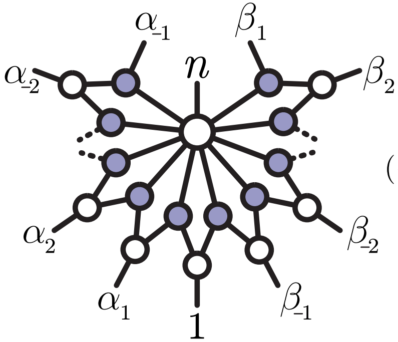

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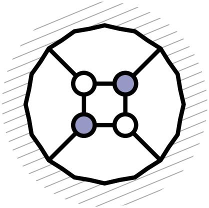

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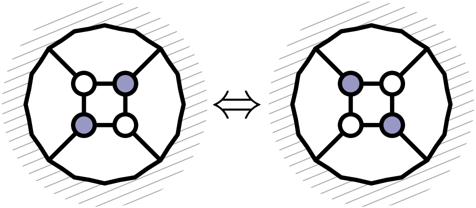

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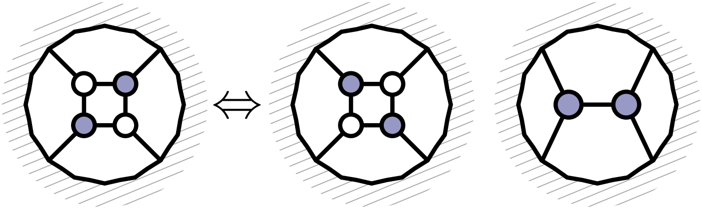

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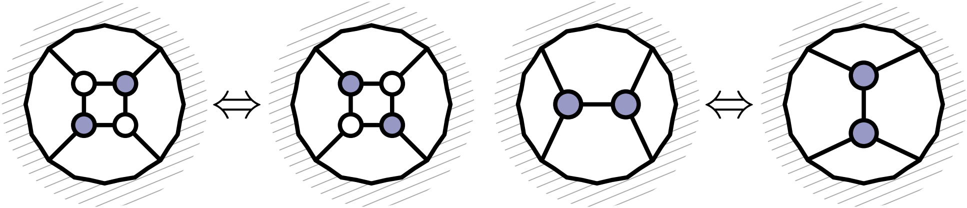

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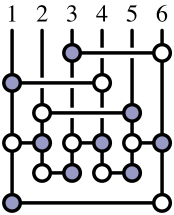

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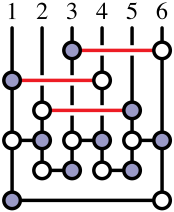

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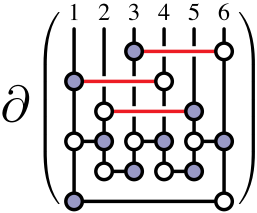

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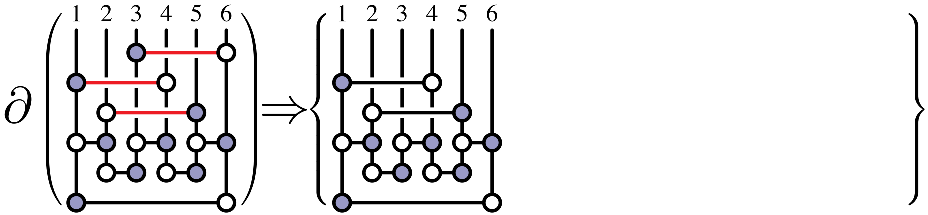

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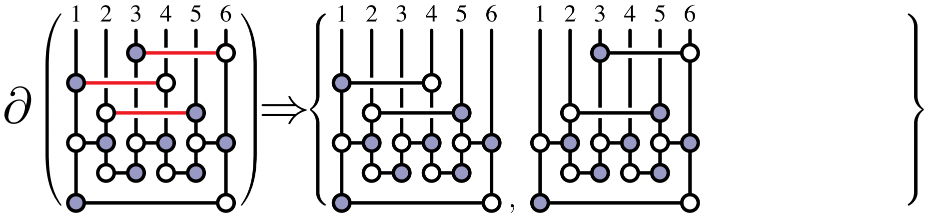

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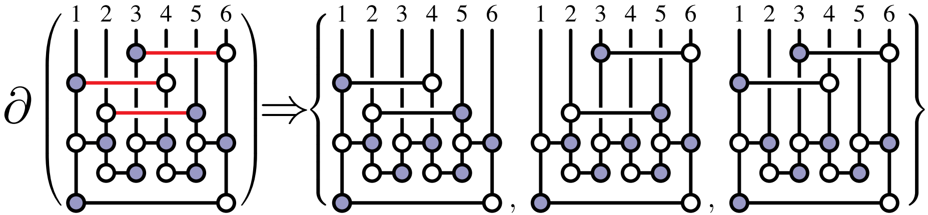

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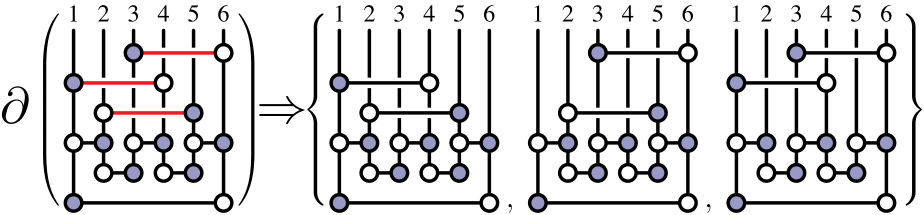

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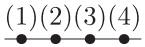

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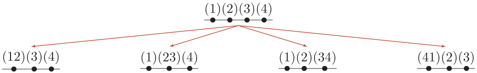

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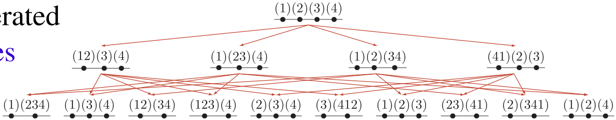

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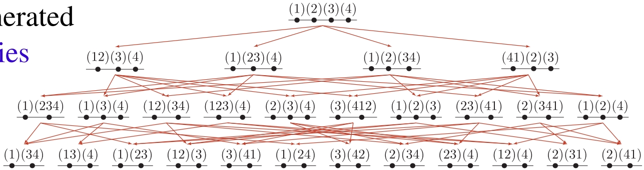

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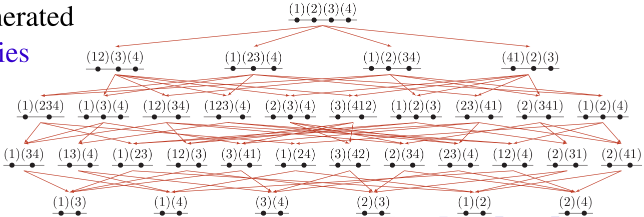

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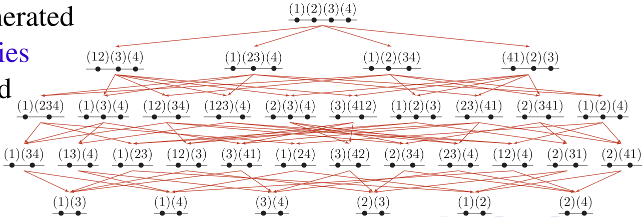

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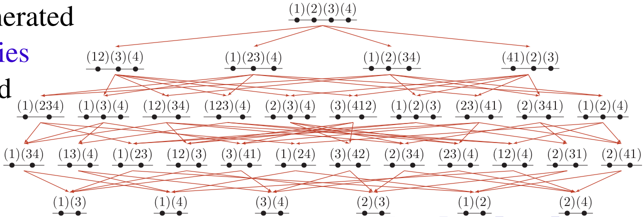

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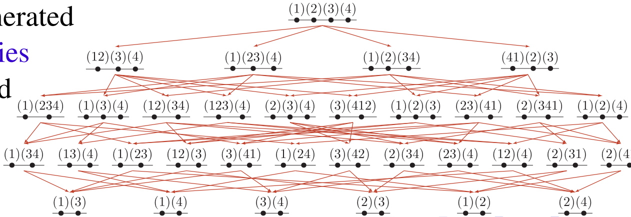

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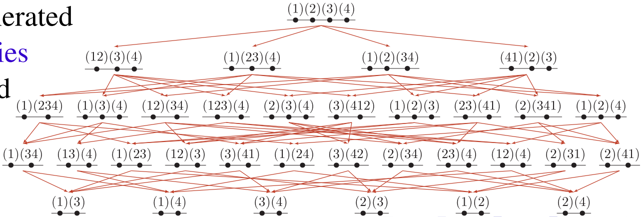

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    - 22 of which are ‘bridge constructible’; 1 is planar
- 10 (equivalence classes of) 8-dimensional varieties (‘leading singularities’)
    - all of which are ‘bridge constructible’; 1 is planar
    - 3,000 distinct functions; spanned by only **434** of them (**3** classes)
- 7 (equivalence classes of) 7-dimensional varieties; 3 planar
- 6 (equivalence classes of) 6-dimensional varieties; 5 planar
- 5 (equivalence classes of) 5-dimensional varieties; 5 planar; 4 ‘prime’
- . . .

Summary of the Classification of On-Shell Varieties of $G(3,6)$

Classification of On-Shell Varieties for 6-Point NMHV ($k = 3$)

- 24 (equivalence classes of) top-dimensional cells
  - each yields an identity among ‘leading singularities’
  - 22 of which are ‘bridge constructible’; 1 is planar
- 10 (equivalence classes of) 8-dimensional varieties (‘leading singularities’)
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- ...

