The Vernacular of the S-Matrix

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Enormous Advances in Understanding Scattering Amplitudes

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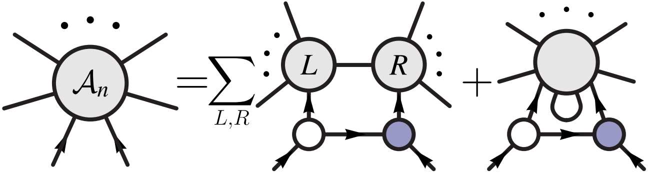

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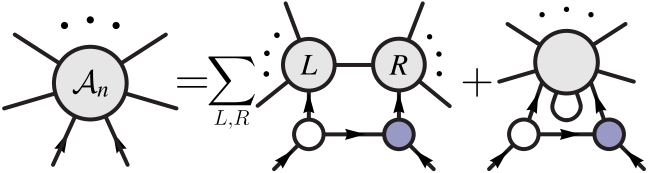

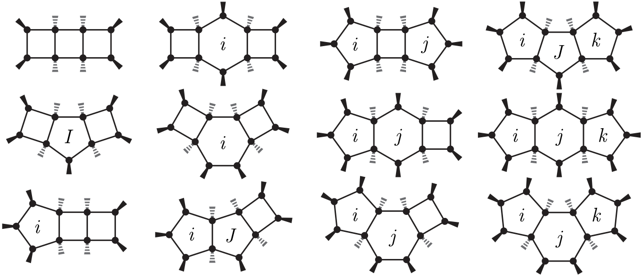

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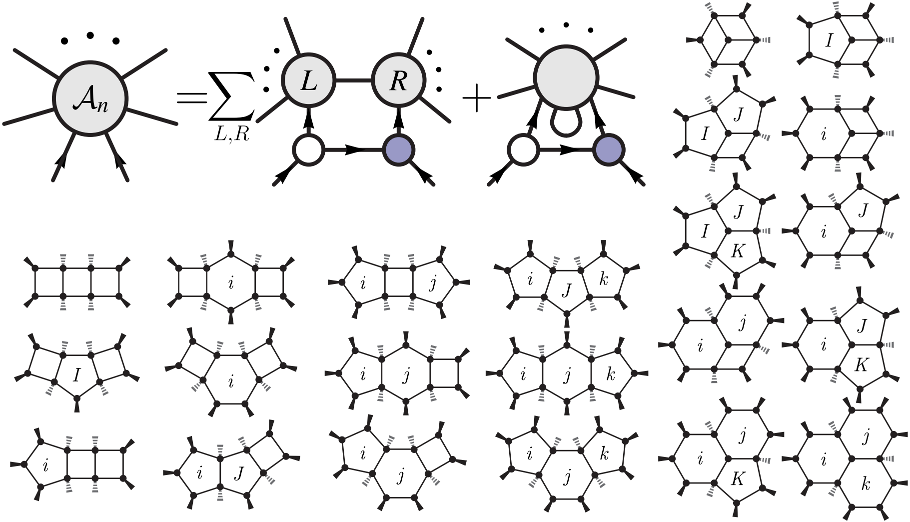

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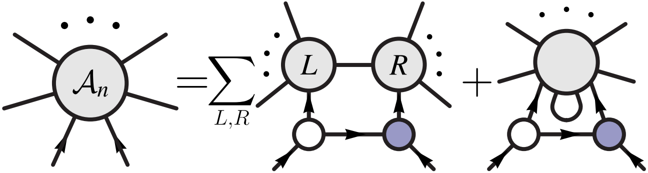

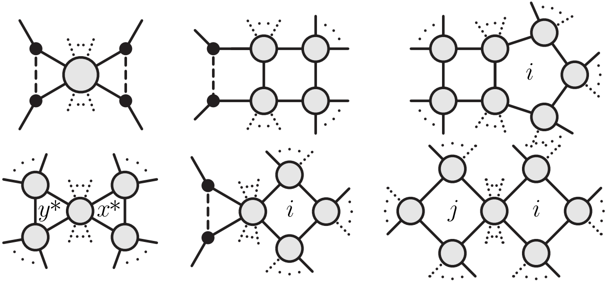

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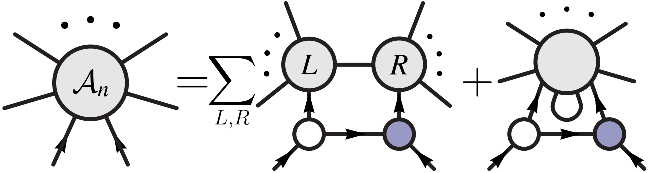

