

The Vernacular of the S-Matrix

Jacob L. Bourjaily

Lecce, Italy

International School on Amplitudes and Cosmology,
Holography and Positive Geometries



The Niels Bohr
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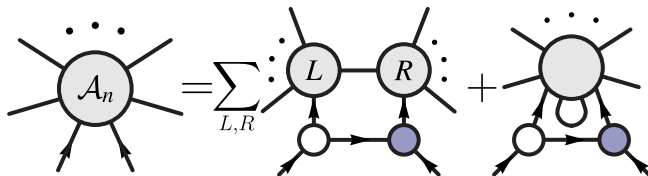
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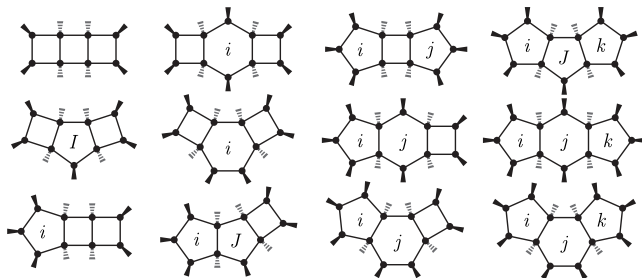
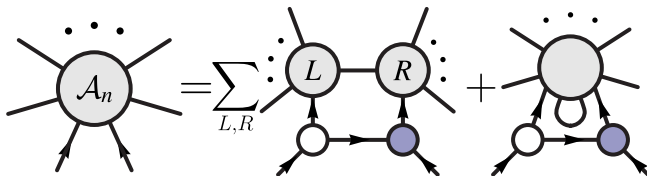
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Enormous Advances in Understanding Scattering Amplitudes

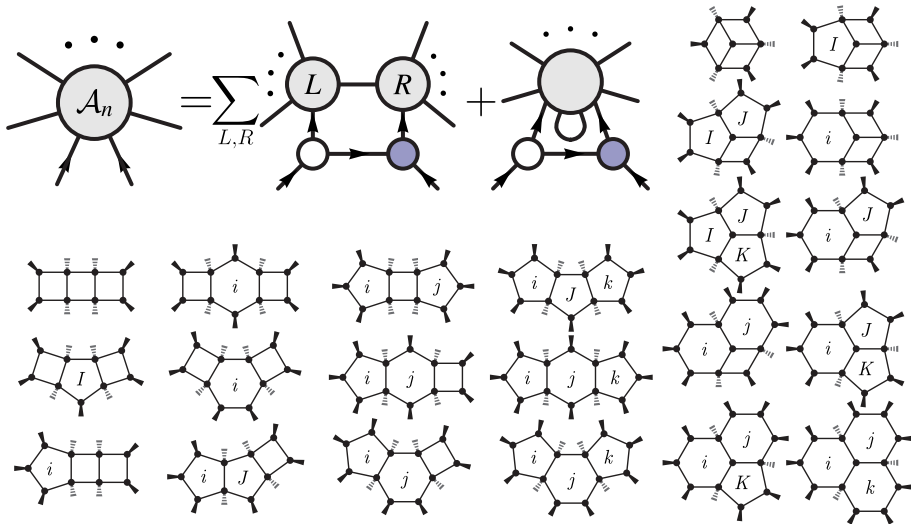
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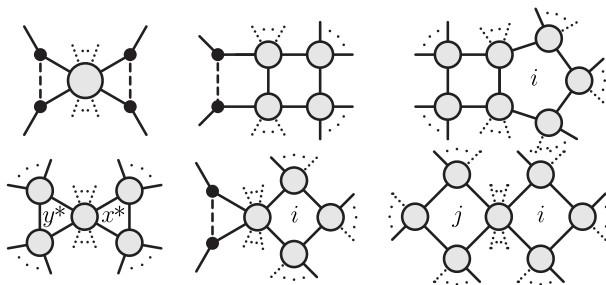
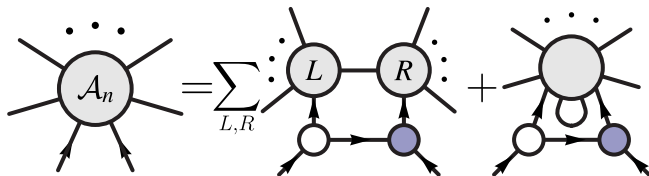
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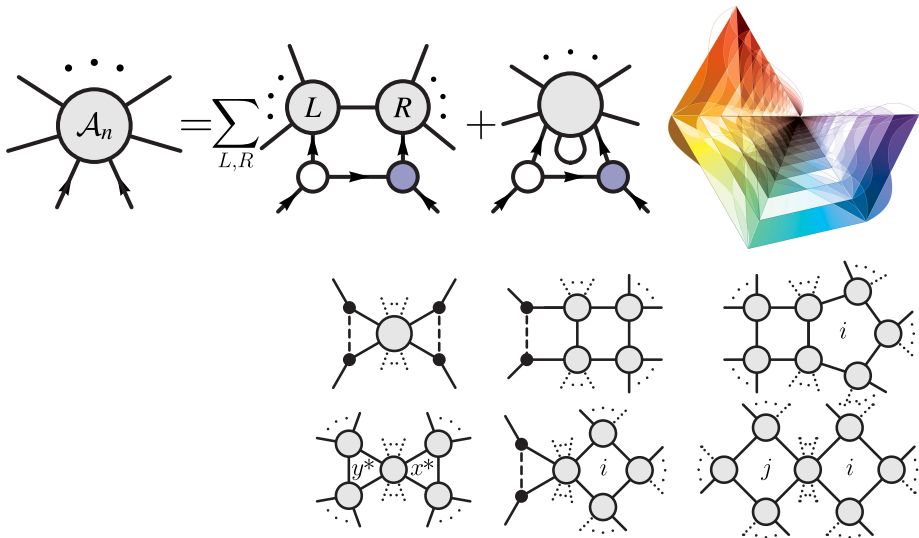
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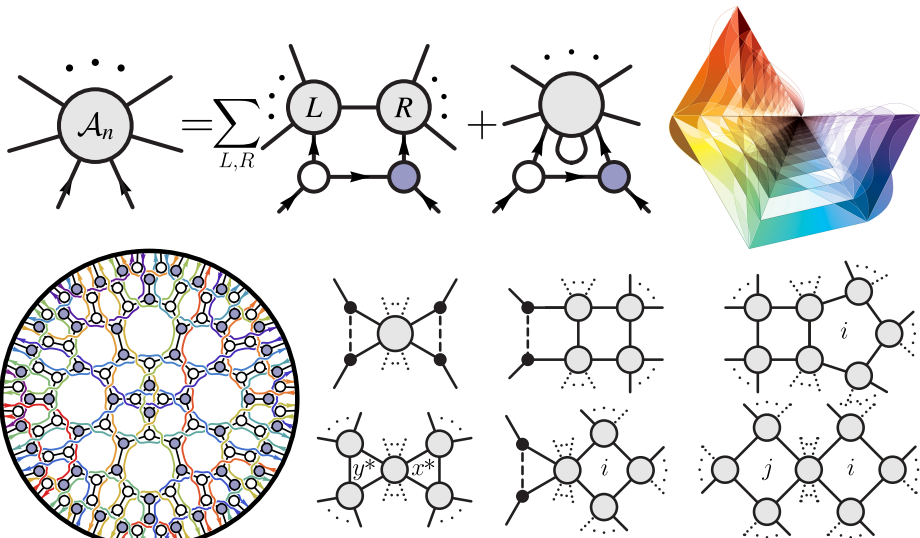
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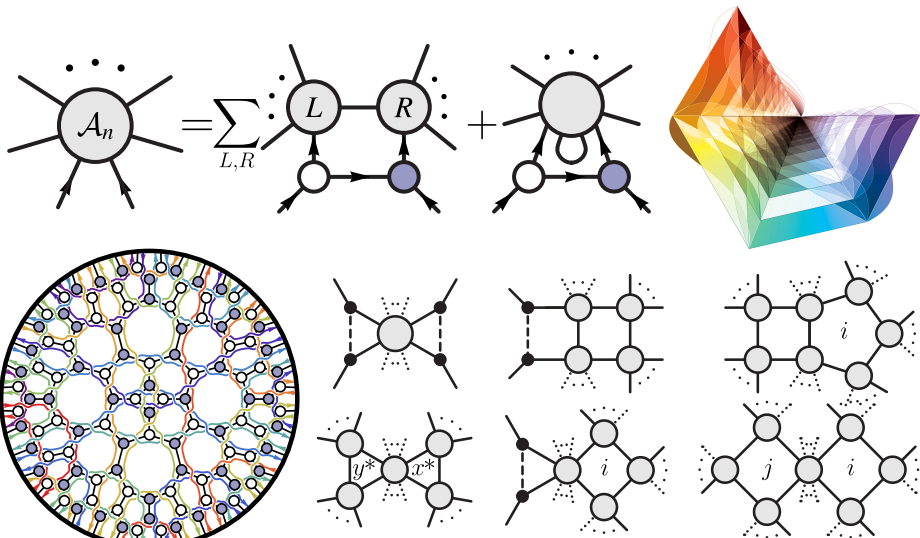
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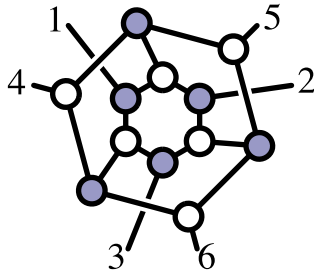


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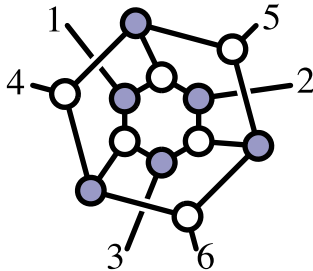
On-Shell Physics/Grassmannian Geometry Correspondence

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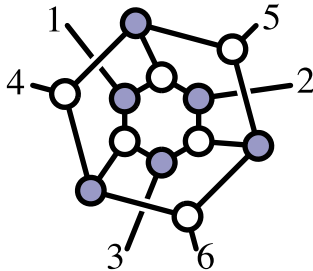
On-Shell Physics/Grassmannian Geometry Correspondence

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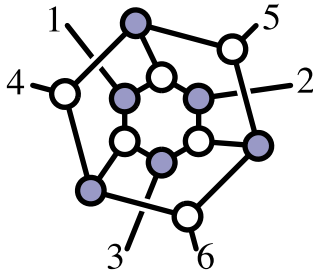
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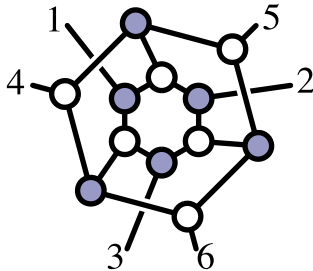
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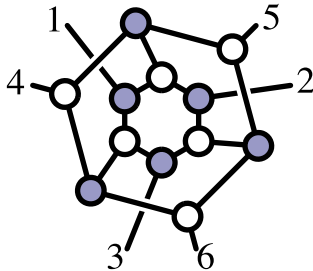
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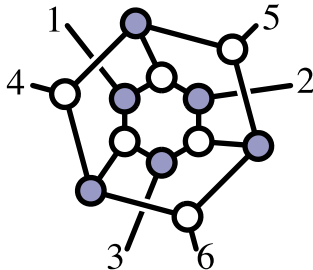
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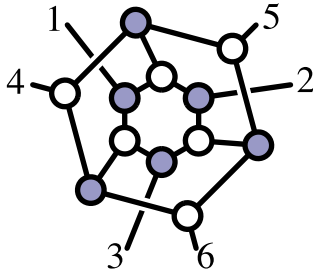
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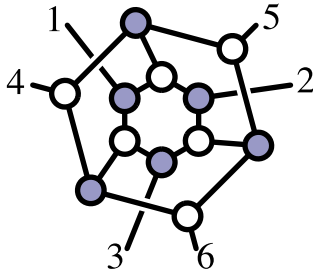
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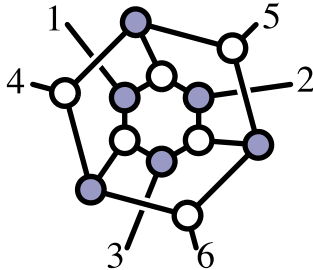
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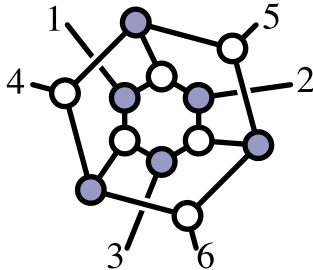
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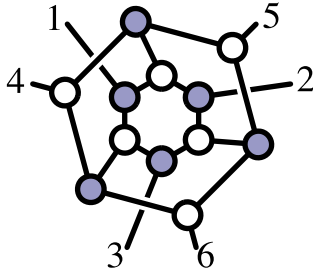
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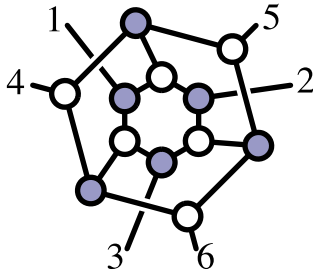
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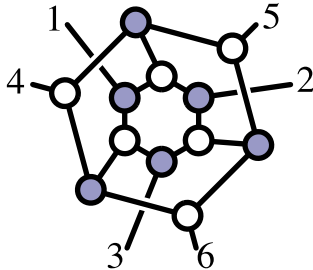
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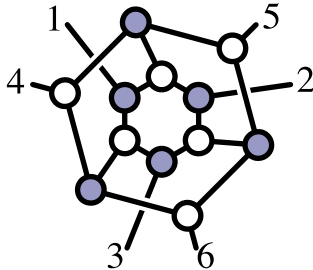
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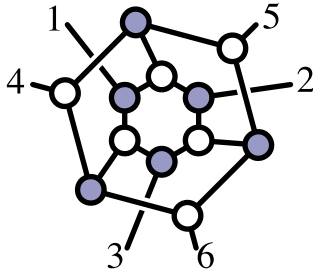
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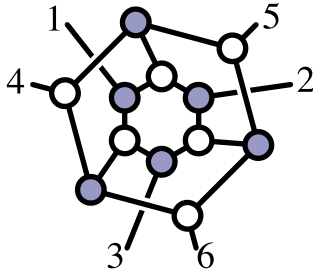
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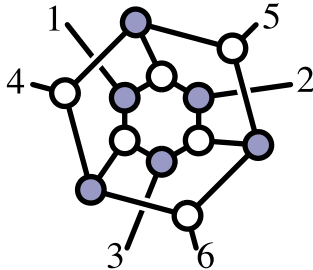
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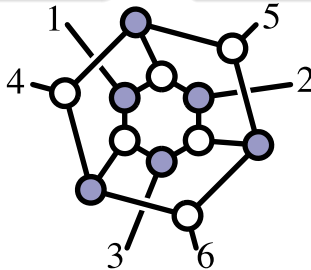


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On-Shell Physics

Grassmannian Geometry



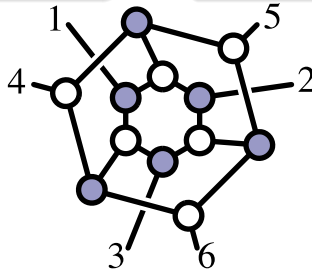
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On-Shell Physics

- on-shell diagrams

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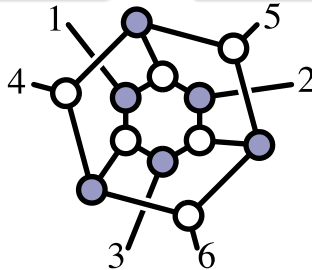
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Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }



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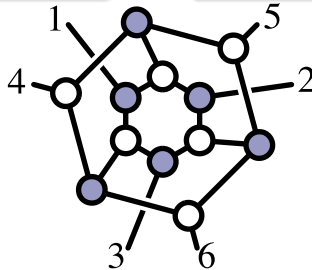
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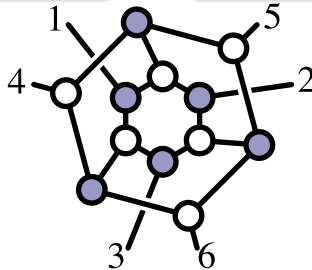
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Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms



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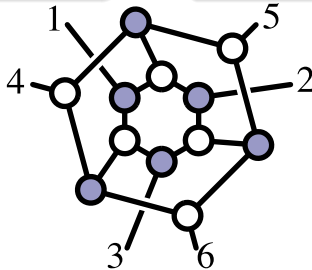
On-Shell Physics

- on-shell diagrams
- physical symmetries



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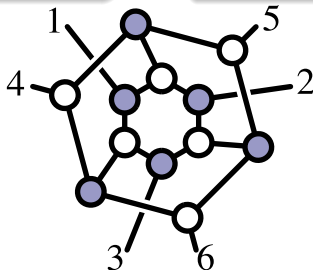
On-Shell Physics

- on-shell diagrams
- physical symmetries
 - trivial symmetries (identities)



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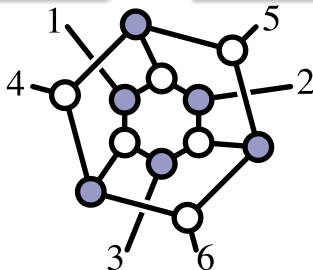
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Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms
 - cluster coordinate mutations



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On-Shell Physics: planar $\mathcal{N}=4$

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Grassmannian Geometry

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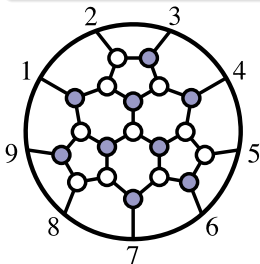
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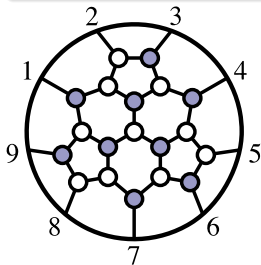
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 - bi-colored
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Grassmannian Geometry

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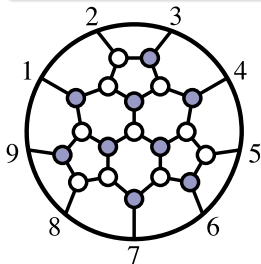
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 - bi-colored, **undirected**
- physical symmetries
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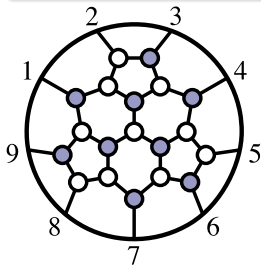
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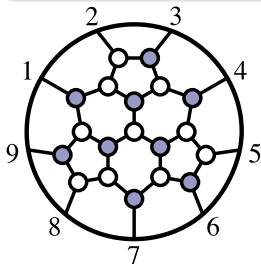
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$$C \equiv \begin{pmatrix} 1 & \alpha_8 & \alpha_5 + \alpha_8 \alpha_{14} & \alpha_5 \alpha_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_{10} & \alpha_4 + \alpha_{10} \alpha_{13} & \alpha_4 \alpha_7 & 0 & 0 \\ \alpha_3 \alpha_9 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_3 + \alpha_6 \alpha_{12} \\ \alpha_9 & 0 & \alpha_1 & \alpha_1 \alpha_{11} & 0 & \alpha_1 \alpha_2 & \alpha_1 \alpha_2 \alpha_7 & 0 & 1 \end{pmatrix}$$

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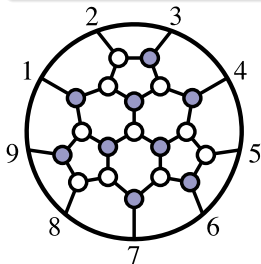
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$$\Omega_C \equiv \left(\frac{d\alpha_1}{\alpha_1} \wedge \cdots \wedge \frac{d\alpha_{14}}{\alpha_{14}} \right)$$

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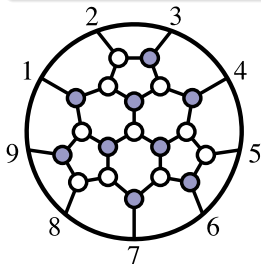
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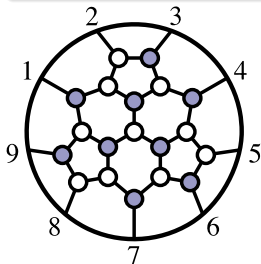
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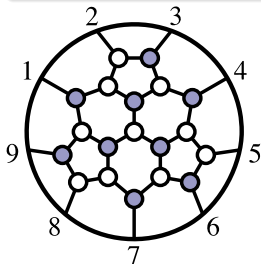
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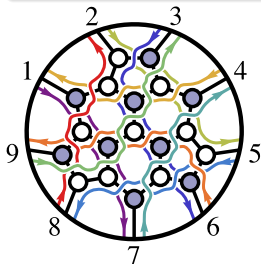
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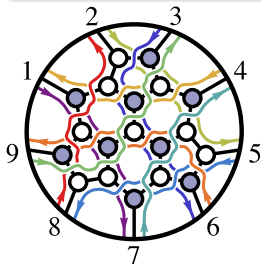
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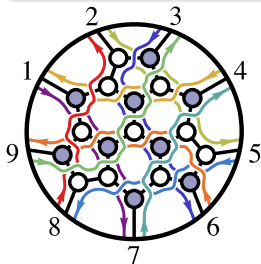
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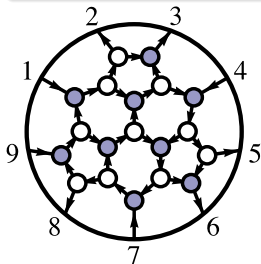
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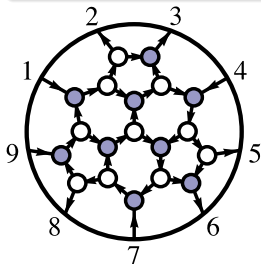
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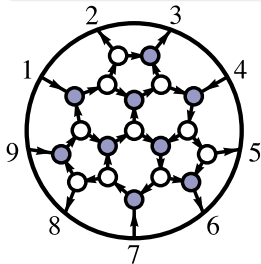
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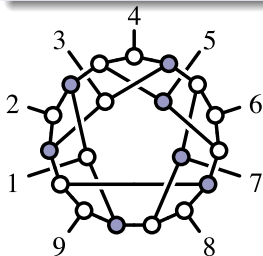
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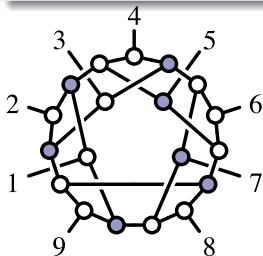
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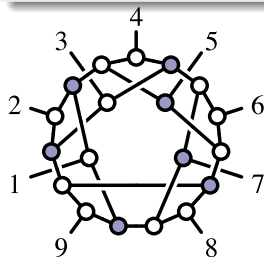
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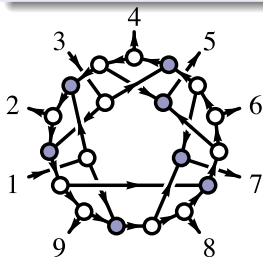
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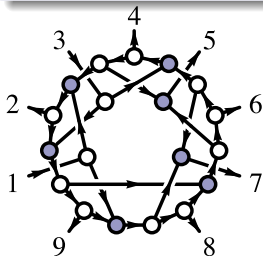
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 - trivial symmetries (identities)



Grassmannian Geometry

- {strata $C \in G(k, n)$, volume-form Ω_C }
- volume-preserving diffeomorphisms
 - cluster coordinate mutations

Important Open Questions (for math *and* physics)

- how many functions exist? (how to name them?)
- what (functional) relations do they satisfy?
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On-Shell Physics/Grassmannian Geometry Correspondence

$$f_{\Gamma} \equiv \prod_i \left(\sum_{h_i, q_i} \int d^3 \text{LIPS}_i \right) \prod_v \mathcal{A}_v \equiv \int \Omega_C \delta(C, p, h)$$

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Organization and Outline

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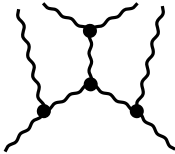
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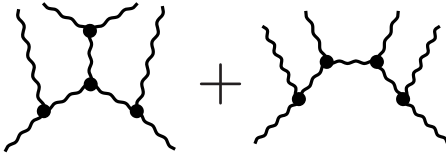
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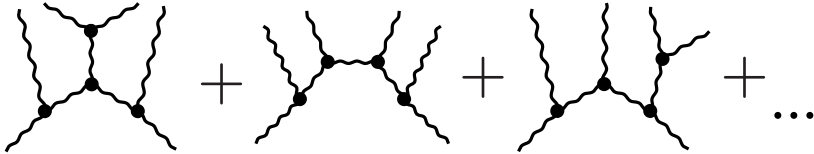
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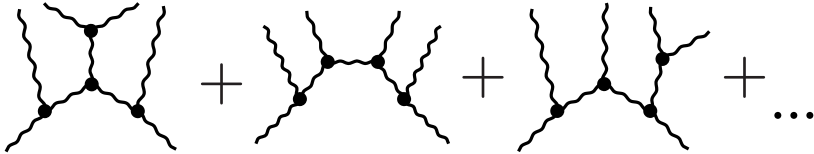
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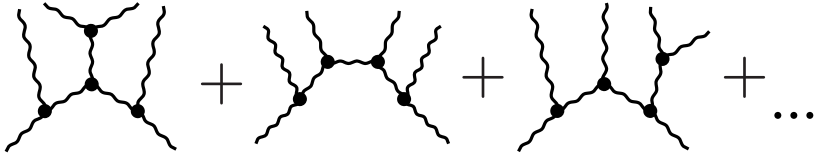
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Supercollider physics

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Eichten *et al.* summarize the motivation for exploring the 1-TeV ($\sim 10^{12}$ eV) energy scale in elementary particle interactions and explore the capabilities of proton-antiproton colliders with beam energies between 1 and 50 TeV. The authors calculate the production rates and characteristics for a number of conventional processes, and discuss their intrinsic physics interest as well as their role as backgrounds to more exotic phenomena. The authors review the theoretical motivation and expected signatures for several new phenomena which may occur on the 1-TeV scale. Their results provide a reference point for the choice of machine parameters and for experiment design.

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Eichten *et al.*: Supercollider physics

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TeV. From Fig. 76 we find the corresponding two-jet cross section (at $p_T = 0.5 \text{ TeV}/c$) to be about $7 \times 10^{-2} \text{ nb/cA}^2$, which is larger by an order of magnitude. Let us next consider the cross section in the neighborhood of the peak in Fig. 80J. The integrated cross section in the bin $0.3 \text{ (GeV/c)} < p_T < 0.4$ is approximately 6.1 nb/cA², with transverse energy given roughly by $\langle E_T \rangle = (1 \text{ TeV}) \times (\text{GeV/c}) = 385 \text{ GeV}$. The corresponding two-jet cross section, again from Fig. 76, is approximately 10 nb/cA², which is larger by 2 orders of magnitude. In fact, we have certainly underestimated $\langle E_T \rangle$ and thus somewhat overestimated the two-jet/three-jet ratio in this second run.

We draw two conclusions from this very casual analysis.

At least at small-to-moderate values of E_T , two-jet events should account for most of the cross section.

The three-jet cross section is large enough that a detailed study of this topology should be possible.

$$\sigma(E_T) \geq \int_{E_T}^{E_T + \Delta E_T} dE_T' \int_{E_T'}^{E_T' + \Delta E_T'} dE_T'' \frac{\sigma(E_T' E_T'')}{E_T' E_T''} (E_T' E_T'' - E_T - E_T') \quad (5.47)$$

where $\sigma(E_T)$ is the two-jet cross section and Δ denotes the minimum E_T required for a discernable two-jet event. For a recent study of double parton scattering at NPS and Tevatron energies, see Rivet and Triandafilidis (1983).

In view of the precision that multiple spectroscopy holds, improving our understanding of the QCD background is an urgent priority for further study.

D. Summary

We conclude this section with a brief summary of the range of jet energy which are accessible for various beam energies and luminosities. We find essentially no differences between pp and $p\bar{p}$ collisions, so only pp results will be given except at $\sqrt{s} = 2.3 \text{ TeV}$ where $p\bar{p}$ runs are expected. Figure 30A shows the E_T range which can be exploited at the level of at least one event per cA² of E_T per unit capacity at 90° in the s.s. (compare Figs. 77–79 and 83). The results are presented in terms of the transverse energy per event E_T , which corresponds to twice the transverse momentum p_T of a jet. In Fig. 105 we plot the values of E_T that distinguish the regions in which the two-photon, quark-gluon, and quark-quark final states are dominant. Comparing with Fig. 106, we find that while the accessible ranges of E_T are impressive, it seems extremely difficult to obtain a clean sample of quark jets. Useful for us is an interesting figure in the total cross section for two jets to interact $\sigma_{2j} = 2p_T^2 / (s - p_T^2)$. For both pp and $p\bar{p}$ at a rapidity interval of -2.5 to $+2.5$. This is shown for pp collisions in Fig. 30B.

It is apparent that these questions are amenable to detailed investigation with the aid of realistic Monte Carlo simulations. Given the elementary two–three cross sections and reasonable parameterizations of the fragmentation functions, this analysis can be carried out with some degree of confidence.

For multi-jet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their non-perturbative regions. The cross sections for the elementary two–four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

Another background source of four-jet events is double parton scattering, as shown in Fig. 103. If all the parton momentum fractions are small, the two interactions may be treated as uncorrelated. The resulting four-jet cross section with transverse energy E_T may then be approximated by

N. ELECTROWEAK PHENOMENA

In this section we discuss the supercollider processes associated with the standard model of the weak and electromagnetic interactions (Glashow, 1961; Weinberg, 1967; Salam, 1968). By “standard model” we understand the SU(3) \times SU(2) \times U(1) theory applied to three quark and lepton doublets, and with the gauge symmetry broken by a single complex Higgs doublet. The particles associated with the electroweak interactions are therefore the left-handed charged intermediate bosons W^\pm , the neutral intermediate

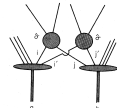


FIG. 30. Four-jet topology arising from two independent parton interactions.

Phys. Med. Phys., Vol. 16, No. 4, October 1983

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For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their leptonic decay. The cross sections for the elementary two- \rightarrow -four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.

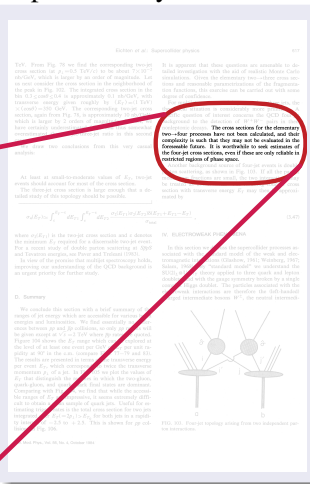
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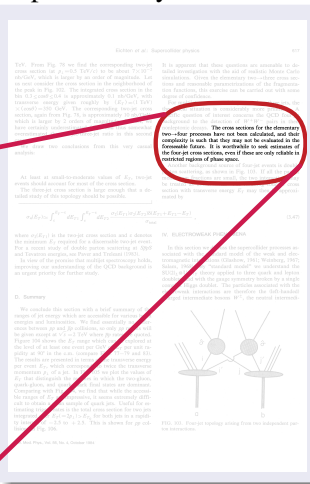


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gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $l = 2$ and replacing the momenta p_3, p_4, p_5, p_6 by $-p_3, -p_4, -p_5, -p_6$.

As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$A_{\Omega}^2(p_1, p_2, p_3, p_4, p_5, p_6) = (\mathcal{B}^T, \mathcal{B}_1^T, \mathcal{B}_2^T, \mathcal{B}_3^T, \mathcal{B}_4^T) \begin{pmatrix} K & K_c & K_u & K_v \\ K_c & K & K_c & K_c \\ K_c & K_c & K & K_c \\ K_c & K_c & K_c & K \end{pmatrix} \begin{pmatrix} \mathcal{B} \\ \mathcal{B}_1 \\ \mathcal{B}_2 \\ \mathcal{B}_3 \\ \mathcal{B}_4 \end{pmatrix}, \quad (6)$$

where $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and \mathcal{B}_4 are 11-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5 and p_6 , and K, K_c, K_u and K_v are constant 11×11 symmetric matrices. The vectors $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ and \mathcal{B}_4 are obtained from the vector \mathcal{B} by the permutations $(p_3 \leftrightarrow p_4), (p_5 \leftrightarrow p_6)$ and $(p_3 \leftrightarrow p_4, p_5 \leftrightarrow p_6)$, respectively, of the momentum variables in \mathcal{B} . The individual components of the vector \mathcal{B} represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K , which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N^2(N^2 - 1)$ and $N^2(N^2 - 1)$, respectively (N is the number of colors, $N = 3$ for QCD):

$$K = \frac{1}{2} g^4 N^2 (N^2 - 1) K^{(12)} + \frac{1}{2} g^4 N^2 (N^2 - 1) K^{(31)}, \quad (7)$$

Here g denotes the gauge coupling constant. The matrices $K^{(12)}$ and $K^{(31)}$ are given in table 1. The vector \mathcal{B} is related to the thirty-three diagrams $D^i (i = 1-33)$ for two-gluon to four-scalar scattering, eleven diagrams $D^i (i = 1-11)$ for two-fermion to four-scalar scattering and sixteen diagrams $D^i (i = 1-16)$ for two-scalar to four-scalar scattering, in the following way:

$$\begin{aligned} \mathcal{B}_0 &= \frac{2g_s}{\sqrt{(1+p_3^2)(1+p_4^2)(1+p_5^2)(1+p_6^2)}} \{ \frac{1}{2} C^{(12)} \cdot D_0^2 - 4i s_{123} E(p_1 + p_2, p_3) C^{(1)} \cdot D_0^4 \\ &\quad - 2i s_{14} G(p_1 + p_2, p_3 + p_4) C^{(1)} \cdot D_0^2 \}, \\ \mathcal{B}_1 &= \frac{2g_s}{2p_3} C^{(12)} \cdot D_1^2, \end{aligned} \quad (8)$$

where the constant matrices $C^{(12)} (11 \times 33)$, $C^{(1)} (11 \times 11)$ and $G^{(1)} (11 \times 16)$ are given in table 2. The Lorentz invariants s_{ij} and t_{ij} are defined as $s_{ij} = (p_i + p_j)^2$, $t_{ij} = (p_i + p_j + p_k)^2$ and the complex functions E and G are given by

$$\begin{aligned} E(p_1, p_2) &= \frac{1}{2} \{ (p_1, p_2)(p_3, p_4) - (p_1, p_3)(p_2, p_4) - (p_1, p_4)(p_2, p_3) + i s_{123} p_1^2 p_2^2 p_3^2 p_4^2 / (p_1, p_2) \}, \\ G(p_1, p_2) &= E(p_1, p_2) E(p_3, p_4). \end{aligned} \quad (9)$$

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TABLE I
Matrix $K(i, j)$ ($i = 1-11, j = 1-11$)

Matrix $K^{(0)}$											Matrix $K^{(1)}$										
8	4	-2	-1	2	0	1	0	0	-1		0	0	0	0	0	0	0	0	0	3	-3
4	8	-1	-1	0	2	1	0	1	-1		0	0	0	0	0	0	0	0	0	3	3
-2	1	4	1	1	2	2	1	2			0	0	0	0	0	0	0	0	0	0	0
2	1	4	2	-1	-1	4	1	1			0	0	0	0	0	0	0	0	0	0	0
-1	1	4	2	8	1	2	4	-2	-1		0	0	0	0	0	0	0	0	0	0	0
0	0	-1	1	8	4	-1	0	1	0		0	0	0	0	0	0	0	0	0	3	3
0	2	1	2	4	4	-2	1	0	0		0	0	0	0	0	0	0	0	0	2	2
1	1	2	4	-1	-2	8	-1	-1	2		0	0	0	0	0	0	0	0	0	0	0
0	0	2	1	-2	0	0	-1	8	-2		2	3	0	0	3	3	0	0	0	0	0
0	1	1	-1	-1	0	-1	4	-8	-1		3	0	0	3	3	0	0	0	0	0	0
-1	-1	2	1	4	-8	2	-2	-1	8		-3	-3	0	-3	-3	0	0	0	0	0	0
Matrix $K^{(2)}$											Matrix $K^{(3)}$										
8	0	0	1	1	0	1	1	0	-1		3	3	0	3	0	4	3	0	0	0	0
0	8	0	2	0	1	1	2	1	-2		3	3	0	0	0	0	0	0	0	0	0
0	0	0	1	1	0	1	1	0	1		0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	2	0	0	1	0		0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	2	0	0	0	-1		0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	1	4	2	0	0	0		0	0	3	0	0	0	0	0	0	0	0
0	1	0	2	2	4	0	0	0	-2		0	0	0	0	0	0	0	0	0	0	0
1	0	2	4	0	0	0	0	0	0		0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	2	-1		0	0	3	3	3	3	0	0	0	0	0
0	1	1	0	0	0	0	2	4	0		0	0	2	0	3	3	0	0	0	0	0
-1	-2	1	0	2	-1	-2	0	-1	0		0	0	0	0	0	0	0	0	0	0	0
Matrix $K^{(4)}$											Matrix $K^{(5)}$										
4	3	0	0	1	0	1	0	0	0		0	0	0	0	0	0	0	0	2	0	0
2	4	0	1	0	0	1	1	0	1		0	0	0	0	0	0	0	0	3	0	0
0	0	4	2	1	1	2	1	0			0	0	0	0	0	0	0	0	3	0	0
1	2	2	0	2	1	0	1	0			0	0	0	0	0	0	0	0	0	3	0
0	0	2	0	0	0	0	4	2	0		0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	1	0			0	0	0	0	0	0	0	0	0	0	0
0	1	1	1	0	0	0	2	4	0		0	0	3	3	3	3	0	0	0	0	0
0	0	0	0	0	0	2	2	0			0	0	0	0	0	0	0	0	0	0	0
0	0	2	1	4	1	2	0	0	-4		3	0	0	0	0	0	0	0	0	0	0
0	0	1	2	2	4	0	0	0	-2		0	0	0	0	0	0	0	0	0	0	0
0	0	0	2	0	0	1	-4	-2	4		-3	0	0	0	-3	0	0	0	0	0	0
Matrix $K^{(6)}$											Matrix $K^{(7)}$										
8	1	-1	-1	1	1	0	1	2	0		3	3	0	0	3	3	0	0	0	0	0
1	8	2	0	1	1	-1	2	0			3	3	0	0	2	3	0	0	0	0	0
-1	2	8	0	1	1	-1	1	0			0	0	0	0	0	0	0	0	0	0	0
-1	2	0	8	0	2	1	0	-1			0	0	2	3	0	0	0	0	0	0	0
1	0	1	1	-1	1	-1	-2	-4	-1		3	3	0	0	3	3	0	0	0	-3	-3
0	1	1	-2	1	0	-1	4	-1			3	3	0	0	3	3	0	0	0	0	0
1	1	1	8	-2	-1	0	2	-2	0		0	0	2	3	3	0	0	0	0	0	0
2	4	-1	-2	2	4	2	1	0	-2		0	0	0	0	0	0	0	0	0	0	0
0	-2	-1	2	4	-2	0	0	0			0	0	0	0	0	0	0	0	0	3	3
0	0	0	0	1	-1	-1	0	-2	0		0	0	3	0	-3	0	0	0	0	0	0

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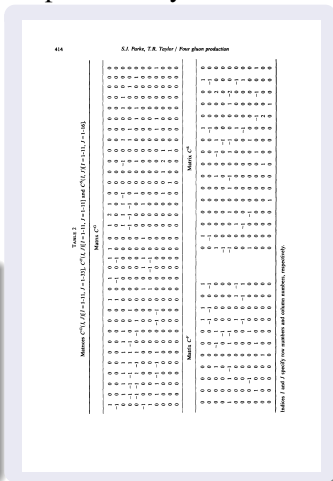
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where ϵ is the totally antisymmetric tensor, $\epsilon_{4444} = 1$. For the future use, we define one more function,

$$P(p_1, p_2) = \{(p_1, p_2)(p_3, p_4) + (p_1, p_3)(p_2, p_4) - (p_1, p_4)(p_2, p_3)\} / (p_1, p_2). \quad (30)$$

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E , F and G on the momenta p_1, p_2, p_3, p_4 .