The Classification of Top-Dim On-Shell Varieties of $G(3,6)$

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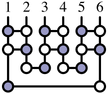

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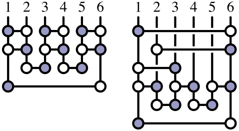

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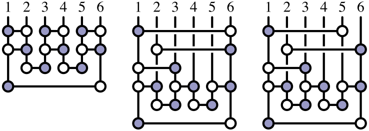

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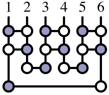

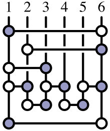

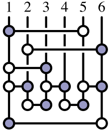

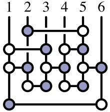

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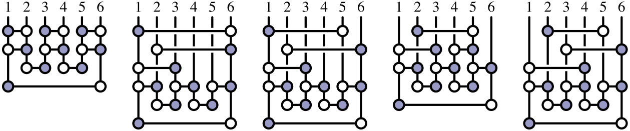

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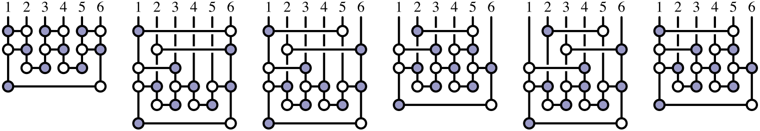

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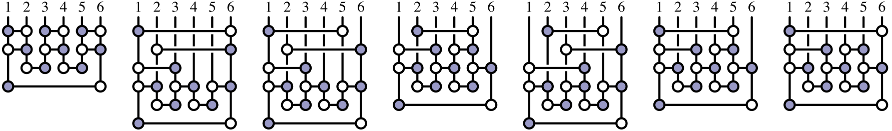

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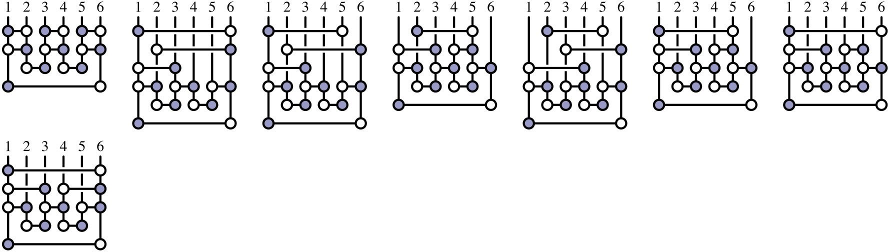

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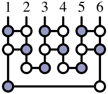
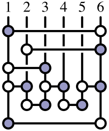
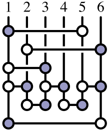
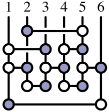
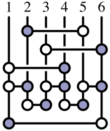
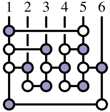


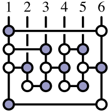


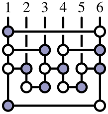


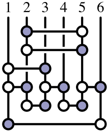




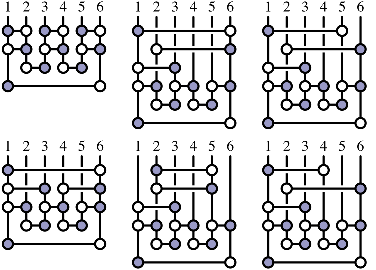








The Classification of Top-Dim On-Shell Varieties of $G(3,6)$



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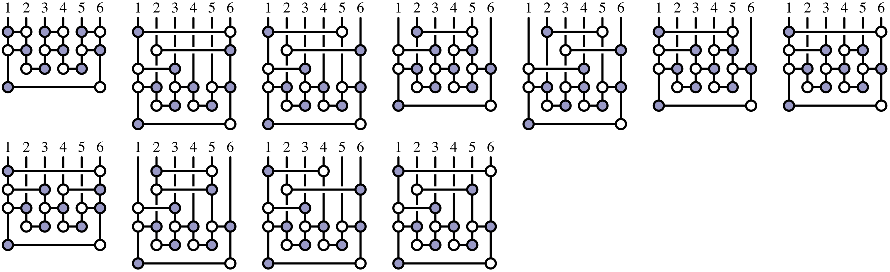

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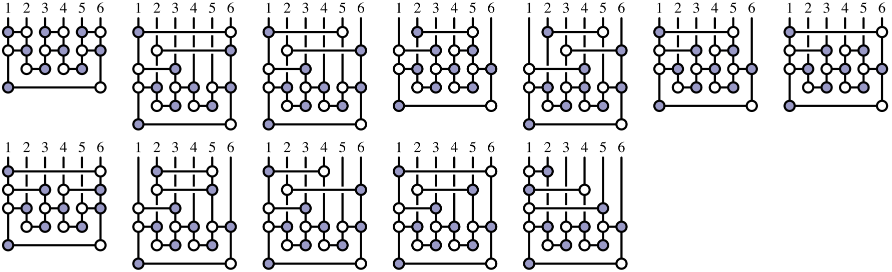

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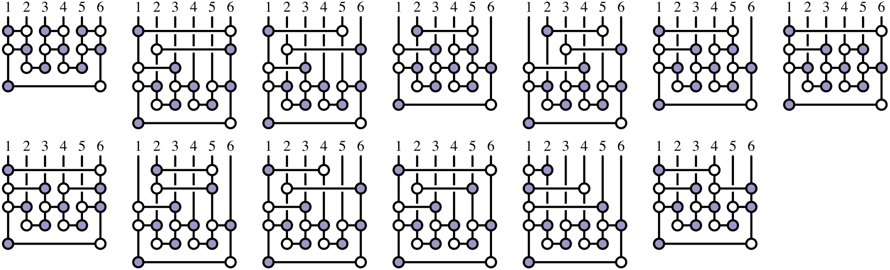

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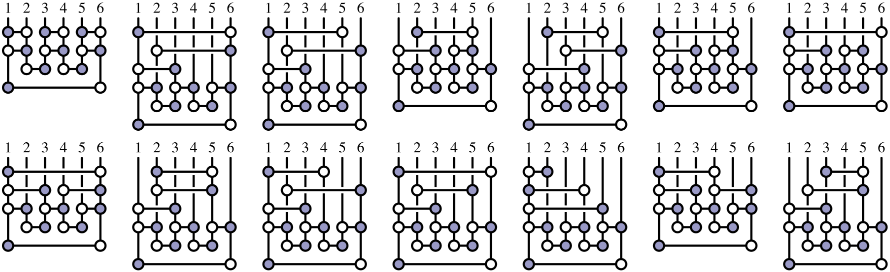

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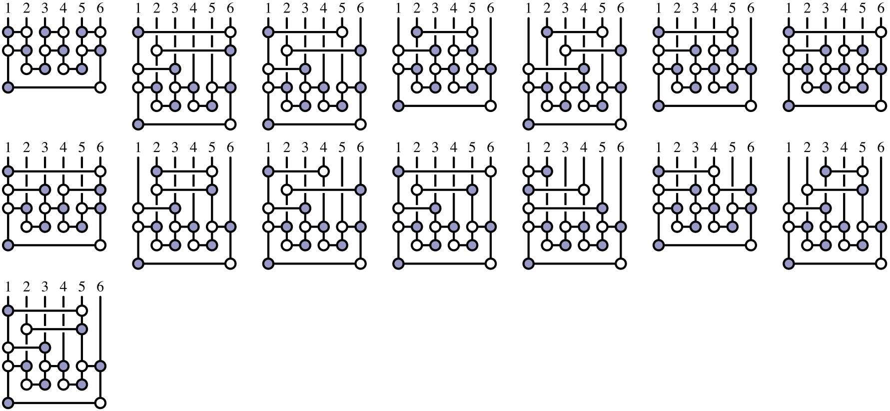

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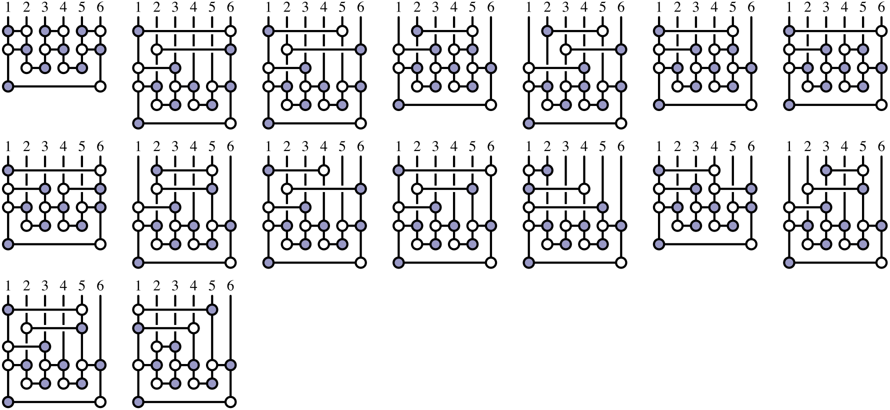

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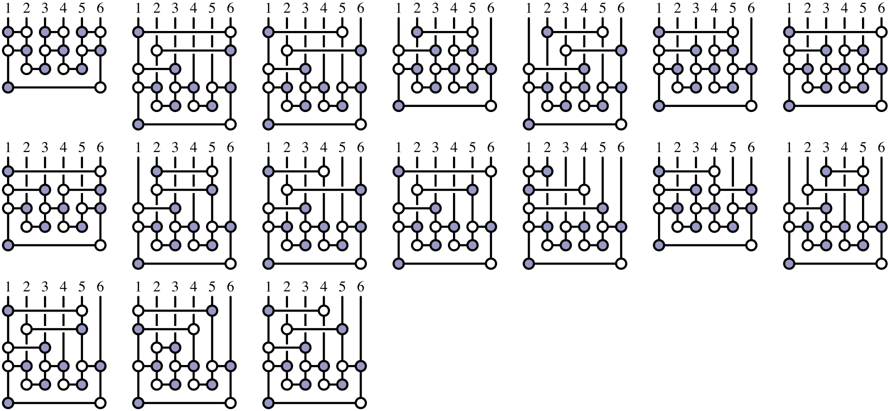

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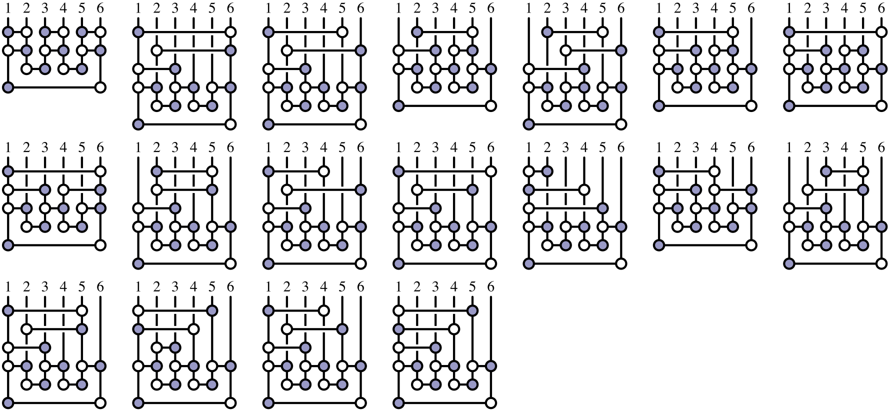