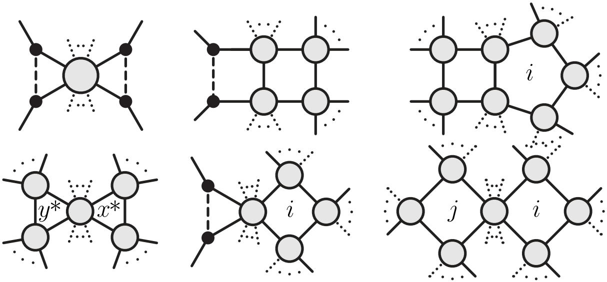

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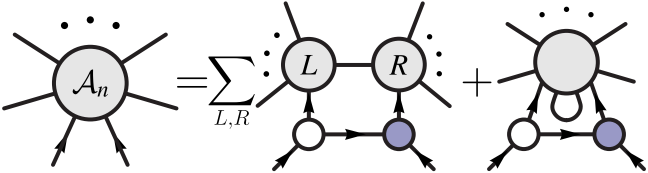

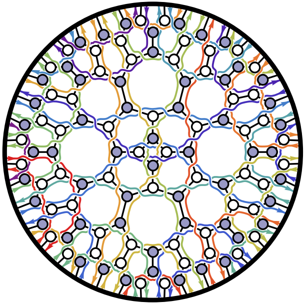

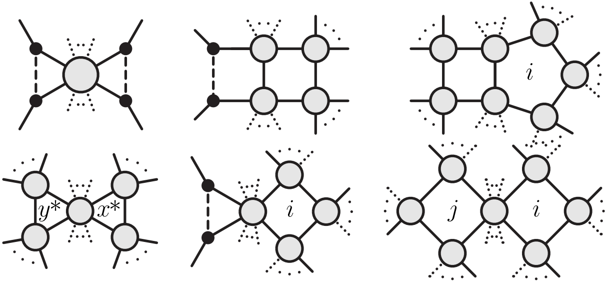

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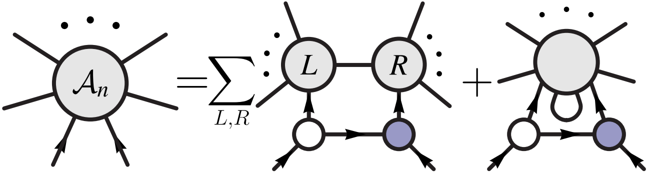

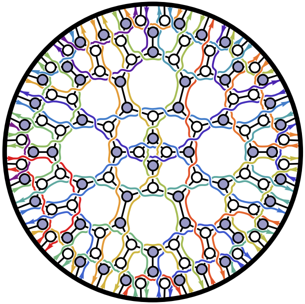

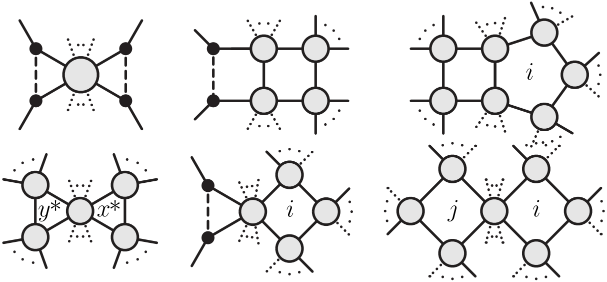

On-Shell Physics/Grassmannian Geometry Correspondence

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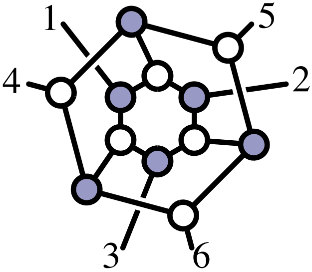
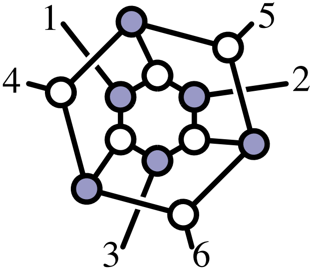


On-Shell Physics/Grassmannian Geometry Correspondence

$$f_\Gamma \equiv \prod_i \Big(\sum_{h_i, q_i} \int d^3\text{LIPS}_i \Big) \prod_v \mathcal{A}_v$$