The diagrams D_i^G are listed below:

$$D_1^G(1) = \frac{\delta_2}{s_{14}s_{23}s_{45}} \{[(p_1 - p_2)(p_3 - p_4)][(p_1 - p_4)(p_2 + p_3)] - [(p_2 - p_3)(p_1 + p_4)] \\ \times [(p_1 - p_4)(p_2 - p_3)] + [(p_2 + p_3)(p_1 - p_4)][(p_1 - p_4)(p_2 - p_3)]\},$$

$$D_2^G(2) = \frac{1}{s_{23}s_{45}} \{2E(p_2 - p_1, p_1 - p_2) - 2E(p_2 - p_4, p_2 - p_1) + \delta_2[(p_1 - p_2)(p_3 - p_4)]\},$$

$$D_3^G(3) = \frac{4}{s_{12}s_{34}s_{123}} \{[(p_1 + p_2 - p_3)(p_4 + p_3 - p_2)]E(p_2, p_1) \\ - [(p_1 + p_2 - p_3)(p_4 - p_3 + p_2)]E(p_2, p_4) \\ - [(p_1 - p_2 + p_3)(p_4 + p_3 - p_2)]E(p_2, p_1) \\ + [(p_1 - p_2 + p_3)(p_4 - p_3 + p_2)]E(p_2, p_4) \\ - [(p_1 - p_2 - p_3)]E(p_1 - p_4, p_3 + p_4) - [p_4(p_2 - p_3)]E(p_1 + p_3, p_1 - p_2) \\ + \delta_2(p_1(p_2 - p_3))[p_4(p_2 - p_3)]\},$$

$$D_4^G(4) = \frac{-2}{s_{45}s_{123}} \{E(p_2 - p_4, p_2 + p_4) - \delta_2 p_4(p_2 - p_4)\},$$

$$D_5^G(5) = \frac{-2}{s_{23}s_{133}} \{E(p_2 + p_4, p_1 - p_3) - \delta_2 p_4(p_2 - p_3)\},$$

$$D_6^G(6) = \frac{\delta_1}{t_{123}},$$

$$D_7^G(7) = \frac{4}{s_{12}s_{34}s_{123}} \{[(p_1 + p_2 - p_3)(p_4 + p_3 - p_2)]E(p_2, p_1) \\ - [(p_1 + p_2 - p_3)(p_4 - p_3 + p_2)]E(p_2, p_4) - [p_4(p_2 - p_3)]E(p_1 - p_2, p_3 - p_4)\},$$

$$D_8^G(8) = \frac{4}{s_{23}s_{123}s_{133}} \{[(p_1 + p_2 - p_3)(p_4 + p_3 - p_2)]E(p_2, p_1) \\ - [(p_1 - p_2 + p_3)(p_4 + p_3 - p_2)]E(p_2, p_4) - [p_1(p_2 - p_3)]E(p_1 - p_2, p_3)\},$$

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$$\begin{aligned} D_1^{(2)}(9) &= \frac{4}{s_{12}s_{34}s_{13}} [(p_1 - p_2 + p_3)(p_1 + p_3 - p_4)] E(p_4, p_3) \\ &\quad - [(p_1 - p_2 + p_3)(p_4 - p_3 + p_4)] E(p_3, p_4) + [(p_4)(p_3 - p_4)] E(p_4, p_3 - p_4), \\ D_1^{(2)}(10) &= \frac{4}{s_{12}s_{34}s_{13}} [(p_1 + p_2 - p_3)(p_4 - p_3 + p_4)] E(p_3, p_4) \\ &\quad - [(p_1 - p_2 + p_3)(p_4 - p_3 + p_4)] E(p_3, p_4) + [(p_1)(p_2 - p_3)] E(p_2 - p_3, p_4), \\ D_1^{(2)}(11) &= \frac{8s_3}{s_{12}s_{34}} [s_{12} - s_{34} + s_{34}], \\ D_1^{(2)}(12) &= \frac{-8s_3}{s_{12}s_{34}} [s_{12} - s_{34} - s_{34}], \\ D_1^{(2)}(13) &= \frac{8s_3}{s_{12}s_{34}s_{13}} [s_{12} - s_{34}] [s_{12} - s_{34} + s_{34}], \\ D_1^{(2)}(14) &= \frac{8s_3}{s_{12}s_{34}s_{13}} [s_{12} - s_{34}] [s_{12} - s_{34} - s_{34}], \\ D_1^{(2)}(15) &= \frac{8s_3}{s_{12}s_{34}} (p_1 - p_3)(p_3 - p_4), \\ D_1^{(2)}(16) &= \frac{-4}{s_{12}s_{34}s_{13}} [s_{12} - s_{34} + s_{34}] E(p_2, p_3), \\ D_1^{(2)}(17) &= \frac{4}{s_{12}s_{34}s_{13}} [s_{12} - s_{34} - s_{34}] E(p_2, p_3), \\ D_1^{(2)}(18) &= \frac{-4}{s_{12}s_{34}s_{13}} [2(p_1 + p_2)(p_3 - p_4) - s_{12}] E(p_3, p_4), \\ D_1^{(2)}(19) &= \frac{-2}{s_{12}s_{34}} E(p_2, p_3 - p_4), \\ D_1^{(2)}(20) &= \frac{2}{s_{12}s_{34}} E(p_2 - p_4, p_3), \\ D_1^{(2)}(21) &= \frac{-4}{s_{12}s_{34}s_{13}} [s_{12} - s_{34} + s_{34}] E(p_3, p_4), \\ D_1^{(2)}(22) &= \frac{4}{s_{12}s_{34}s_{13}} [s_{12} - s_{34} - s_{34}] E(p_3, p_4), \\ D_1^{(2)}(23) &= \frac{4}{s_{12}s_{34}s_{13}} [2(p_1 + p_2)(p_3 - p_4) + s_{12}] E(p_3, p_4), \end{aligned}$$

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$$\begin{aligned} D_1^G(24) &= \frac{-2}{t_{12}s_{34}} E(p_1, p_2, p_3), \\ D_1^G(25) &= \frac{2}{s_{14}s_{31}} E(p_0, p_1, p_2), \\ D_1^G(26) &= \frac{-2}{s_{12}t_{12}} E(p_0, p_2, p_3), \\ D_1^G(27) &= \frac{2}{s_{04}t_{12}} E(p_1, p_0, p_3), \\ D_1^G(28) &= \frac{2}{s_{13}t_{12}} E(p_0, p_1, p_2), \\ D_1^G(29) &= \frac{-2}{s_{34}t_{12}} E(p_1, p_0, p_3), \\ D_1^G(30) &= \frac{4}{t_{12}s_{34}t_{123}} [(p_1 + p_2 - p_3)(p_0 + p_1 - p_3) - t_{123}] E(p_0, p_3), \\ D_1^G(31) &= \frac{4}{t_{12}s_{04}t_{123}} [(p_1 + p_2 - p_3)(p_0 - p_1 + p_3) + t_{123}] E(p_0, p_3), \\ D_1^G(32) &= \frac{4}{t_{12}s_{04}t_{123}} [(p_1 - p_2 + p_3)(p_0 + p_1 - p_3) + t_{123}] E(p_0, p_3), \\ D_1^G(33) &= \frac{4}{t_{12}s_{04}t_{123}} [(p_1 - p_2 + p_3)(p_0 - p_1 + p_3) - t_{123}] E(p_0, p_3), \end{aligned} \quad (11)$$

where $\delta_0 = 1$.
The diagrams D_1^G are obtained from D_2^G by replacing δ_2 by $\delta_0 = 0$ and the functions $E(p_i, p_j)$ by $G(p_i, p_j)$.

The diagrams D_2^G are listed below:

$$\begin{aligned} D_2^G(1) &= \frac{4}{t_{12}s_{34}t_{123}} \{ [F(p_0, p_3)E(p_0, p_1) - F(p_0, p_2)E(p_0, p_3)] \\ &\quad + [F(p_0, p_2) + s_{34}]E(p_0, p_1) \}, \\ D_2^G(2) &= \frac{-4}{s_{04}s_{34}} \{ [F(p_0, p_1) + s_{34}]E(p_0, p_2) \\ &\quad + [F(p_0, p_2) + s_{34}]E(p_0, p_3) - F(p_0, p_1)E(p_0, p_3) \}, \\ D_2^G(3) &= \frac{4}{s_{12}s_{34}t_{123}} \{ [F(p_0, p_3)E(p_0, p_1) - F(p_0, p_2)E(p_0, p_3)] \\ &\quad - [F(p_0, p_1) - s_{34} + s_{04}]E(p_0, p_2) \}, \end{aligned}$$

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$$\begin{aligned} D_1^G(4) &= \frac{4}{s_{12}s_{34}s_{13}} [F(p_2, p_3)E(p_1, p_4) - F(p_1, p_3)E(p_2, p_4) \\ &\quad + \{F(p_1, p_3) - \frac{1}{2}s_{12} - \frac{1}{2}s_{13} + \frac{1}{2}s_{14}\}E(p_1, p_4)], \\ D_1^G(5) &= \frac{2}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} + s_{14}]E(p_1, p_4), \\ D_1^G(6) &= \frac{2}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} - \frac{1}{2}s_{12} + \frac{1}{2}s_{13}]E(p_1, p_4), \\ D_1^G(7) &= \frac{4}{s_{12}s_{34}s_{13}} \{ [F(p_2, p_3) - \frac{1}{2}s_{12} - \frac{1}{2}s_{13} + \frac{1}{2}s_{14}]E(p_1, p_4) \\ &\quad + [F(p_1, p_3) + \frac{1}{2}s_{12}]E(p_1, p_4) - [F(p_1, p_3) + \frac{1}{2}s_{13}]E(p_1, p_4) \}, \\ D_1^G(8) &= \frac{1}{s_{12}s_{34}} E(p_1 - p_3, p_4), \\ D_1^G(9) &= \frac{2}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} + s_{14}]E(p_1, p_4), \\ D_1^G(10) &= \frac{2}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} - \frac{1}{2}s_{12} + \frac{1}{2}s_{13}]E(p_1, p_4), \\ D_1^G(11) &= \frac{1}{2s_{12}s_{34}} \{ [s_{12} + s_{13} - s_{14}]E(p_1 - p_3, p_4) \\ &\quad - [s_{12} + s_{13} - s_{14}]E(p_1 - p_3, p_4) - [s_{12} + s_{13} - s_{14}]E(p_1 + p_3, p_4) \}, \\ D_1^G(12) &= \frac{1}{2s_{12}s_{34}} \{ [s_{12} + s_{13} - s_{14}]E(p_1 - p_3, p_4) \\ &\quad - [s_{12} + s_{13} - s_{14}]E(p_1 - p_3, p_4) - [s_{12} + s_{13} - s_{14}]E(p_1 + p_3, p_4) \}, \end{aligned} \quad (12)$$

The diagrams D_i^G are listed below:

$$\begin{aligned} D_1^G(1) &= \frac{1}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} + s_{14}]E(p_1 - p_3 - p_4), \\ D_1^G(2) &= \frac{1}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} + s_{14}]E(p_1 - p_3 + p_4), \\ D_1^G(3) &= \frac{1}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} + s_{14}]E(p_1 - p_3 - p_4), \\ D_1^G(4) &= \frac{1}{s_{12}s_{34}s_{13}} [s_{12} + s_{13} - s_{14}]E(p_1 - p_3 + p_4), \\ D_1^G(5) &= \frac{1}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} - \frac{1}{2}s_{12} + \frac{1}{2}s_{13}]E(p_1 - p_3 - p_4), \\ D_1^G(6) &= \frac{1}{s_{12}s_{34}s_{13}} [s_{12} - s_{13} - \frac{1}{2}s_{12} + \frac{1}{2}s_{13}]E(p_1 - p_3 + p_4), \end{aligned}$$

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$$D_1^0(7) = \frac{1}{f_{13}f_{14}f_{15}} [s_{14} - s_{16} + s_{12}][s_{12} - s_{15} - s_{23}],$$

$$D_1^0(8) = \frac{1}{s_{14}f_{13}f_{15}} [s_{12} + s_{15} - s_{23}][s_{14} - s_{16} + s_{12}],$$

$$D_1^0(9) = \frac{1}{s_{23}f_{14}f_{15}} [s_{14} + s_{16} - s_{12}][s_{15} - s_{16} + s_{23}],$$

$$D_1^0(10) = \frac{1}{s_{23}s_{14}} (p_1 - p_3)(p_1 - p_4),$$

$$D_1^0(11) = \frac{1}{s_{14}s_{16}} (p_1 - p_4)(p_1 - p_3),$$

$$D_1^0(12) = \frac{1}{s_{14}s_{15}} (p_1 - p_3)(p_2 - p_3),$$

$$D_1^0(13) = \frac{1}{s_{13}s_{14}} (p_1 - p_3)(p_1 - p_4),$$

$$D_1^0(14) = \frac{1}{s_{23}s_{14}} (p_1 - p_3)(p_1 - p_4),$$

$$D_1^0(15) = \frac{1}{s_{14}s_{15}s_{16}} \{ [(p_2 + p_3)(p_1 - p_4)][(p_1 - p_4)(p_2 - p_3)] \\ + [(p_2 - p_3)(p_1 - p_4)][(p_1 - p_4)(p_2 + p_3)] \\ + [(p_1 + p_4)(p_2 - p_3)][(p_1 - p_4)(p_2 - p_3)] \},$$

$$D_1^0(16) = \frac{2}{s_{14}s_{15}s_{16}} \{ [(p_2 - p_3)(p_1 + p_4)][(p_1 - p_4)(p_2 - p_3)] \\ + [(p_1 + p_4)(p_2 - p_3)][(p_1 - p_4)(p_2 - p_3)] \\ + [(p_1 - p_4)(p_2 + p_3)][(p_1 - p_4)(p_2 - p_3)] \}. \quad (13)$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eq. (11)–(13). The result is substituted to eq. (8) to obtain the vectors \mathcal{Q}_1 and \mathcal{Q}_2 . After generating the vectors \mathcal{Q}_{1a} , \mathcal{Q}_{1b} , \mathcal{Q}_{2a} , \mathcal{Q}_{2b} , and \mathcal{Q}_3 , by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_4 and A_2 . Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates on Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multi-gluon amplitudes are tested by checking the gauge invariance. Due to the specifics

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of our calculation, the most powerful test does not rely on the gauge symmetry, but on the appropriate permutation symmetries. The function $A(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1, p_2, p_3) and separately, (p_4, p_5, p_6) , whereas the function $A(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under the permutations of (p_1, p_2, p_3, p_4) and separately, (p_5, p_6) . This test is extremely powerful, because the required permutation symmetries are hidden in our supersymmetry relations, eqs. (1) and (3), and in the structure of amplitudes involving different species of particles. Another, very important test relies on the absence of the double poles of the form $(s_{ij})^{-2}$ in the cross section, as required by general arguments based on the helicity conservation. Further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the two goes to three cross section [3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has successfully passed both these numerical checks.

Details of the calculation, together with a full exposition of our techniques, will be given in a forthcoming article. Furthermore, we hope to obtain a simple analytic form for the answer, making our result not only an experimentalist's, but also a theorist's delight.

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References

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The Discovery of Incredible, Unanticipated Simplicity

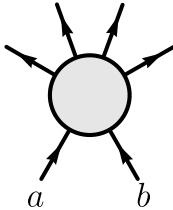
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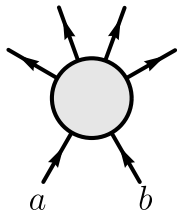
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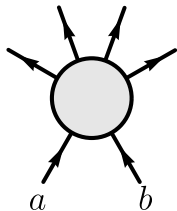
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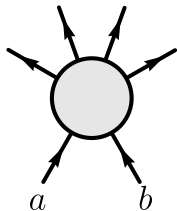
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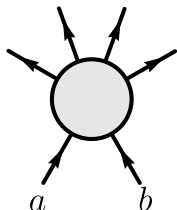
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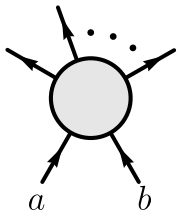
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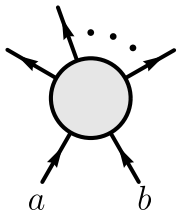


A Feynman diagram consisting of a central grey circle. Four external lines with arrows pointing away from the circle are shown: one at the top-left, one at the top-right, one at the bottom-left labeled 'a', and one at the bottom-right labeled 'b'. Three dots are placed above the top-right arrow, indicating an arbitrary number of external lines.

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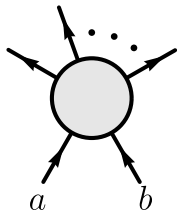


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A Feynman diagram consisting of a central grey circle with n external lines. Two lines at the bottom are labeled a and b with arrows pointing towards the circle. The other $n-2$ lines have arrows pointing away from the circle. Three dots between the top two outgoing lines indicate an arbitrary number of external legs.

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On What *Data* Does a Scattering Amplitude Depend?

A scattering amplitude, \mathcal{A}_n , can be a generally complicated(?) function of all the *physically observable data* describing each of the particles involved.

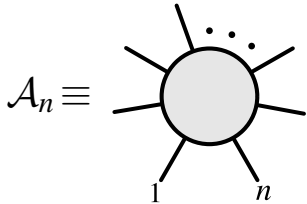
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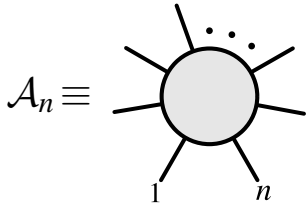
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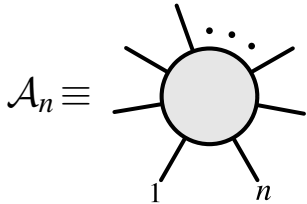
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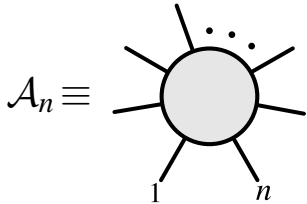
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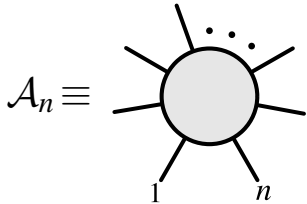


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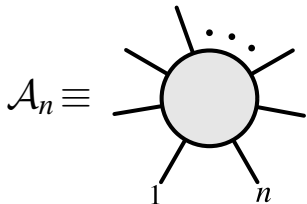


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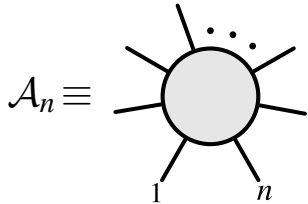


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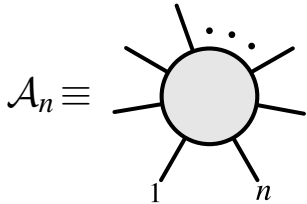


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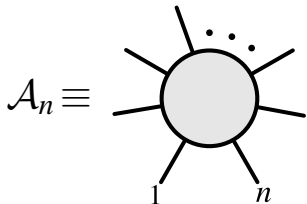


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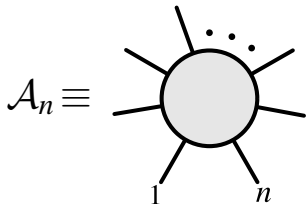


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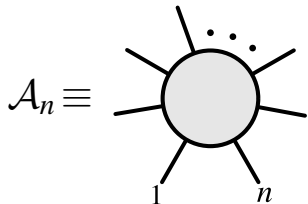


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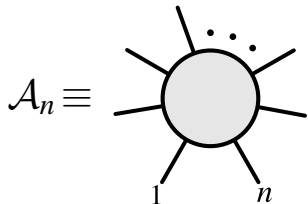


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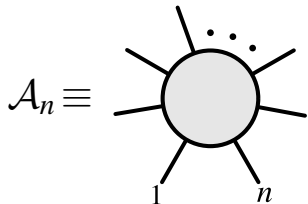


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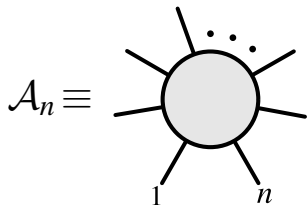


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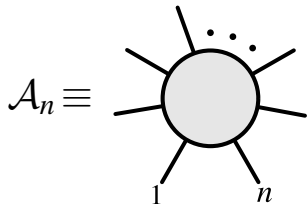


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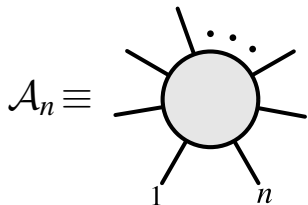
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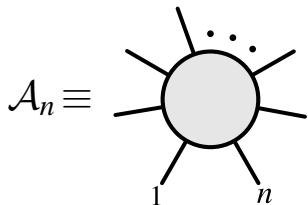
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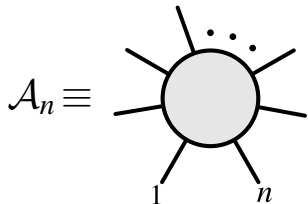
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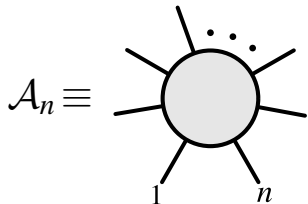
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A scattering amplitude, \mathcal{A}_n , can be a generally complicated(?) function of all the *physically observable data* describing each of the particles involved.



Physical data for the a^{th} particle: $|a\rangle$

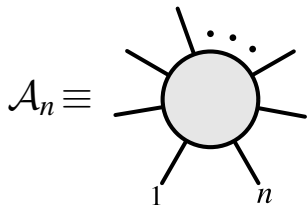
- p_a^μ momentum, *on-shell*: $p_a^2 - m_a^2 = 0$
- σ_a spin, helicity $h_a = \pm \sigma_a$ ($m_a = 0$)
- q_a all the *non-kinematical* quantum numbers of a (color, flavor, ...)

Although a Lagrangian formalism requires that we use **polarization tensors**, it is *impossible* to continuously define polarizations for each helicity state without introducing *unobservable (gauge)* redundancy—*e.g.* for $\sigma_a = 1$:

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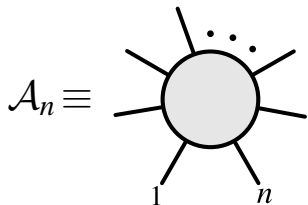
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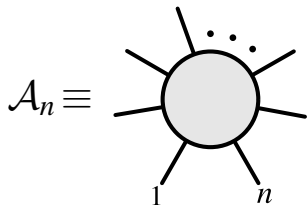
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Broadening the Class of Physically Meaningful Functions

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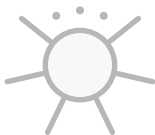
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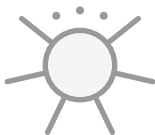
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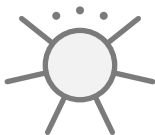
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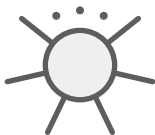
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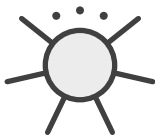
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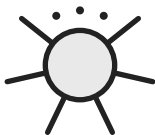
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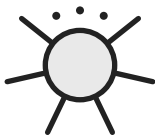
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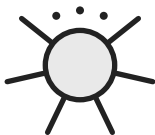
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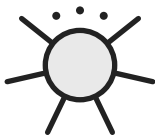
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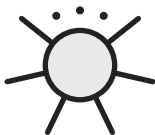
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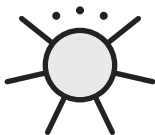
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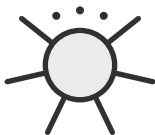
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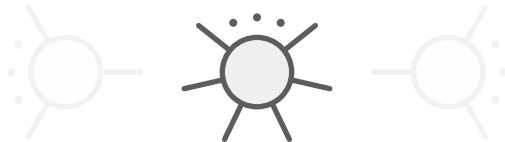
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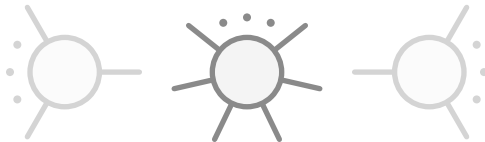
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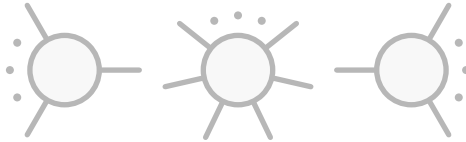
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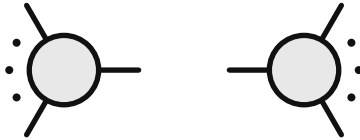
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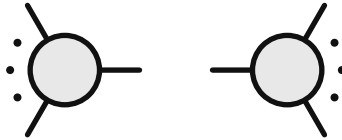
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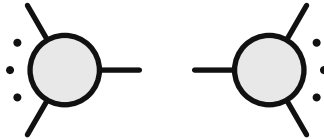
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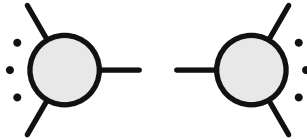
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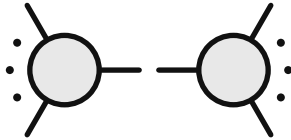
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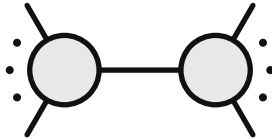
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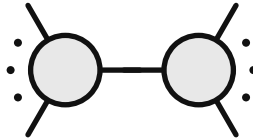
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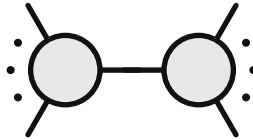
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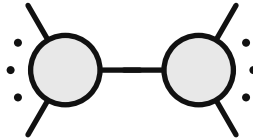
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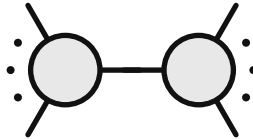
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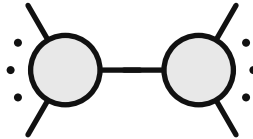
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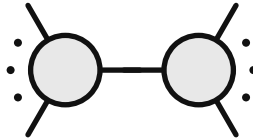
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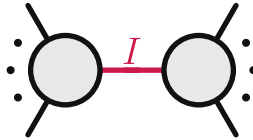
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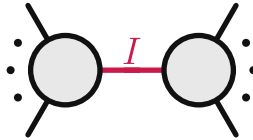
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Internal Particles:

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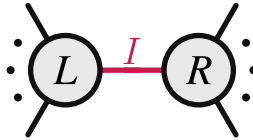
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Internal Particles: **locality** dictates that we multiply each amplitude,

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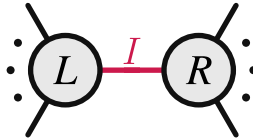


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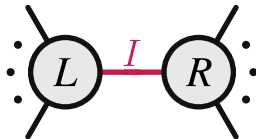


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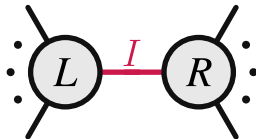


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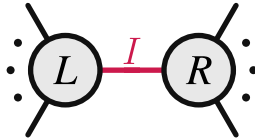


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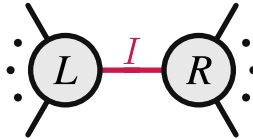


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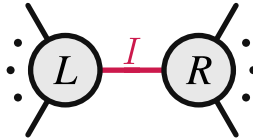


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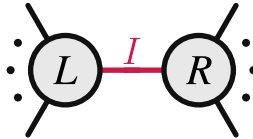


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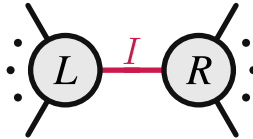


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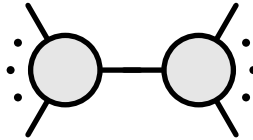
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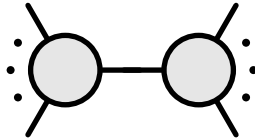
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On-Shell Functions:

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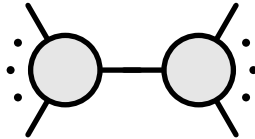
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On-Shell Functions: networks of amplitudes, \mathcal{A}_v , connected by any number of internal particles, $i \in I$, forming a graph Γ called an “**on-shell diagram**”.

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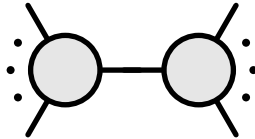


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$$f_{\Gamma} \equiv \prod_{i \in I} \left(\sum_{\substack{h_i, q_i, \\ m_i, \dots}} \int d^{d-1} \text{LIPS}_i \right) \prod_v \mathcal{A}_v$$

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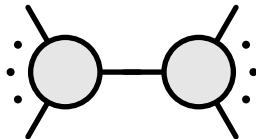


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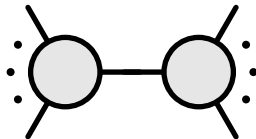
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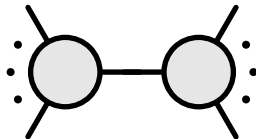
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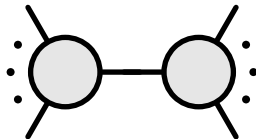
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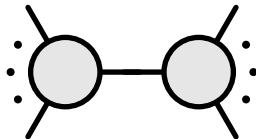
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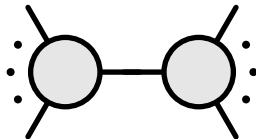
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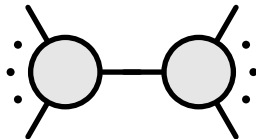
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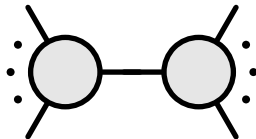
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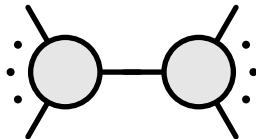
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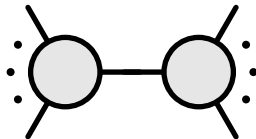
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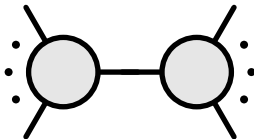
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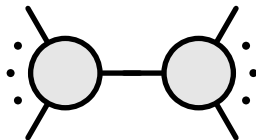
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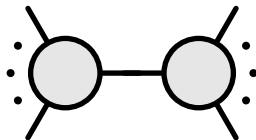
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Making *Masslessness* Manifest: Spinor-Helicity Variables

To avoid *constraining* each particle's momentum to be null, *van der Waerden* introduced (in 1929!) **spinor-helicity** variables to make this always trivial.

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the *span* of 2 vectors $\lambda^\alpha \in \mathbb{C}^n$

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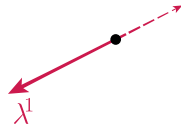
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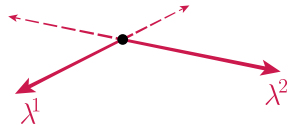
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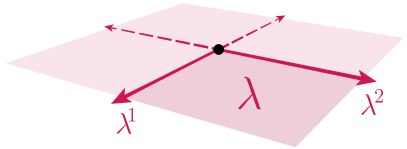
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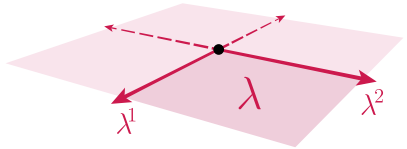
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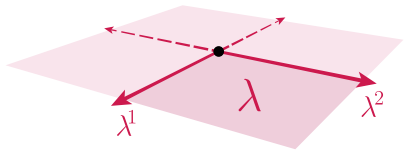
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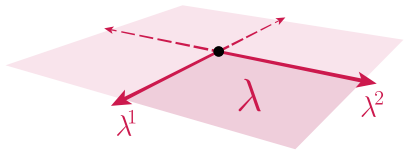
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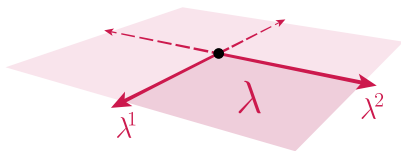
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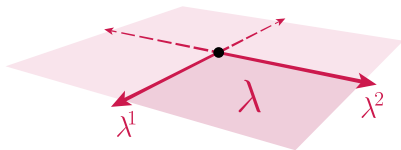
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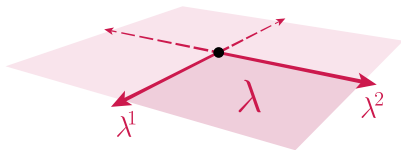
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The Grassmannian Geometry of Kinematical Constraints

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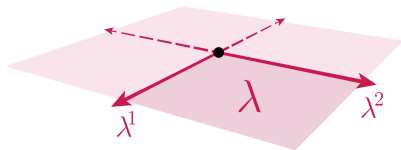
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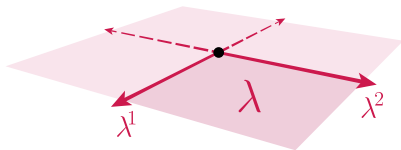
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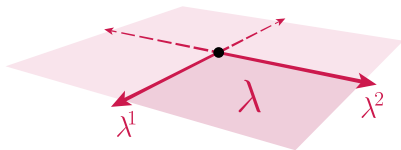
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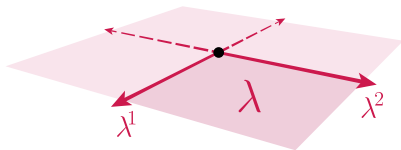
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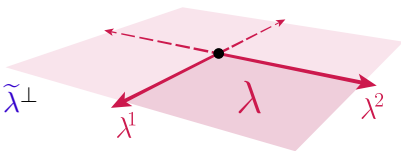
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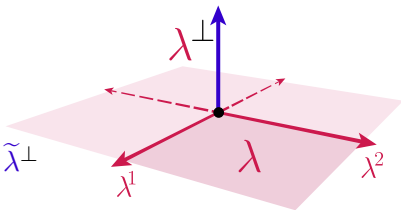
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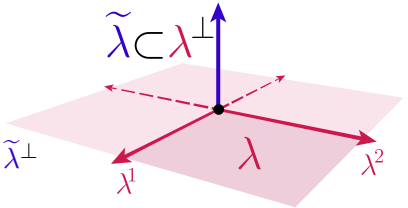
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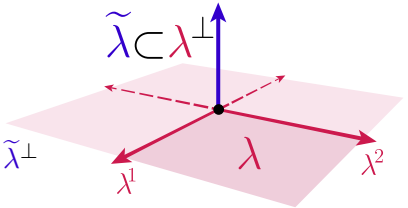
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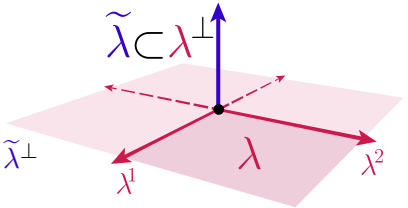
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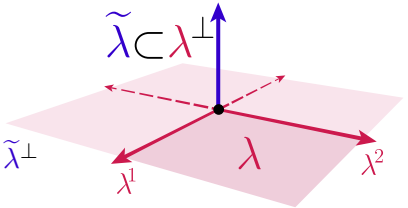
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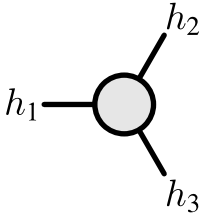


Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

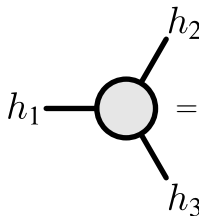
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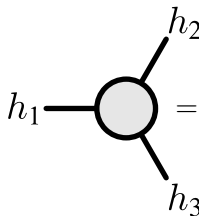


A Feynman diagram representing a three-point vertex. It consists of a central gray circle with three external lines. One line extends horizontally to the left and is labeled h_1 . Another line extends upwards and to the right, labeled h_2 . The third line extends downwards and to the right, labeled h_3 .

$$h_1 \text{---} \bigcirc = f(\lambda_1 \tilde{\lambda}_1, \lambda_2 \tilde{\lambda}_2, \lambda_3 \tilde{\lambda}_3) \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda})$$

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A Feynman diagram showing a central gray circle with three external lines. The left line is horizontal and labeled h_1 . The top-right line is labeled h_2 . The bottom-right line is labeled h_3 .

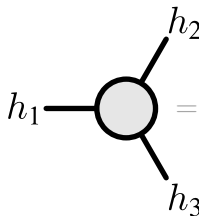
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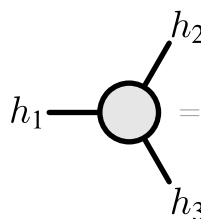
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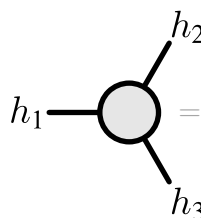
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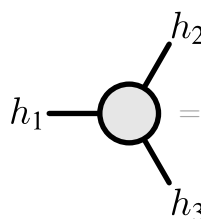
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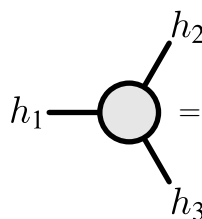
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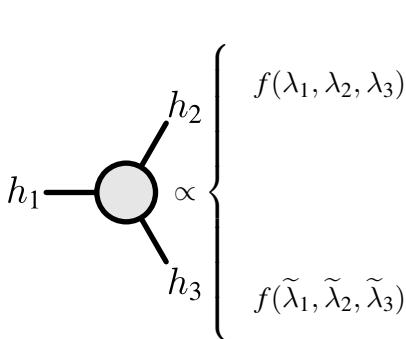
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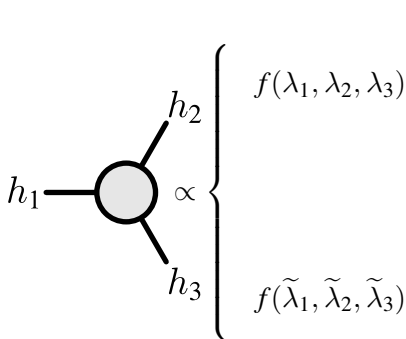
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$$\lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}$$

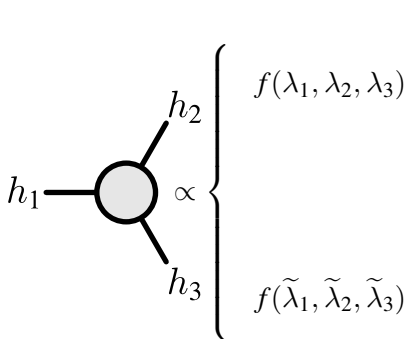
or

$$\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix}$$

$$\tilde{\lambda}^\perp \equiv ([23] [31] [12]) \supset \lambda$$

Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).



$$h_1 \text{---} \bigcirc \begin{matrix} \nearrow h_2 \\ \searrow h_3 \end{matrix} \propto \left\{ \begin{array}{l} f(\lambda_1, \lambda_2, \lambda_3) \\ f(\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3) \end{array} \right.$$

$$\lambda^\perp \equiv (\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle) \supset \tilde{\lambda}$$

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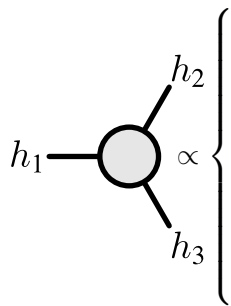
or

$$\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix}$$

$$\tilde{\lambda}^\perp \equiv ([23] [31] [12]) \supset \lambda$$

Building Blocks: the S-Matrix for Three Massless Particles

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A Feynman diagram showing a central shaded circle vertex with three external lines. The left line is labeled h_1 , the top-right line is labeled h_2 , and the bottom-right line is labeled h_3 . The diagram is part of a larger equation structure.

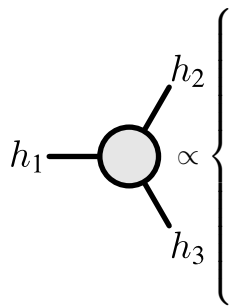
$$\left\{ \begin{array}{l} \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \\ [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2} \end{array} \right. \propto \left\{ \begin{array}{l} \lambda^\perp \equiv (\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle) \triangleright \tilde{\lambda} \\ \lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \end{array} \right.$$

or

$$\left\{ \begin{array}{l} \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix} \\ \tilde{\lambda}^\perp \equiv ([23] [31] [12]) \triangleright \lambda \end{array} \right.$$

Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).