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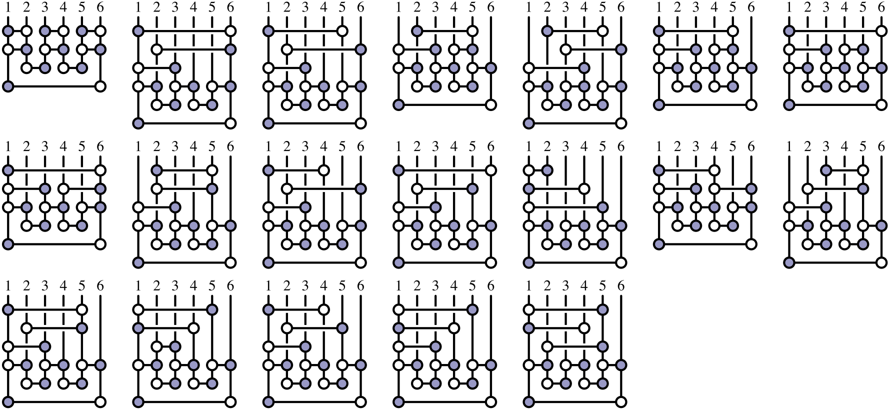

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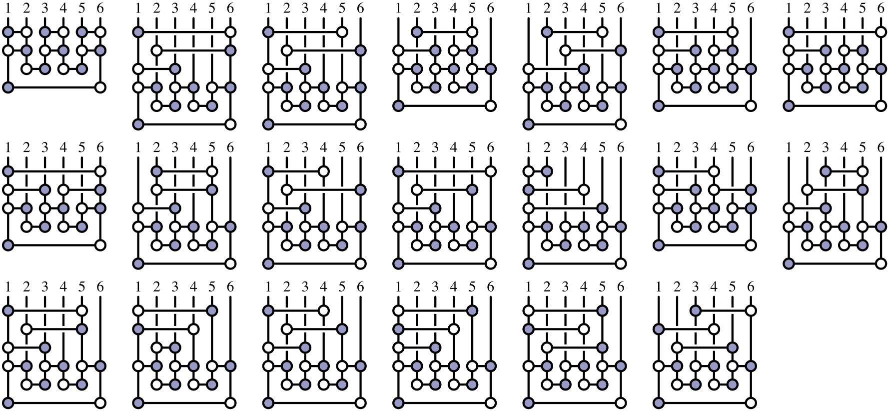

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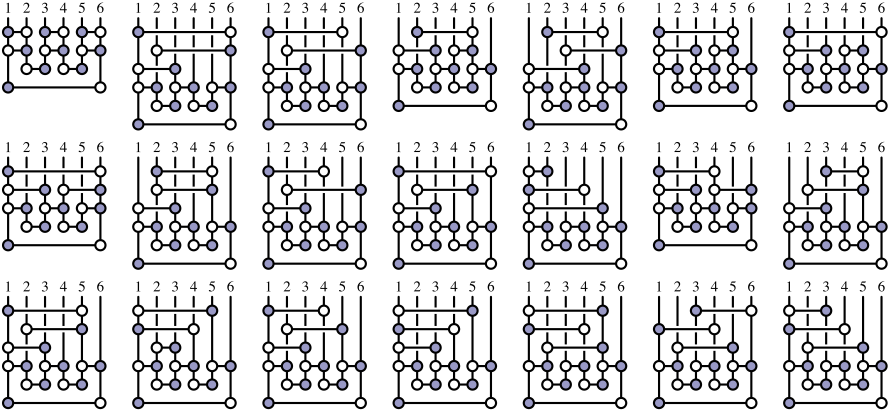

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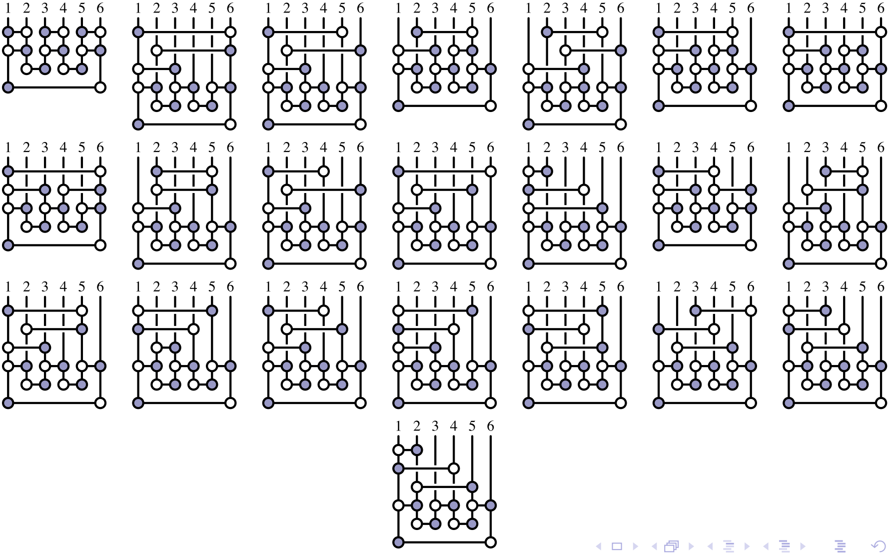

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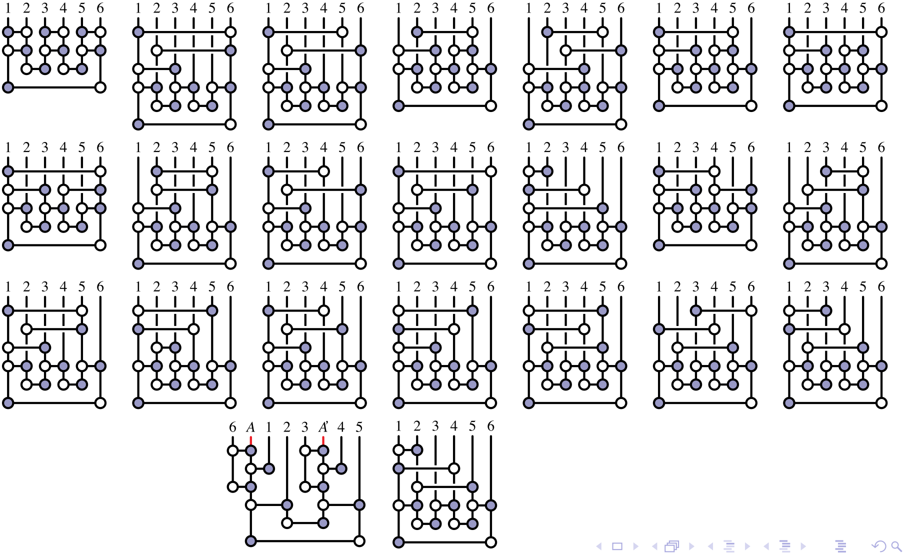

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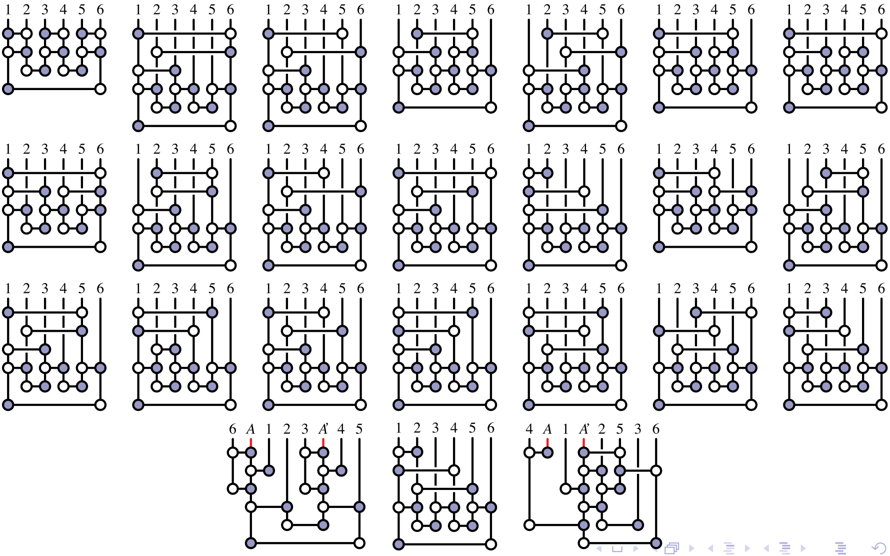

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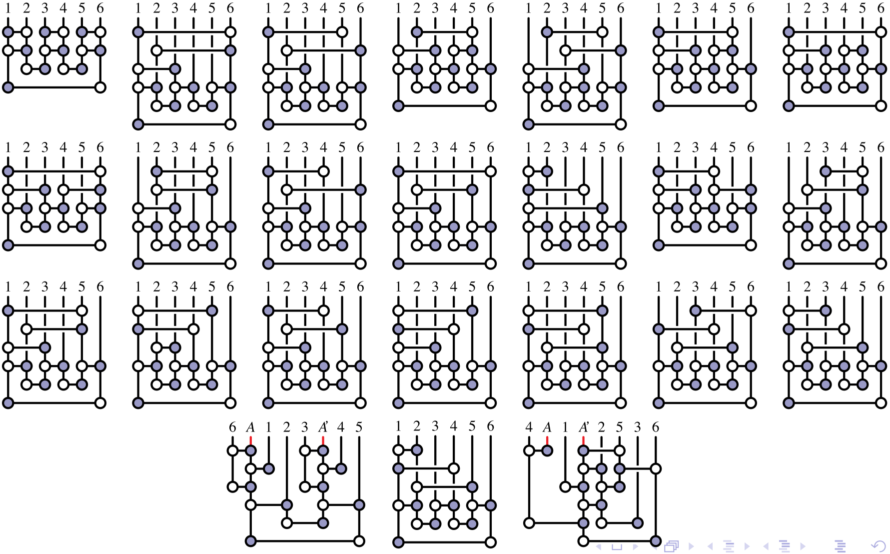