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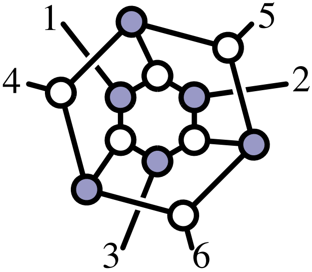

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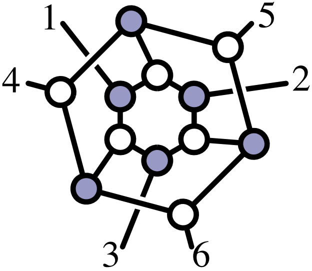

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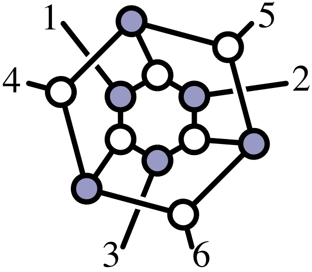

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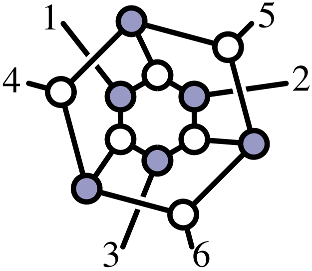

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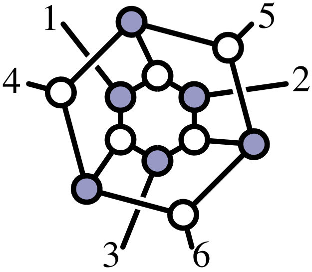

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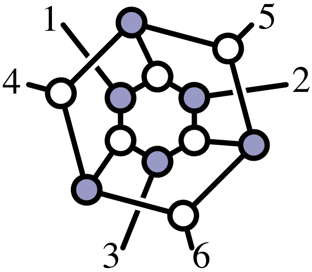

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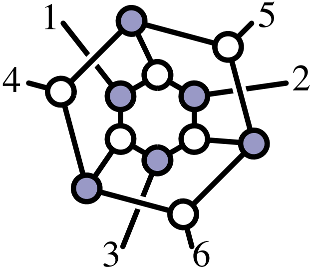

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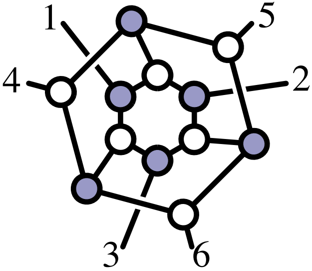

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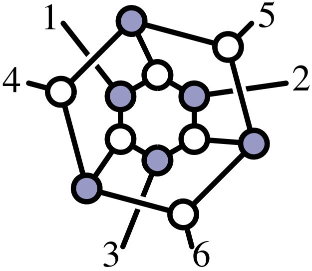

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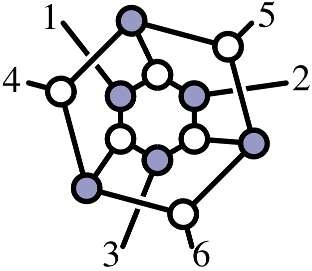

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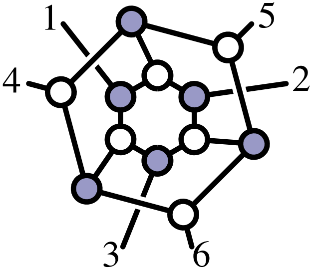

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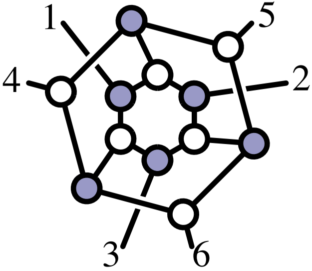

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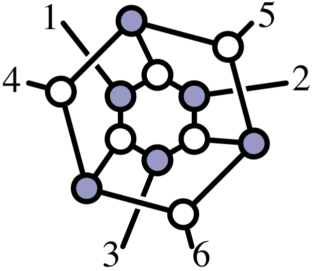

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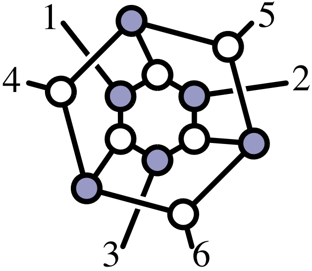