A Feynman diagram showing a central grey circle representing a vertex. Three lines extend from the vertex: one to the left labeled h_1 , one to the top-right labeled h_2 , and one to the bottom-right labeled h_3 . To the right of the vertex is a large curly brace containing two expressions, with a proportionality symbol \propto between them.

$$\propto \left\{ \begin{array}{l} \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \\ [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2} \end{array} \right.$$

$$\lambda^\perp \equiv (\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle) \supset \tilde{\lambda}$$

$$\lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}$$

or

$$\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix}$$

$$\tilde{\lambda}^\perp \equiv ([23] [31] [12]) \supset \lambda$$

Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

$$\begin{array}{c}
\text{\scriptsize h_1} \text{---} \bullet \left(\begin{array}{l} \text{\scriptsize h_2} \\ \text{\scriptsize h_3} \end{array} \right) \propto \left\{ \begin{array}{ll} \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} & \lambda^\perp \equiv (\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle) \supset \tilde{\lambda} \\ \xrightarrow[\langle ab \rangle \rightarrow \mathcal{O}(\epsilon)]{} \mathcal{O}(\epsilon^{-(h_1+h_2+h_3)}) & \lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \\ \\ [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2} & \text{or} \\ \xrightarrow[a b \rightarrow \mathcal{O}(\epsilon)]{} \mathcal{O}(\epsilon^{(h_1+h_2+h_3)}) & \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix} \\ & \tilde{\lambda}^\perp \equiv ([23] [31] [12]) \supset \lambda \end{array} \right.
\end{array}$$

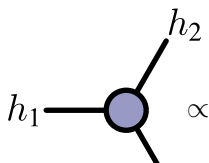
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

$$\begin{array}{lcl}
\begin{array}{c} h_2 \\ \diagup \\ \text{---} \bigcirc \text{---} \\ \diagdown \\ h_3 \end{array} \begin{array}{l} h_1 \\ \text{---} \end{array} \propto \left\{ \begin{array}{l} \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \\ \xrightarrow{\langle ab \rangle \rightarrow \mathcal{O}(\epsilon)} \mathcal{O}(\epsilon^{-(h_1+h_2+h_3)}) \end{array} \right. & \begin{array}{l} \lambda^\perp \equiv (\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle) \supset \tilde{\lambda} \\ \lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix} \end{array} \\
\begin{array}{c} [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2} \\ \xrightarrow{[ab] \rightarrow \mathcal{O}(\epsilon)} \mathcal{O}(\epsilon^{(h_1+h_2+h_3)}) \end{array} & \begin{array}{l} \text{or} \\ \tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^i & \tilde{\lambda}_2^i & \tilde{\lambda}_3^i \\ \tilde{\lambda}_1^{\dot{2}} & \tilde{\lambda}_2^{\dot{2}} & \tilde{\lambda}_3^{\dot{2}} \end{pmatrix} \\ \tilde{\lambda}^\perp \equiv ([23] [31] [12]) \supset \lambda \end{array}
\end{array}$$

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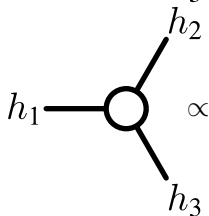
$$\propto \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1}$$

$$h_1 + h_2 + h_3 \leq 0$$

$$\lambda^\perp \equiv (\langle 23 \rangle \langle 31 \rangle \langle 12 \rangle) \triangleright \tilde{\lambda}$$

$$\lambda \equiv \begin{pmatrix} \lambda_1^1 & \lambda_2^1 & \lambda_3^1 \\ \lambda_1^2 & \lambda_2^2 & \lambda_3^2 \end{pmatrix}$$

or



$$\propto [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2}$$

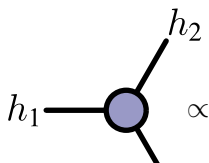
$$h_1 + h_2 + h_3 \geq 0$$

$$\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix}$$

$$\tilde{\lambda}^\perp \equiv ([23] [31] [12]) \triangleright \lambda$$

Building Blocks: the S-Matrix for Three Massless Particles

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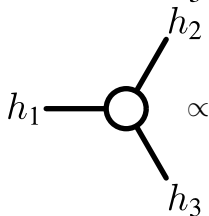
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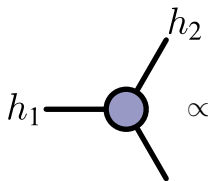
$$h_1 + h_2 + h_3 \geq 0$$

$$\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix}$$

$$\tilde{\lambda}^\perp \equiv ([23] [31] [12]) \triangleright \lambda$$

Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).



\propto

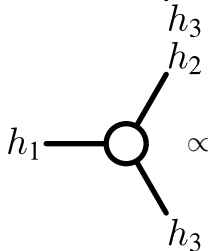
$$\langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1}$$

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or



\propto

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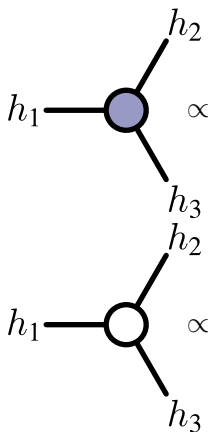
$$h_1 + h_2 + h_3 \geq 0$$

$$\tilde{\lambda} \equiv \begin{pmatrix} \tilde{\lambda}_1^1 & \tilde{\lambda}_2^1 & \tilde{\lambda}_3^1 \\ \tilde{\lambda}_1^2 & \tilde{\lambda}_2^2 & \tilde{\lambda}_3^2 \end{pmatrix}$$

$$\tilde{\lambda}^\perp \equiv ([23] [31] [12]) \triangleright \lambda$$

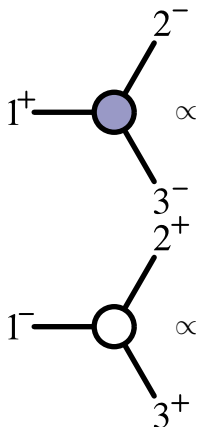
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).


$$\begin{aligned} &\propto \langle 12 \rangle^{h_3-h_1-h_2} \langle 23 \rangle^{h_1-h_2-h_3} \langle 31 \rangle^{h_2-h_3-h_1} \\ &\quad h_1 + h_2 + h_3 \leq 0 \end{aligned}$$
$$\begin{aligned} &\propto [12]^{h_1+h_2-h_3} [23]^{h_2+h_3-h_1} [31]^{h_3+h_1-h_2} \\ &\quad h_1 + h_2 + h_3 \geq 0 \end{aligned}$$

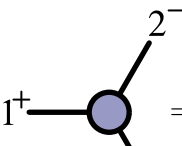
Building Blocks: the S-Matrix for Three Massless Particles

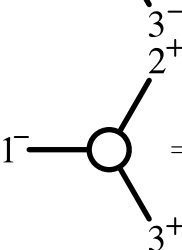
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Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).


$$= \frac{\langle 23 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda})$$


$$= \frac{[23]^4}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda})$$

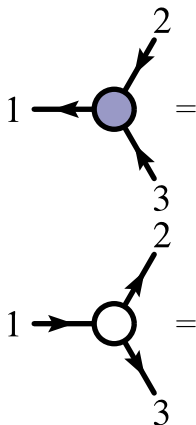
Building Blocks: the S-Matrix for Three Massless Particles

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$$\begin{aligned} \text{Diagram 1: } 1^+ \text{ incoming, } 2^- \text{ and } 3^- \text{ outgoing} &= \frac{\langle 23 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \\ \text{Diagram 2: } 1^- \text{ incoming, } 2^+ \text{ and } 3^+ \text{ outgoing} &= \frac{[23]^4}{[12] [23] [31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \end{aligned}$$

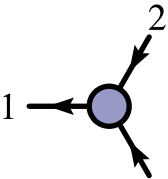
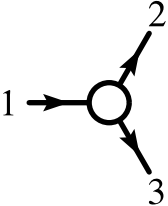
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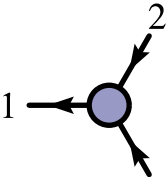
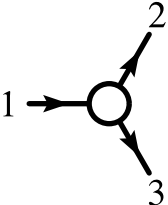
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$$= \frac{\langle 23 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3(+, -, -)$$

$$= \frac{[23]^4}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3(-, +, +)$$

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Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

$$\begin{aligned}
 & \text{Diagram 1: A blue circle with an incoming line 1 from the left, an outgoing line 2 to the top-right, and an outgoing line 3 to the bottom-right.} \\
 & = \frac{\langle 3\,1\rangle\langle 2\,3\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle} \delta^{2\times 2}(\lambda\cdot\tilde{\lambda}) \equiv \mathcal{A}_3(+\tfrac{1}{2}, -\tfrac{1}{2}, -) \\
 \\
 & \text{Diagram 2: A white circle with an incoming line 1 from the left, an outgoing line 2 to the top-right, and an outgoing line 3 to the bottom-right.} \\
 & = \frac{[3\,1][2\,3]^3}{[1\,2][2\,3][3\,1]} \delta^{2\times 2}(\lambda\cdot\tilde{\lambda}) \equiv \mathcal{A}_3(-\tfrac{1}{2}, +\tfrac{1}{2}, +)
 \end{aligned}$$

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$$\begin{aligned}
 & \text{Diagram 1: A vertex (blue circle) with three external lines labeled 1, 2, and 3. Line 1 is horizontal to the left, line 2 is diagonal up-right, and line 3 is diagonal down-right.} \\
 & = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3^{(2)} \\
 \\
 & \text{Diagram 2: A vertex (white circle) with three external lines labeled 1, 2, and 3. Line 1 is horizontal to the left, line 2 is diagonal up-right, and line 3 is diagonal down-right.} \\
 & = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[1\,2] [2\,3] [3\,1]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3^{(1)}
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 & = \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3^{(2)} \\
 \\
 & \text{Diagram 2: A vertex (white circle) with three external lines labeled 1, 2, and 3. Line 1 is horizontal to the left, line 2 is diagonal up-right, and line 3 is diagonal down-right.} \\
 & = \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[1\,2] [2\,3] [3\,1]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3^{(1)}
 \end{aligned}$$

Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

$$\begin{aligned}
\text{Diagram 1} &= \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3^{(2)} \\
\text{Diagram 2} &= \frac{\delta^{1 \times 4}(\tilde{\lambda}^\perp \cdot \tilde{\eta})}{[12][23][31]} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3^{(1)}
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Extracting Components from Super-Amplitudes

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Coupling Constant Constraints: Scaling and Spin/Statistics

The coupling constants f^{q_1, q_2, q_3} are quantum-number-dependent **constants** which define the theory.

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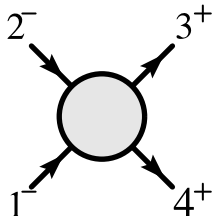
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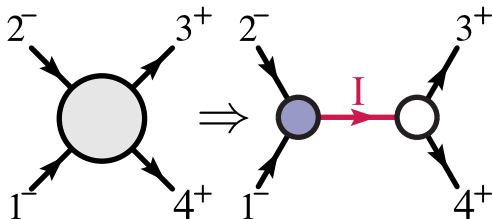
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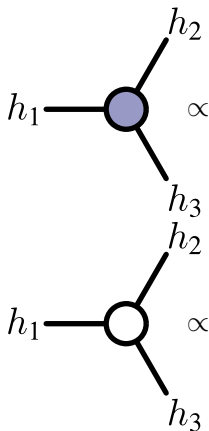
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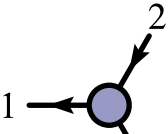
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Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).


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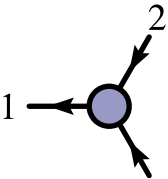
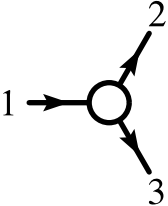
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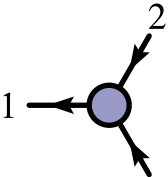
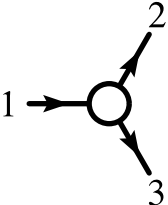
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 & = \frac{\langle 3\,1\rangle\langle 2\,3\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle} \delta^{2\times 2}(\lambda\cdot\tilde{\lambda}) \equiv \mathcal{A}_3(+\tfrac{1}{2}, -\tfrac{1}{2}, -) \\
 \\
 & \text{Diagram 2: A white circle with an incoming line from the left labeled 1, an outgoing line to the top-right labeled 2, and a wavy outgoing line to the bottom-right labeled 3.} \\
 & = \frac{[3\,1][2\,3]^3}{[1\,2][2\,3][3\,1]} \delta^{2\times 2}(\lambda\cdot\tilde{\lambda}) \equiv \mathcal{A}_3(-\tfrac{1}{2}, +\tfrac{1}{2}, +)
 \end{aligned}$$

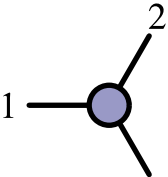
Building Blocks: the S-Matrix for Three Massless Particles

Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

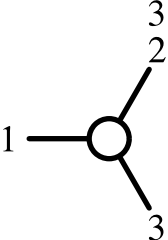
$$\begin{aligned}
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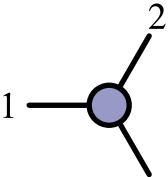
$$= \frac{\delta^{2 \times 4}(\lambda \cdot \tilde{\eta})}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle} \delta^{2 \times 2}(\lambda \cdot \tilde{\lambda}) \equiv \mathcal{A}_3^{(2)}$$