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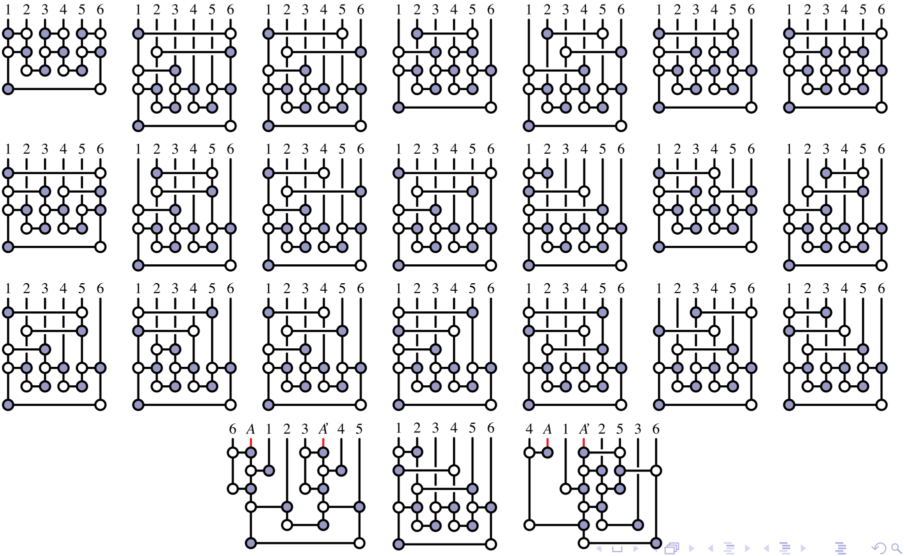

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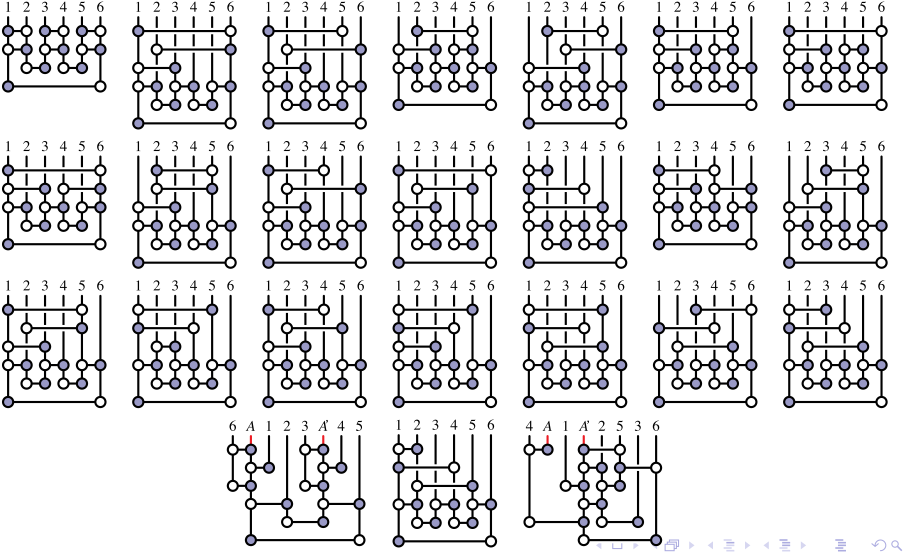


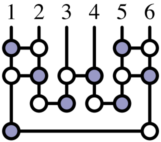
Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$

$$f_1 \equiv \oint\limits_{(123)=0} \Omega_1 = \left.\frac{\delta^{3\times4}\big(C^*\!\cdot\widetilde{\eta}\big)\,\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{(234)(345)(456)(561)(612)}\right|_{C^*}$$

$$= \frac{\delta^{3\times4}\big(C^*\!\cdot\widetilde{\eta}\big)\,\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 23\rangle\,[56]\,\langle 3|4{+}5|6]s_{456}\langle 1|5{+}6|4]\langle 12\rangle\,[45]}$$

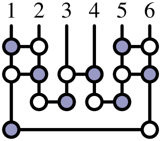
$$C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$



Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$

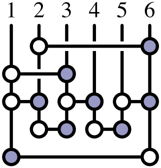
$$f_1 \equiv \oint\limits_{(123)=0} \Omega_1 = \left.\frac{\delta^{3\times4}(C^*\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(234)(345)(456)(561)(612)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\delta^{3\times4}(C^*\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle23\rangle\,[56]\,\langle3|4{+}5|6]\,s_{456}\,\langle1|5{+}6|4]\,\langle12\rangle\,[45]}$$



$$f_2 \equiv \oint\limits_{(123)=0} \Omega_2 = \left.\frac{(235)\,\delta^{3\times4}(C^*\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(136)(156)(234)(245)(256)(345)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

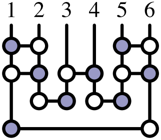
$$= \frac{\langle23\rangle\,[64]\,\delta^{3\times4}(C^*\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle13\rangle\,[45]\,\langle1|5{+}6|4]\,\langle23\rangle\,[56]\,\langle2|4{+}5|6]\,\langle2|5{+}6|4]\,\langle3|4{+}5|6]}$$



Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$

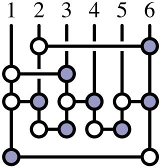
$$f_1 \equiv \oint_{(123)=0} \Omega_1 = \left.\frac{\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(234)(345)(456)(561)(612)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 23\rangle\,[56]\,\langle 3|4{+}5|6]s_{456}\langle 1|5{+}6|4]\langle 12\rangle\,[45]}$$



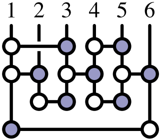
$$f_2 \equiv \oint_{(123)=0} \Omega_2 = \left.\frac{(235)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(136)(156)(234)(245)(256)(345)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\langle 23\rangle\,[64]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 13\rangle\,[45]\,\langle 1|5{+}6|4]\langle 23\rangle\,[56]\,\langle 2|4{+}5|6]\langle 2|5{+}6|4]\langle 3|4{+}5|6]}$$



$$f_3 \equiv \oint_{(123)=0} \Omega_4 = \left.\frac{(145)\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{(124)(136)(156)(245)(345)(456)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

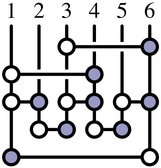
$$= \frac{\langle 1|4{+}5|6]\;\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\,\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 12\rangle\,[56]\,\langle 13\rangle\,[45]\,\langle 1|5{+}6|4]\langle 2|4{+}5|6]\langle 3|4{+}5|6]s_{456}}$$



Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$

$$f_4 \equiv \oint_{(123)=0} \Omega_5 = \left.\frac{(135)\,\delta^{3\times4}\big(C^{*}\!\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{(124)(145)(156)(236)(345)(356)}\right|_{C^{*}} \qquad C^{*} \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

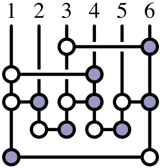
$$= \frac{\langle 13\rangle\,[64]\;\delta^{3\times4}\big(C^{*}\!\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 12\rangle\,[56]\,\langle 1|4{+}5|6]\langle 1|5{+}6|4]\langle 23\rangle\,[45]\,\langle 3|4{+}5|6]\langle 3|5{+}6|4]}$$



Enumeration of All (ten) 'Leading Singularities' of $G(3,6)$

$$f_4 \equiv \oint\limits_{(123)=0} \Omega_5 = \left.\frac{(135)\,\delta^{3\times4}\big(C^{*}\!\cdot\!\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\!\cdot\!\widetilde{\lambda}\big)}{(124)(145)(156)(236)(345)(356)}\right|_{C^{*}} \qquad C^{*} \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\langle 13\rangle\,[64]\;\delta^{3\times4}\big(C^{*}\!\cdot\!\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\!\cdot\!\widetilde{\lambda}\big)}{\langle 12\rangle\,[56]\,\langle 1|4{+}5|6]\langle 1|5{+}6|4]\langle 23\rangle\,[45]\,\langle 3|4{+}5|6]\langle 3|5{+}6|4]}$$



$$f_5 \equiv \oint\limits_{(123)=0} \Omega_9 = \left.\frac{(125)\,\delta^{3\times4}\big(C^{*}\!\cdot\!\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\!\cdot\!\widetilde{\lambda}\big)}{(134)(156)(245)(256)(16(25)\bigcap(34))}\right|_{C^{*}} \qquad C^{*} \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

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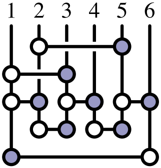

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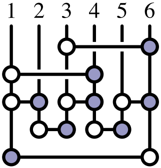

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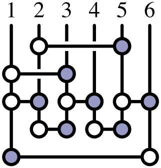
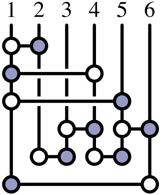


$$C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$f_6 \equiv \oint_{(123)=0} \Omega_{12} = \left.\frac{(134)^2(456)\,\delta^{3\times4}\big(C^{*}\!\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{(124)(145)(146)(156)(234)(345)(346)(356)}\right|_{C^*}$$

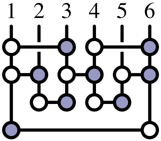
$$= \frac{\langle 13\rangle^2 s_{456}\,\delta^{3\times4}\big(C^{*}\!\cdot\widetilde{\eta}\big)\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 12\rangle\langle 1|4{+}5|6]\langle 1|4{+}6|5]\langle 1|5{+}6|4]\langle 23\rangle\langle 3|4{+}5|6]\langle 3|4{+}6|5]\langle 3|5{+}6|4]}$$



Enumeration of All (ten) ‘Leading Singularities’ of $G(3,6)$

$$f_7 \equiv \oint_{(123)=0} \Omega_{13} = \left.\frac{(145)^2\,\delta^{3\times4}\big(C^*\!\cdot\widetilde{\eta}\big)\,\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{(125)(134)(146)(156)(245)(345)(456)}\right|_{C^*} \qquad C^* \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 & \lambda_4^1 & \lambda_5^1 & \lambda_6^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 & \lambda_4^2 & \lambda_5^2 & \lambda_6^2 \\ 0 & 0 & 0 & [56] & [64] & [45] \end{pmatrix}$$

$$= \frac{\langle 1|4{+}5|6]^2\,\delta^{3\times4}\big(C^*\!\cdot\widetilde{\eta}\big)\,\delta^{2\times2}\big(\lambda\cdot\widetilde{\lambda}\big)}{\langle 12\rangle\,[64]\,\langle 13\rangle\,[56]\,\langle 1|4{+}6|5]\langle 1|5{+}6|4]\langle 2|4{+}5|6]\langle 3|4{+}5|6]s_{456}}$$



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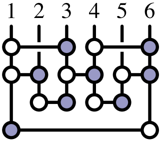
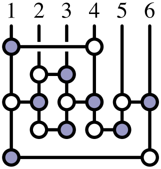


$$f_8 \equiv \oint\limits_{(14(23)\cap(56))=0} \Omega_{16} = \int \frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_8}{\alpha_8}\;\delta^{3\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}(\alpha)\big)$$

$$C(\alpha)\equiv\begin{pmatrix}1 & \alpha_6 & \alpha_6\,\alpha_7 & 0 & 0 & \alpha_1\\ 0 & 1 & \alpha_5{+}\alpha_7 & 0 & \alpha_2 & \alpha_2\,\alpha_4\\ \alpha_8 & 0 & 0 & 1 & \alpha_3 & \alpha_3\,\alpha_4\end{pmatrix}$$