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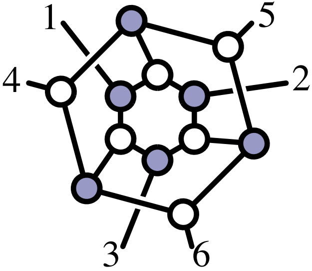

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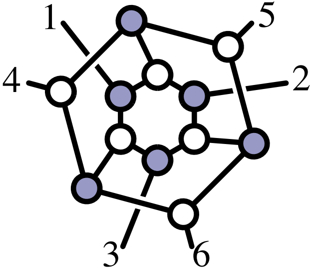

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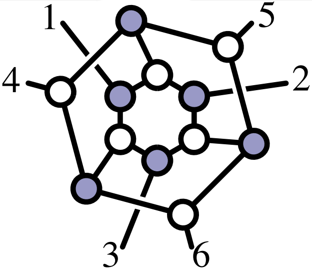
On-Shell Physics

$\Leftrightarrow$

Grassmannian Geometry



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On-Shell Physics

- on-shell diagrams

$\Leftrightarrow$

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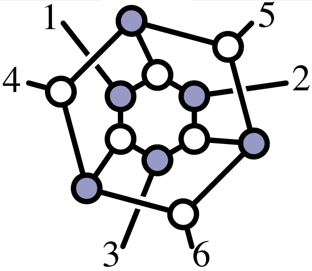

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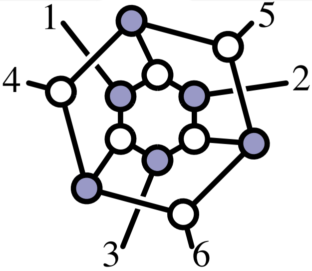
On-Shell Physics

- on-shell diagrams

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$



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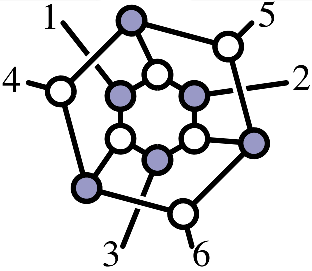

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On-Shell Physics

- on-shell diagrams

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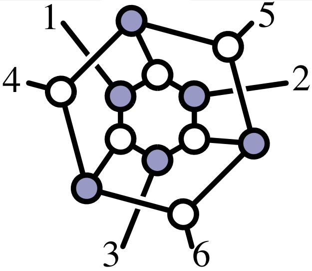
Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms



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On-Shell Physics

- on-shell diagrams

- physical symmetries

$\Leftrightarrow$

Grassmannian Geometry

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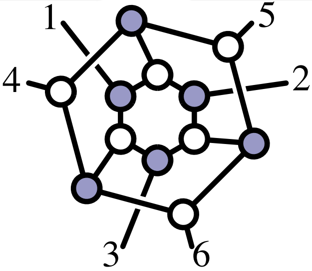

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On-Shell Physics

- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

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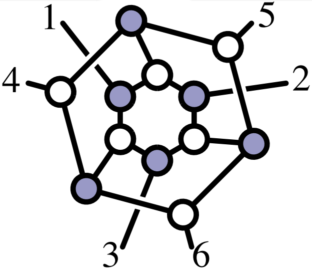

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On-Shell Physics

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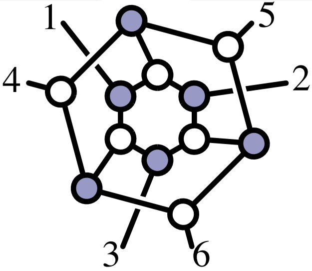
$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations



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On-Shell Physics: planar $\mathcal{N}=4$

- on-shell diagrams
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k, n)$, volume-form $\Omega_C\}$
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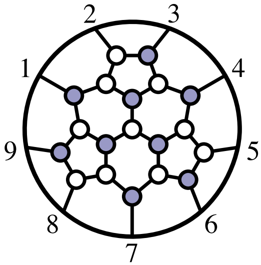

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On-Shell Physics: planar $\mathcal{N}=4$

- on-shell diagrams
  - bi-colored
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

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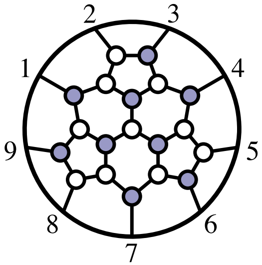
- volume-preserving diffeomorphisms
  - cluster coordinate mutations