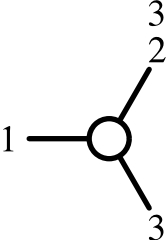
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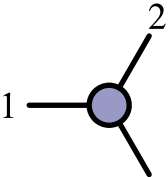
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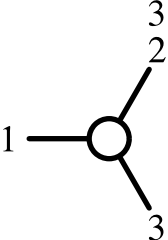
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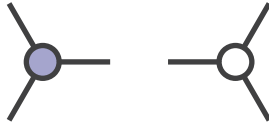
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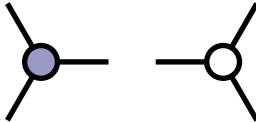
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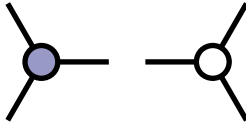
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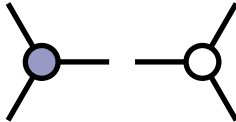
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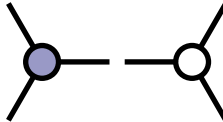
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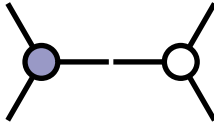
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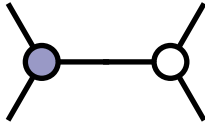
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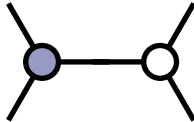
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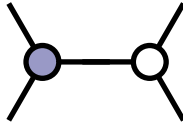
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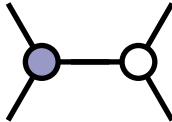
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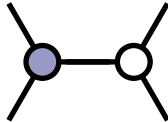
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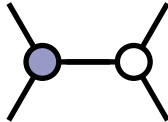
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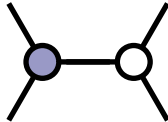
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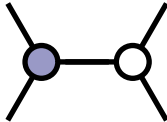
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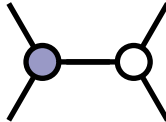
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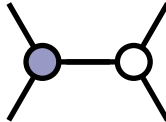
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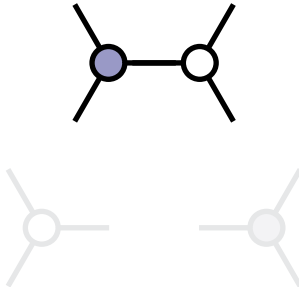
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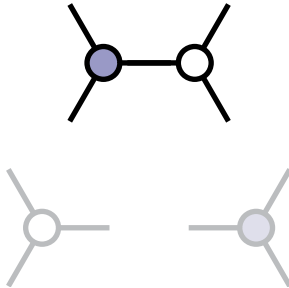
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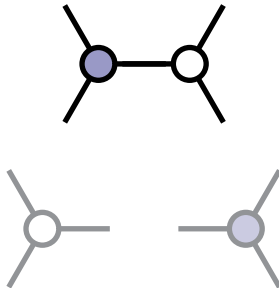
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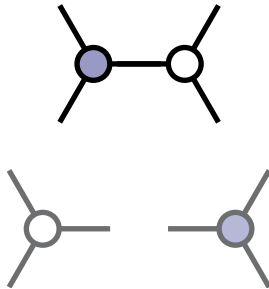
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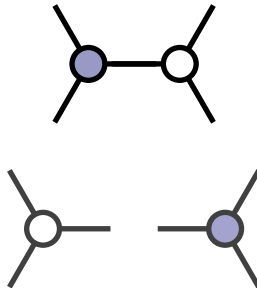
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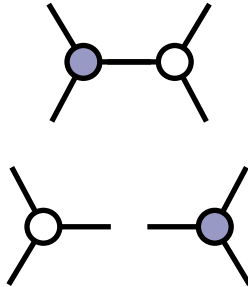
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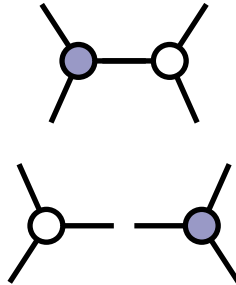
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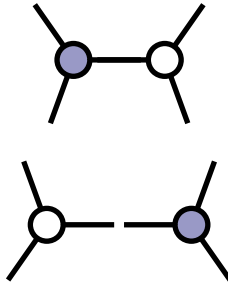
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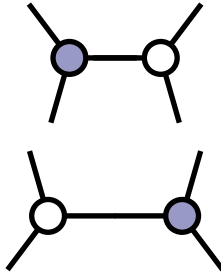
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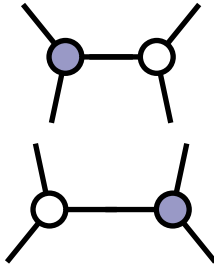
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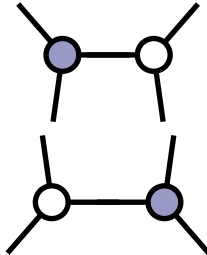
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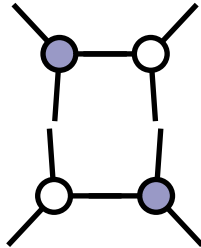
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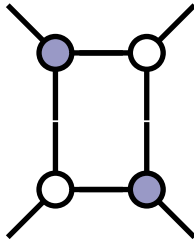
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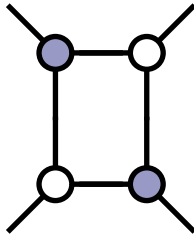
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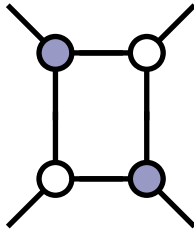
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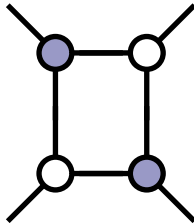
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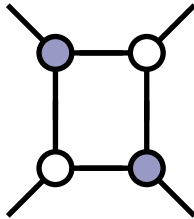
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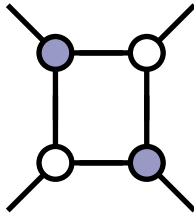
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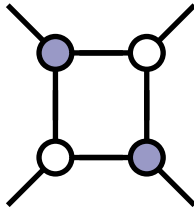
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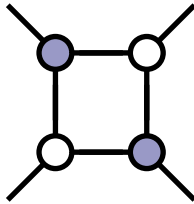
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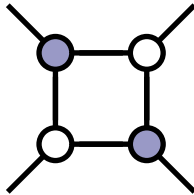
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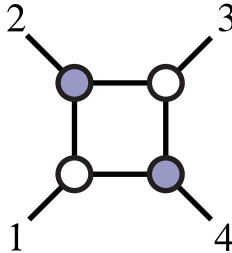
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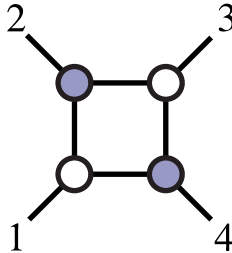
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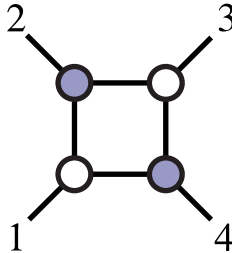
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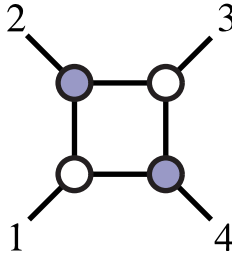
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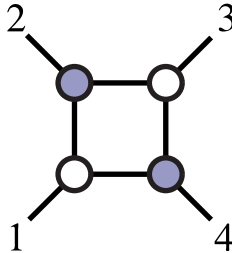
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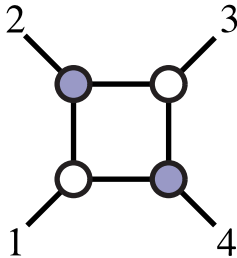
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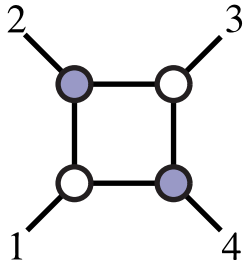
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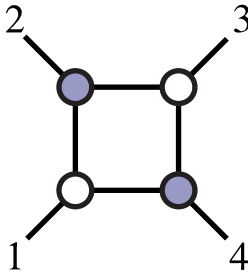
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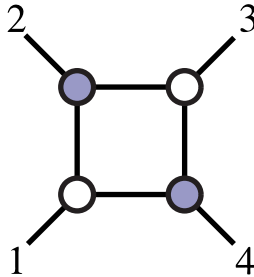
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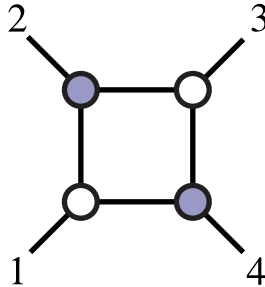
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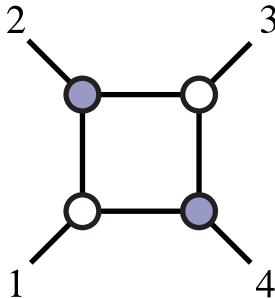
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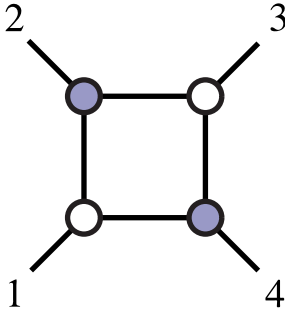
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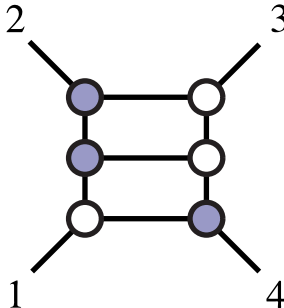
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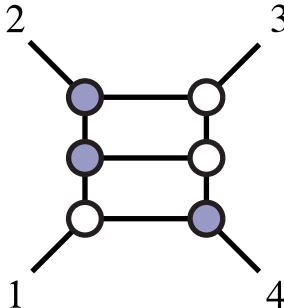
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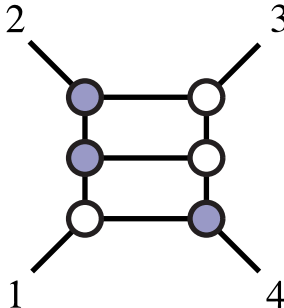
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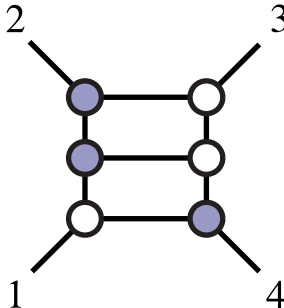
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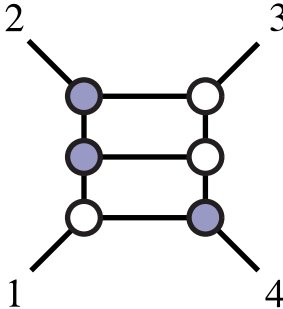
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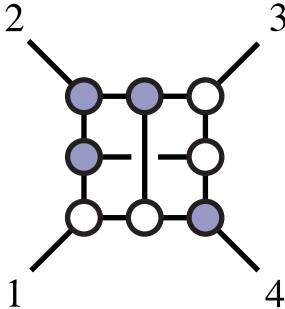
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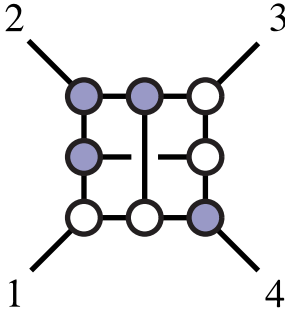
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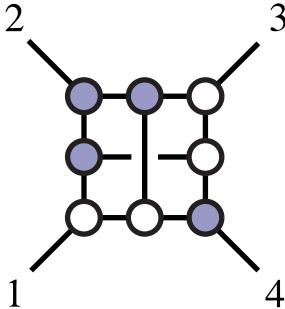
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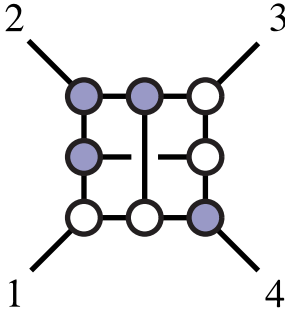
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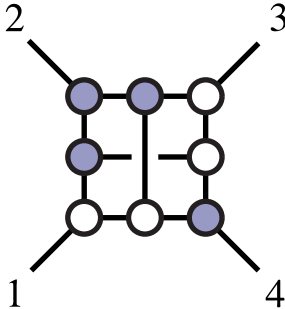
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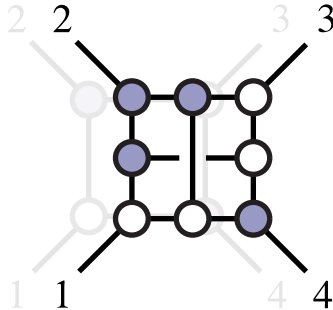
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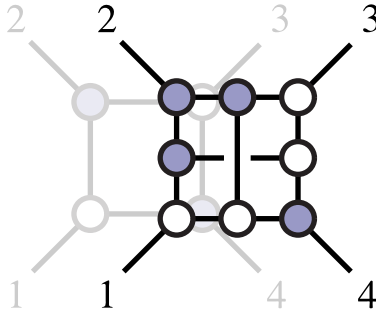
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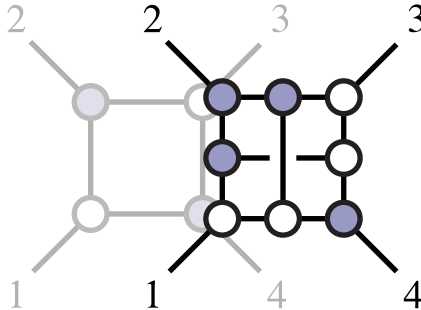
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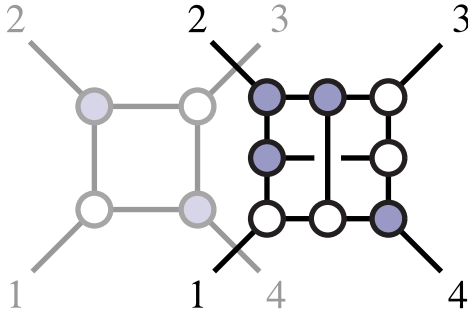
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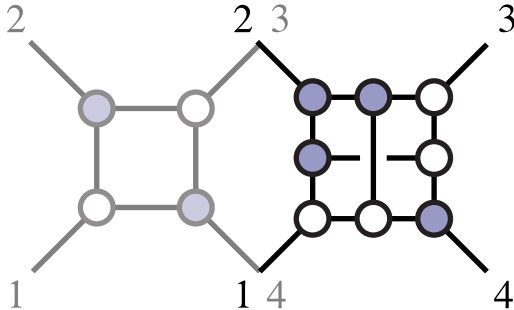
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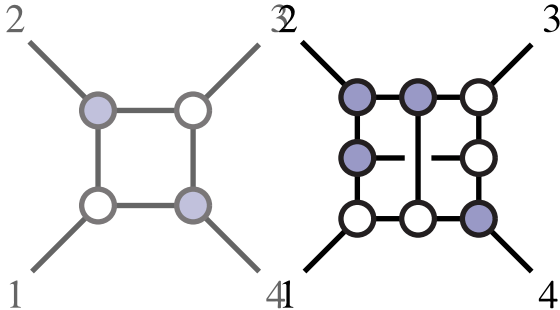
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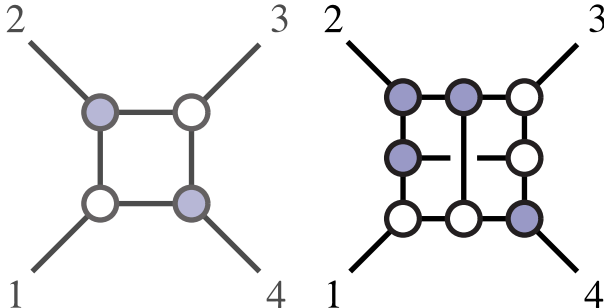
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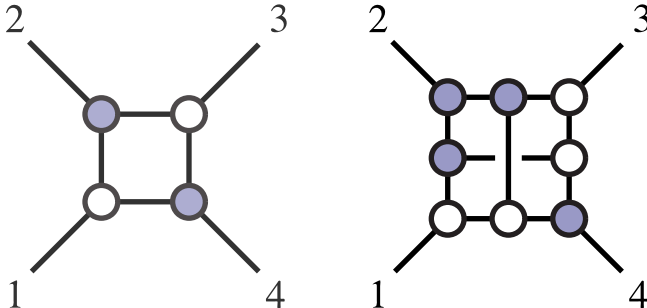
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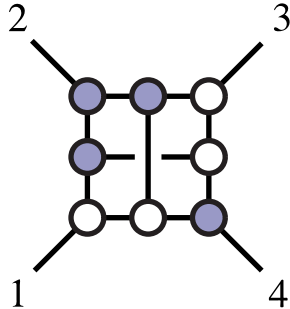
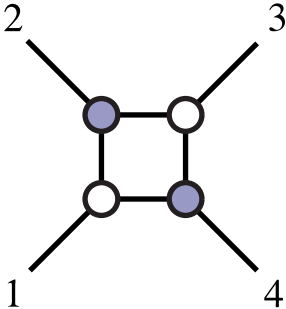
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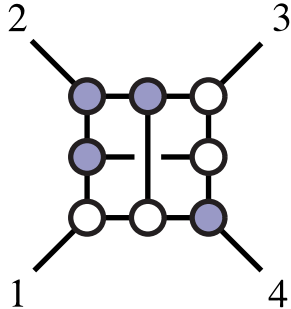
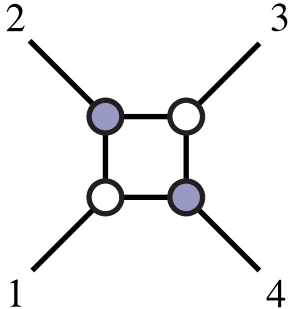
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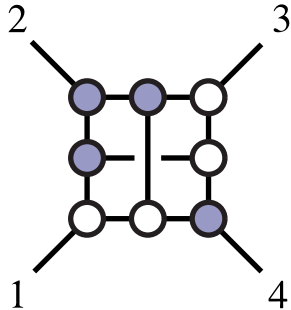
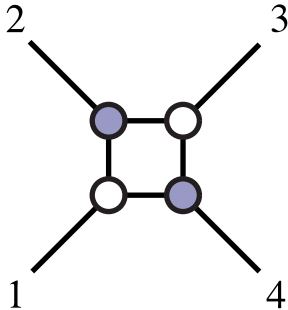
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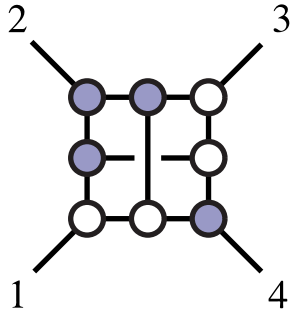
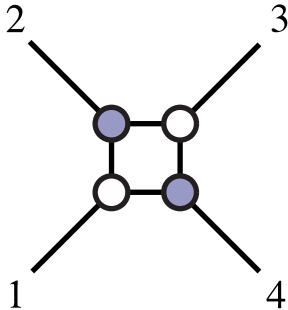
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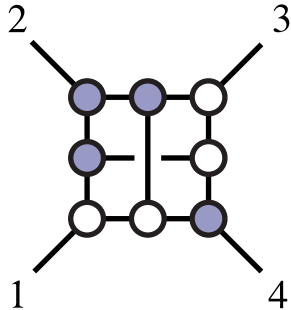
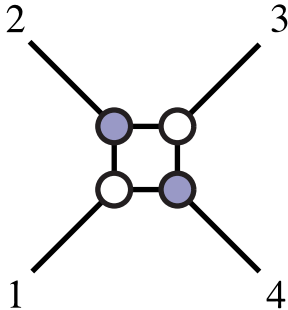
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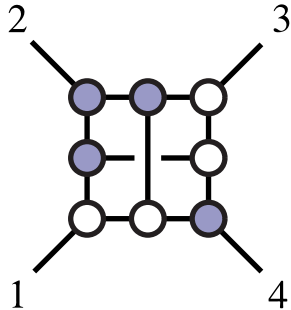
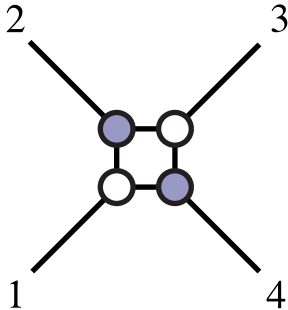
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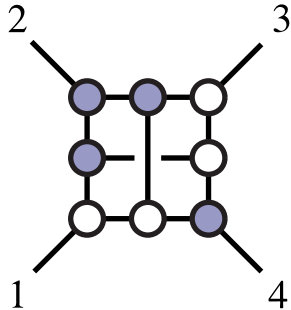
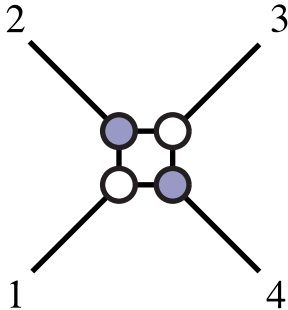
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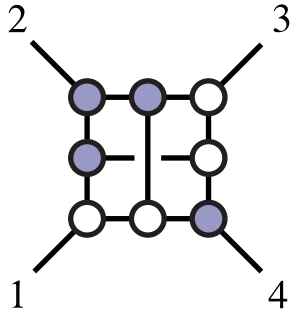
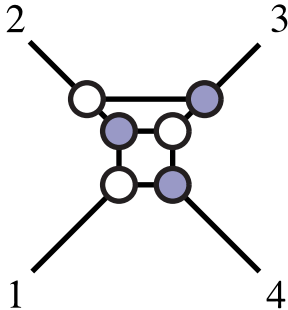
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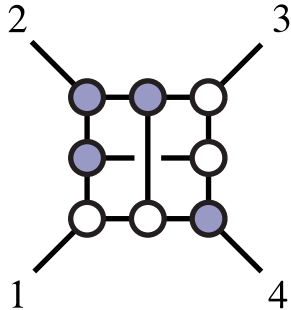
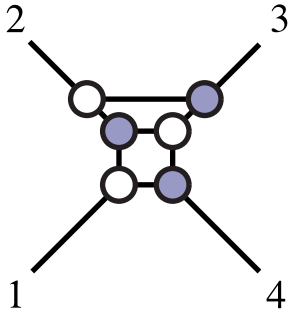
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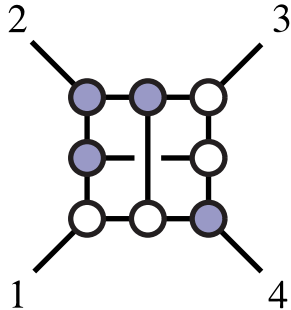
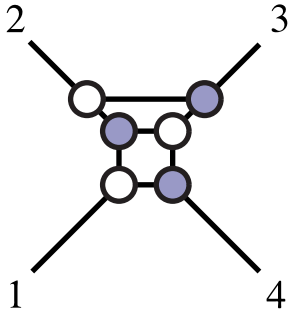
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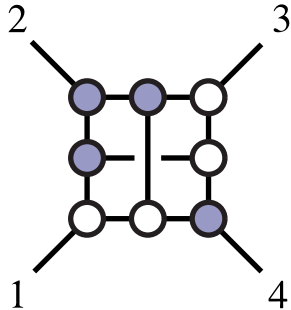
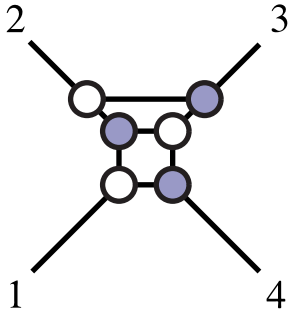
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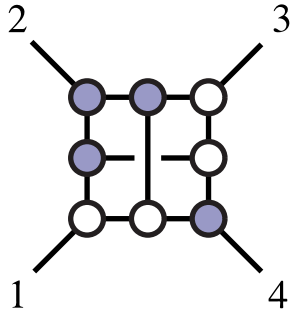
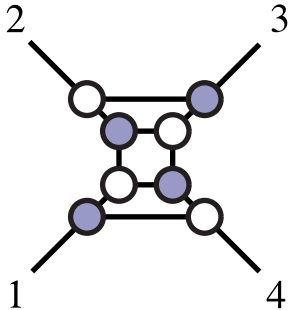
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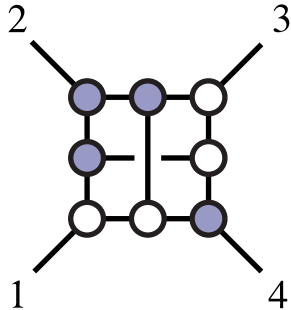
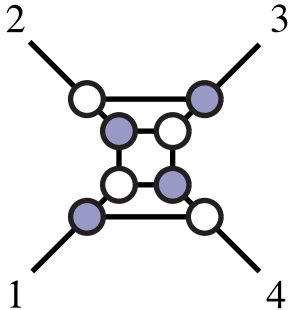
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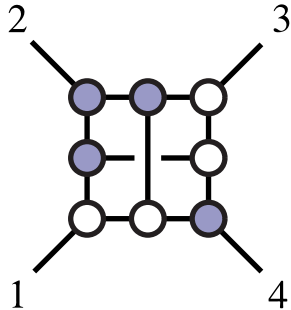
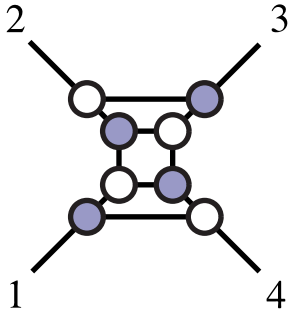
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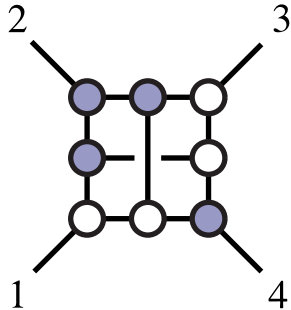
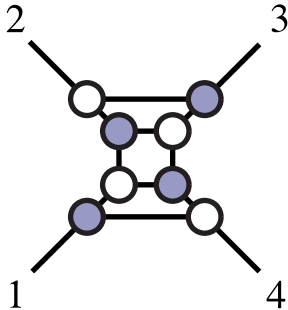
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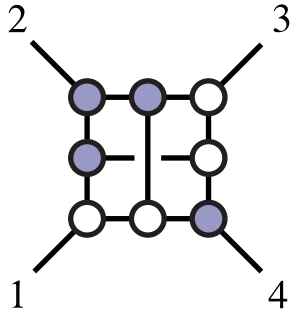
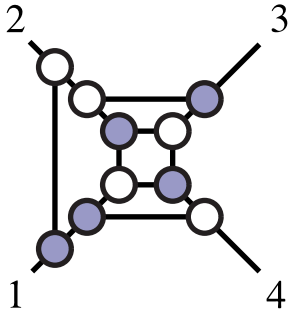
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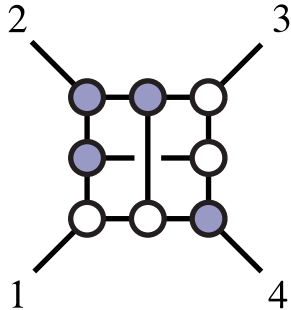
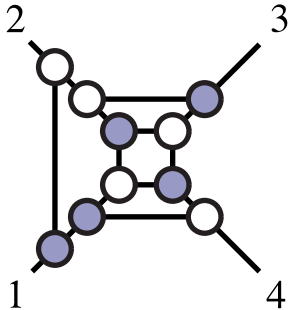
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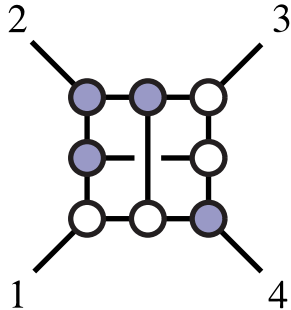
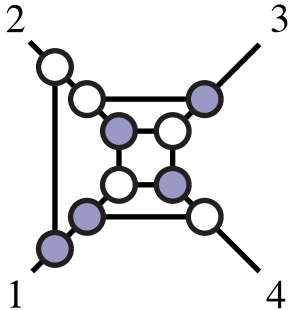
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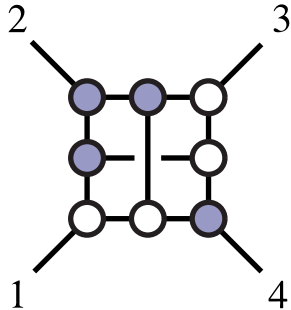
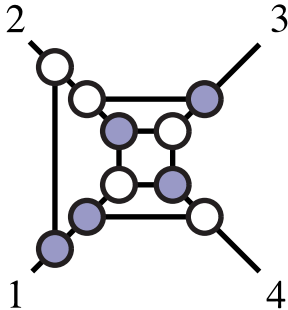
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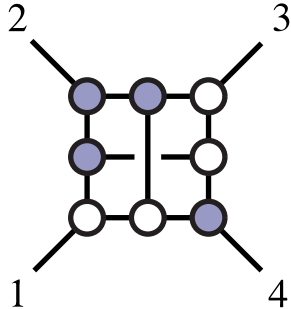
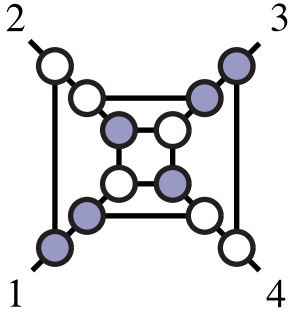
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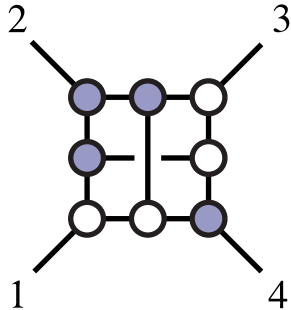
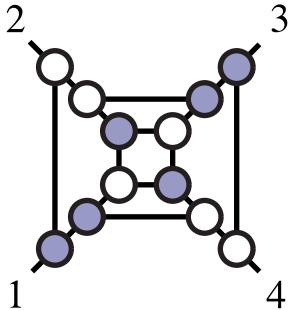
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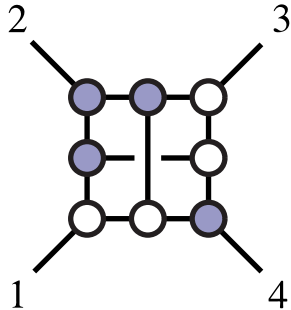
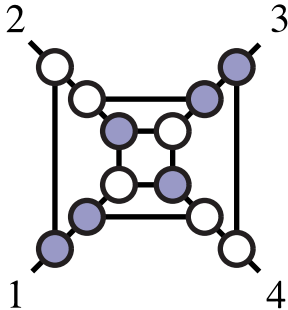
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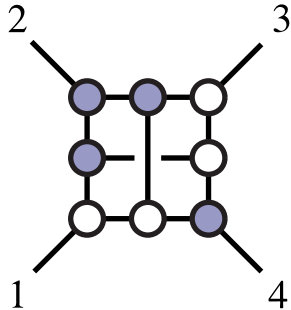
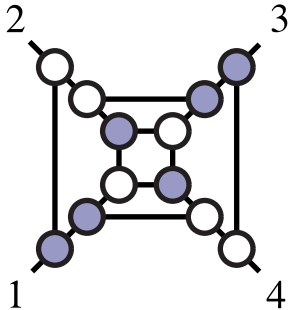
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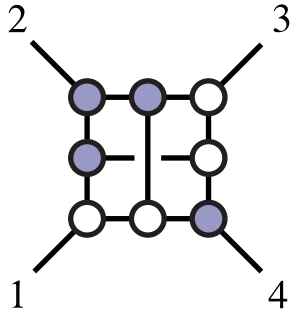
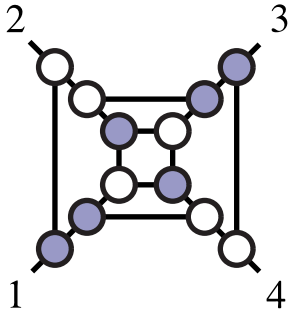
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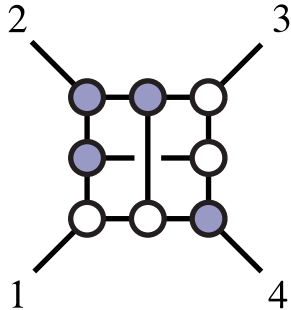
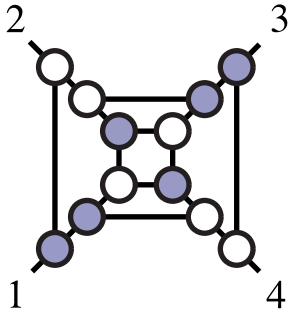
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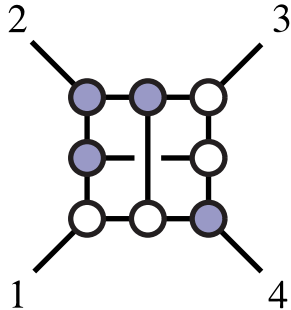
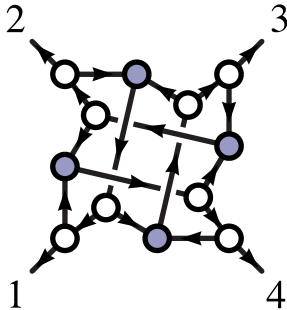
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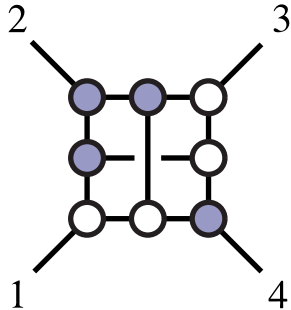
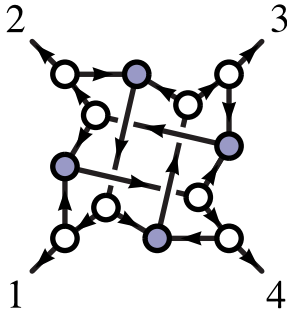
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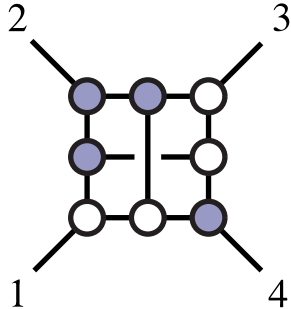
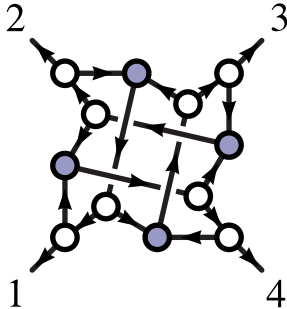
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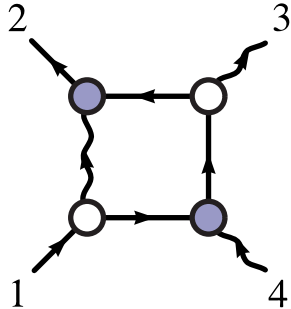
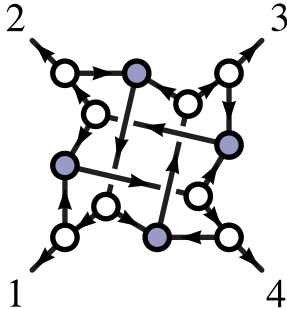
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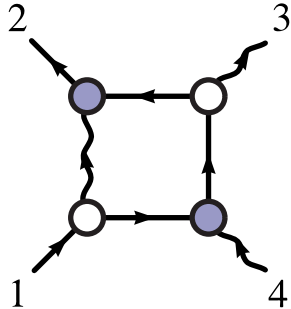
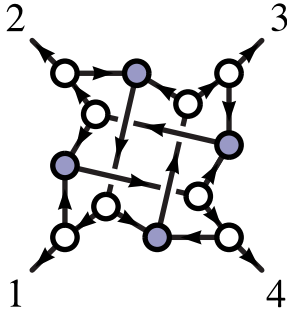
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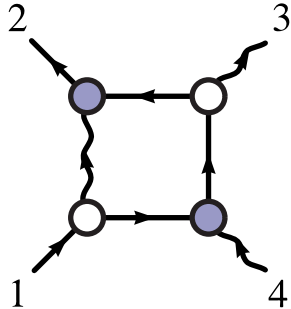
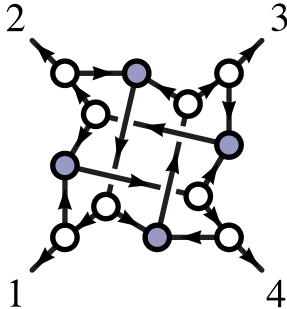
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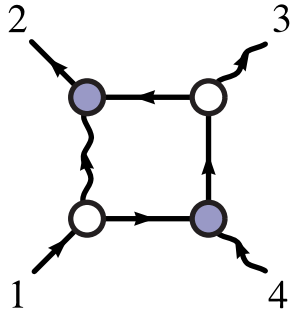
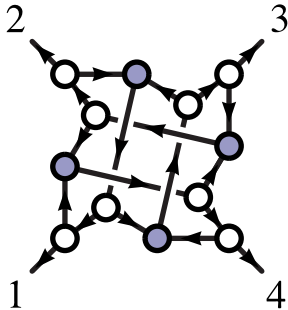
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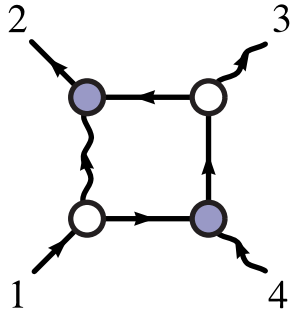
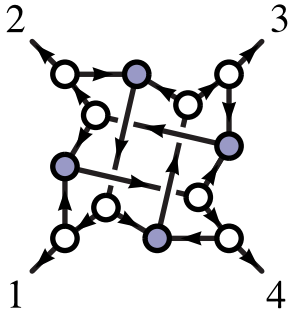
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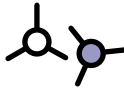
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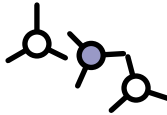
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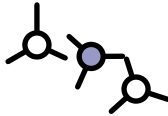
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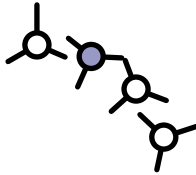
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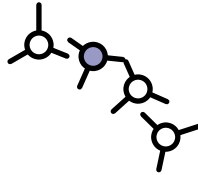
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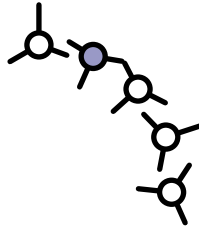
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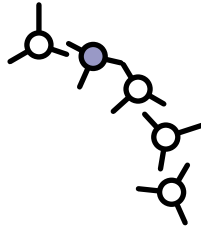
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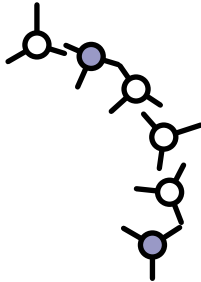
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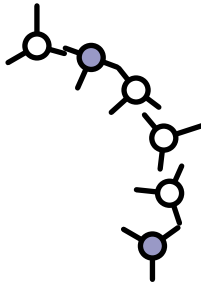
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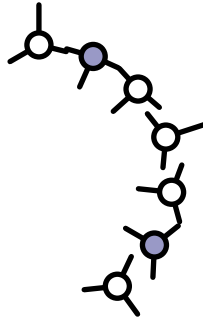
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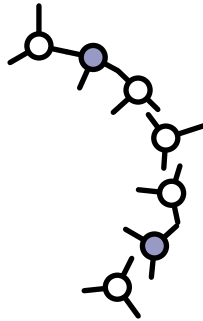
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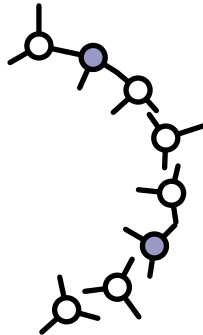
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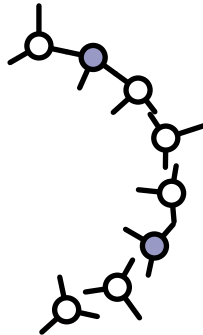
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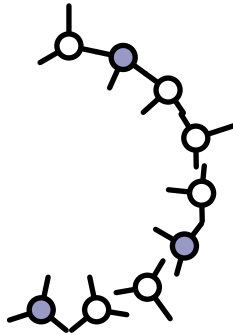
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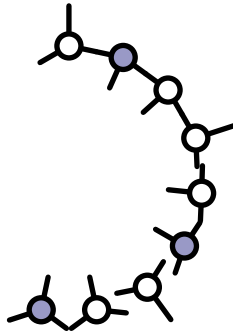
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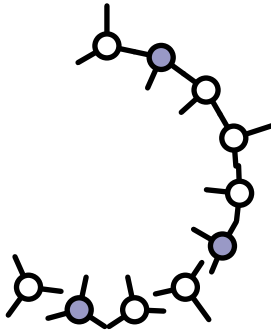
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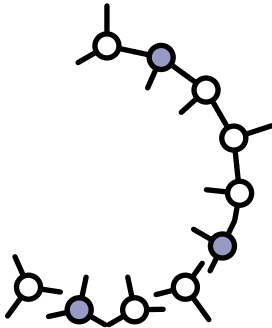
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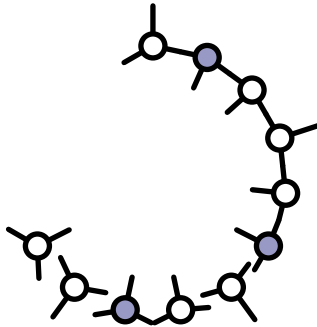
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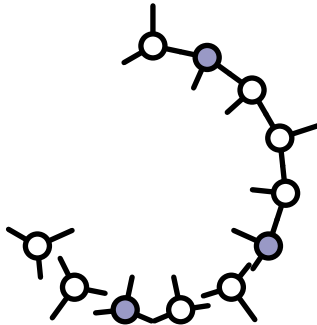
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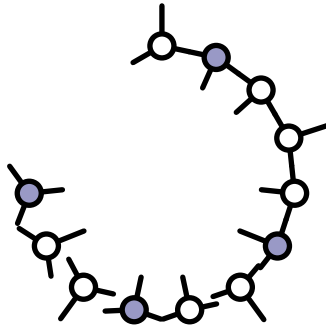
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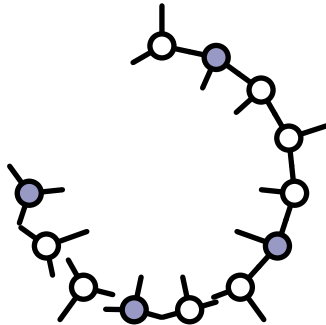
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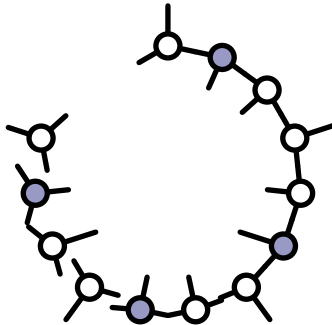
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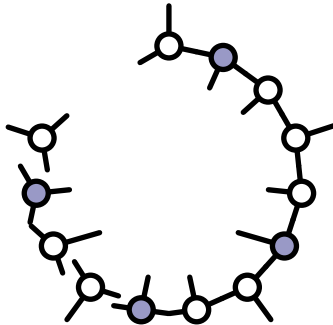
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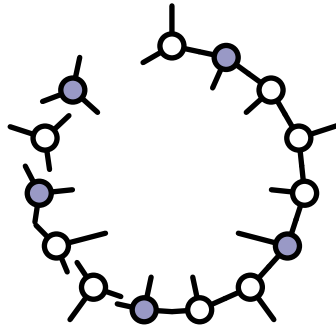
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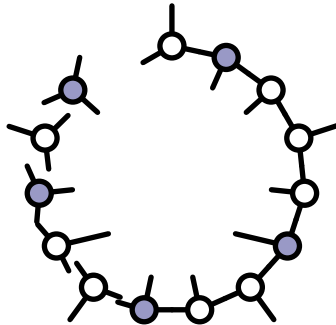
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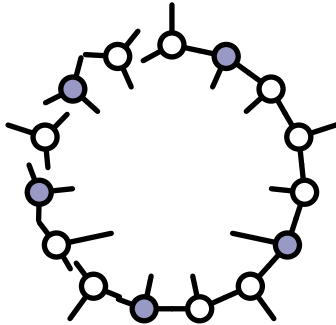
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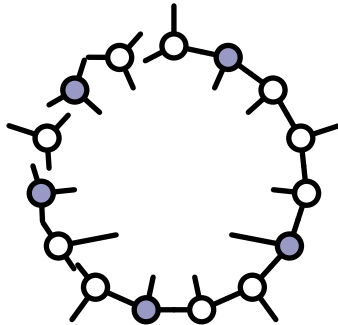
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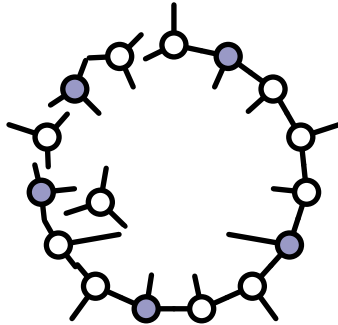
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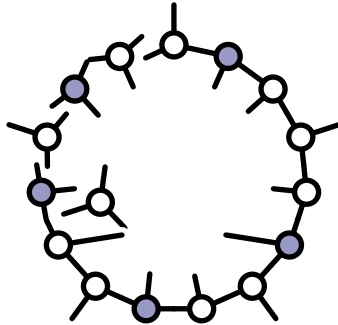
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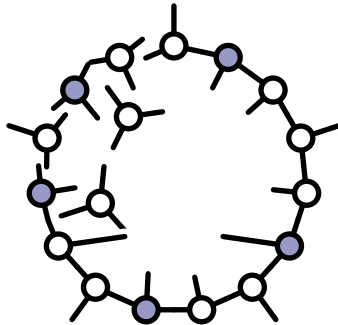
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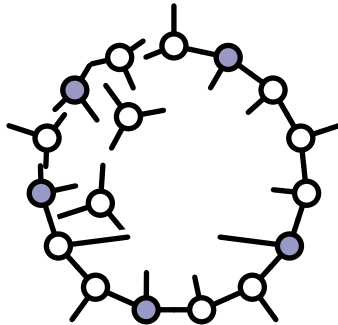
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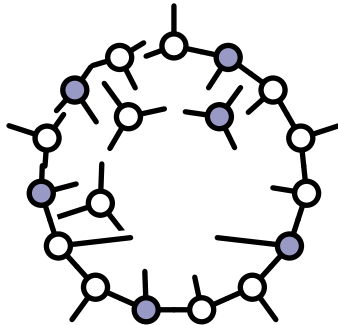
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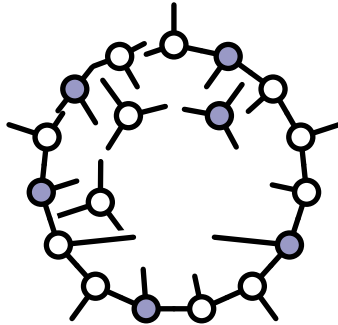
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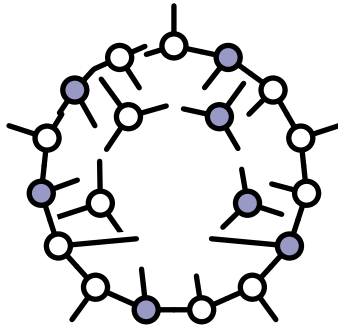
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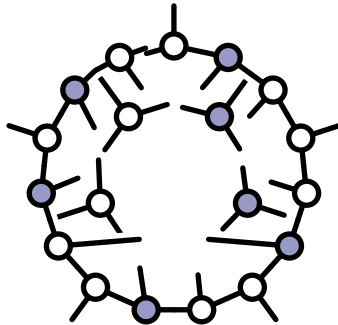
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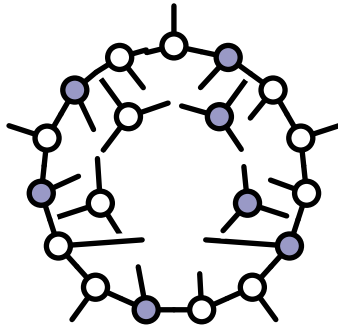
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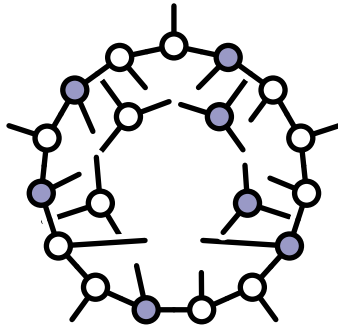
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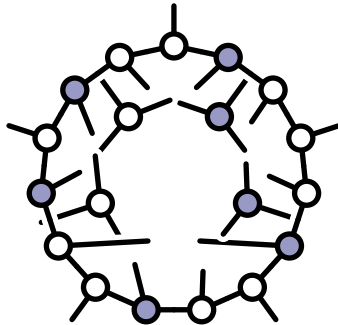
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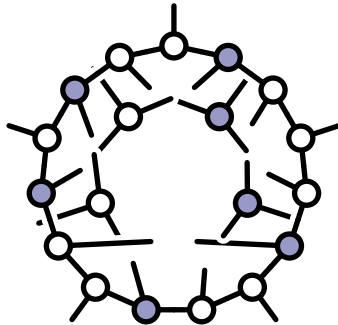
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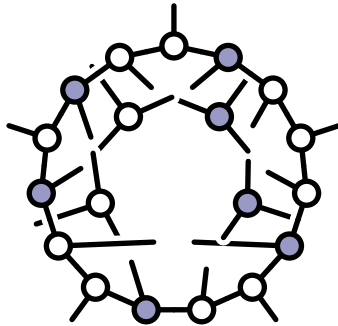
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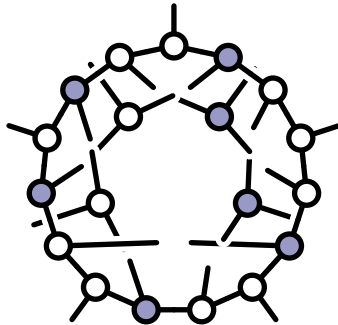
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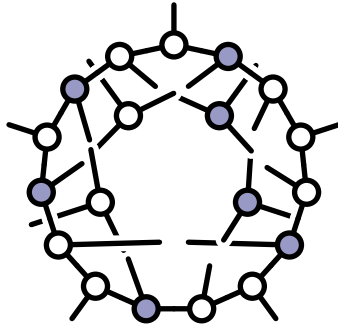
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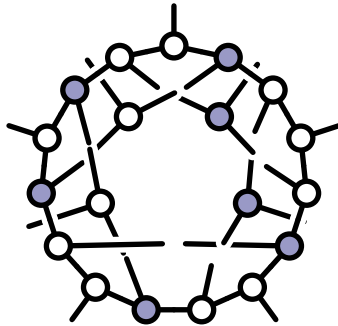
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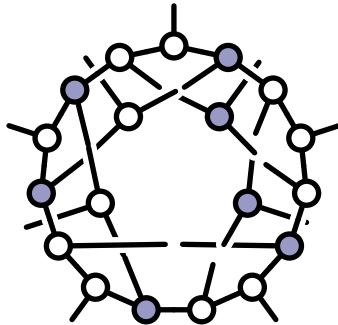
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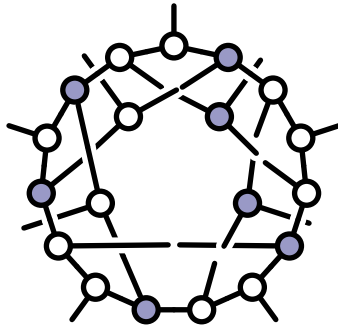
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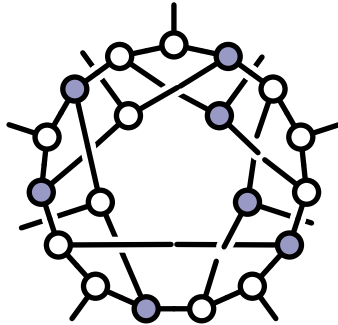
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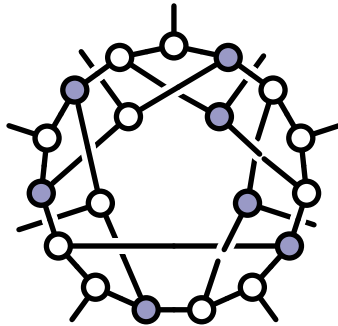
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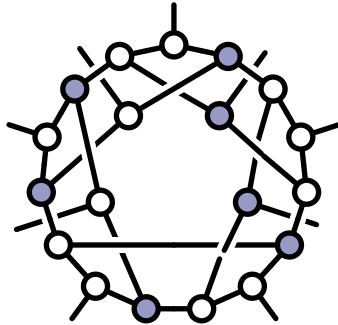
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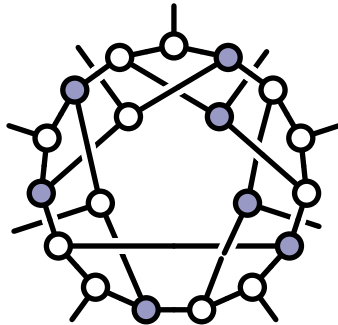
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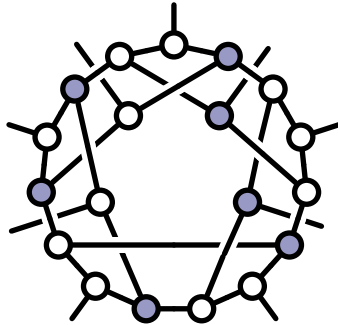
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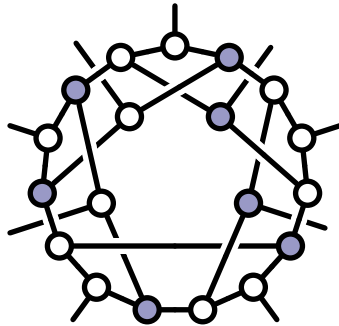
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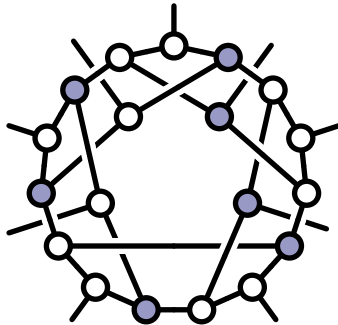
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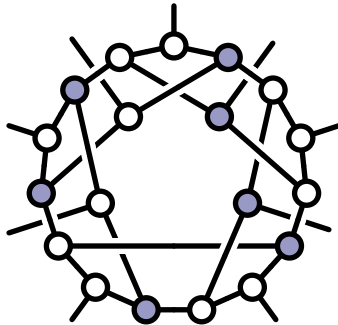
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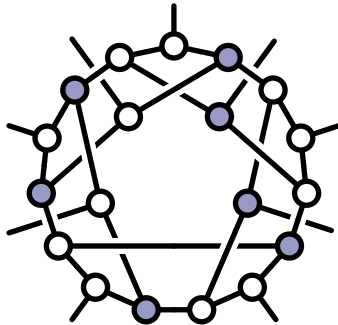
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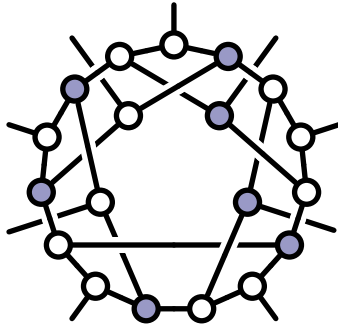
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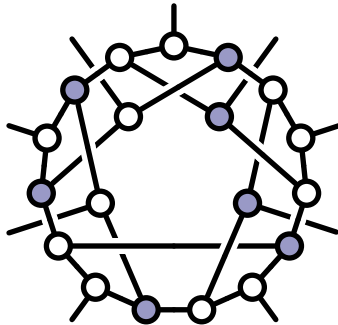
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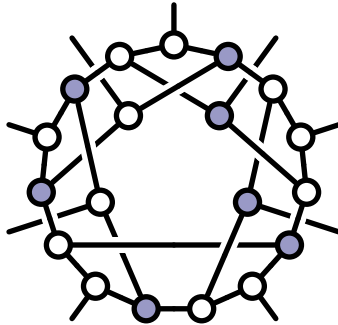
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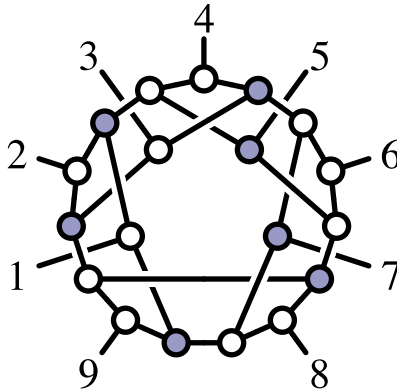
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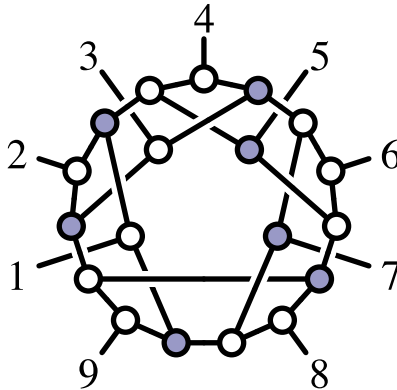
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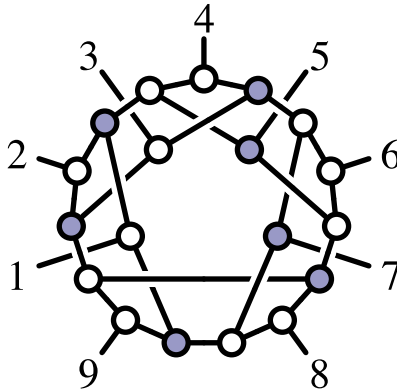
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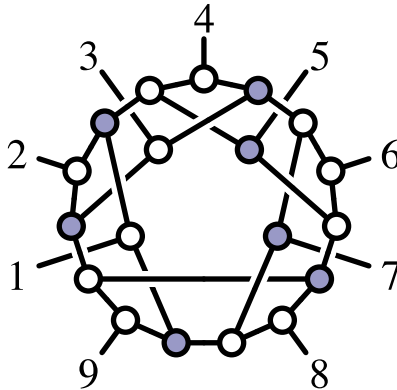
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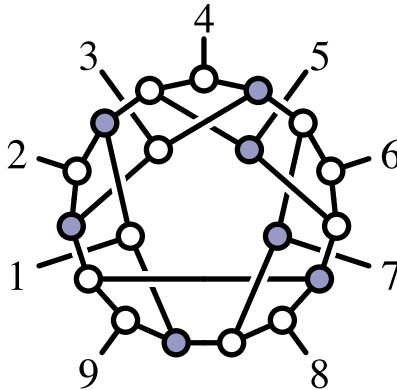
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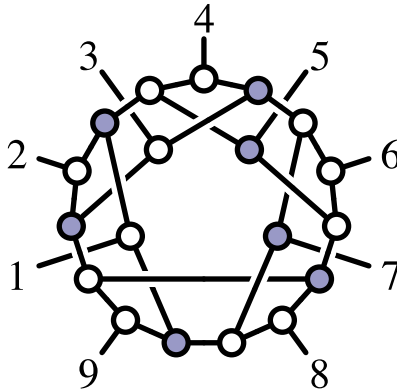
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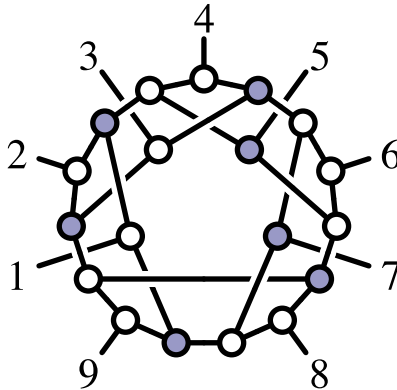
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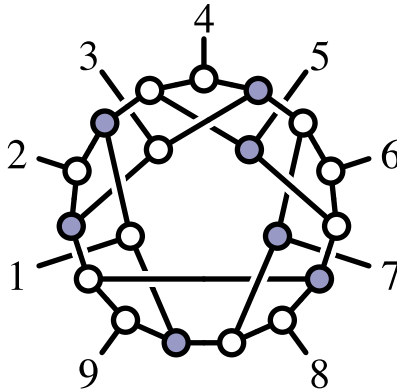
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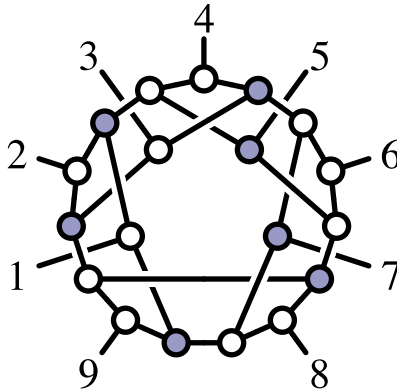
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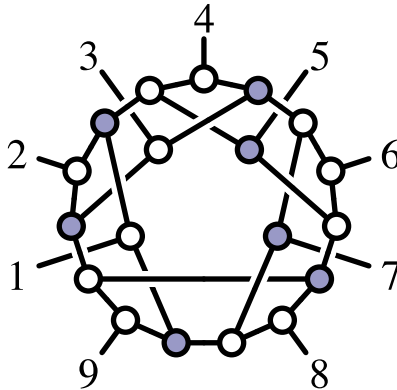
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Building-Up On-Shell Diagrams with “BCFW” Bridges

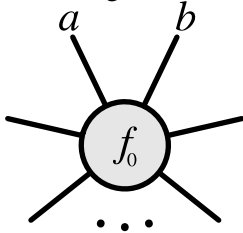
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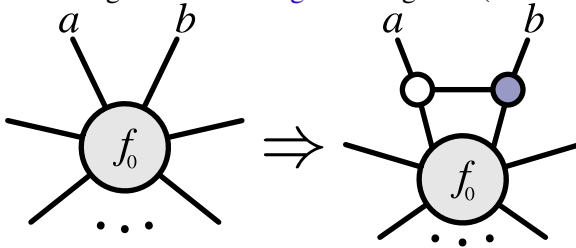
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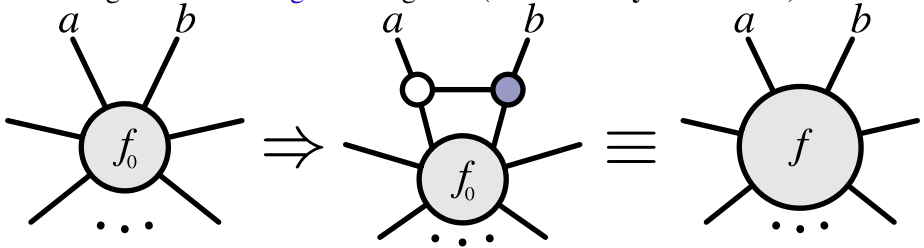
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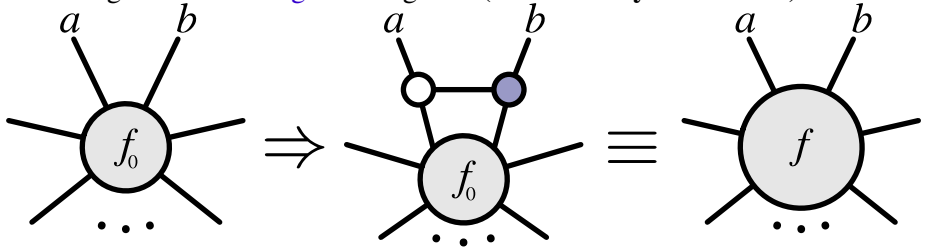
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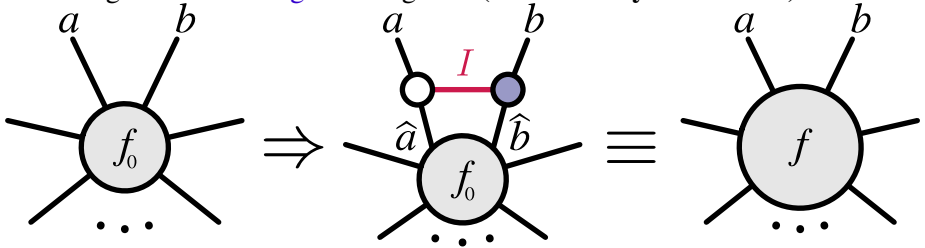
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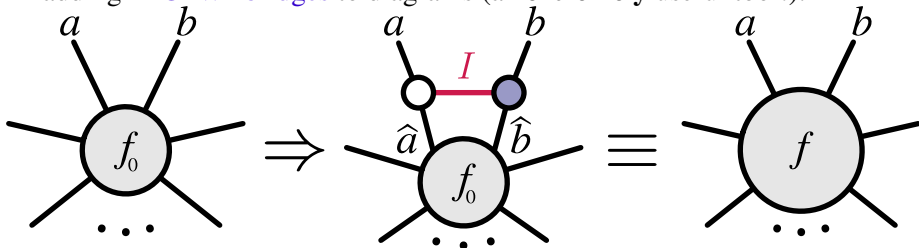
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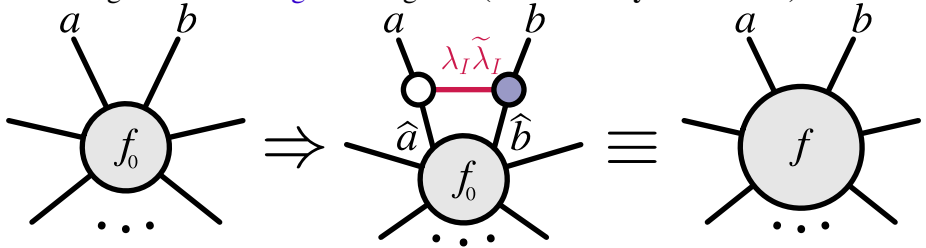


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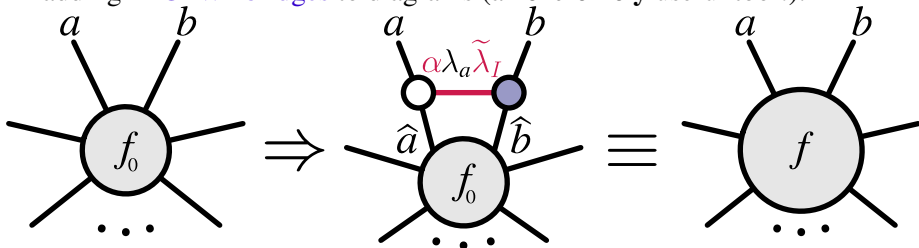