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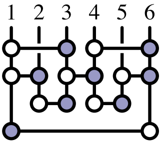
$$= \frac{\langle 1|4{+}5|6]^2\,\delta^{3\times4}(C^*\!\cdot\widetilde{\eta})\delta^{2\times2}(\lambda\cdot\widetilde{\lambda})}{\langle 12\rangle\,[64]\,\langle 13\rangle\,[56]\,\langle 1|4{+}6|5]\langle 1|5{+}6|4]\langle 2|4{+}5|6]\langle 3|4{+}5|6]s_{456}}$$



$$f_8 \equiv \oint\limits_{(14(23)\bigcap(56))=0} \Omega_{16} = \int \frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_8}{\alpha_8}\;\delta^{3\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}(\alpha)\big)$$

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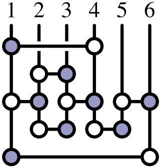
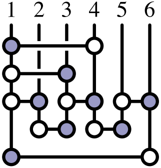


$$f_9 \equiv \oint\limits_{(14(23)\bigcap(56))=0} \Omega_{18} = \int \frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_8}{\alpha_8}\;\delta^{3\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}(\alpha)\big)$$

$$C(\alpha) \equiv \begin{pmatrix} 1 & \alpha_5 & \alpha_7 & 0 & 0 & \alpha_1 \\ 0 & 1 & \alpha_4 & 0 & \alpha_2 & \alpha_2\,\alpha_6 \\ \alpha_8 & 0 & 0 & 1 & \alpha_3 & \alpha_3\,\alpha_6 \end{pmatrix}$$



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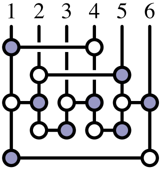
$$f_{10} \equiv \oint_{z=0} \Omega_{20} = \int \frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_8}{\alpha_8}\; \delta^{3\times4}\big(C(\alpha)\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C(\alpha)\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}(\alpha)\big)$$

$$C(\alpha) \equiv \begin{pmatrix} \alpha_6\,\alpha_8 & \alpha_1 & 1 & \alpha_6 & \alpha_1\,\alpha_7 & 0 \\ \alpha_8 & 0 & 0 & 1 & \alpha_5 & \alpha_4 \\ \alpha_3 & \alpha_2 & 0 & 0 & \alpha_2\,\alpha_7 & 1 \end{pmatrix}$$



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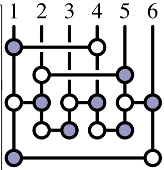

On-Shell Physics/Grassmannian Geometry Correspondence

$$f_\Gamma \equiv \prod_i \Big(\sum_{h_i, q_i} \int d^3\mathrm{LIPS}_i \Big) \prod_v \mathcal{A}_v \equiv \int \Omega_C \; \delta(C, p, h)$$

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On-Shell Physics

- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
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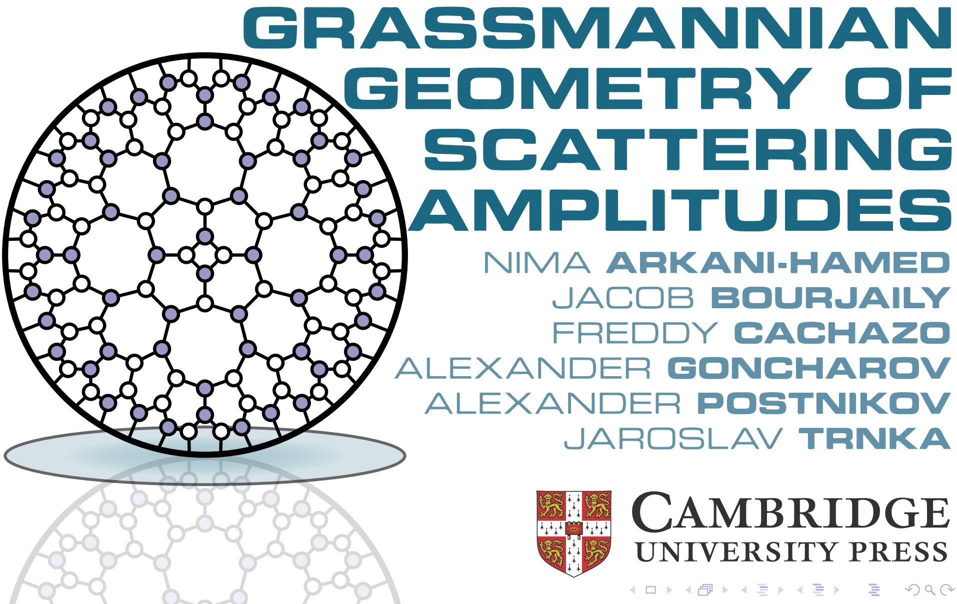
  - do these extend to entire *amplitudes*?
- do loop-level recursion relations exist?



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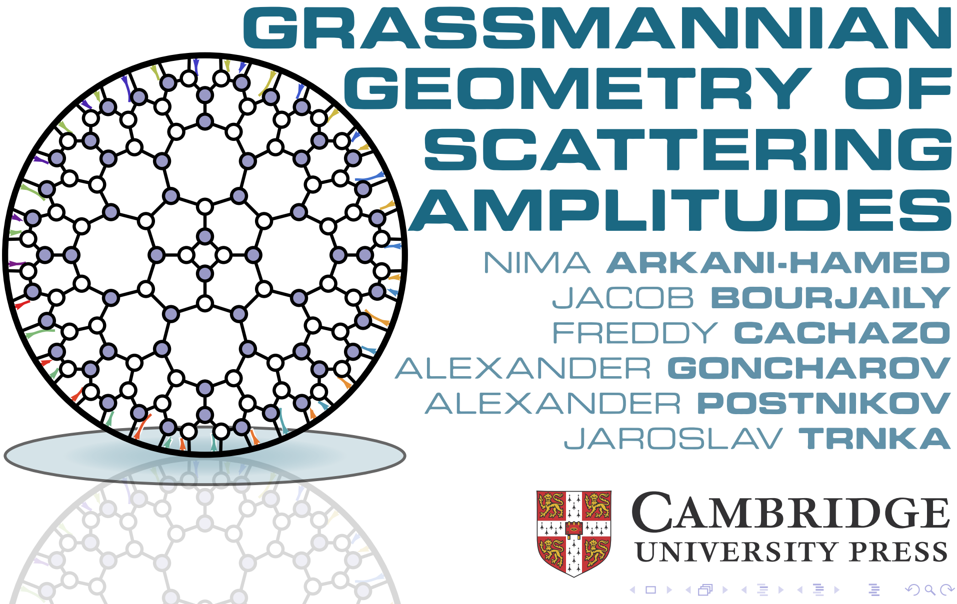

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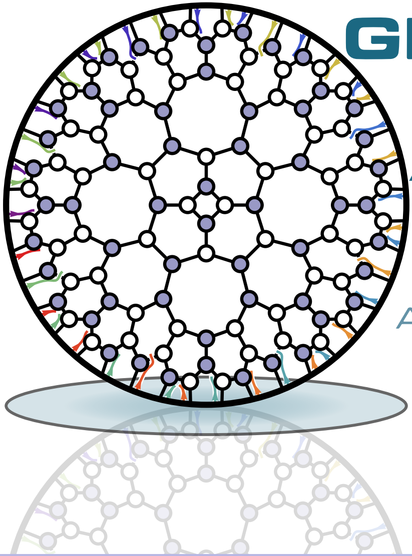

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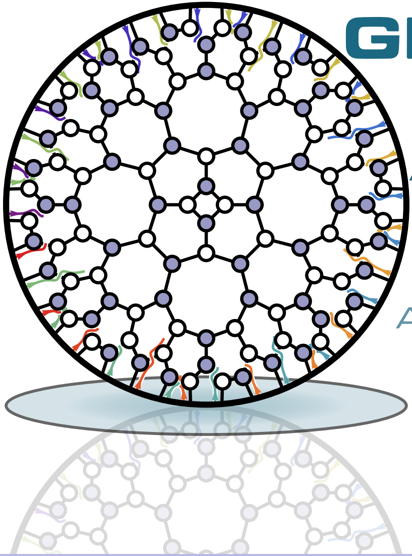

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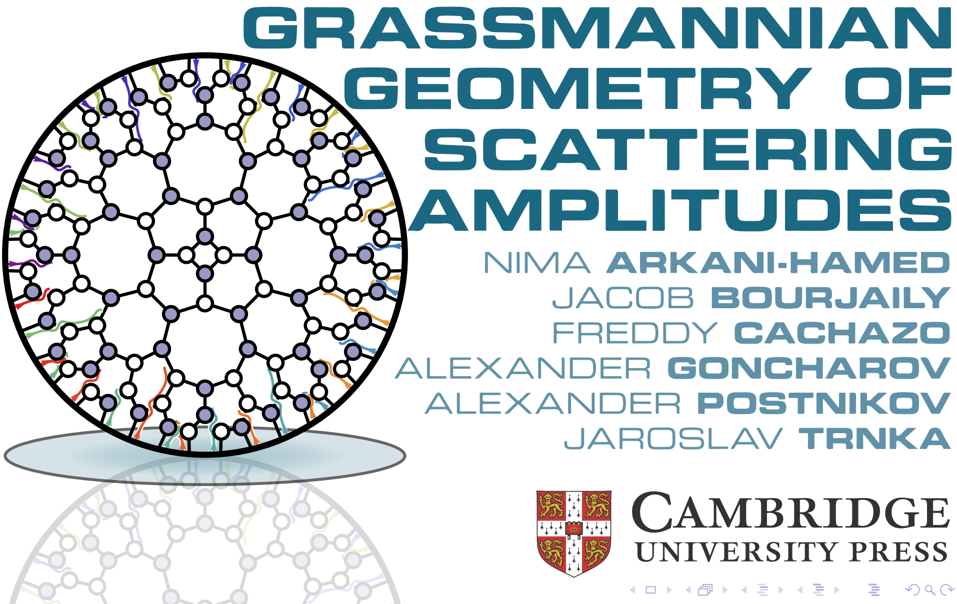