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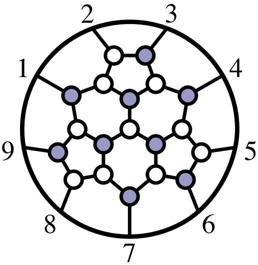
On-Shell Physics: planar $\mathcal{N}\!=\!4$

- on-shell diagrams
  - bi-colored, **un**directed
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C\in G(k,n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations



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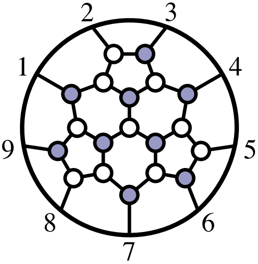
On-Shell Physics: planar $\mathcal{N}=4$

- on-shell diagrams
  - bi-colored, **un**directed, **planar**
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
- volume-preserving diffeomorphisms
  - cluster coordinate mutations



On-Shell Physics/Grassmannian Geometry Correspondence

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On-Shell Physics: planar $\mathcal{N}=4$

- on-shell diagrams
  - bi-colored, **un**directed, **planar**
- physical symmetries
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
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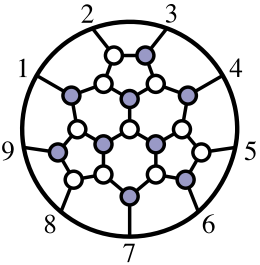

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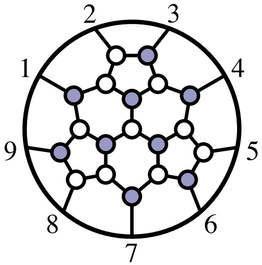

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Grassmannian Geometry

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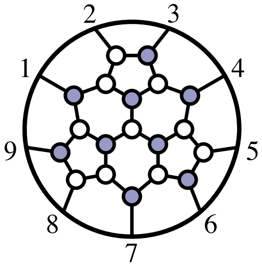

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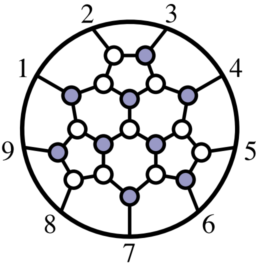

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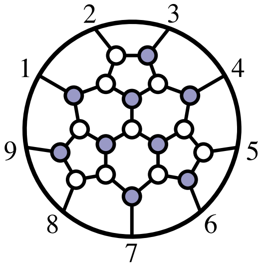

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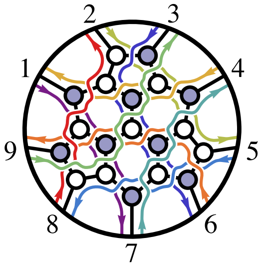

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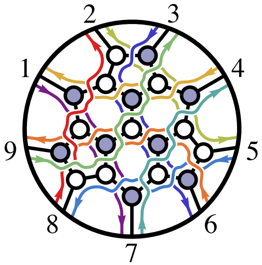

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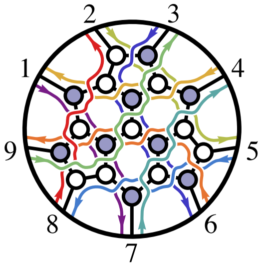

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On-Shell Physics: planar $\mathcal{N}<4$

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Grassmannian Geometry

- $\{$strata $C\in G(k,n)$, volume-form $\Omega_C\}$
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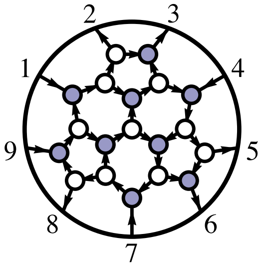

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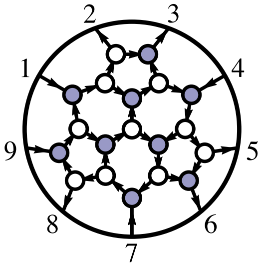

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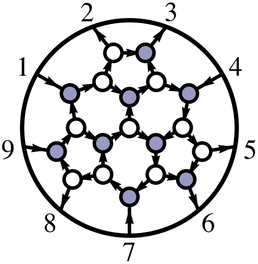