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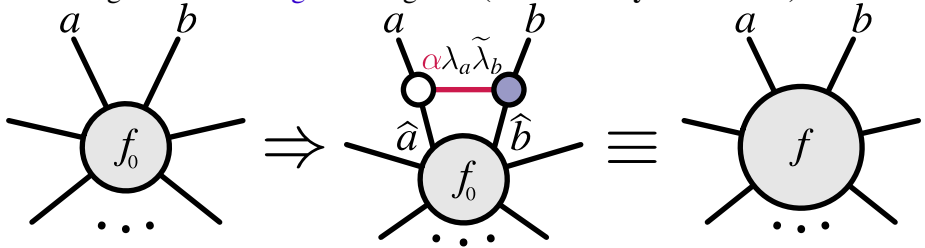


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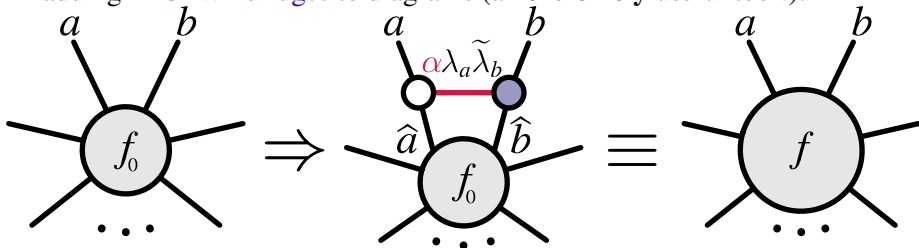


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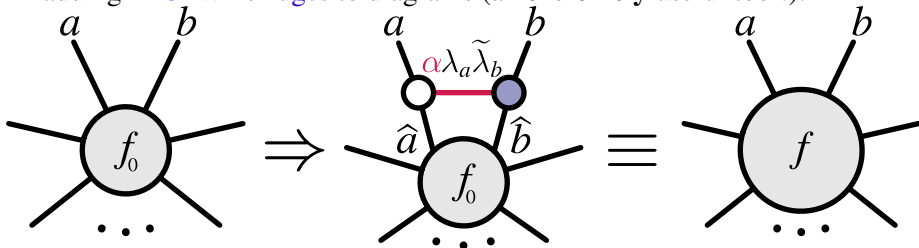


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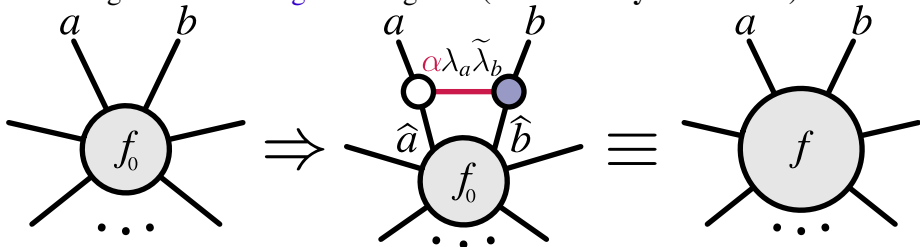


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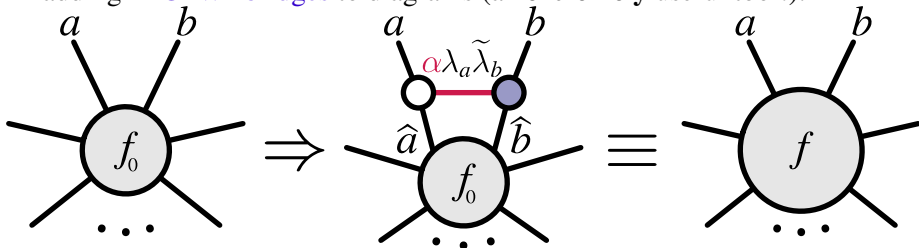
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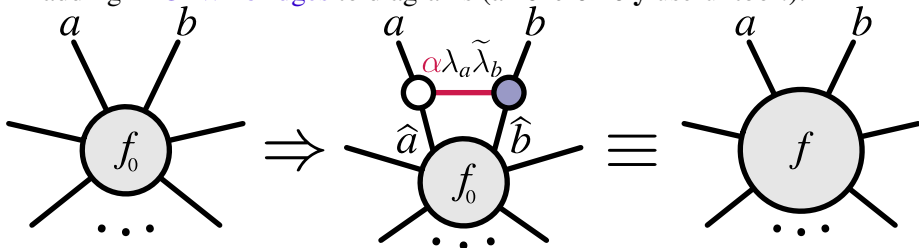
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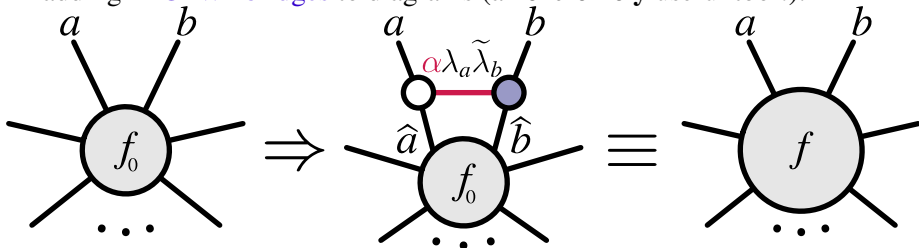
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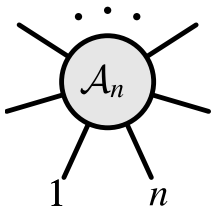
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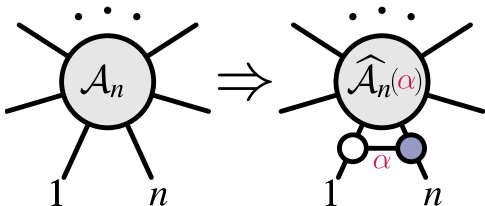
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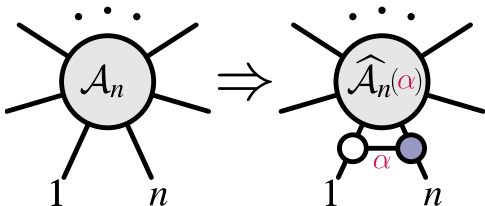
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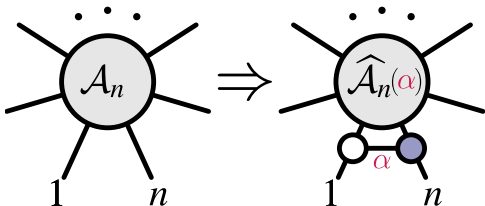
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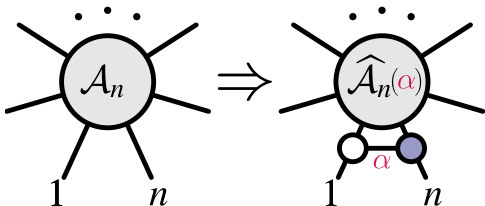


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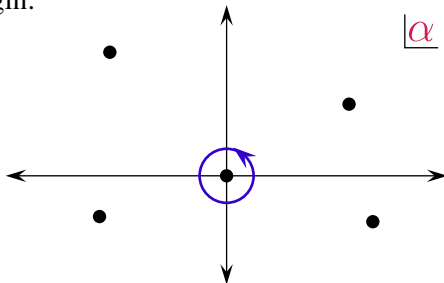
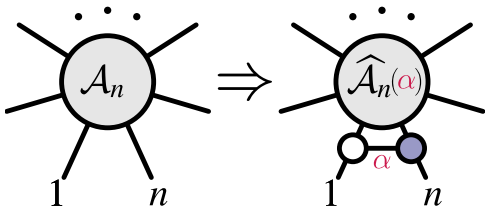


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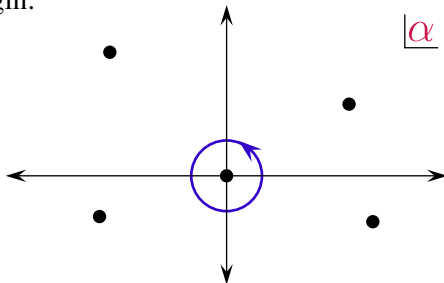
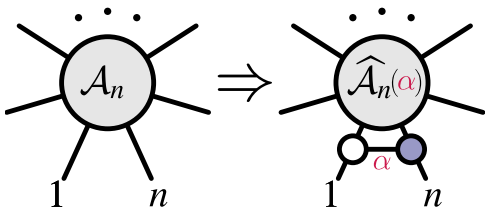


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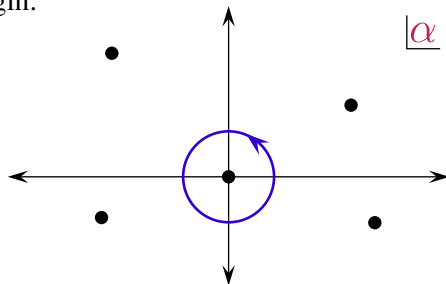
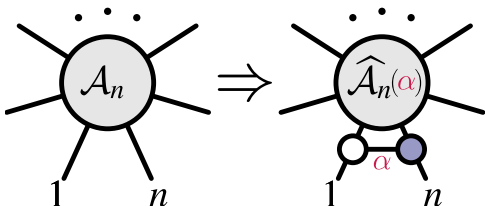


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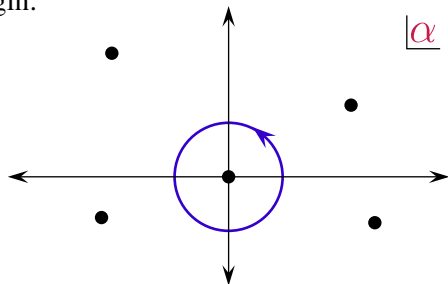
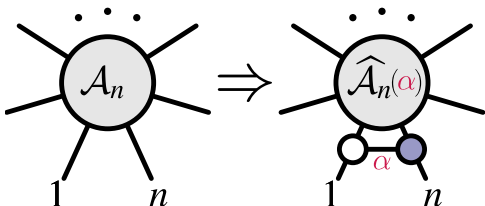


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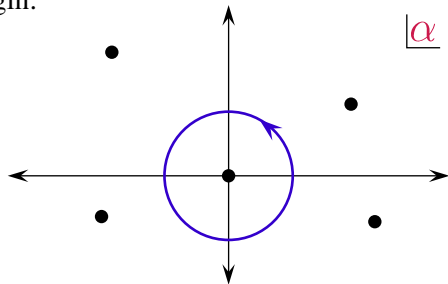
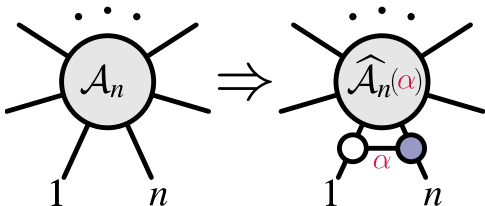


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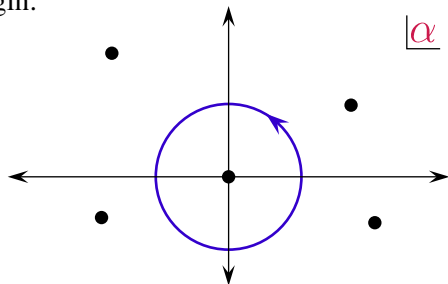
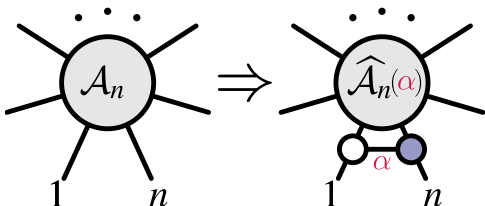


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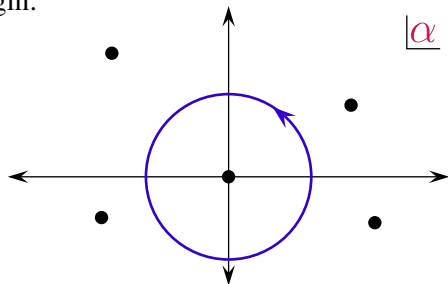
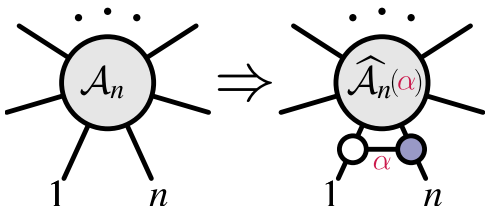


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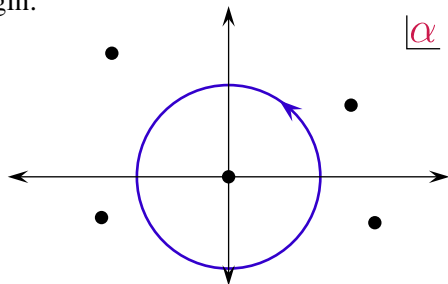
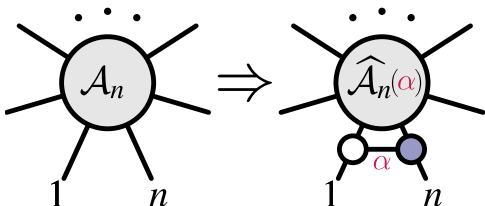


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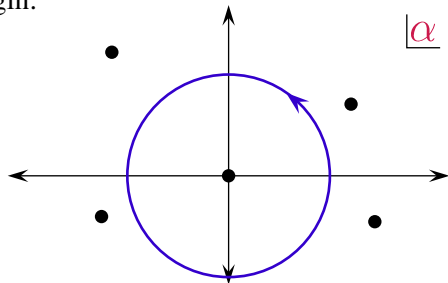
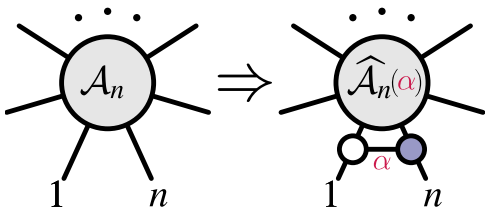


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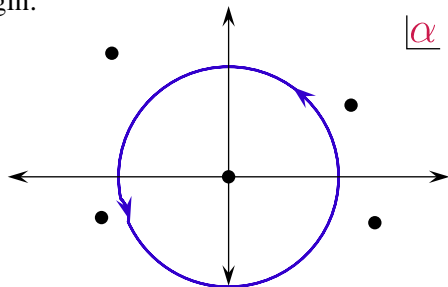
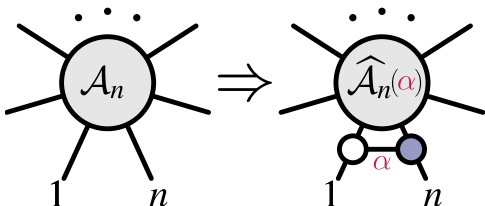


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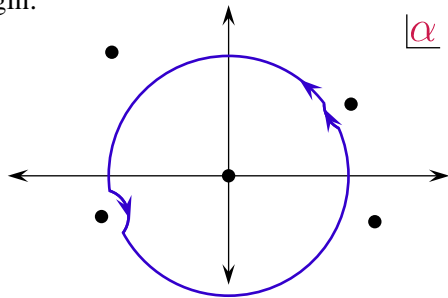
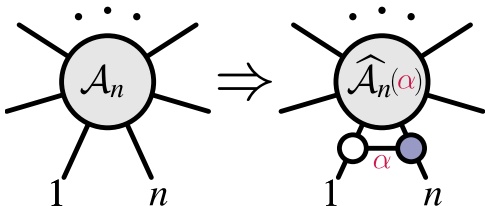


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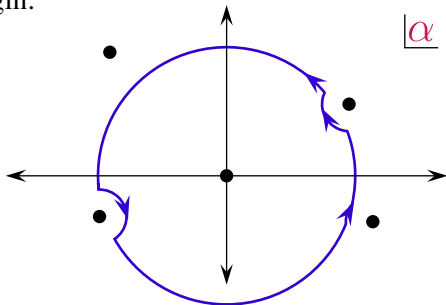
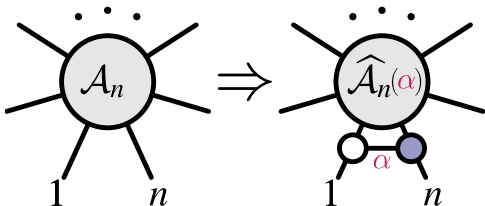


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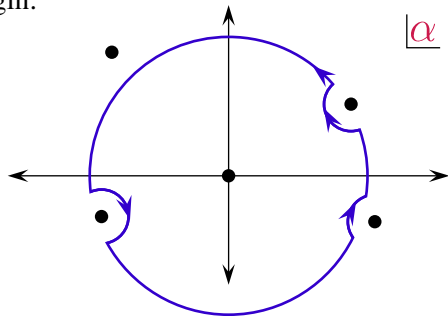
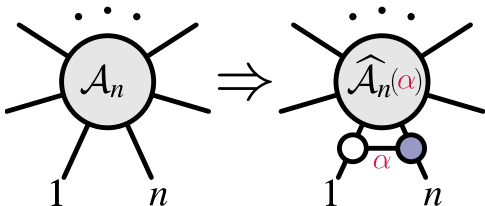


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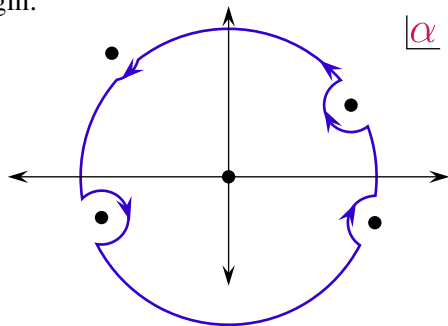
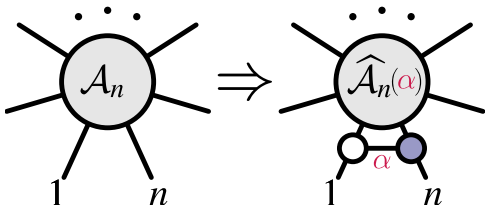


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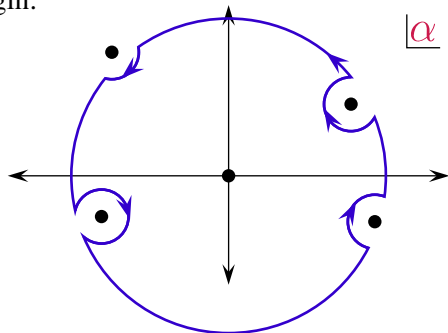
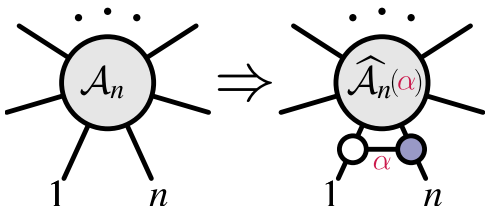


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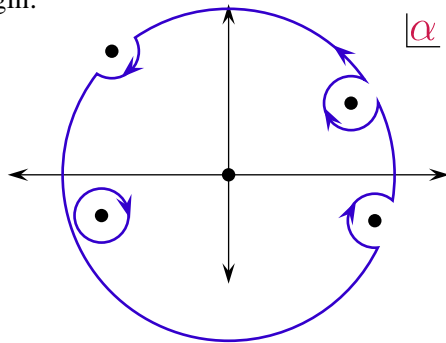
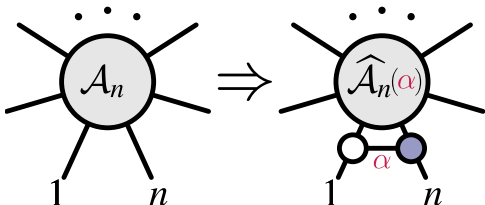


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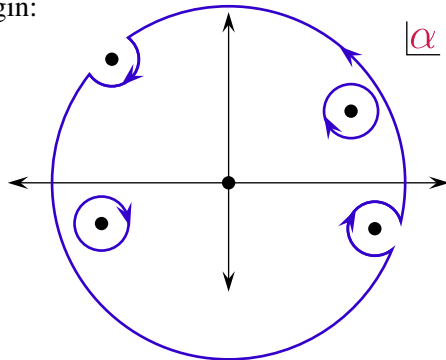
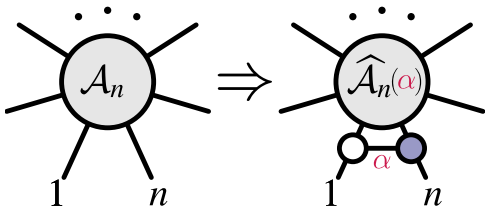


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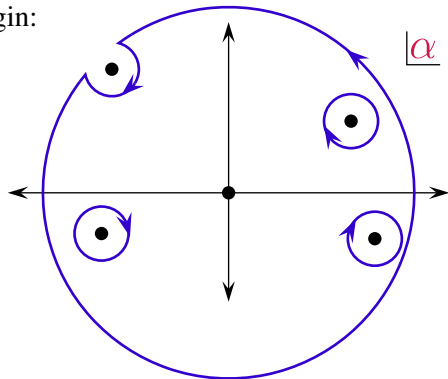
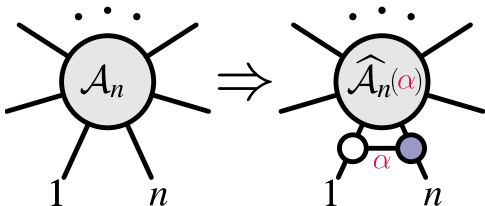


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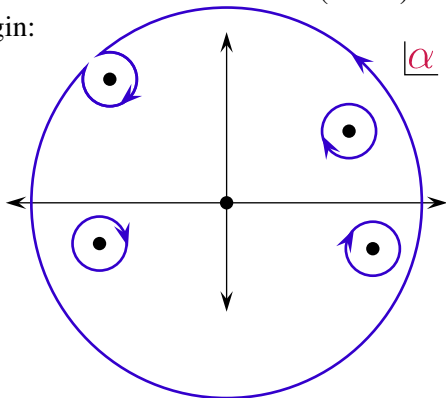
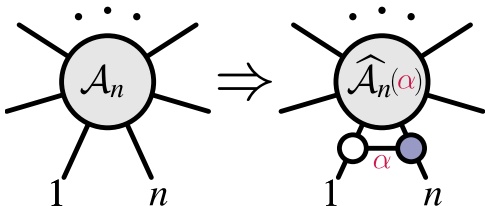


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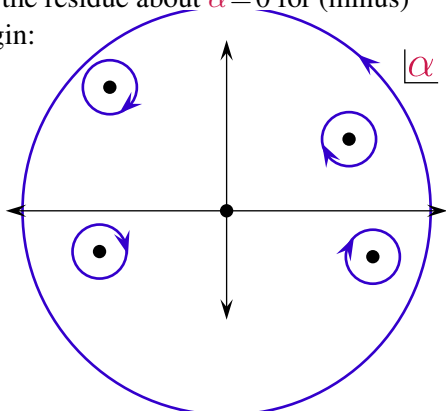
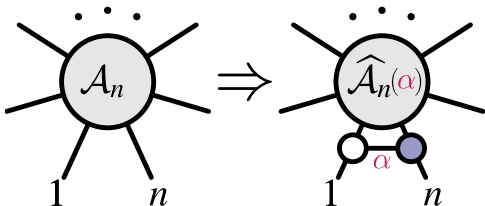


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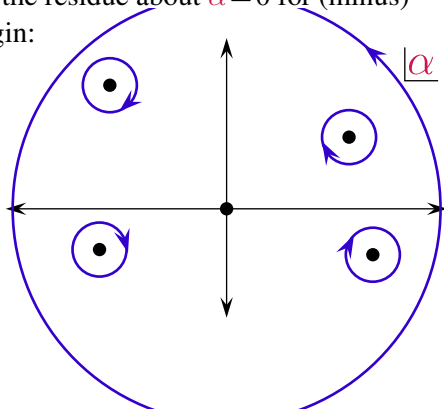
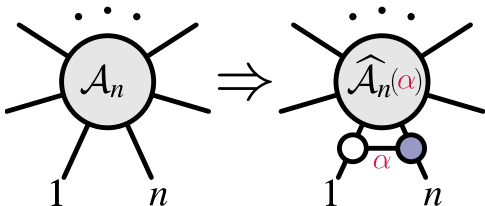


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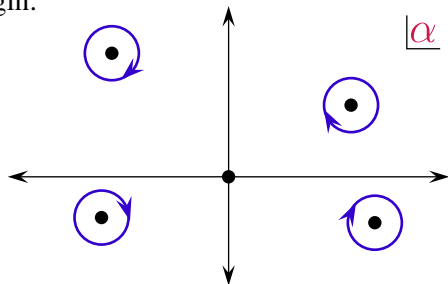
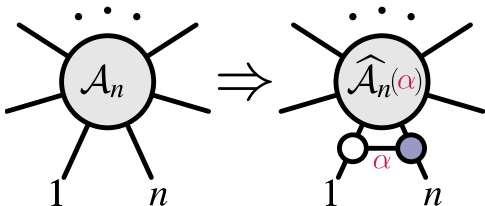


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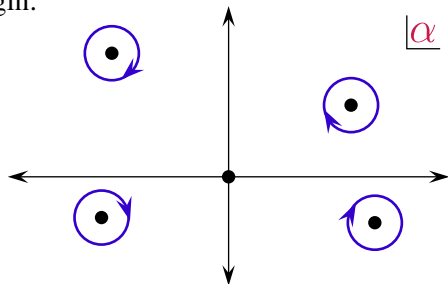
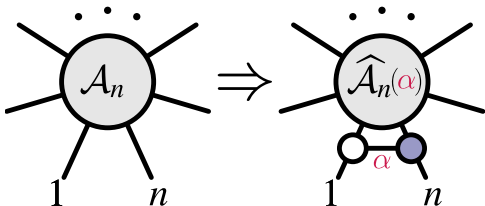


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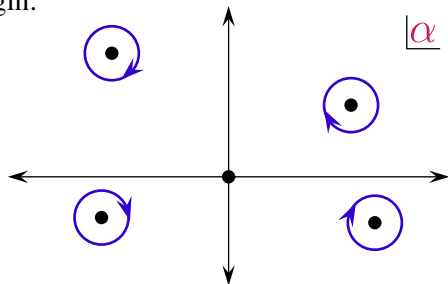
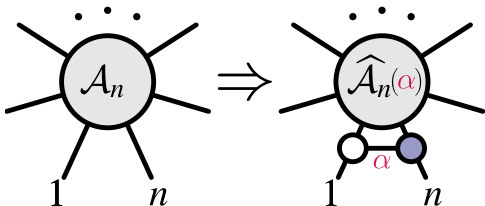


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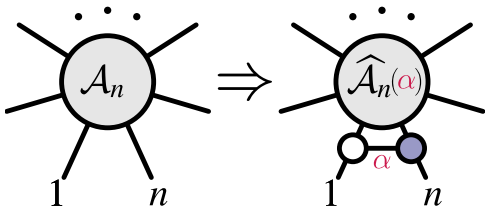


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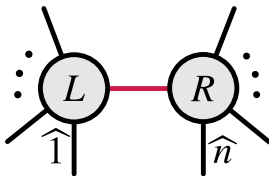


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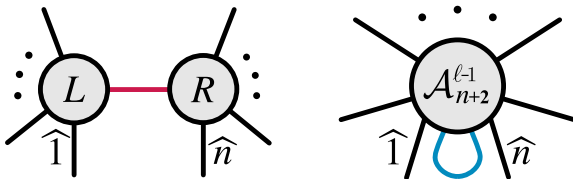


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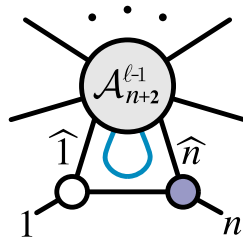
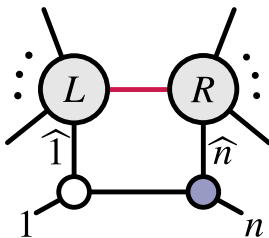


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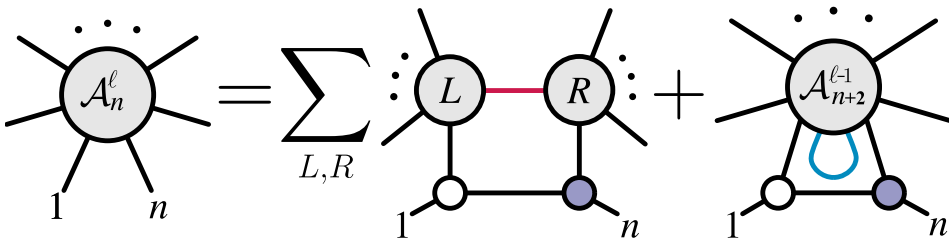


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Consider adding a BCFW bridge to the full n -particle scattering amplitude the **undeformed** amplitude \mathcal{A}_n is recovered as the **residue** about $\alpha=0$:

$$\mathcal{A}_n = \hat{\mathcal{A}}_n(\alpha \rightarrow 0) \propto \oint_{\alpha=0} \frac{d\alpha}{\alpha} \hat{\mathcal{A}}_n(\alpha)$$

We can use **Cauchy's theorem** to trade the residue about $\alpha=0$ for (minus) the sum of residues away from the origin—these come in two types: **factorization-channels** and **forward-limits**



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The diagram illustrates the analytic bootstrap recursion relation for all-loop amplitudes. On the left, a circular node labeled A_n^ℓ has n external legs, with the first and last legs labeled 1 and n respectively. This node is equal to a sum over all possible partitions L, R of the external legs. The summand consists of two circular nodes, L and R , connected by a red horizontal line. Node L has legs 1 and n (represented by a white circle), and node R has legs 1 and n (represented by a blue circle). The sum is then added to a third term, which is a circular node labeled $A_{n+2}^{\ell-1}$ with a blue loop connecting its legs 1 and n (represented by a white circle and a blue circle respectively).

$$A_n^\ell = \sum_{L,R} \left(\text{Diagram with nodes } L \text{ and } R \text{ connected by a red line} \right) + \left(\text{Diagram with node } A_{n+2}^{\ell-1} \text{ and a blue loop} \right)$$

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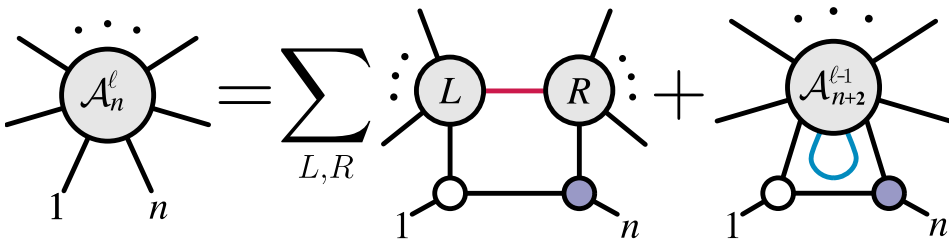
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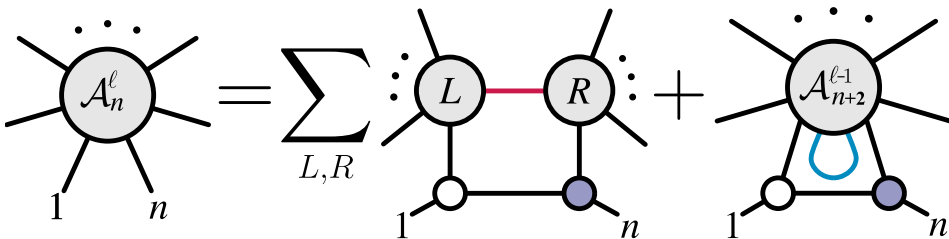
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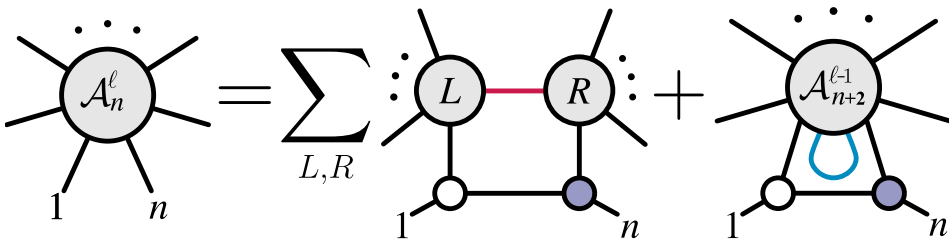
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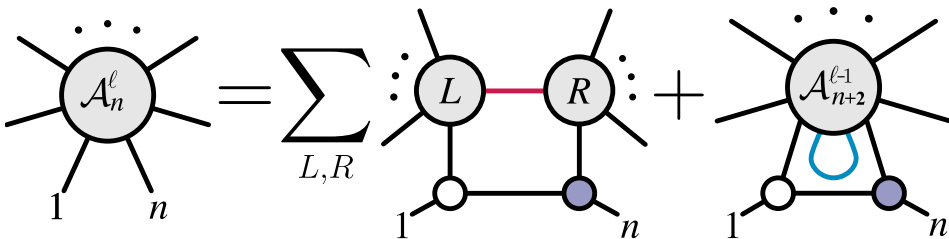
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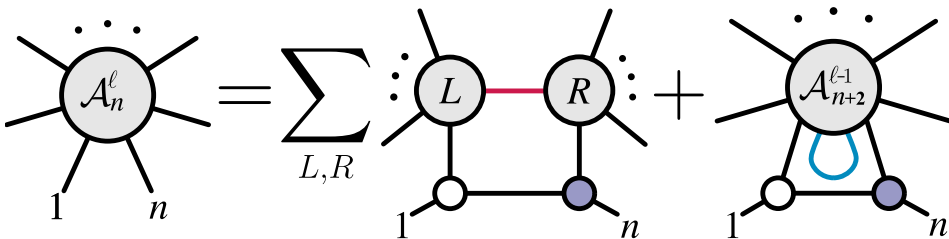
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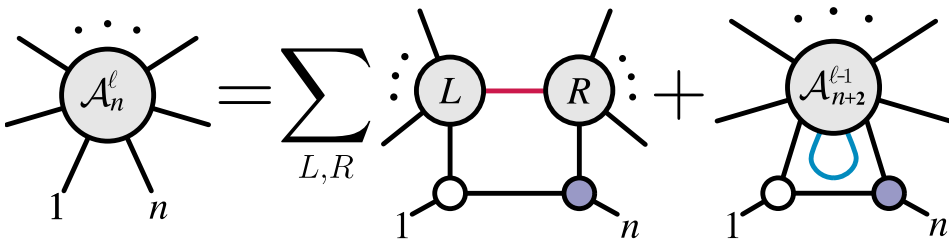
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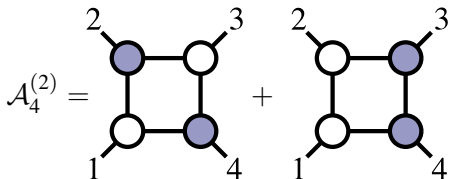
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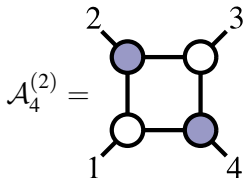
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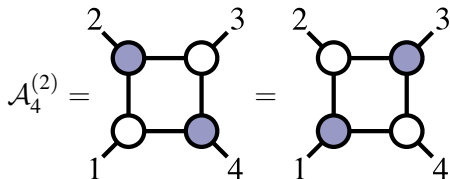
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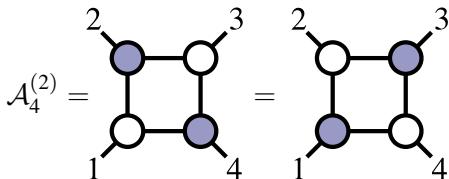
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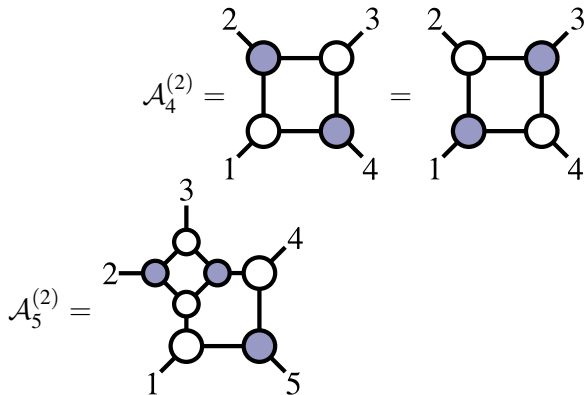
The diagram illustrates the definition of $\mathcal{A}_4^{(2)}$ as the sum of two square graphs. The first square has blue nodes at the top-left and bottom-right positions, with edges labeled 1, 2, 3, and 4. The second square has blue nodes at the top-right and bottom-left positions, with edges labeled 1, 2, 3, and 4. The equation is shown as $\mathcal{A}_4^{(2)} =$ followed by the two squares separated by a plus sign.

The diagram shows a graph labeled $\mathcal{A}_5^{(2)} =$ on the left. The graph consists of four nodes arranged in a square. The top-left node is a grey circle labeled $\mathcal{A}_4^{(2)}$. The top-right node is a white circle. The bottom-left node is a white circle. The bottom-right node is a blue circle. The nodes are connected by edges forming the square. Additionally, there are five external edges labeled 1 through 5: edge 1 connects the bottom-left node to the bottom-right node; edge 2 connects the top-left node to the left; edge 3 connects the top-left node to the top; edge 4 connects the top-right node to the top-right; edge 5 connects the bottom-right node to the bottom-right.