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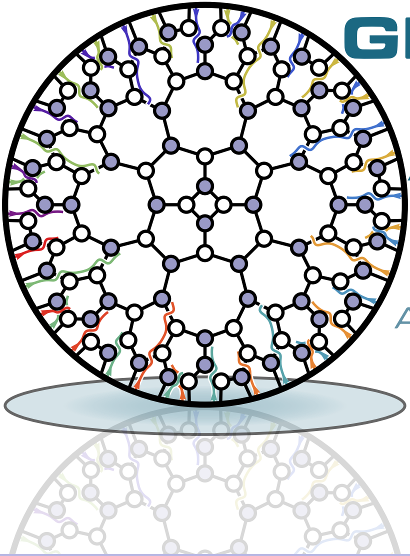

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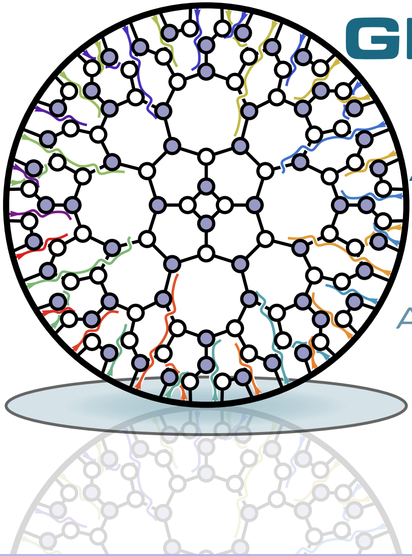

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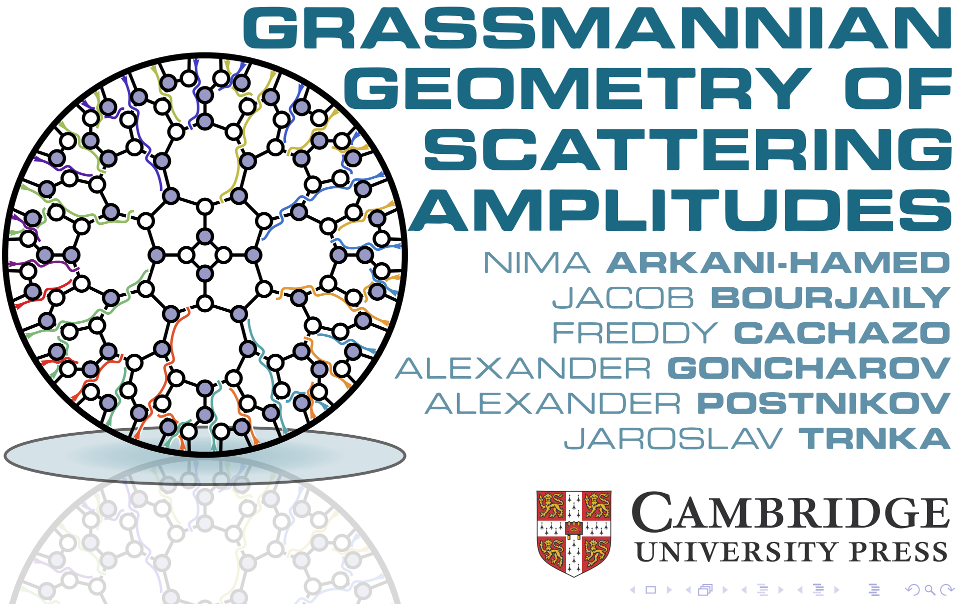

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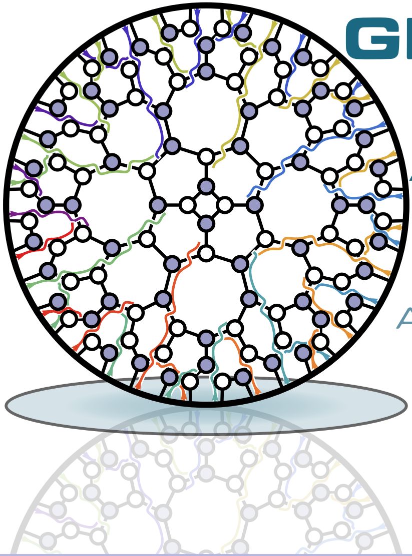

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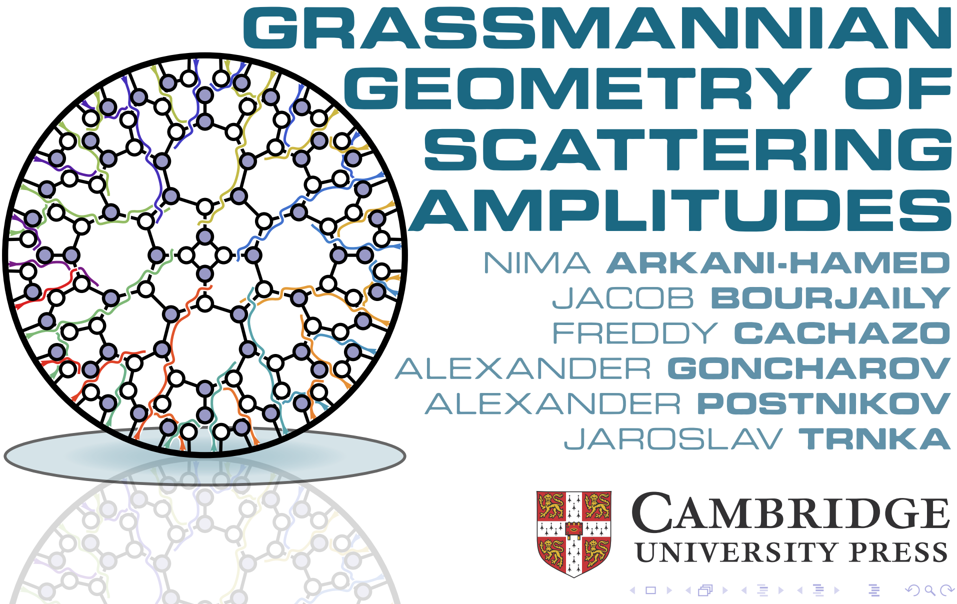

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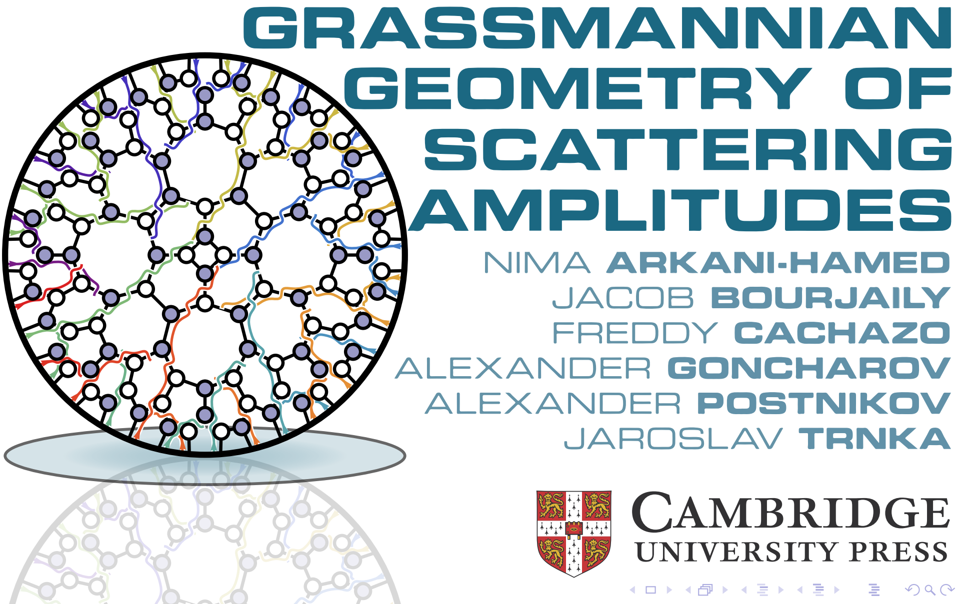

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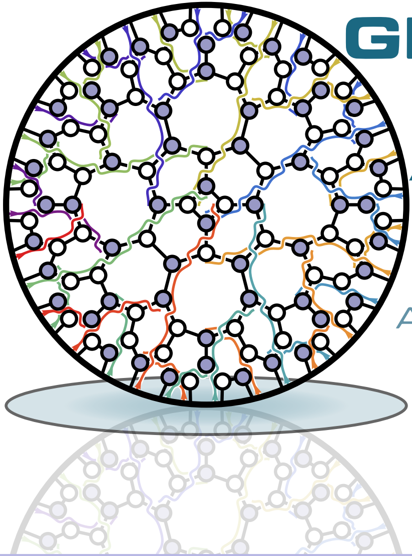

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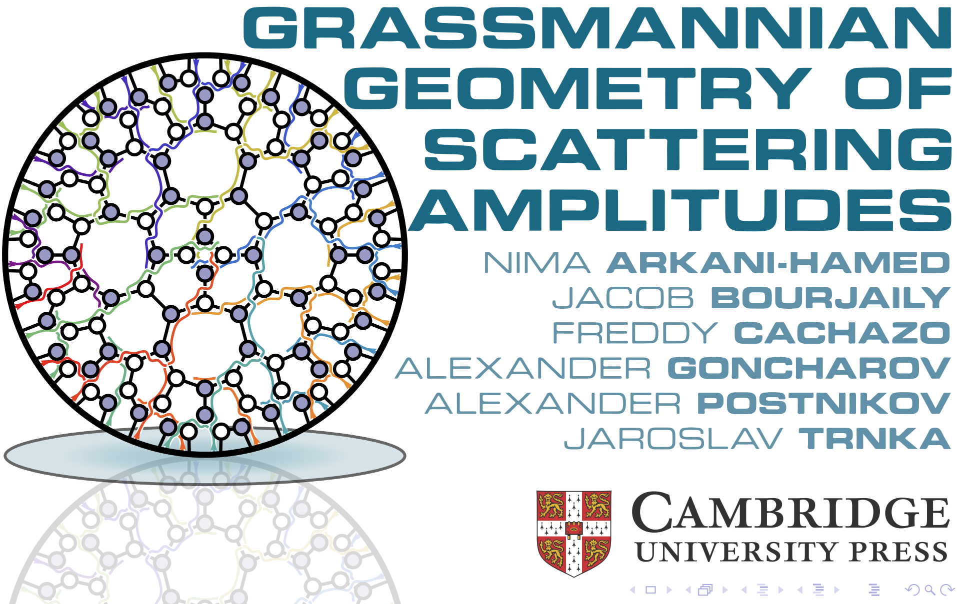

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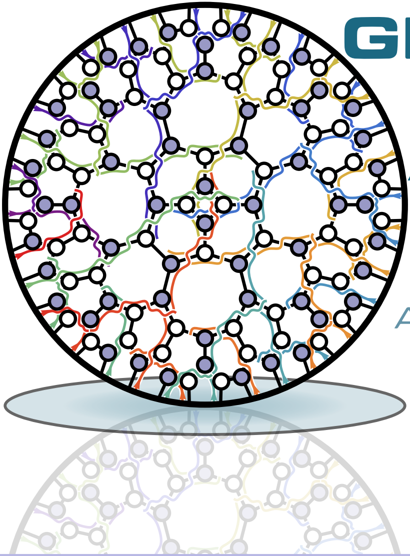

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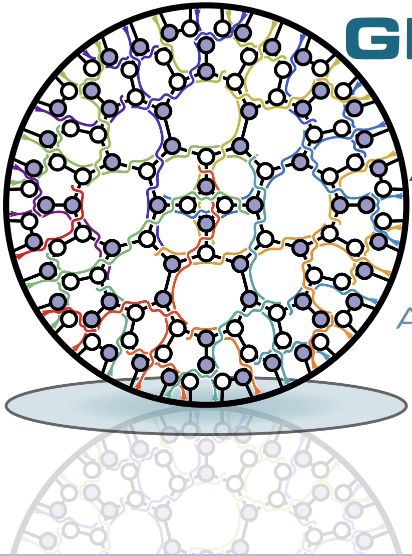

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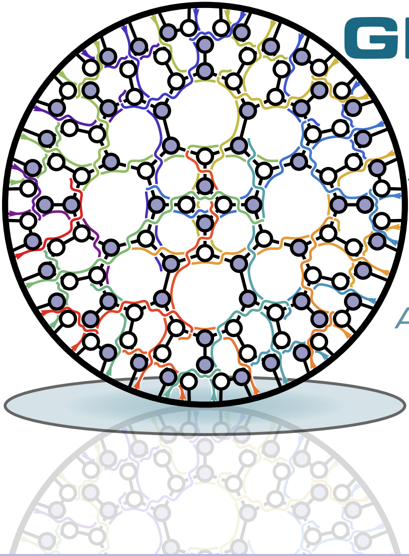

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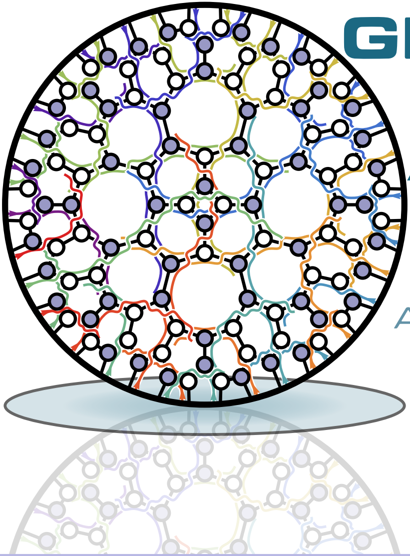

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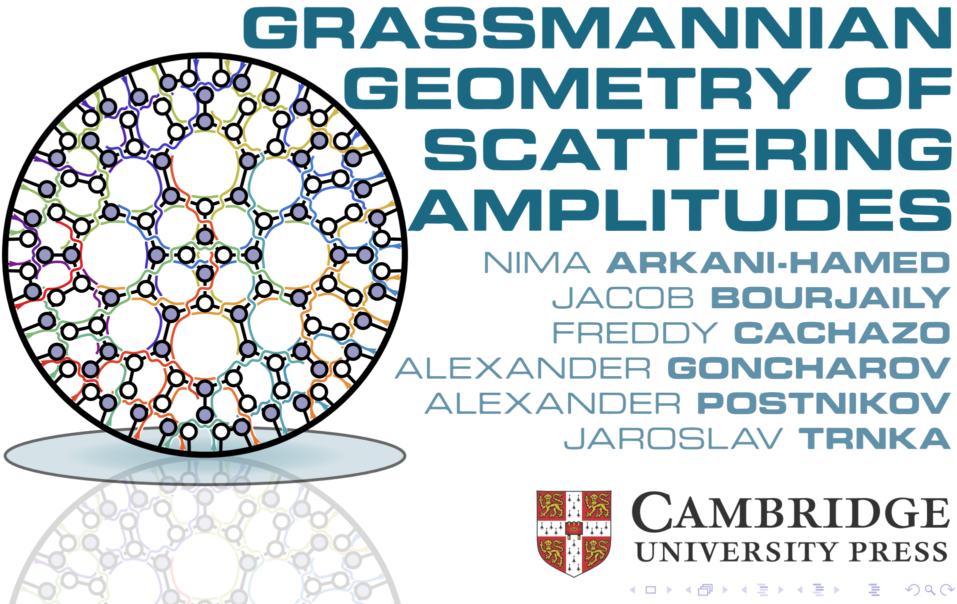

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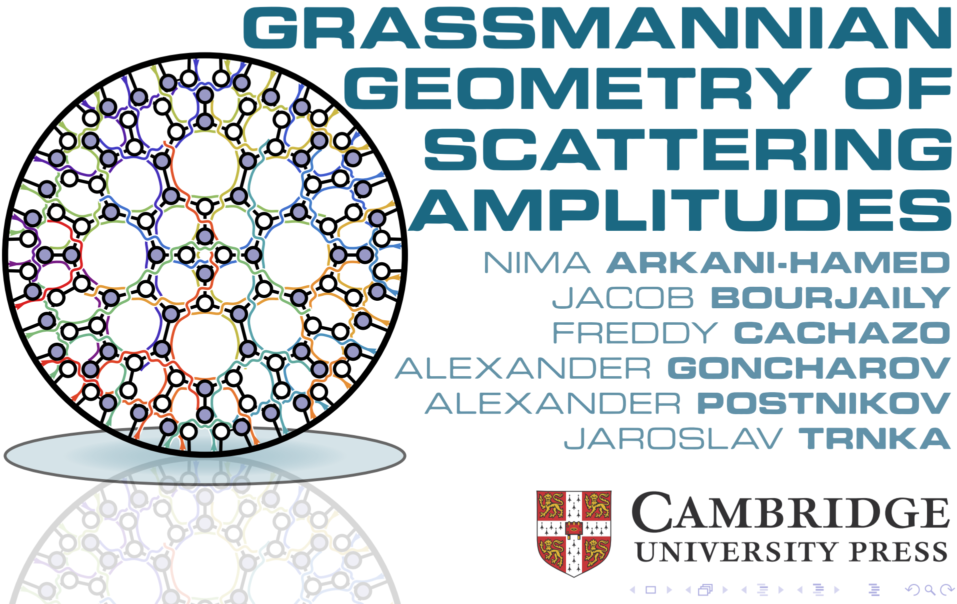

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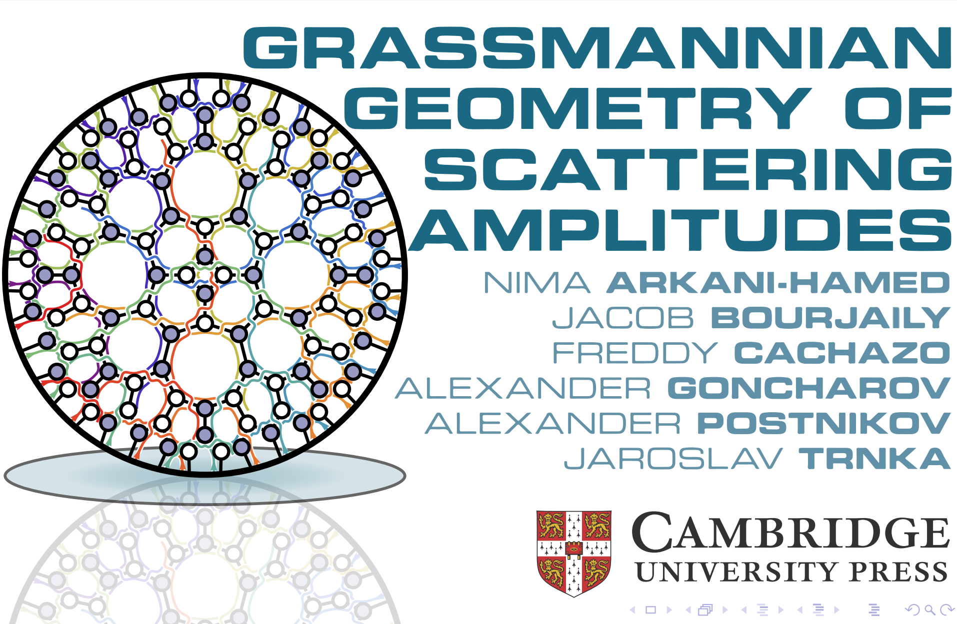

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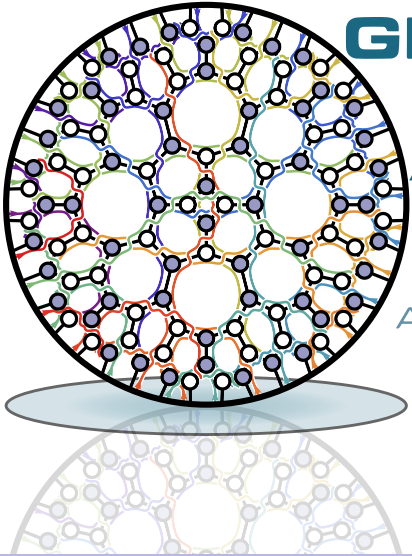

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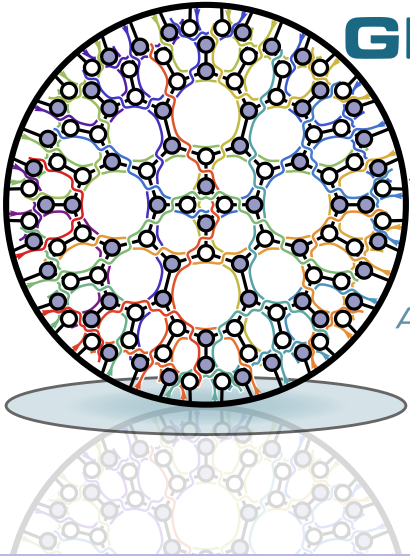