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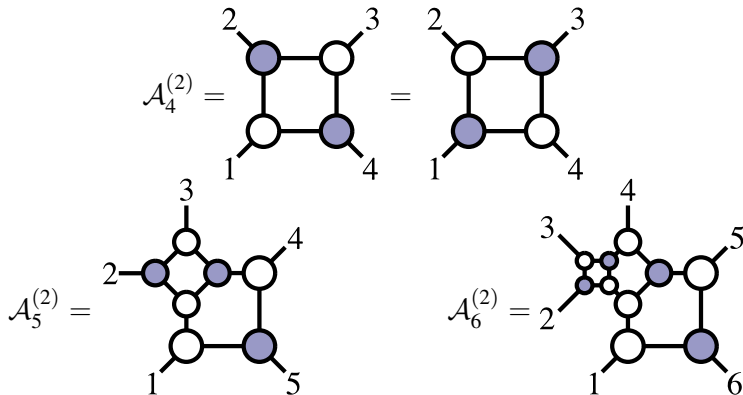
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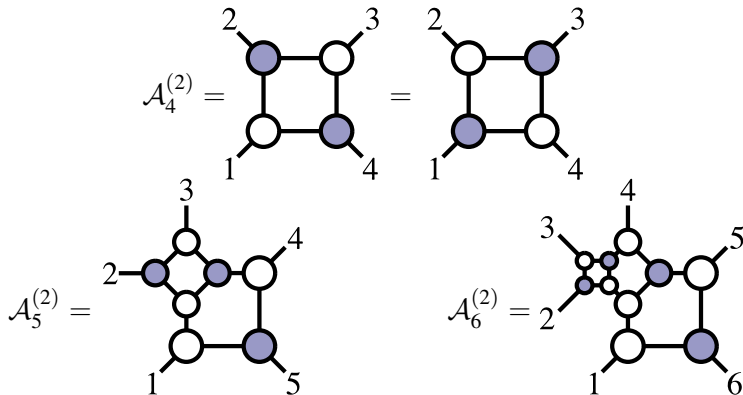
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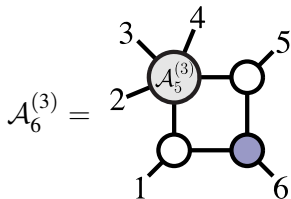
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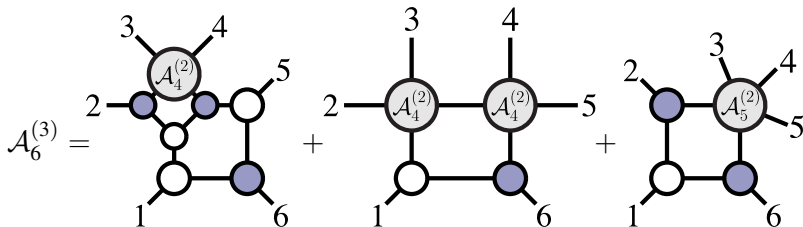
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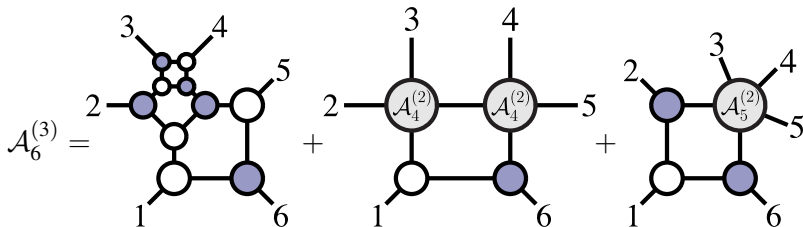
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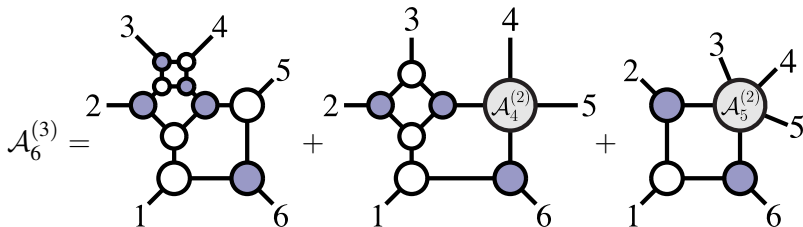
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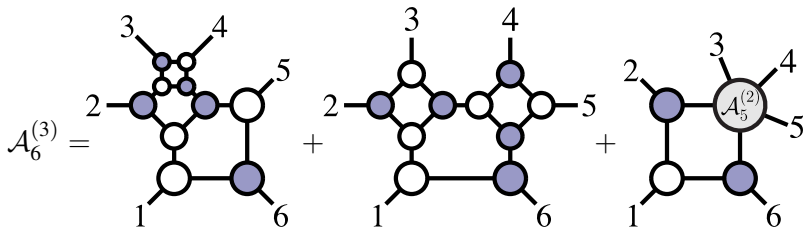
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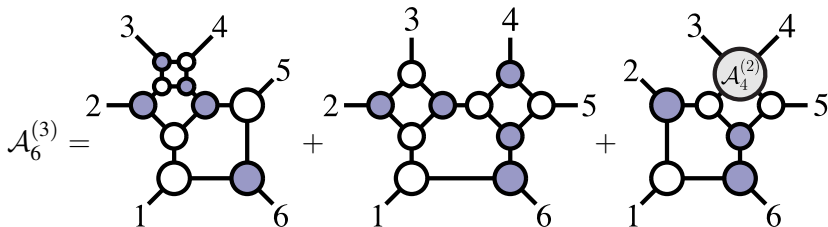
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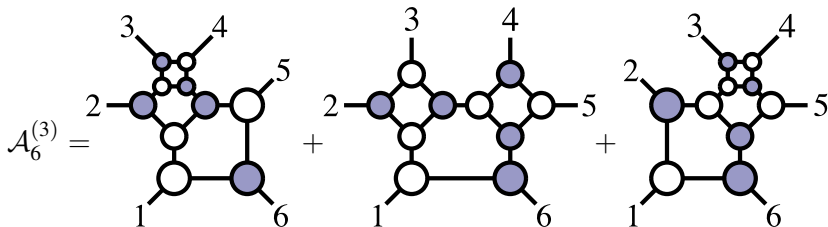
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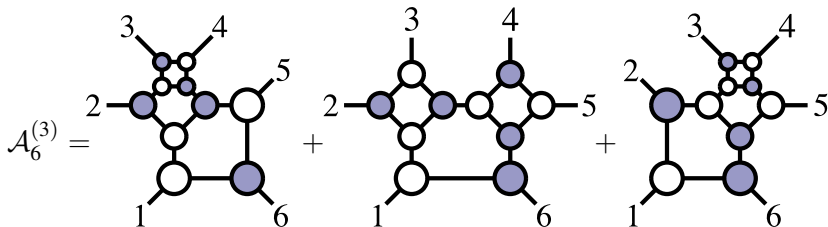
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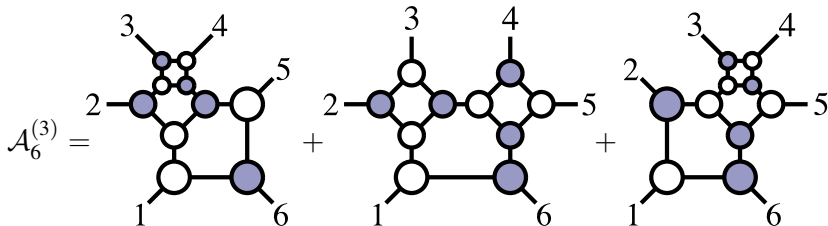


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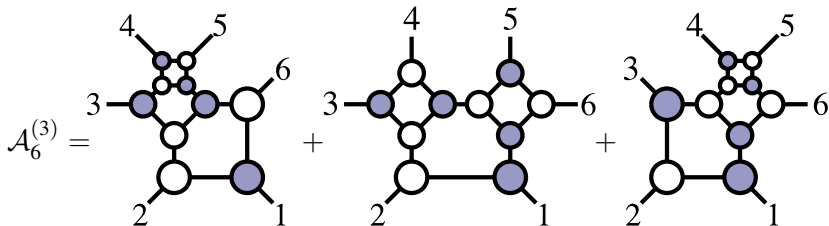


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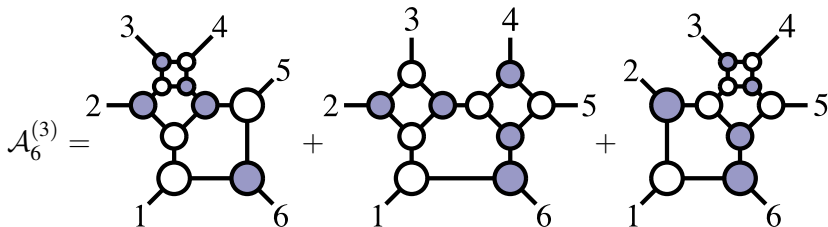
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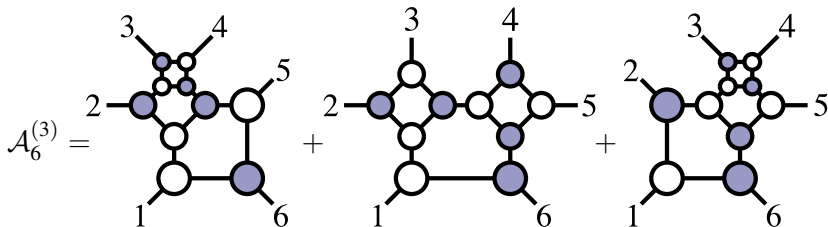


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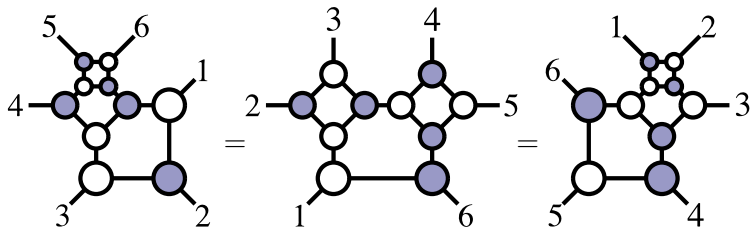


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- on-shell diagrams can often be related in surprising ways

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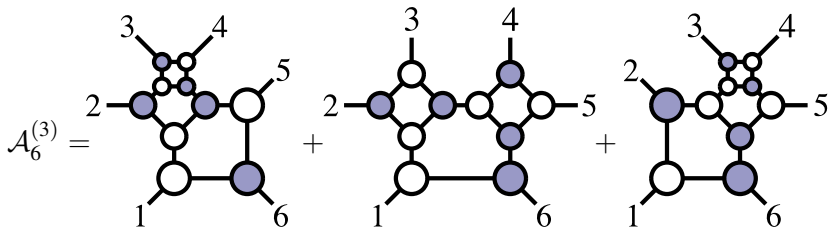


Observations regarding recursed representations of scattering amplitudes:

- varying recursion ‘schema’ can generate *many* ‘BCFW formulae’
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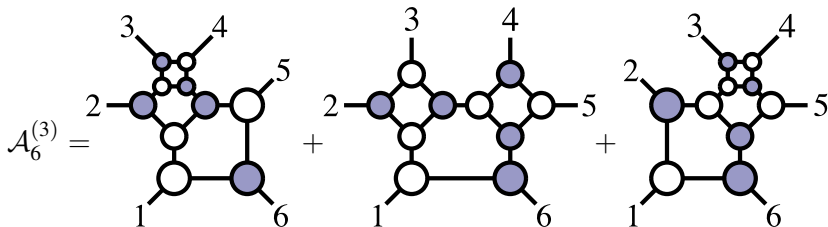


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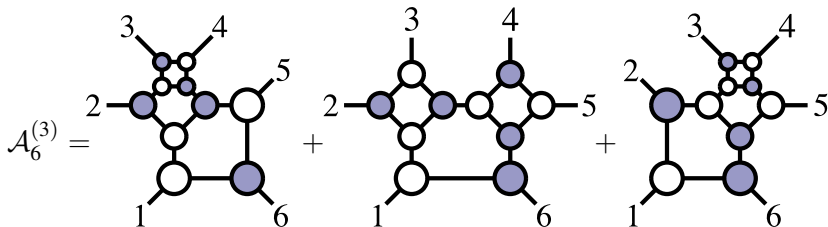
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On-Shell Recursion of Loop-Amplitude Integrands

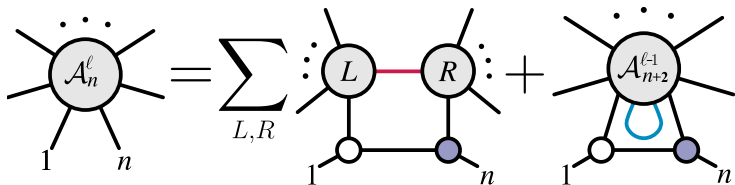
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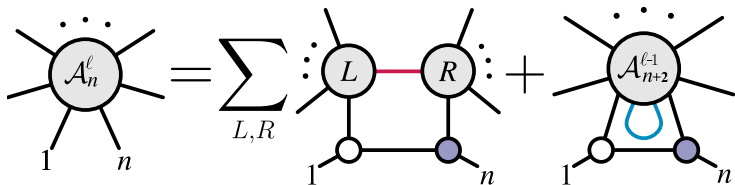
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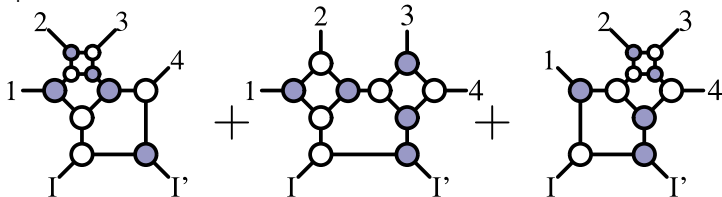
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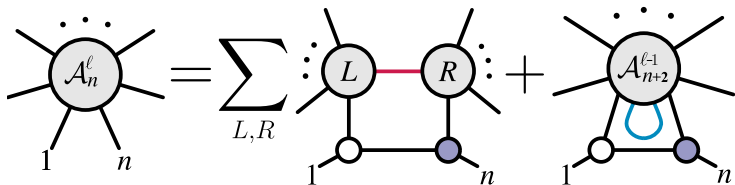
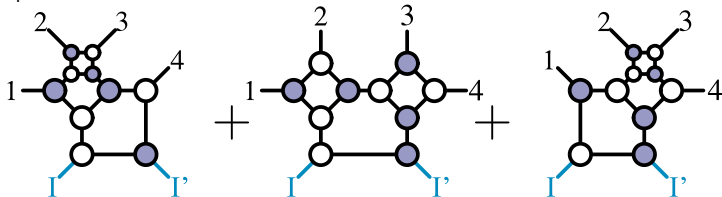
$$\mathcal{A}_n^\ell = \sum_{L,R} \text{Diagram}(L,R) + \mathcal{A}_{n+2}^{\ell-1}$$

The diagram illustrates a general recursion formula for loop amplitudes. A loop amplitude \mathcal{A}_n^ℓ is equal to a sum over L and R of a diagram with two vertices L and R connected by a red line, plus a diagram with a loop amplitude $\mathcal{A}_{n+2}^{\ell-1}$ and a blue loop.

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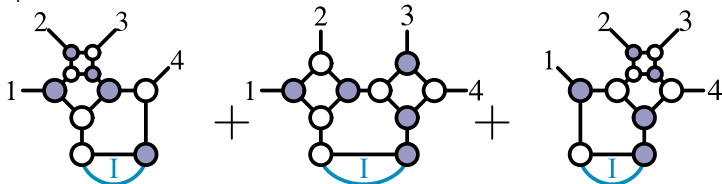
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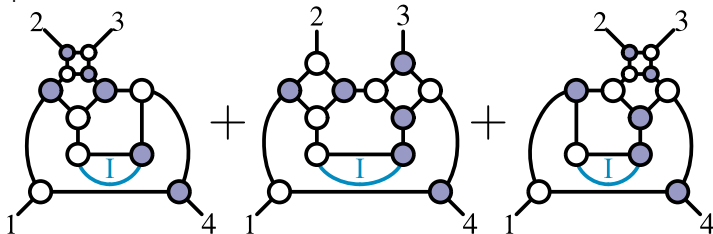


$$\begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} \mathcal{A}_n^\ell \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array} \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} = \sum_{L,R} \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} L \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array} R \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} + \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} \mathcal{A}_{n+2}^{\ell-1} \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array} \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array} \begin{array}{c} \vdots \\ \bullet \\ \vdots \end{array}$$

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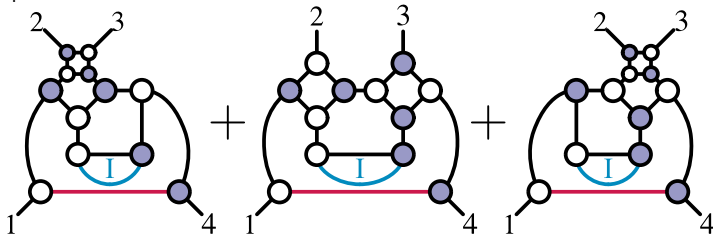
$$\mathcal{A}_n^\ell = \sum_{L,R} \mathcal{A}_L^{\ell-1} \mathcal{A}_R^{\ell-1} + \mathcal{A}_{n+2}^{\ell-1}$$

The diagram illustrates a general recursion formula for loop amplitudes. On the left is a circle with n external legs, labeled \mathcal{A}_n^ℓ . This is equal to a sum over all possible splits L and R of the product of two lower-loop amplitudes $\mathcal{A}_L^{\ell-1}$ and $\mathcal{A}_R^{\ell-1}$, plus a term $\mathcal{A}_{n+2}^{\ell-1}$ which represents a bubble diagram with two internal legs.

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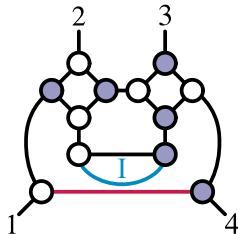


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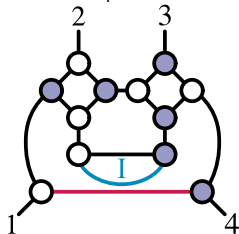


The diagram shows the decomposition of a vertex A_n^ℓ into a sum over two types of vertices, L and R , and a self-energy correction term. The first term is a sum over L and R vertices, where L and R are connected by a red line. The second term is a self-energy correction term, where the vertex $A_{n+2}^{\ell-1}$ is connected to itself by a blue loop. The diagram is labeled with 1 and n at the bottom, indicating the external legs.

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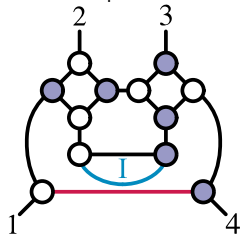
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The diagram illustrates the on-shell recursion relation for a loop amplitude \mathcal{A}_n^ℓ . The left side shows a circle with n external legs, labeled 1 and n , and a summation over all possible factorizations L, R . The right side shows the sum of two terms: a tree-level amplitude $\mathcal{A}_{n+2}^{\ell-1}$ with $n+2$ external legs, and a diagram representing the forward limit of a tree amplitude, where the external legs are connected by a red line and a blue arc.

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Let's look at an example of how loop amplitudes are represented by recursion.

For $\mathcal{A}_4^{(2),1}$, the only terms come from the 'forward limit' of the tree $\mathcal{A}_6^{(3),0}$:



$$\int_{\ell \in \mathbb{R}^{3,1}} d^4 \ell$$

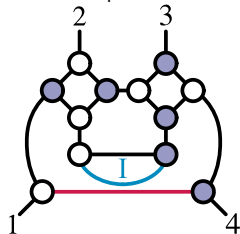
$$\text{Diagram of } \mathcal{A}_n^\ell = \sum_{L,R} \text{Diagram of } L \text{ and } R \text{ connected by a red line} + \text{Diagram of } \mathcal{A}_{n+2}^{\ell-1}$$

The diagram illustrates the on-shell recursion relation for a loop amplitude \mathcal{A}_n^ℓ . On the left is a circle with n external legs, labeled 1 and n , and ℓ internal lines. This is equal to a sum over all possible partitions L and R of the external legs. The first term in the sum is a diagram where two sub-amplitudes L and R are connected by a red line, with external legs 1 and n at the bottom. The second term is a diagram where a sub-amplitude $\mathcal{A}_{n+2}^{\ell-1}$ is connected to external legs 1 and n by a blue arc labeled 'I'.

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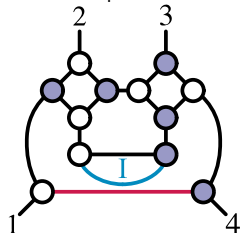
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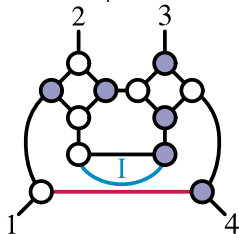
$$\ell \equiv (\lambda_I \tilde{\lambda}_I + \alpha \lambda_1 \tilde{\lambda}_4) \in \mathbb{R}^{3,1}$$

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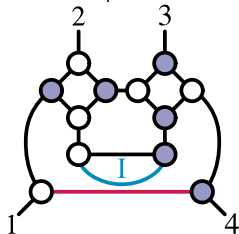
$$\int_{\ell \in \mathbb{R}^{3,1}} d^4 \ell \quad \Longleftrightarrow \quad \int_{\ell \equiv (\lambda_I \tilde{\lambda}_I + \alpha \lambda_I \tilde{\lambda}_\alpha) \in \mathbb{R}^{3,1}} \frac{d^2 \lambda_I d^2 \tilde{\lambda}_I}{\text{vol}(GL_1)} d\alpha \langle \mathbf{I1} \rangle [n\mathbf{I}]$$

$$\mathcal{A}_4^{(2),0} \times \int_{\ell \in \mathbb{R}^{3,1}} d\log\left(\frac{\ell^2}{(\ell - \ell^*)^2}\right) d\log\left(\frac{(\ell + p_1)^2}{(\ell - \ell^*)^2}\right) d\log\left(\frac{(\ell + p_1 + p_2)^2}{(\ell - \ell^*)^2}\right) d\log\left(\frac{(\ell - p_4)^2}{(\ell - \ell^*)^2}\right)$$

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$$= \mathcal{A}_4^{(2),0} \times \int_{\ell \in \mathbb{R}^{3,1}} d^4 \ell \frac{(p_1 + p_2)^2 (p_3 + p_4)^2}{\ell^2 (\ell + p_1)^2 (\ell + p_1 + p_2)^2 (\ell - p_4)^2}$$

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On-shell diagrams can be altered without changing their associated functions

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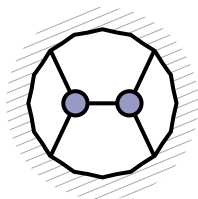
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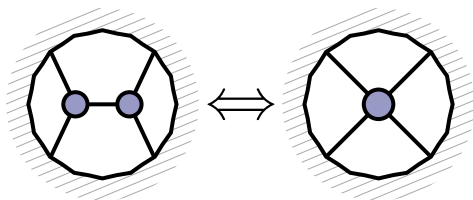
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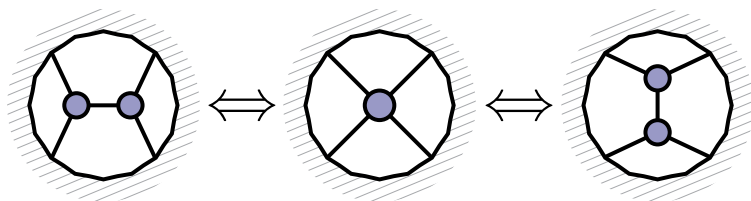
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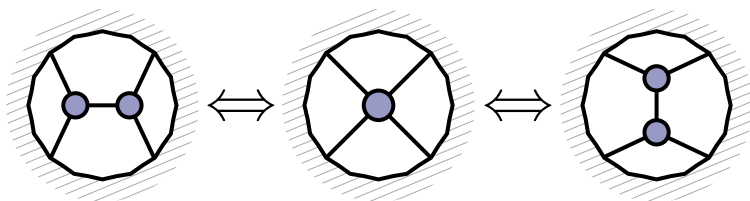
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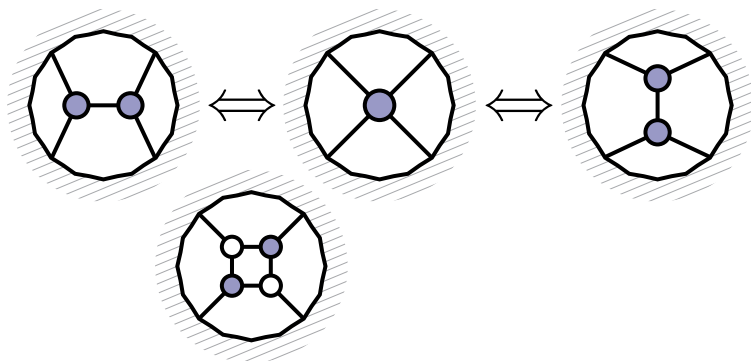
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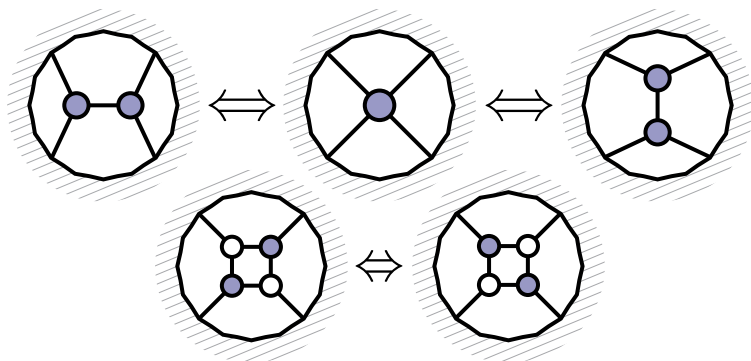
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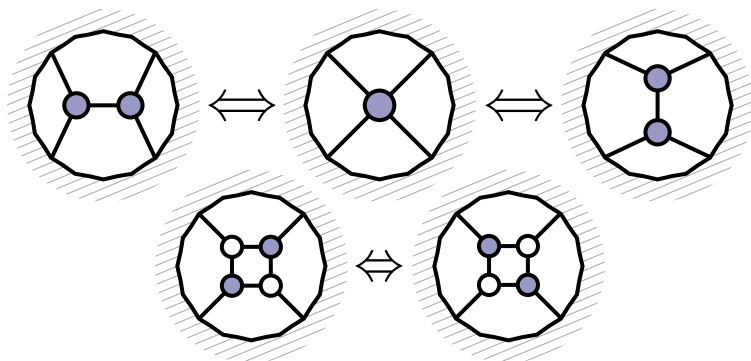
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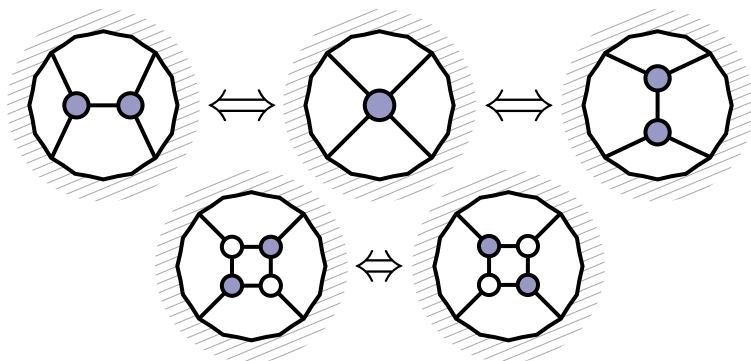
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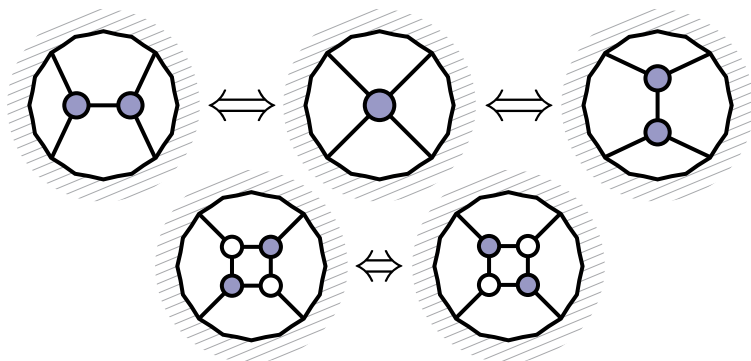


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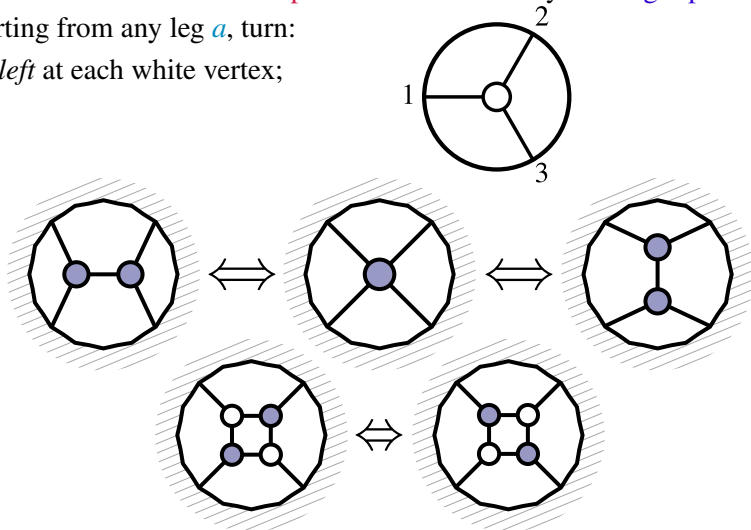


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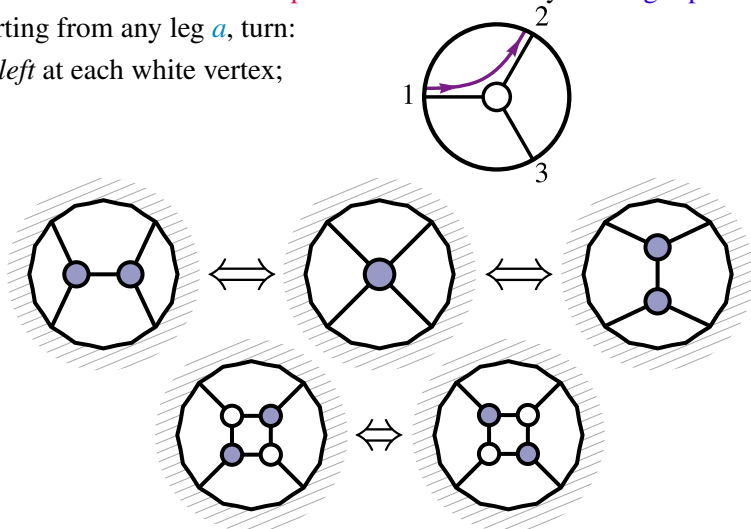


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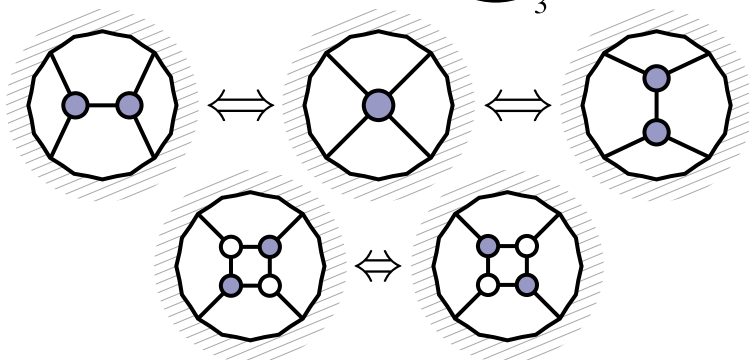
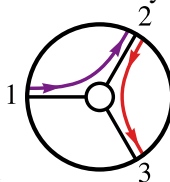


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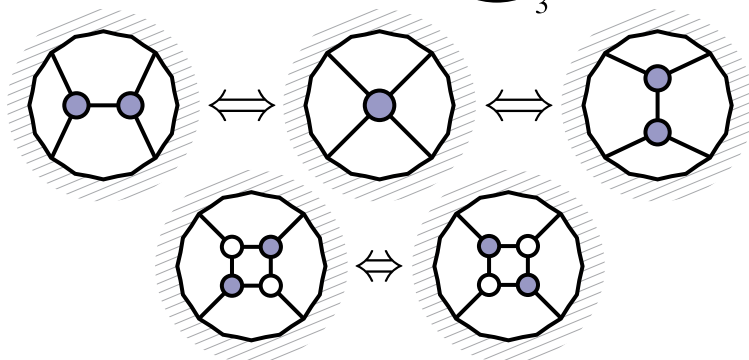
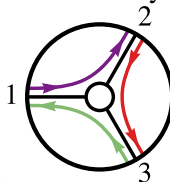


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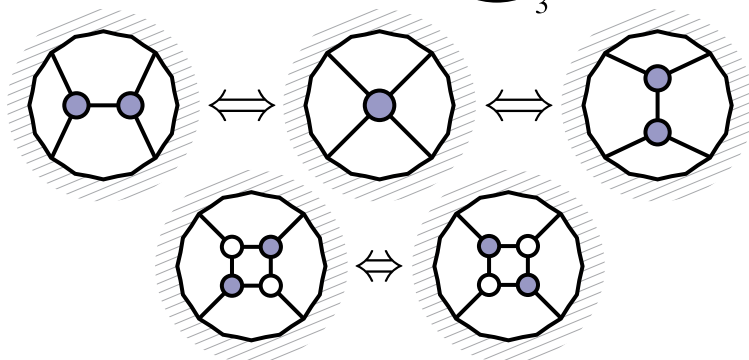
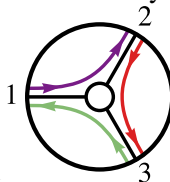


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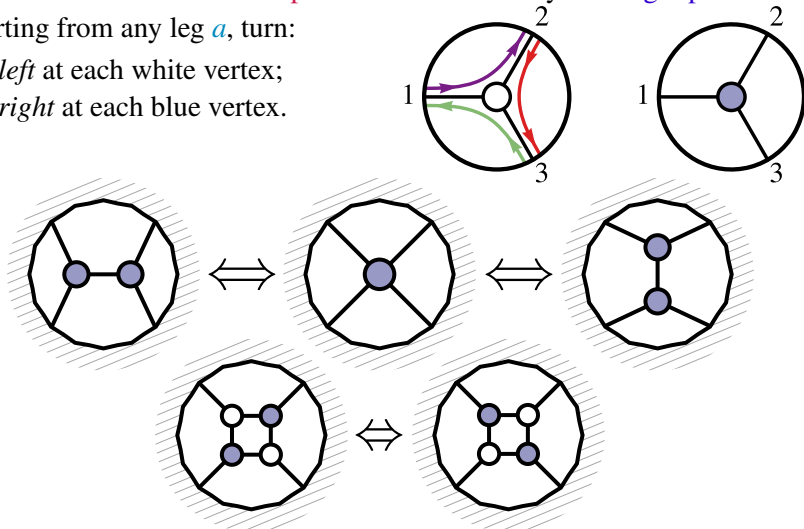


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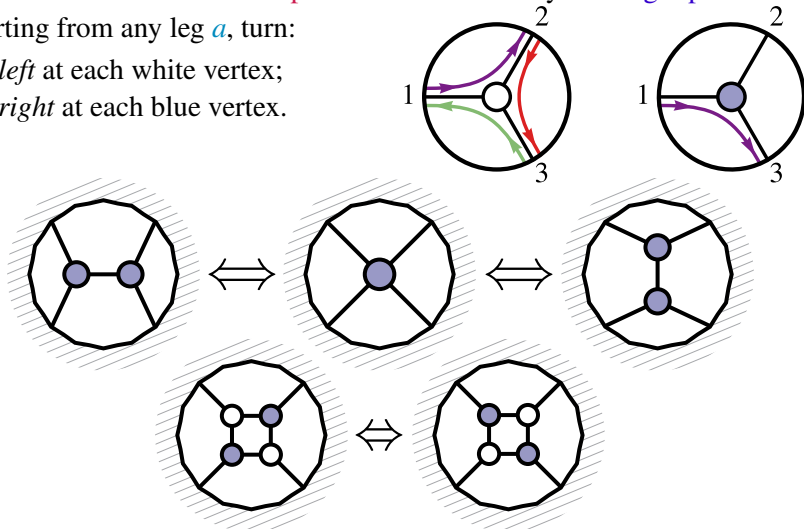


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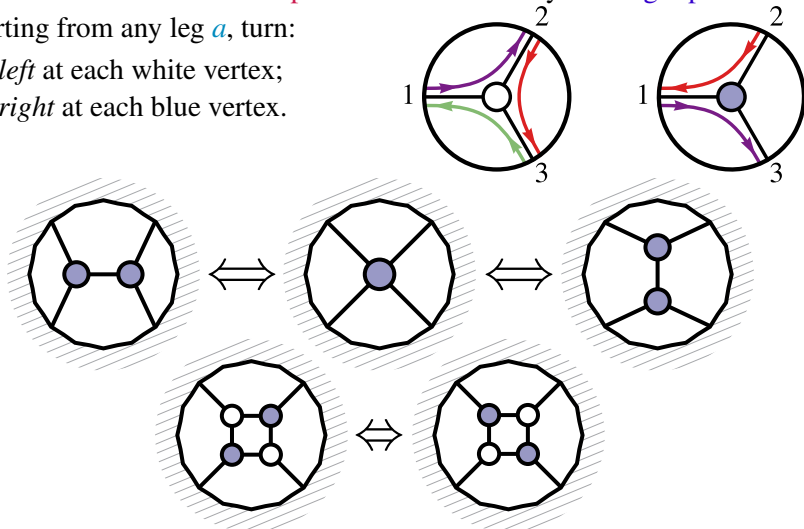


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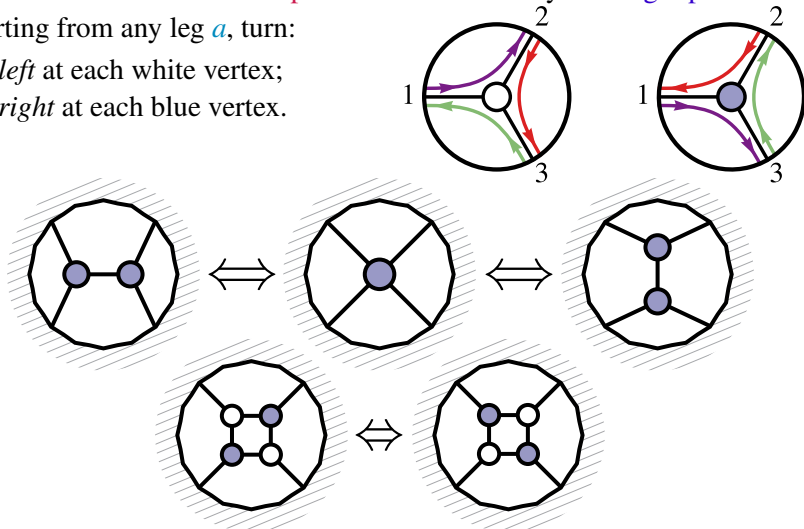


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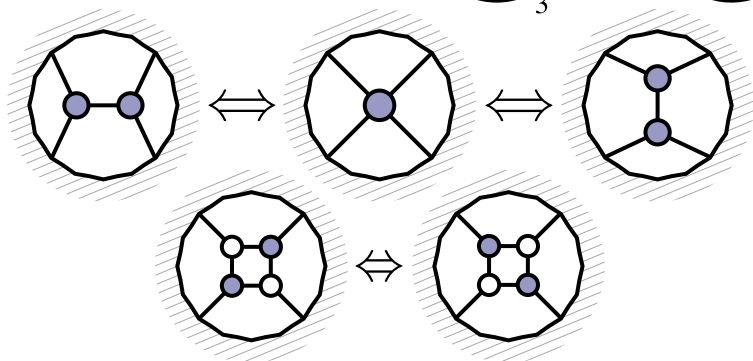
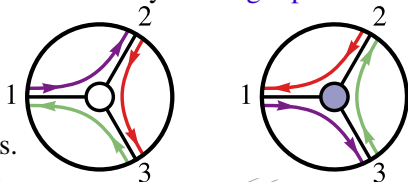
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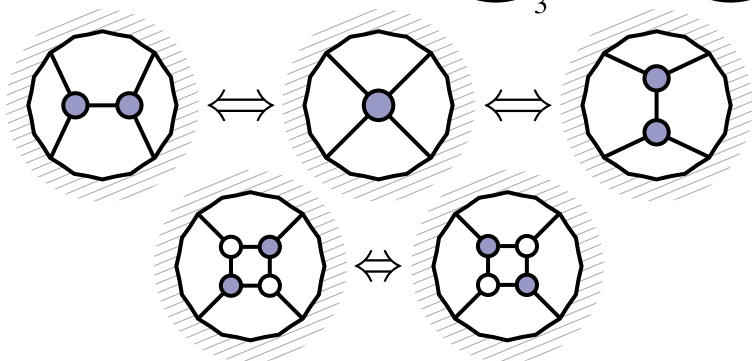
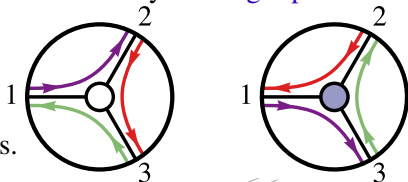
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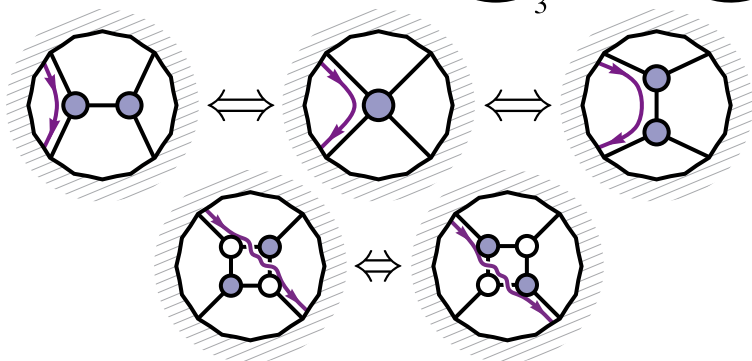
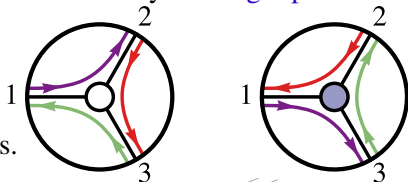
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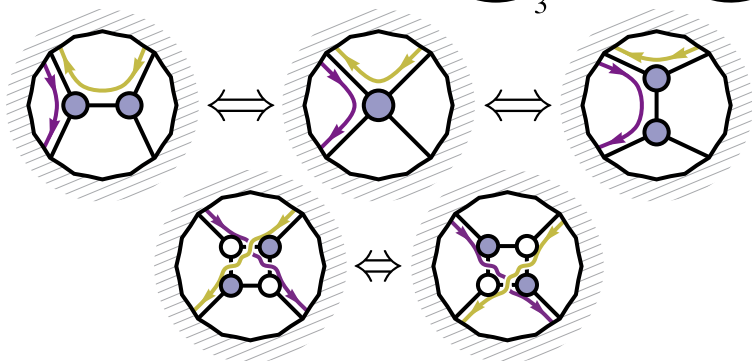
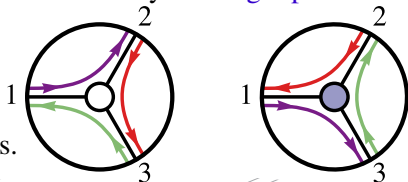
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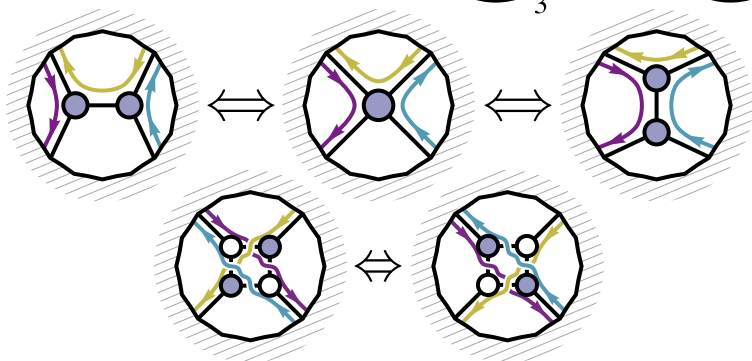
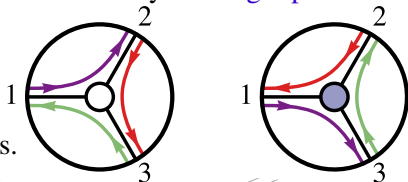
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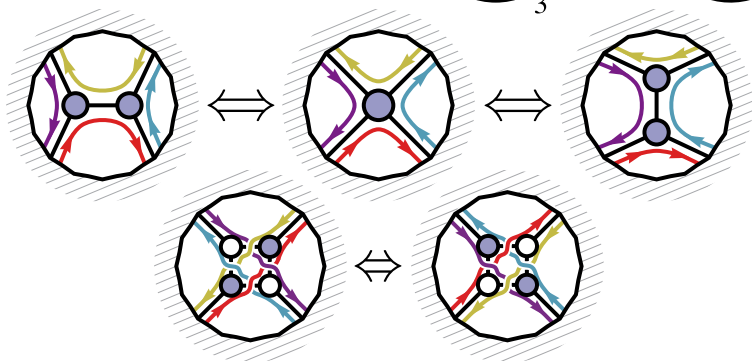
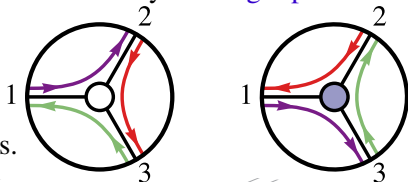
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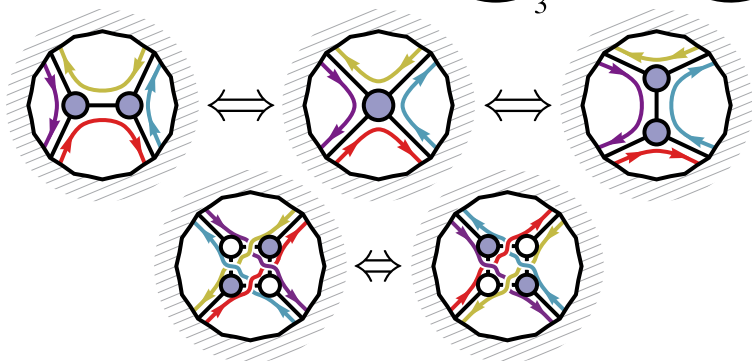
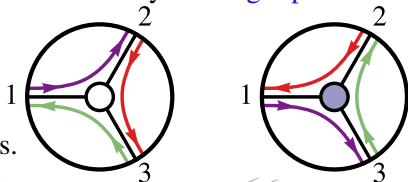
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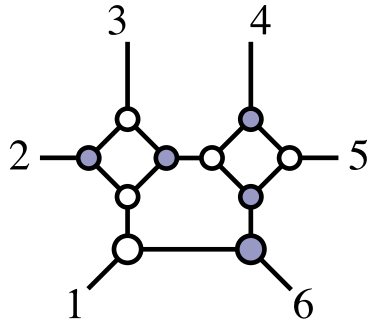
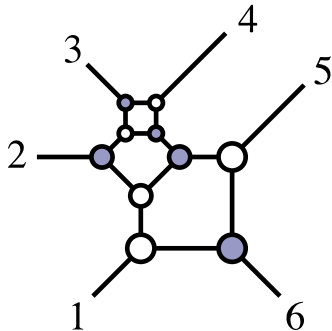
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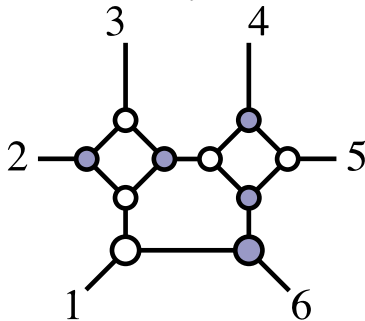
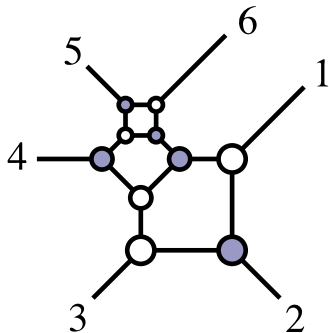
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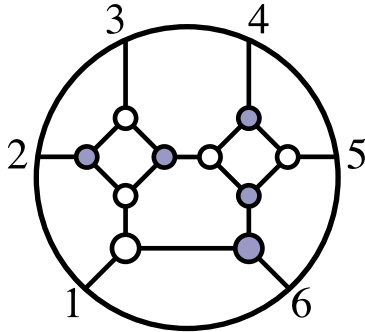
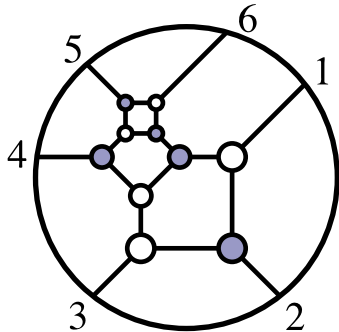
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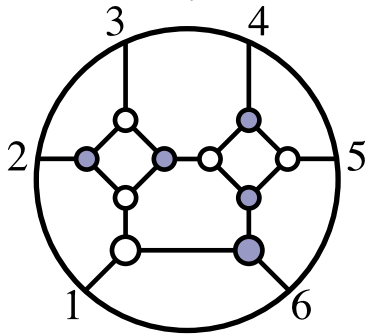
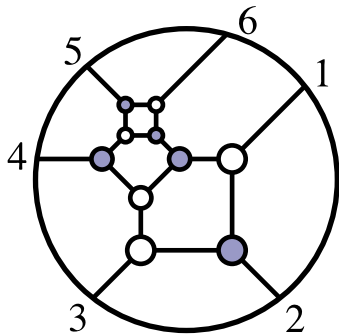
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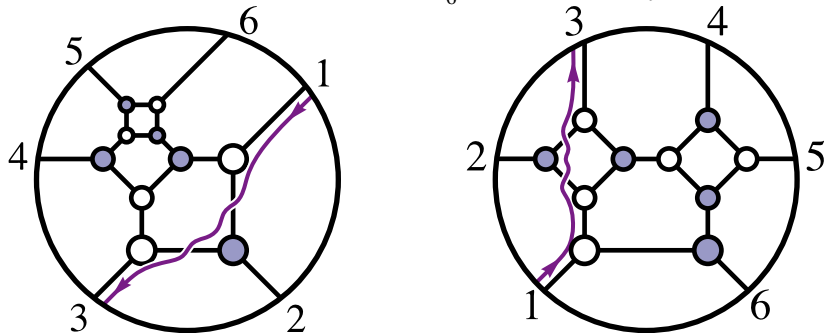
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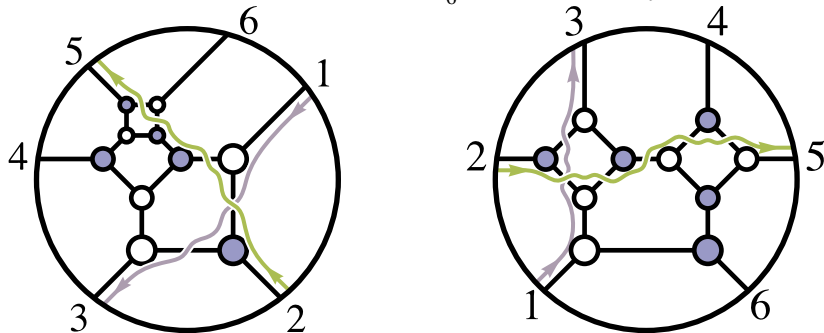
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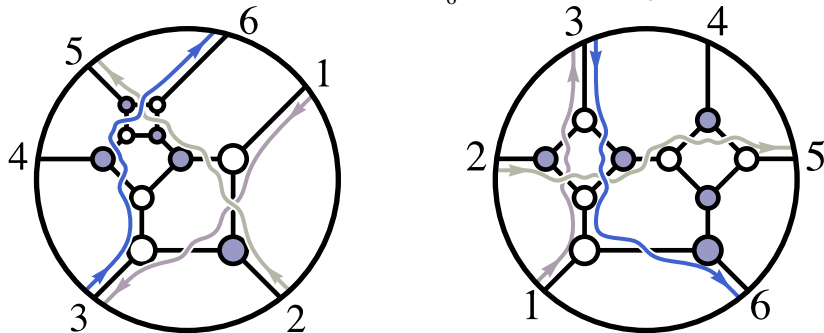
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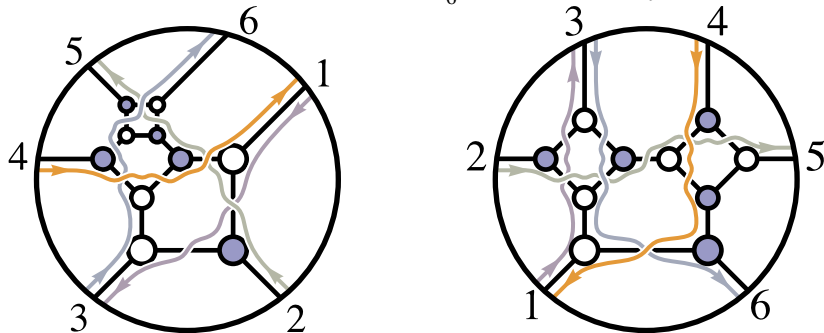
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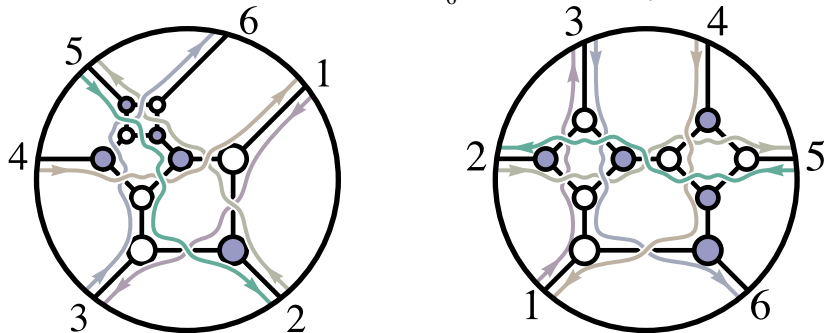
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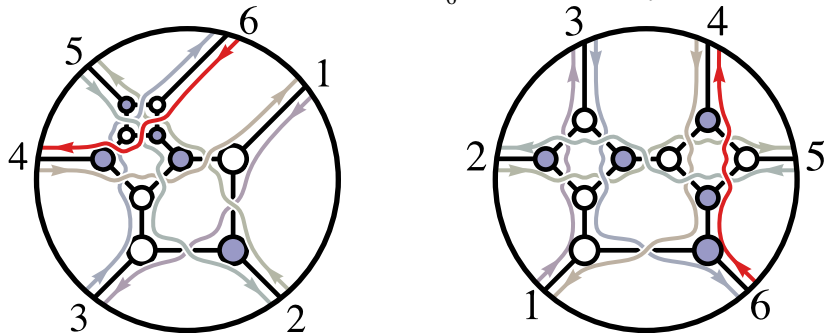
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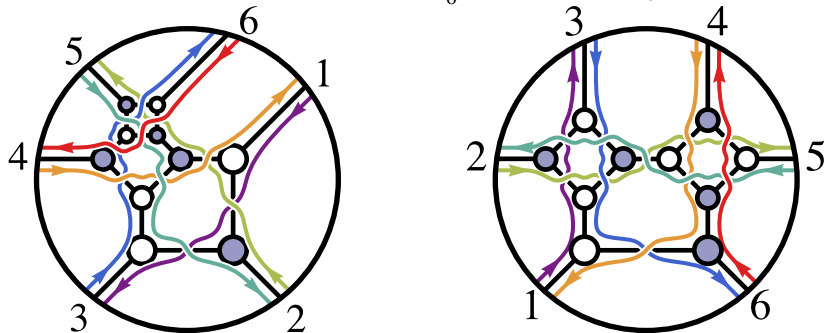
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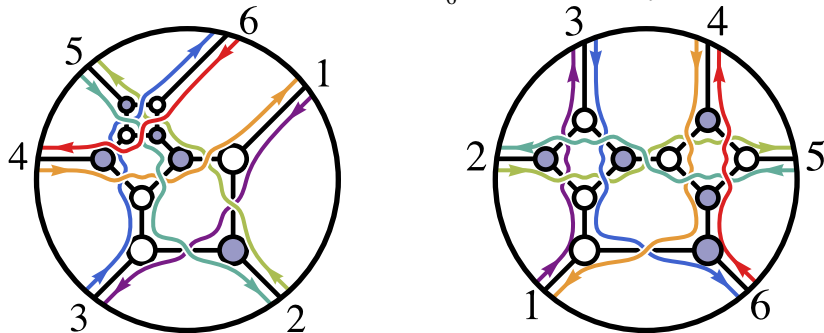
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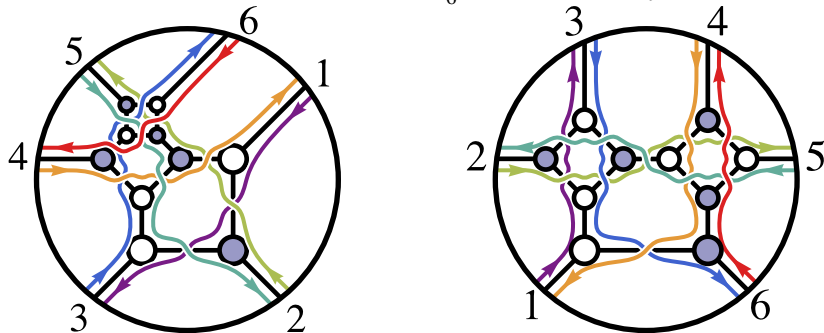
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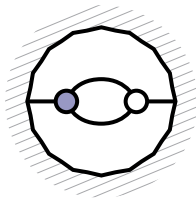
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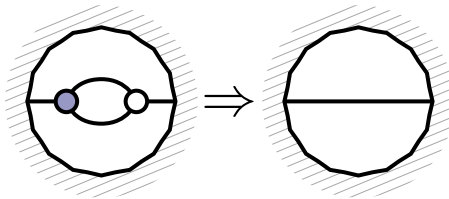
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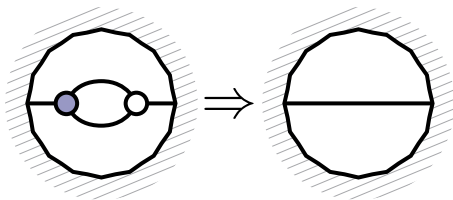
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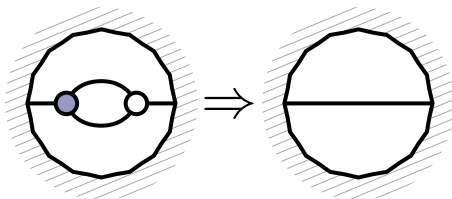


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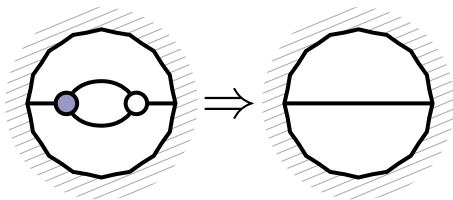


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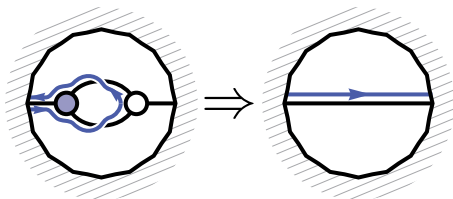


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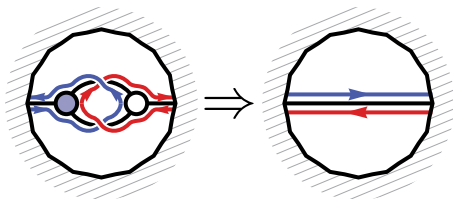


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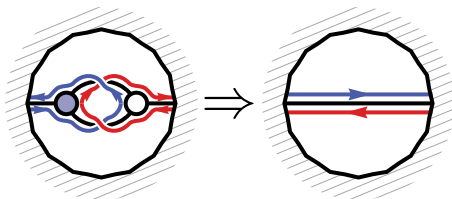
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Such factors of $d\alpha/\alpha$ arising from bubble deletion encode **loop integrands**!



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Recall that attaching ‘BCFW bridges’ can lead to very rich on-shell diagrams.

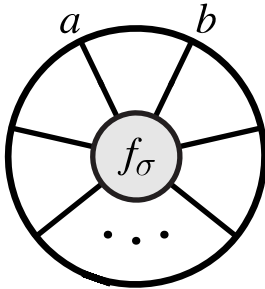
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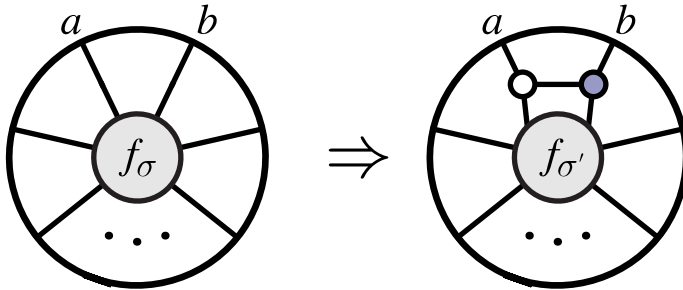
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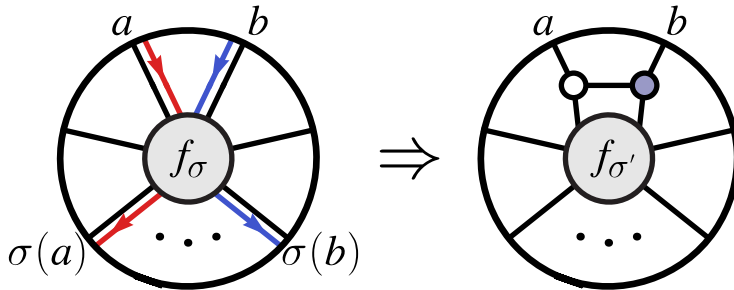
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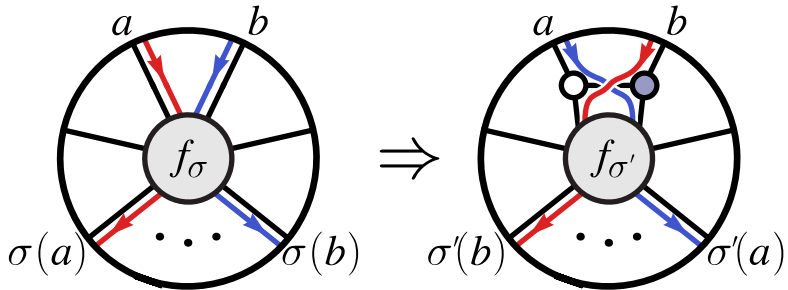
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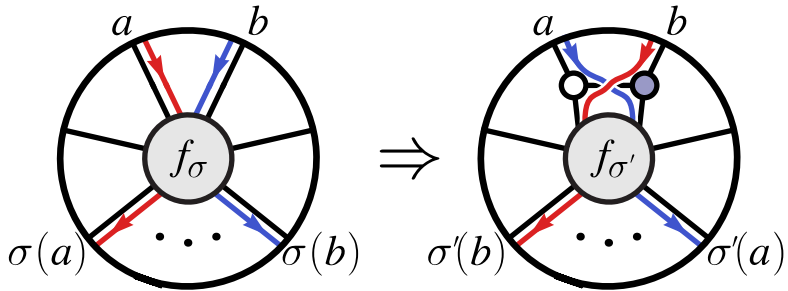
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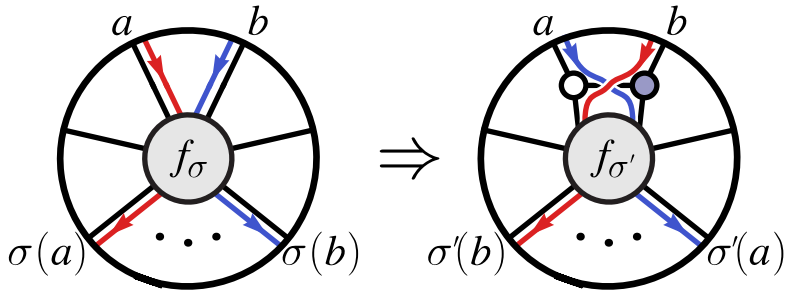
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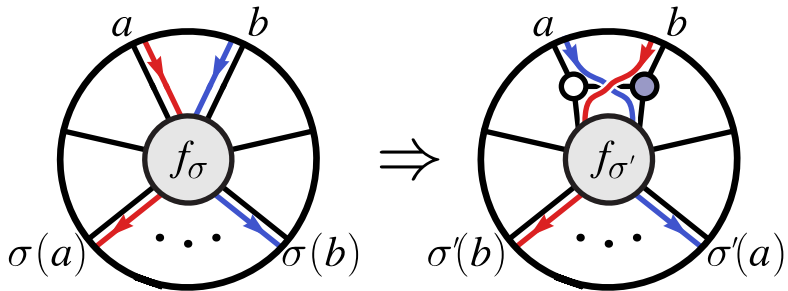
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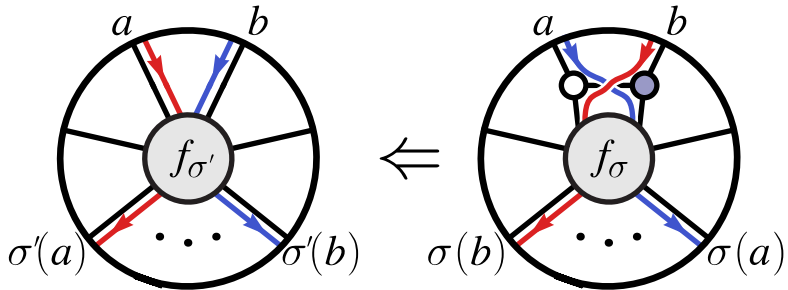
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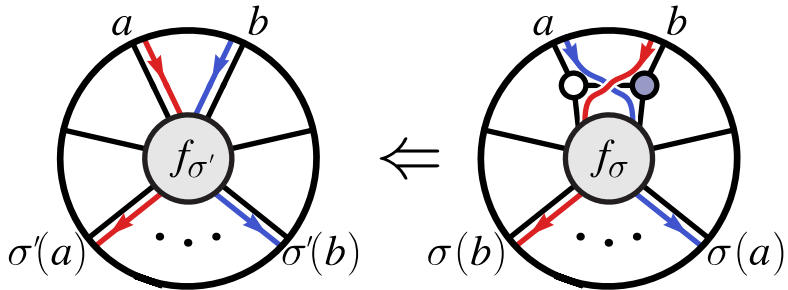
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Read the other way, we can ‘peel-off’ bridges and thereby **decompose** a permutation into transpositions according to $\sigma = (ab) \circ \sigma'$



Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions

'Bridge' Decomposition

$$\sigma: \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 3 & 5 & 6 & 7 & 8 & 10 \end{pmatrix}$$

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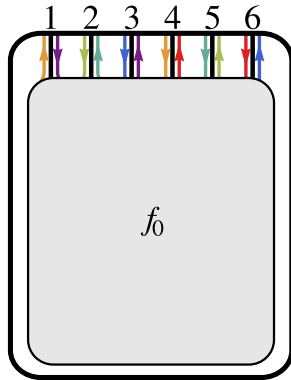
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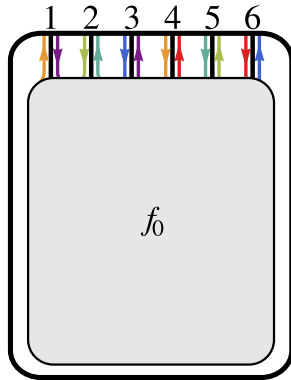


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	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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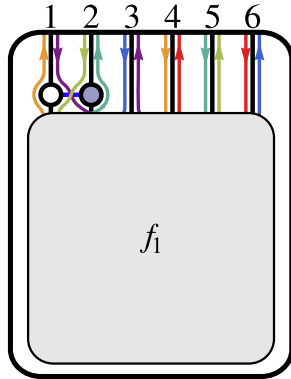
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$$f_0 \left\{ \begin{array}{c} \textcolor{red}{1} \textcolor{red}{2} \textcolor{blue}{3} \textcolor{blue}{5} \\ \downarrow \downarrow \\ \textcolor{blue}{3} \textcolor{blue}{5} \end{array} \right. \left\{ \begin{array}{c} 3 \textcolor{blue}{6} \textcolor{blue}{7} \textcolor{blue}{8} \textcolor{blue}{10} \\ \downarrow \downarrow \downarrow \downarrow \downarrow \\ 6 \textcolor{blue}{7} \textcolor{blue}{8} \textcolor{blue}{10} \end{array} \right\} \tau \textcolor{blue}{(12)}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (a\ b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1} f_1$$



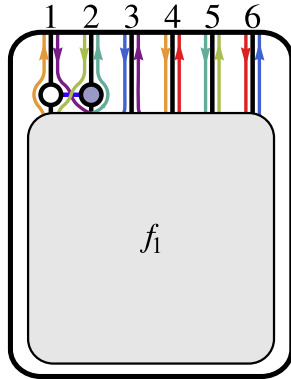
'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	
f_1	{5	3	6	7	8	10}	(1 2)

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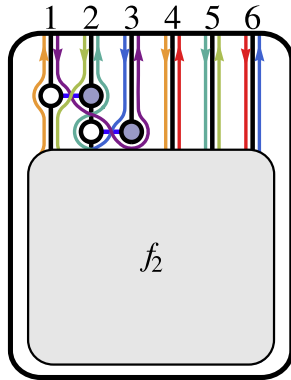
'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)

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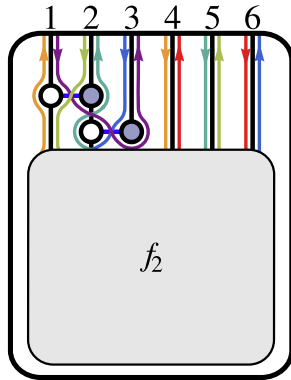
'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	

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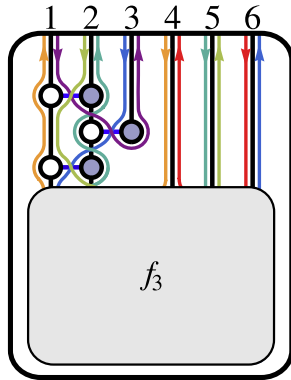
'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)

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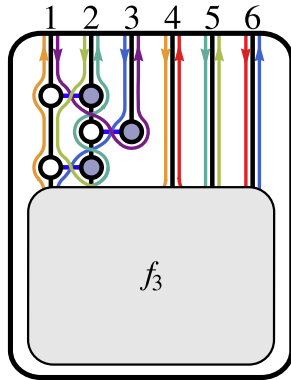
'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	
f_1	{5	3	6	7	8	10}	(1 2)
f_2	{5	6	3	7	8	10}	(2 3)
f_3	{6	5	3	7	8	10}	(1 2)

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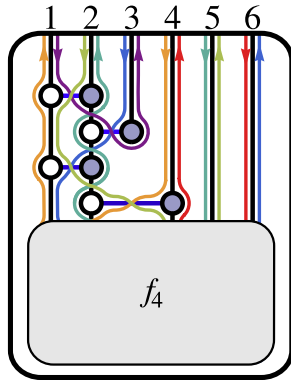
'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)

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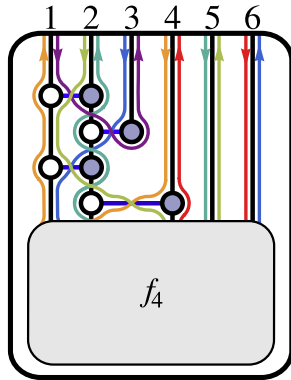
'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	

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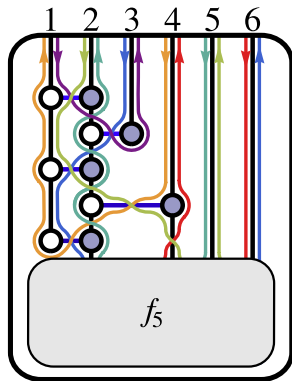
'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)

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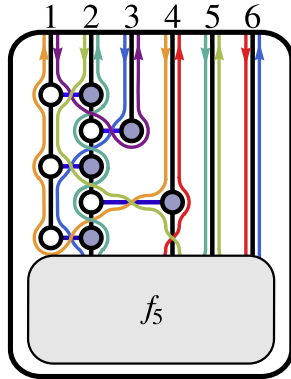
'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	

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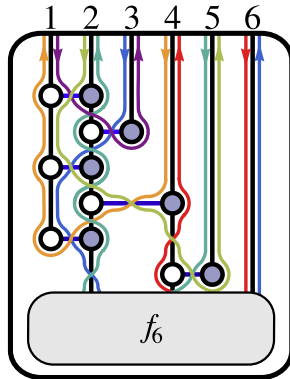
'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)

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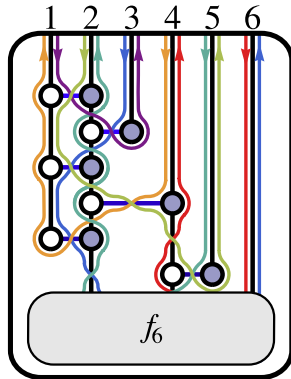
'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	

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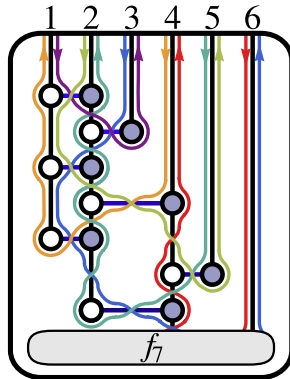
'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)

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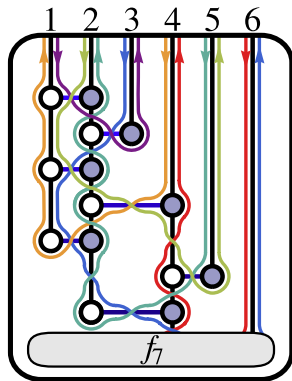
'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	

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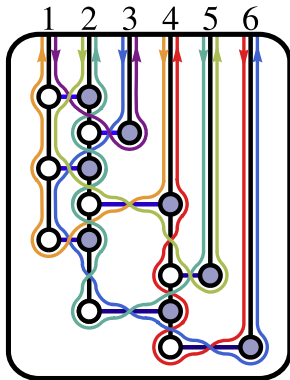
'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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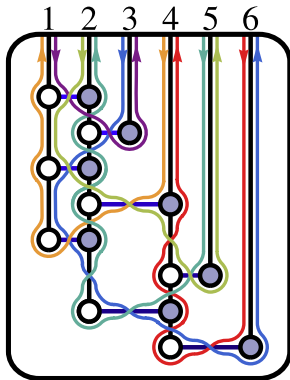
'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

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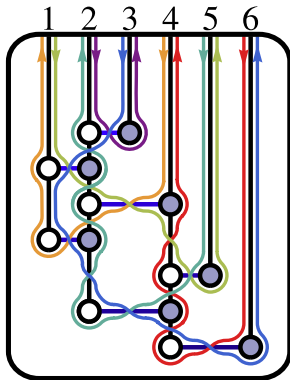
'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

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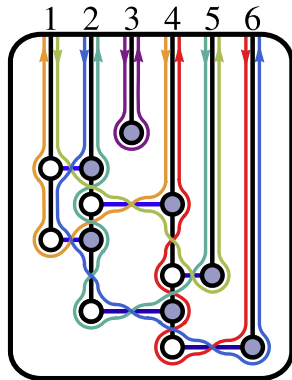
'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_1	{5	3	6	7	8	10}	(23)
f_2	{5	6	3	7	8	10}	(12)
f_3	{6	5	3	7	8	10}	(24)
f_4	{6	7	3	5	8	10}	(12)
f_5	{7	6	3	5	8	10}	(45)
f_6	{7	6	3	8	5	10}	(24)
f_7	{7	8	3	6	5	10}	(46)
f_8	{7	8	3	10	5	6}	

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'Bridge' Decomposition

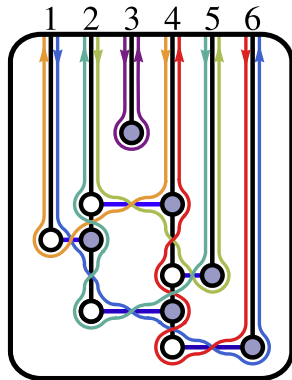
1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

f_2	{	5	6	3	7	8	10	}	(1 2)
f_3	{	6	5	3	7	8	10	}	(2 4)
f_4	{	6	7	3	5	8	10	}	(1 2)
f_5	{	7	6	3	5	8	10	}	(4 5)
f_6	{	7	6	3	8	5	10	}	(2 4)
f_7	{	7	8	3	6	5	10	}	(4 6)
f_8	{	7	8	3	10	5	6	}	

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'Bridge' Decomposition

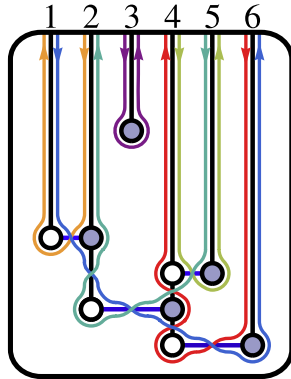
1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

f_3	{6	5	3	7	8	10}	(24)
f_4	{6	7	3	5	8	10}	(12)
f_5	{7	6	3	5	8	10}	(45)
f_6	{7	6	3	8	5	10}	(24)
f_7	{7	8	3	6	5	10}	(46)
f_8	{7	8	3	10	5	6}	

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'Bridge' Decomposition

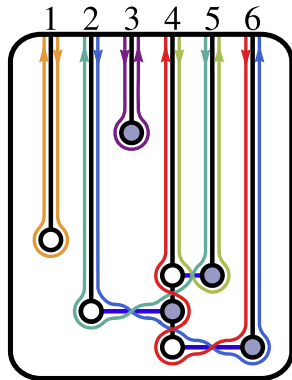
1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

f_4	{	6	7	3	5	8	10	}	(1 2)
f_5	{	7	6	3	5	8	10	}	(4 5)
f_6	{	7	6	3	8	5	10	}	(2 4)
f_7	{	7	8	3	6	5	10	}	(4 6)
f_8	{	7	8	3	10	5	6	}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (a\ b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \frac{d\alpha_5}{\alpha_5} \frac{d\alpha_6}{\alpha_6} \frac{d\alpha_7}{\alpha_7} \frac{d\alpha_8}{\alpha_8} f_8$$



'Bridge' Decomposition

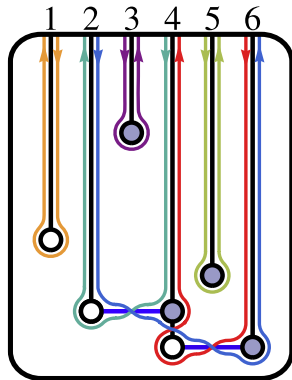
1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

$$\begin{aligned}
 f_5 & \{7 \ 6 \ 3 \ 5 \ 8 \ 10\} \\
 f_6 & \{7 \ 6 \ 3 \ 8 \ 5 \ 10\} \\
 f_7 & \{7 \ 8 \ 3 \ 6 \ 5 \ 10\} \\
 f_8 & \{7 \ 8 \ 3 \ 10 \ 5 \ 6\}
 \end{aligned}
 \begin{matrix}
 (4\ 5) \\
 (2\ 4) \\
 (4\ 6)
 \end{matrix}$$

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'Bridge' Decomposition

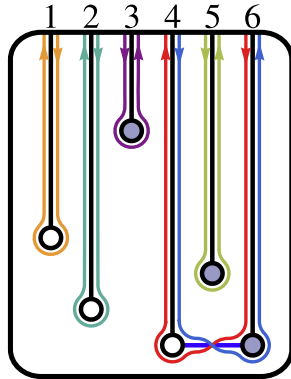
1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

$$\begin{aligned}
 f_6 & \{7 \quad 6 \quad 3 \quad 8 \quad 5 \quad 10\} \\
 f_7 & \{7 \quad 8 \quad 3 \quad 6 \quad 5 \quad 10\} \\
 f_8 & \{7 \quad 8 \quad 3 \quad 10 \quad 5 \quad 6\}
 \end{aligned}
 \begin{matrix}
 (2\ 4) \\
 (4\ 6)
 \end{matrix}$$

Canonical Coordinates for Computing On-Shell Functions

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'Bridge' Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

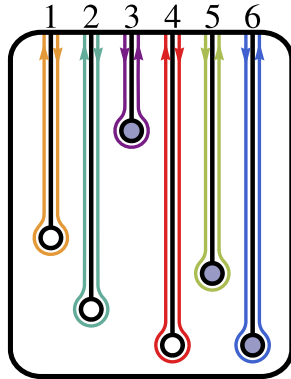
$$f_7 \{7 \ 8 \ 3 \ 6 \ 5 \ 10\} (46)$$

$$f_8 \{7 \ 8 \ 3 \ 10 \ 5 \ 6\}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (a\ b)$ such that $\sigma(a) < \sigma(b)$:

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'Bridge' Decomposition

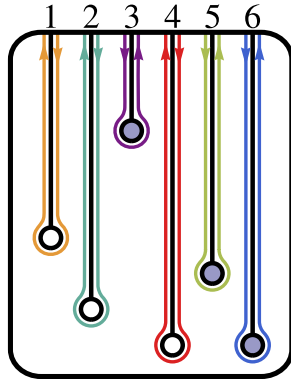
1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

$f_8 \{7\ 8\ 3\ 10\ 5\ 6\}$

Canonical Coordinates for Computing On-Shell Functions

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'Bridge' Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

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$$f_8 = \prod_{a=\sigma(a)+n} \left(\delta^4(\tilde{\eta}_a) \delta^2(\tilde{\lambda}_a) \right) \prod_{b=\sigma(b)} \left(\delta^2(\lambda_b) \right)$$

‘Bridge’ Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

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Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (a\ b)$ such that $\sigma(a) < \sigma(b)$:

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$$C \equiv \begin{pmatrix} \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{blue}{3} & \textcolor{red}{4} & \textcolor{blue}{5} & \textcolor{blue}{6} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

‘Bridge’ Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

$f_8 \{ \textcolor{red}{7} \ \textcolor{red}{8} \ \textcolor{blue}{3} \ \textcolor{red}{10} \ \textcolor{blue}{5} \ \textcolor{blue}{6} \}$

Canonical Coordinates for Computing On-Shell Functions

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$$f_8 = \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{blue}{3} & \textcolor{red}{4} & \textcolor{blue}{5} & \textcolor{blue}{6} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

‘Bridge’ Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

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‘Bridge’ Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

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$$f_7 = \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{pmatrix}$$

$(4\ 6): c_6 \mapsto c_6 + \alpha_8 c_4$

'Bridge' Decomposition

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \quad \tau$$

$$\begin{array}{l} f_7 \{7\ 8\ 3\ 6\ 5\ 10\} \\ f_8 \{7\ 8\ 3\ 10\ 5\ 6\} \end{array} (4\ 6)$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (a\ b)$ such that $\sigma(a) < \sigma(b)$:

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$$f_6 = \frac{d\alpha_7}{\alpha_7} \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{pmatrix}$$