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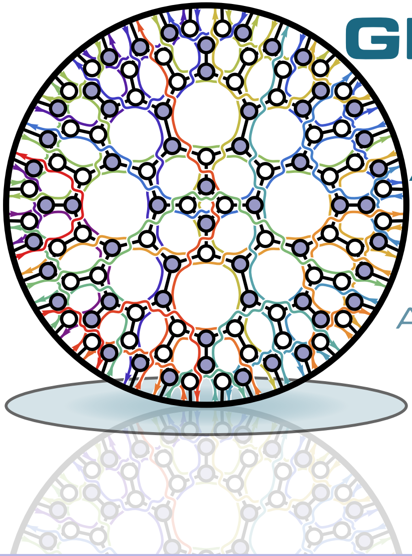

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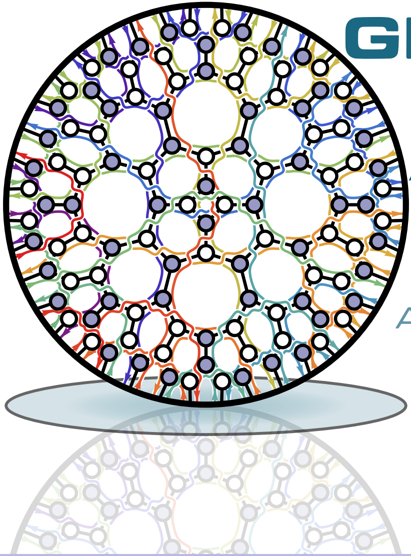

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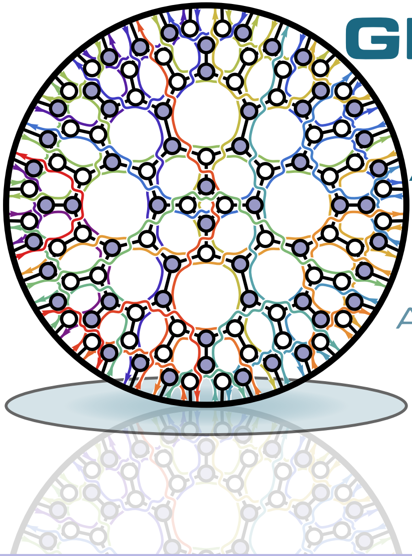

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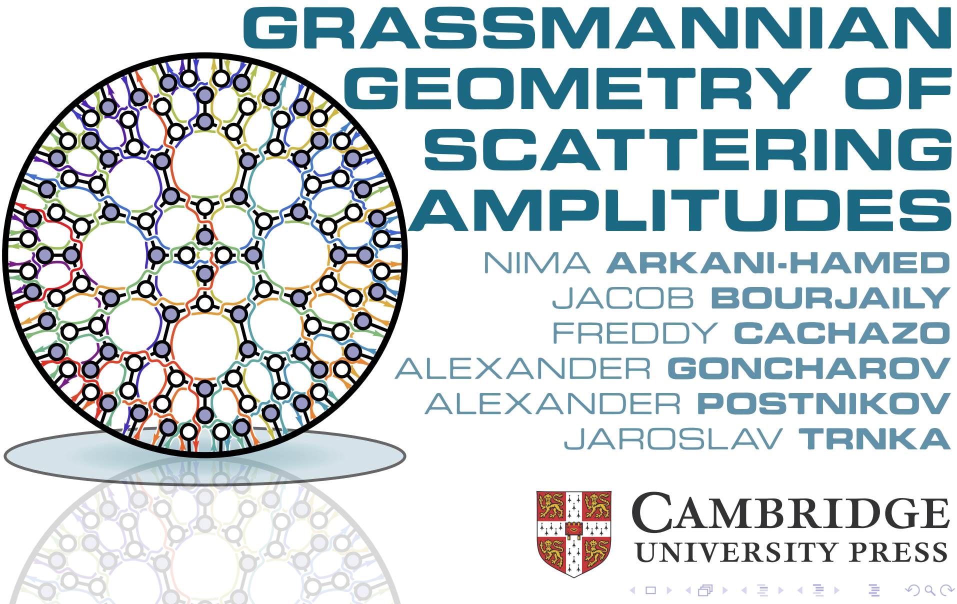

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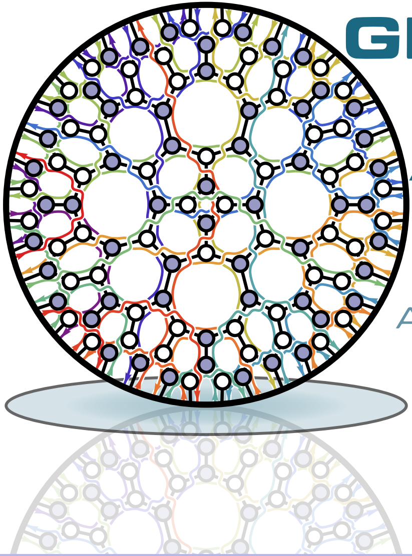

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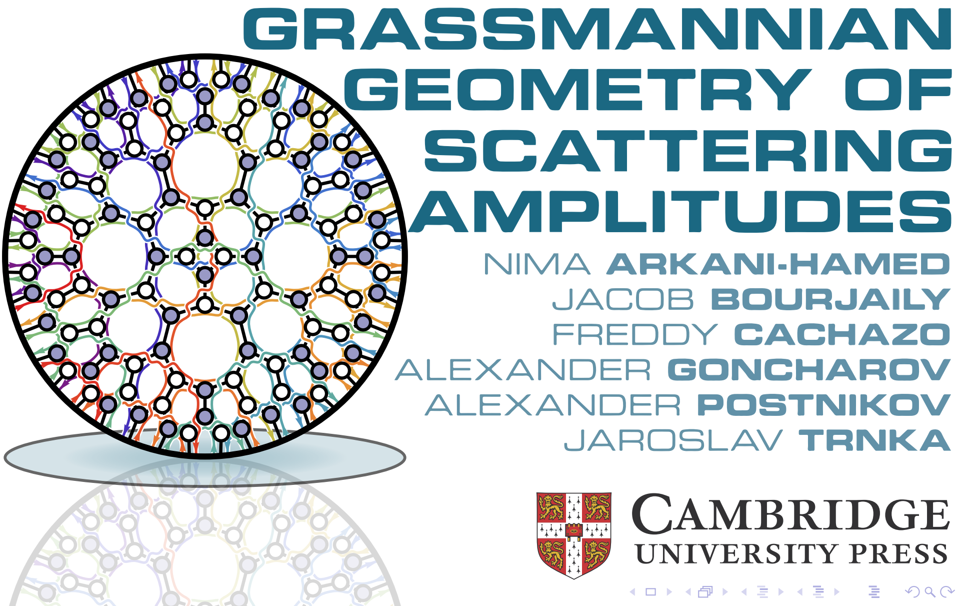

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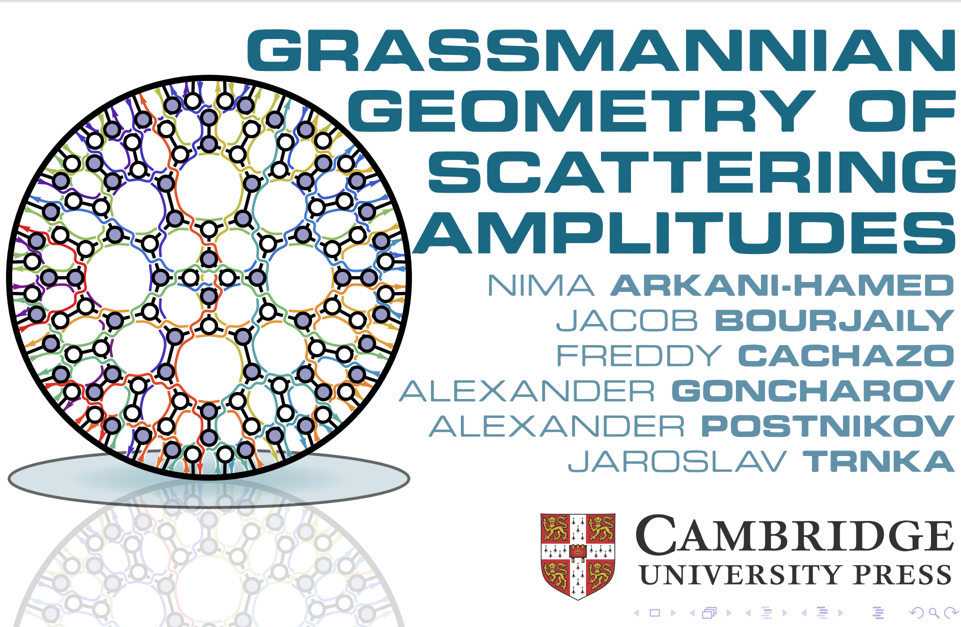

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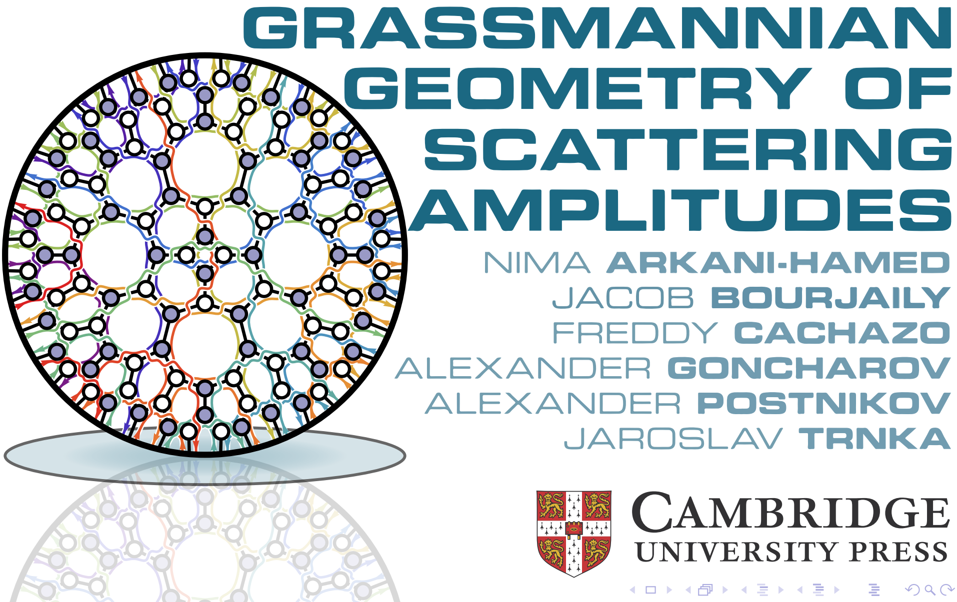

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