(24): $c_4 \mapsto c_4 + \alpha_7 c_2$

'Bridge' Decomposition

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \end{array} \quad \tau$$

$$\begin{array}{l} f_6 \{7 \ 6 \ 3 \ 8 \ 5 \ 10\} \\ f_7 \{7 \ 8 \ 3 \ 6 \ 5 \ 10\} \\ f_8 \{7 \ 8 \ 3 \ 10 \ 5 \ 6\} \end{array} \begin{array}{l} (24) \\ (46) \end{array}$$

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (ab)$ such that $\sigma(a) < \sigma(b)$:

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$$f_5 = \frac{d\alpha_6}{\alpha_6} \frac{d\alpha_7}{\alpha_7} \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6 \alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

$(45): c_5 \mapsto c_5 + \alpha_6 c_4$

'Bridge' Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

f_5	{7	6	3	5	8	10}	(45)
f_6	{7	6	3	8	5	10}	
f_7	{7	8	3	6	5	10}	(46)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (ab)$ such that $\sigma(a) < \sigma(b)$:

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$$f_4 = \frac{d\alpha_5}{\alpha_5} \dots \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6 \alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

$(12): c_2 \mapsto c_2 + \alpha_5 c_1$

'Bridge' Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

f_4	{	6	7	3	5	8	10	}	(1 2)
f_5	{	7	6	3	5	8	10	}	(4 5)
f_6	{	7	6	3	8	5	10	}	(2 4)
f_7	{	7	8	3	6	5	10	}	(4 6)
f_8	{	7	8	3	10	5	6	}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (a\ b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \frac{d\alpha_5}{\alpha_5} \frac{d\alpha_6}{\alpha_6} \frac{d\alpha_7}{\alpha_7} \frac{d\alpha_8}{\alpha_8} f_8$$

$$f_3 = \frac{d\alpha_4}{\alpha_4} \dots \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \alpha_5 & 0 & \alpha_4 \alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4 + \alpha_7) \alpha_6 \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

(24): $c_4 \mapsto c_4 + \alpha_4 c_2$

'Bridge' Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

f_3	{6	5	3	7	8	10}	(24)
f_4	{6	7	3	5	8	10}	(12)
f_5	{7	6	3	5	8	10}	(45)
f_6	{7	6	3	8	5	10}	(24)
f_7	{7	8	3	6	5	10}	(46)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (a\ b)$ such that $\sigma(a) < \sigma(b)$:

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$$f_2 = \frac{d\alpha_3}{\alpha_3} \dots \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & (\alpha_3 + \alpha_5) & 0 & \alpha_4 \alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4 + \alpha_7) \alpha_6 \alpha_7 & 0 & \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

$(1\ 2): c_2 \mapsto c_2 + \alpha_3 c_1$

'Bridge' Decomposition

1	2	3	4	5	6	
↓	↓	↓	↓	↓	↓	τ

f_2	{	5	6	3	7	8	10	}	(1 2)
f_3	{	6	5	3	7	8	10	}	(2 4)
f_4	{	6	7	3	5	8	10	}	(1 2)
f_5	{	7	6	3	5	8	10	}	(4 5)
f_6	{	7	6	3	8	5	10	}	(2 4)
f_7	{	7	8	3	6	5	10	}	(4 6)
f_8	{	7	8	3	10	5	6	}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—e.g., always choose the **first** transposition $\tau \equiv (ab)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \frac{d\alpha_5}{\alpha_5} \frac{d\alpha_6}{\alpha_6} \frac{d\alpha_7}{\alpha_7} \frac{d\alpha_8}{\alpha_8} f_8$$

$$f_1 = \frac{d\alpha_2}{\alpha_2} \dots \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & (\alpha_3 + \alpha_5) & \alpha_2(\alpha_3 + \alpha_5) & \alpha_4 \alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4 + \alpha_7) & \alpha_6 \alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

$(23): c_3 \mapsto c_3 + \alpha_2 c_2$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_1	{5	3	6	7	8	10}	(23)
f_2	{5	6	3	7	8	10}	(12)
f_3	{6	5	3	7	8	10}	(24)
f_4	{6	7	3	5	8	10}	(12)
f_5	{7	6	3	5	8	10}	(45)
f_6	{7	6	3	8	5	10}	(24)
f_7	{7	8	3	6	5	10}	(46)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (ab)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1} \frac{d\alpha_2}{\alpha_2} \frac{d\alpha_3}{\alpha_3} \frac{d\alpha_4}{\alpha_4} \frac{d\alpha_5}{\alpha_5} \frac{d\alpha_6}{\alpha_6} \frac{d\alpha_7}{\alpha_7} \frac{d\alpha_8}{\alpha_8} f_8$$

$$f_0 = \frac{d\alpha_1}{\alpha_1} \dots \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & (\alpha_1 + \alpha_3 + \alpha_5) & \alpha_2(\alpha_3 + \alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4 + \alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

$(12): c_2 \mapsto c_2 + \alpha_1 c_1$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(12)
f_1	{5	3	6	7	8	10}	(23)
f_2	{5	6	3	7	8	10}	(12)
f_3	{6	5	3	7	8	10}	(24)
f_4	{6	7	3	5	8	10}	(12)
f_5	{7	6	3	5	8	10}	(45)
f_6	{7	6	3	8	5	10}	(24)
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f_8	{7	8	3	10	5	6}	

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$$f_0 = \frac{d\alpha_1}{\alpha_1} \dots \frac{d\alpha_8}{\alpha_8} \delta^{3 \times 4} (C \cdot \tilde{\eta}) \delta^{3 \times 2} (C \cdot \tilde{\lambda}) \delta^{2 \times 3} (\lambda \cdot C^\perp)$$

$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & (\alpha_1 + \alpha_3 + \alpha_5) & \alpha_2(\alpha_3 + \alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4 + \alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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Canonical Coordinates for Computing On-Shell Functions

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'Bridge' Decomposition							
	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
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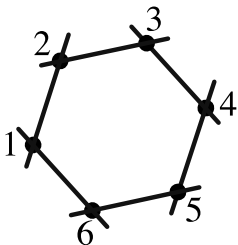
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(6 1): $c_1 \mapsto c_1 + \alpha_0 c_6$

'Bridge' Decomposition							
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f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
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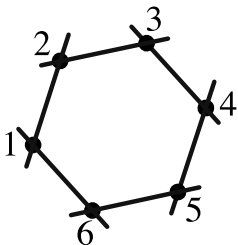
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(12)
f_1	{5	3	6	7	8	10}	(23)
f_2	{5	6	3	7	8	10}	(12)
f_3	{6	5	3	7	8	10}	(24)
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f_5	{7	6	3	5	8	10}	(45)
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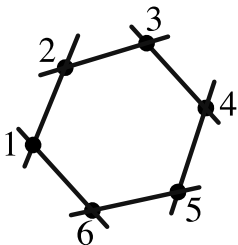
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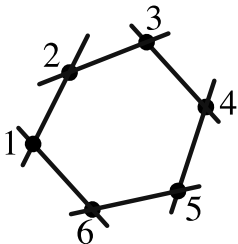
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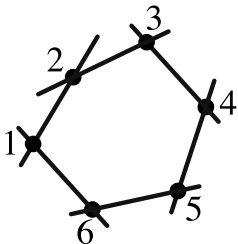
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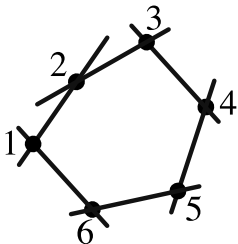
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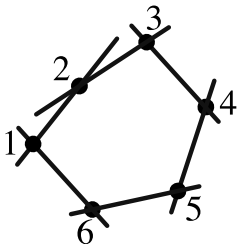
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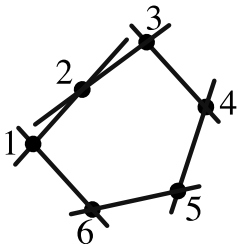
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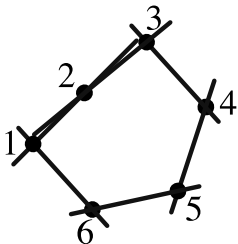
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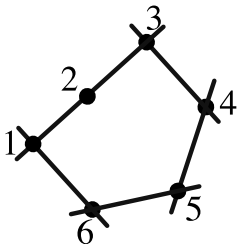
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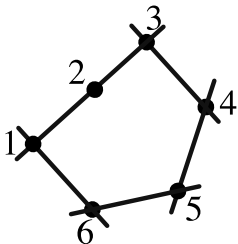
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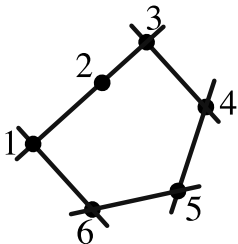
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f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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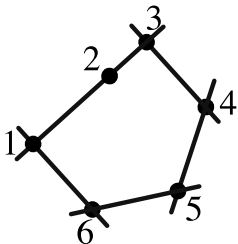
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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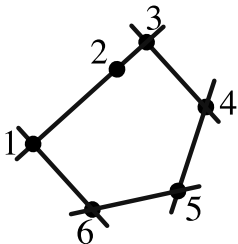
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	↓	↓	↓	↓	↓	↓	τ
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
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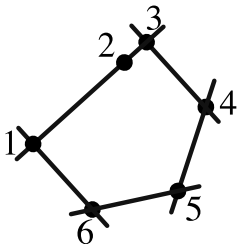
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	↓	↓	↓	↓	↓	↓	τ
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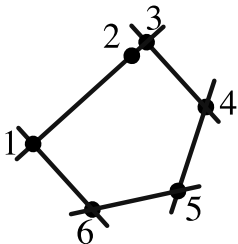
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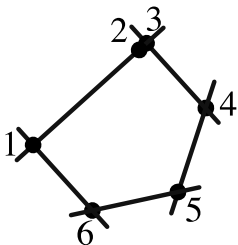
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	↓	↓	↓	↓	↓	↓	τ
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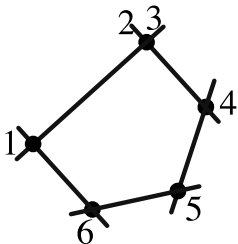
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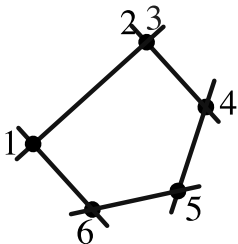
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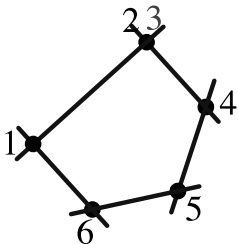
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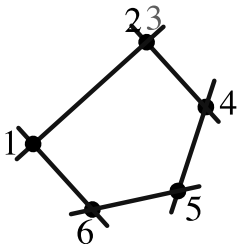
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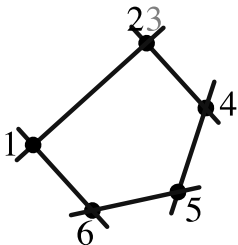
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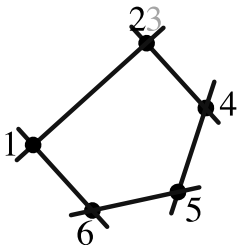
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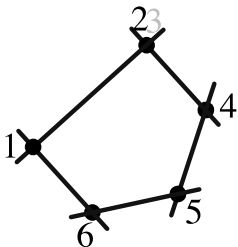
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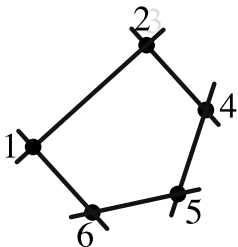
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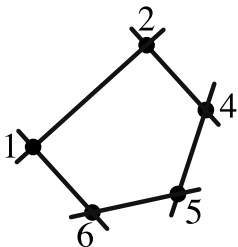
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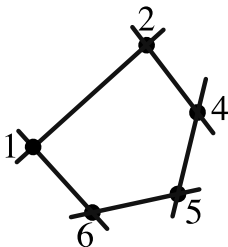
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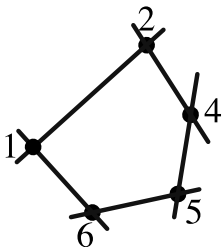
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f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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Canonical Coordinates for Computing On-Shell Functions

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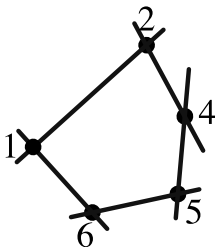
$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & (\alpha_3 + \alpha_5) & 0 & \alpha_4 \alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4 + \alpha_7) & \alpha_6 \alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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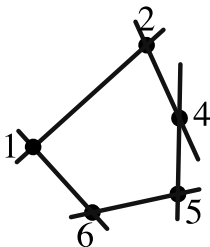
$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_3 + \alpha_5) & 0 & \alpha_4 \alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4 + \alpha_7) & \alpha_6 \alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

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	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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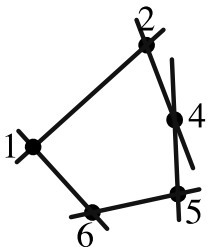
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	1	2	3	4	5	6	
	↓	↓	↓	↓	↓		τ
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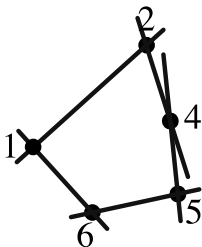
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	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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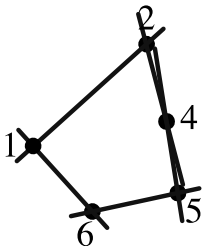
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	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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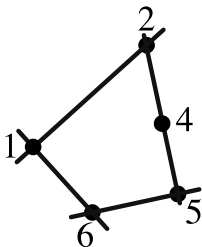
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	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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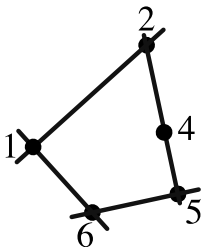
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 2 \\ \alpha_5 \\ 1 \end{matrix} & \begin{matrix} 3 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 4 \\ \alpha_4\alpha_5 \\ (\alpha_4 + \alpha_7) \\ 1 \end{matrix} & \begin{matrix} 5 \\ 0 \\ \alpha_6\alpha_7 \\ \alpha_6 \end{matrix} & \begin{matrix} 6 \\ 0 \\ 0 \\ \alpha_8 \end{matrix} \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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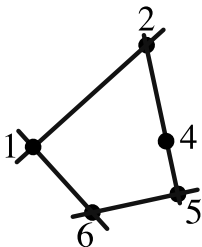
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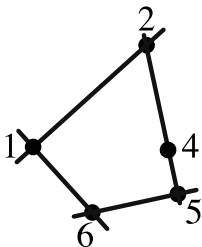
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & 1 & \alpha_5 & 0 & \alpha_4 \alpha_5 & 0 & 0 \\ \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & 0 & 1 & 0 & (\alpha_4 + \alpha_7) \alpha_6 \alpha_7 & 0 & 0 \\ \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

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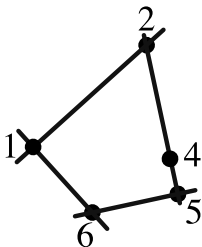
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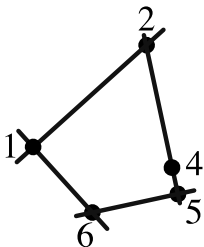
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	↓	↓	↓	↓	↓	↓	τ
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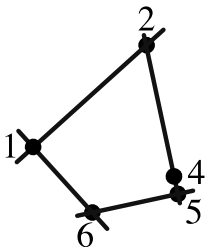
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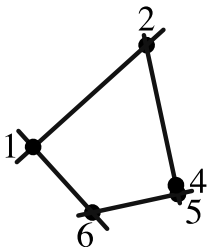
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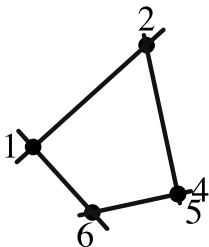
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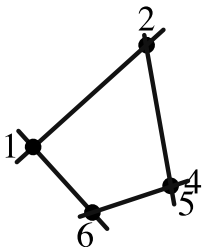
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \alpha_7 & \alpha_6 \alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 & \end{pmatrix}$$

'Bridge' Decomposition

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	↓	↓	↓	↓	↓	↓	τ
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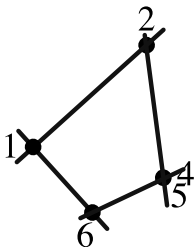
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & \alpha_7 & \alpha_6 \alpha_7 & 0 \\ 3 & 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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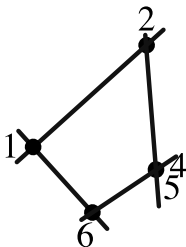
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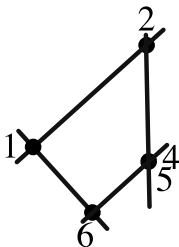
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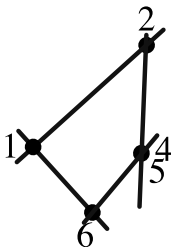
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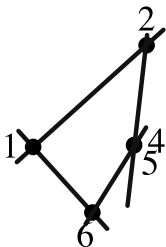
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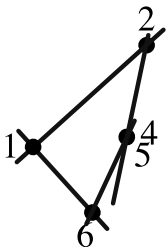
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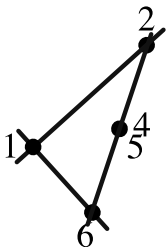
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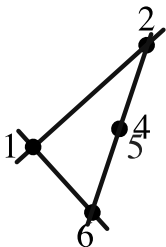
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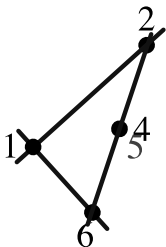
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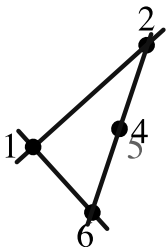
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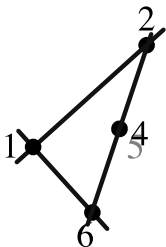
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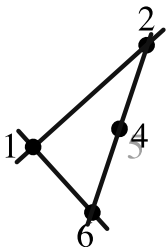
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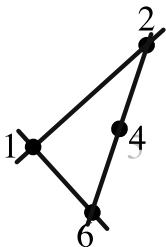
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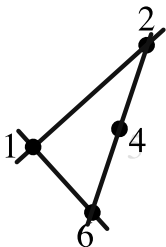
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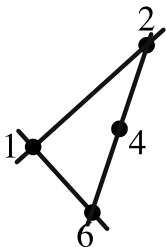
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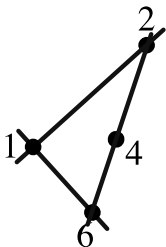
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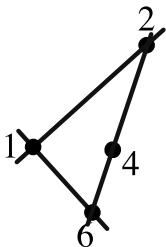
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & \alpha_8 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

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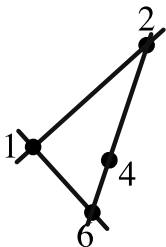
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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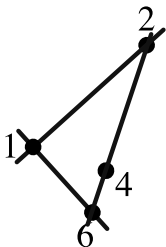
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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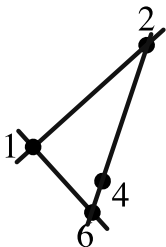
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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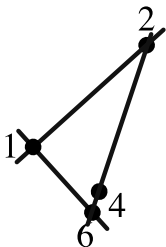
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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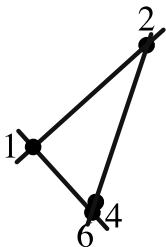
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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f_5	{7	6	3	5	8	10}	(4 5)
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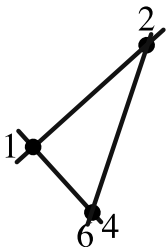
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 & 0 & \alpha_8 \\ 4 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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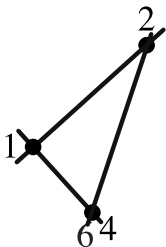


$$C \equiv \begin{pmatrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 0 \\ 0 \\ 0 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{pmatrix} \end{pmatrix}$$

‘Bridge’ Decomposition							
	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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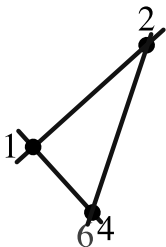
$$C \equiv \begin{pmatrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{pmatrix} \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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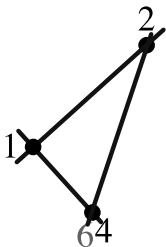
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ \alpha_8 \end{matrix} \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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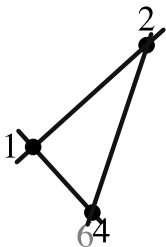


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‘Bridge’ Decomposition							
	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
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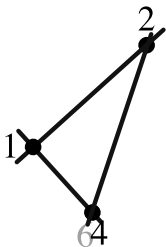


$$C \equiv \begin{pmatrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix} \end{pmatrix}$$

‘Bridge’ Decomposition							
	1	2	3	4	5	6	τ
	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
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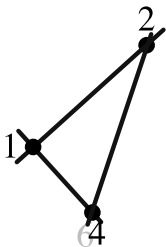
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'Bridge' Decomposition

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	↓	↓	↓	↓	↓	↓	τ
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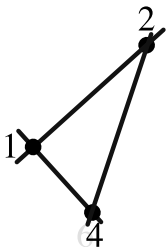


$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 & \alpha_8 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

'Bridge' Decomposition							
	1	2	3	4	5	6	τ
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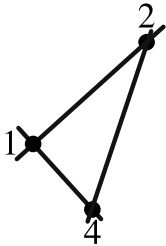


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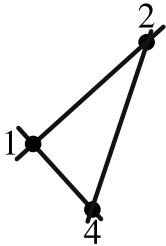
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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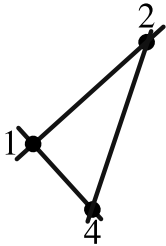
$$C \equiv \begin{pmatrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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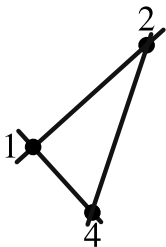
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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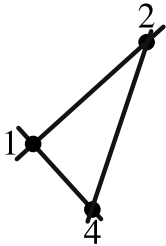


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'Bridge' Decomposition							
	1	2	3	4	5	6	τ
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f_4	{6	7	3	5	8	10}	(1 2)
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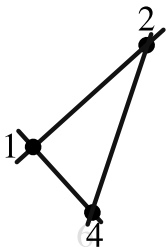
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	↓	↓	↓	↓	↓	↓	τ
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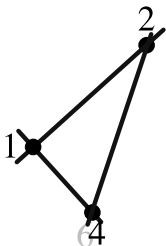
$$(46): c_6 \mapsto c_6 + \alpha_8 c_4$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
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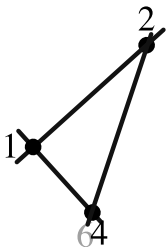
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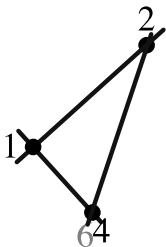
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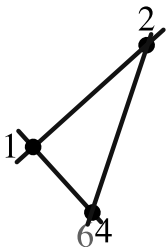
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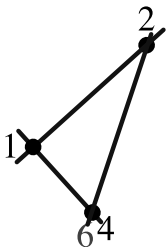
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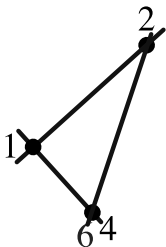
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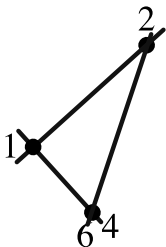
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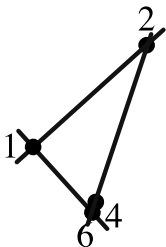
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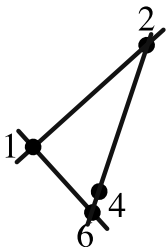
$$(24): c_4 \mapsto c_4 + \alpha_7 c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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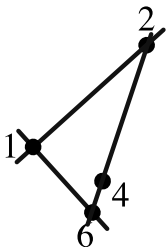
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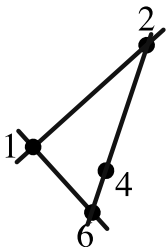
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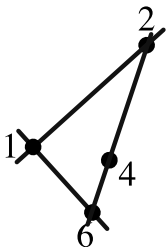
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	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the **first** transposition $\tau \equiv (ab)$ such that $\sigma(a) < \sigma(b)$:



$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 1 & 0 & \alpha_8 \\ 5 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

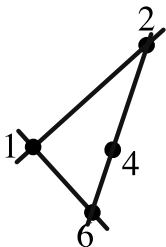
$$(24): c_4 \mapsto c_4 + \alpha_7 c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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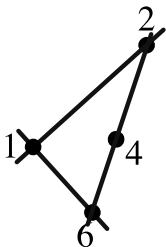
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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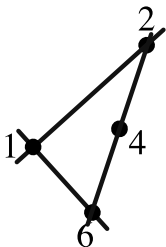
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'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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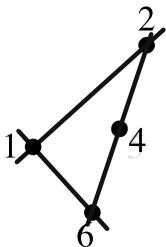
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	1	2	3	4	5	6	
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f_4	{6	7	3	5	8	10}	(1 2)
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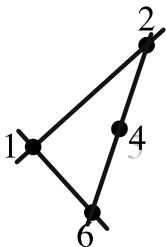
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'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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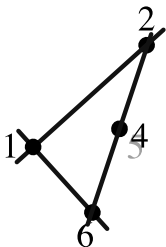
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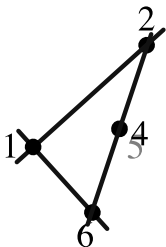
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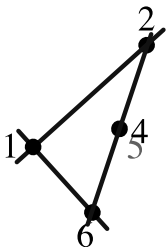
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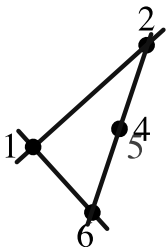
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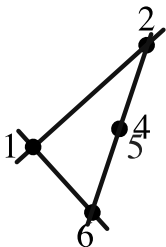
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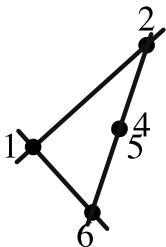
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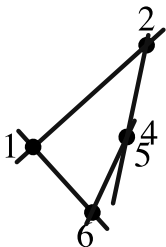
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	↓	↓	↓	↓	↓	↓	τ
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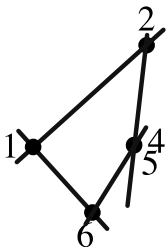
$(12): c_2 \mapsto c_2 + \alpha_5 c_1$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	(12)
f_1	{5	3	6	7	8	10}	(23)
f_2	{5	6	3	7	8	10}	(12)
f_3	{6	5	3	7	8	10}	(24)
f_4	{6	7	3	5	8	10}	(12)
f_5	{7	6	3	5	8	10}	(45)
f_6	{7	6	3	8	5	10}	(24)
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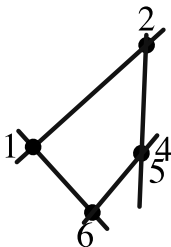
$(1\ 2): c_2 \mapsto c_2 + \alpha_5 c_1$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
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f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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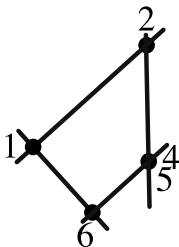
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$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} & \begin{matrix} \alpha_5 \\ 1 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ \alpha_7 \\ 1 \end{matrix} & \begin{matrix} 0 \\ \alpha_6 \alpha_7 \\ \alpha_6 \end{matrix} & \begin{matrix} 0 \\ 0 \\ \alpha_8 \end{matrix} \end{pmatrix}$$

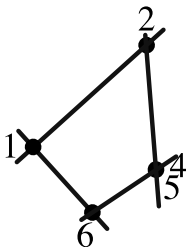
$(1\ 2): c_2 \mapsto c_2 + \alpha_5 c_1$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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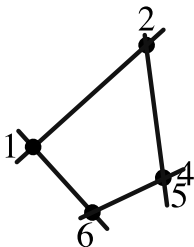
$(12): c_2 \mapsto c_2 + \alpha_5 c_1$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
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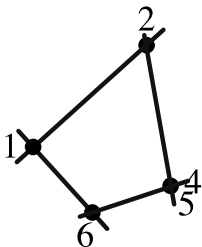
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'Bridge' Decomposition

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f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(12)
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f_3	{6	5	3	7	8	10}	(24)
f_4	{6	7	3	5	8	10}	(12)
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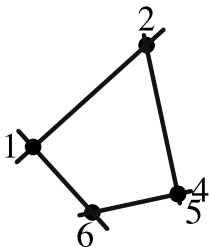
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	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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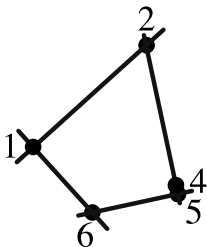
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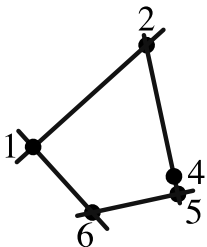
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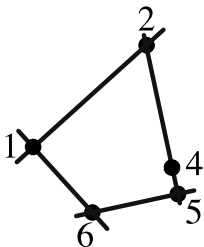
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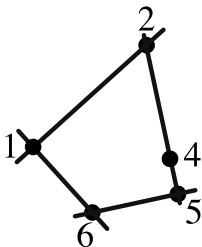
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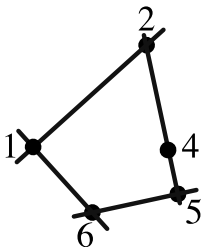
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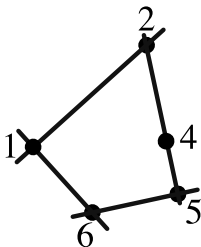
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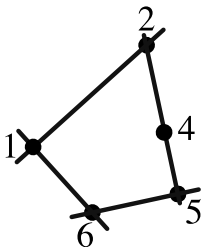
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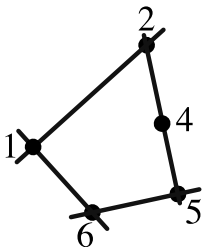
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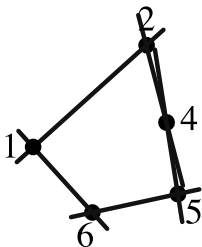
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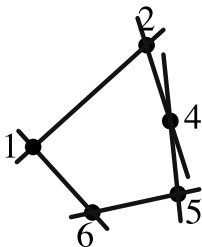
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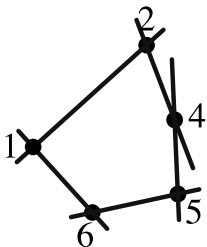
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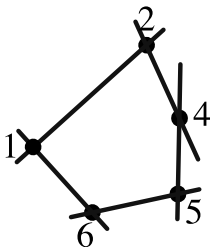
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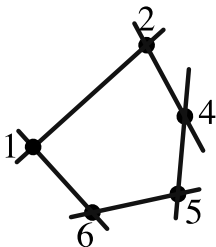
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'Bridge' Decomposition

	1	2	3	4	5	6	τ
f_0	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	
f_0	{3	5	6	7	8	10}	(12)
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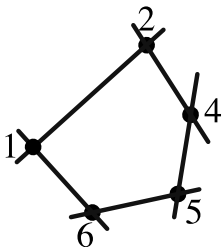
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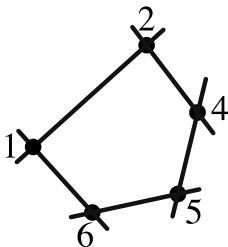
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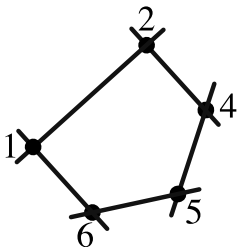
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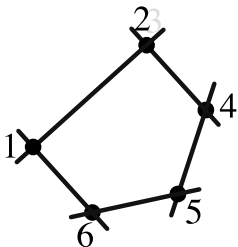
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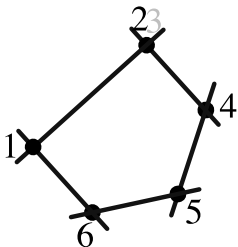
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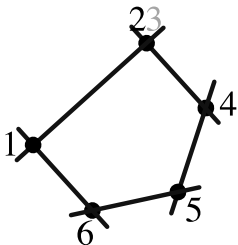
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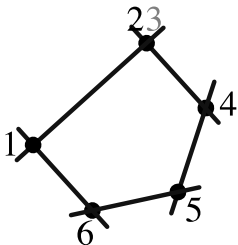
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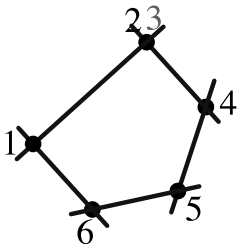
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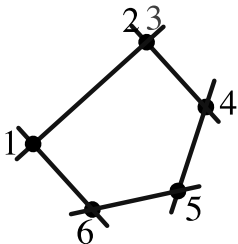
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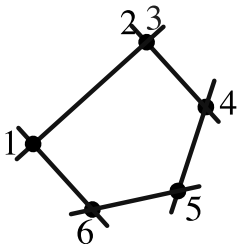
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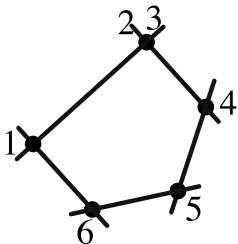
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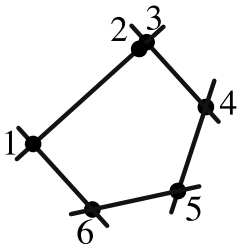
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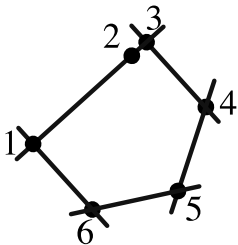
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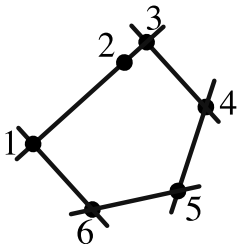
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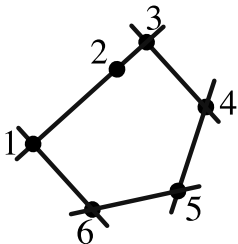
$(12): c_2 \mapsto c_2 + \alpha_1 c_1$

'Bridge' Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
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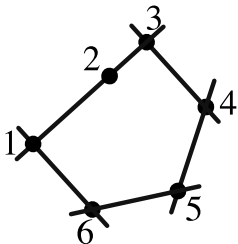
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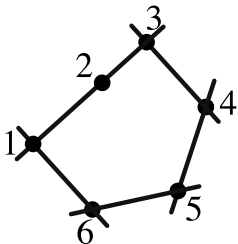
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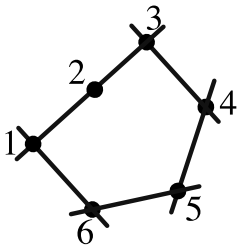
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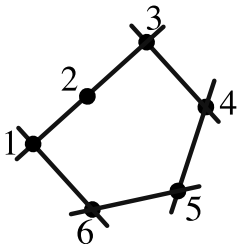
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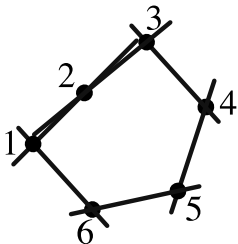
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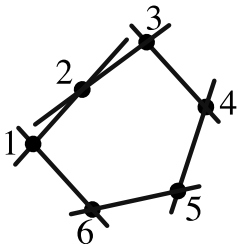
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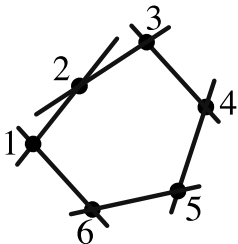
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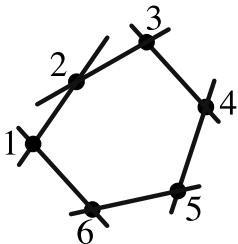
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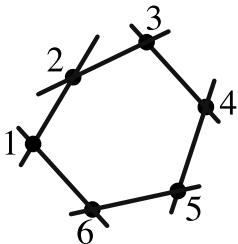
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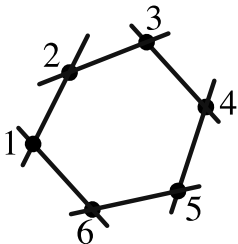
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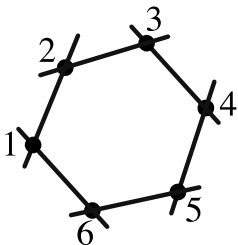
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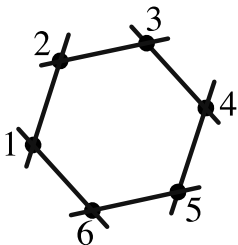
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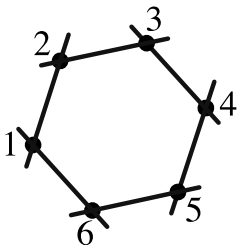
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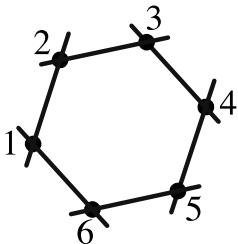
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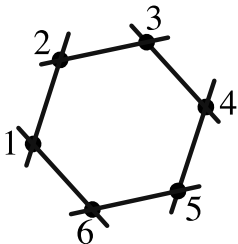
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'Bridge' Decomposition

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	↓	↓	↓	↓	↓	↓	τ
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f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
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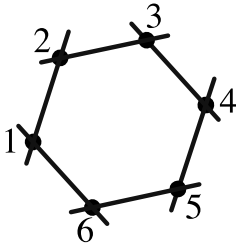


$$C \equiv \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_1 + \alpha_3 + \alpha_5) & \alpha_2(\alpha_3 + \alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4 + \alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{pmatrix}$$

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Canonical Coordinates for Computing On-Shell Functions



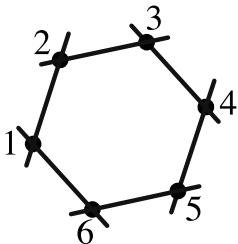
$$C \equiv \begin{pmatrix} & 1 & 2 & 3 & 4 & 5 & 6 \\ \begin{matrix} 1 \\ 0 \\ \alpha_0\alpha_8 \end{matrix} & \begin{matrix} (\alpha_1 + \alpha_3 + \alpha_5) \\ 1 \\ 0 \end{matrix} & \begin{matrix} \alpha_2(\alpha_3 + \alpha_5) \\ \alpha_2 \\ 0 \end{matrix} & \begin{matrix} \alpha_4\alpha_5 \\ (\alpha_4 + \alpha_7) \\ 1 \end{matrix} & \begin{matrix} 0 \\ \alpha_6\alpha_7 \\ \alpha_6 \end{matrix} & \begin{matrix} 0 \\ 0 \\ \alpha_8 \end{matrix} \end{pmatrix}$$

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	1	2	3	4	5	6	τ
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Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{6,3} \equiv \frac{d\alpha_0}{\alpha_0} \cdots \frac{d\alpha_8}{\alpha_8}$$



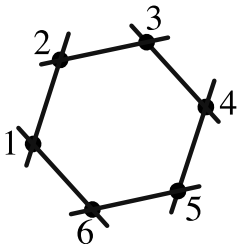
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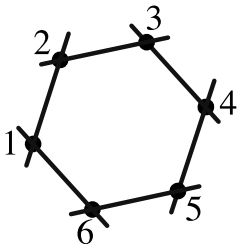
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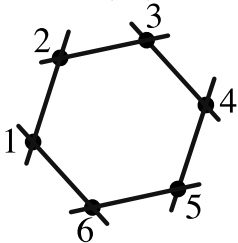
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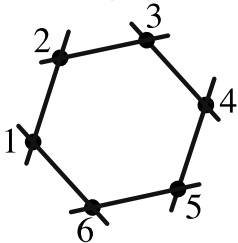
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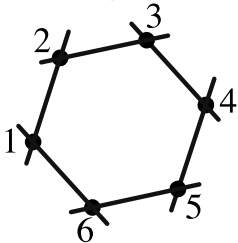
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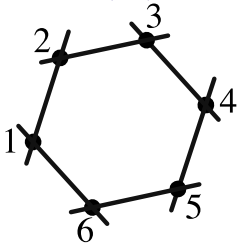
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