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Grassmannian Geometry

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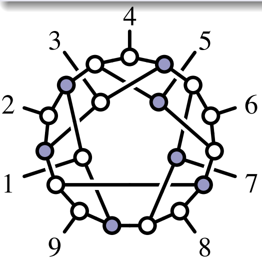

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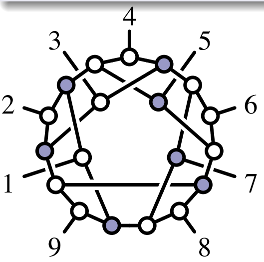

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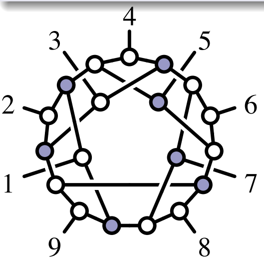

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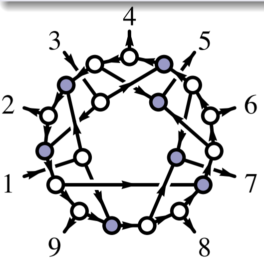

$$C^\perp \equiv \begin{pmatrix} \alpha_1 & 1 & 0 & \alpha_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_3 & 1 & 0 & \alpha_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_5 & 1 & \alpha_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & \alpha_7 & \alpha_8 \\ 0 & 0 & 0 & 0 & 0 & \alpha_9 & 1 & \alpha_{10} & 0 \\ \alpha_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \alpha_{12} \\ 0 & \alpha_{13} & \alpha_{14} & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Omega_C \equiv \left(\frac{d\alpha_1}{\alpha_1}\wedge\cdots\wedge\frac{d\alpha_{14}}{\alpha_{14}}\right)\times\big(1+\alpha_2\alpha_4\alpha_{13}(\alpha_8+\alpha_7\alpha_{12})\big)^{\mathcal{N}-4}$$

On-Shell Physics/Grassmannian Geometry Correspondence

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On-Shell Physics: non-planar $\mathcal{N} < 4$

- on-shell diagrams
  - bi-colored, **directed** , **non**-planar
- physical symmetries: **?**
  - trivial symmetries (identities)

$\Leftrightarrow$

Grassmannian Geometry

- $\{$strata $C \in G(k,n)$, volume-form $\Omega_C\}$
  - cluster variety , $\big(\prod_i \frac{d\alpha_i}{\alpha_i}\big) \times \mathcal{J}^{\mathcal{N}-4}$
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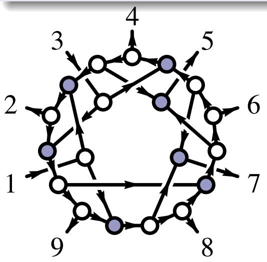

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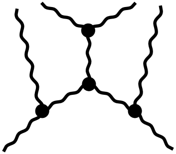

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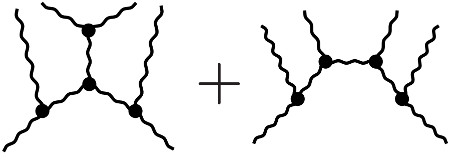

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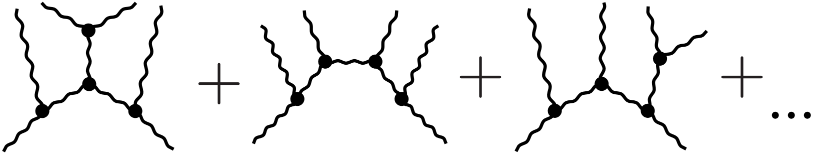

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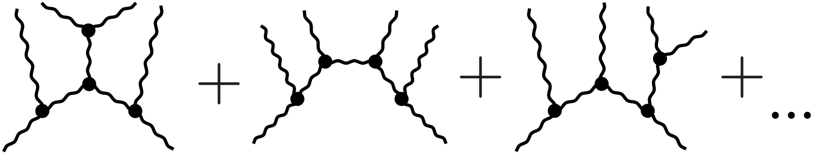

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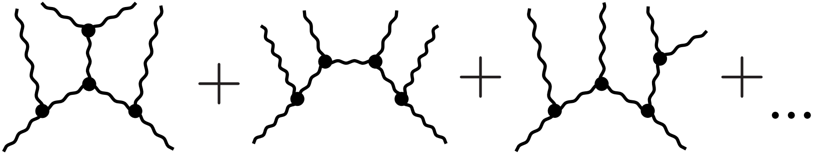

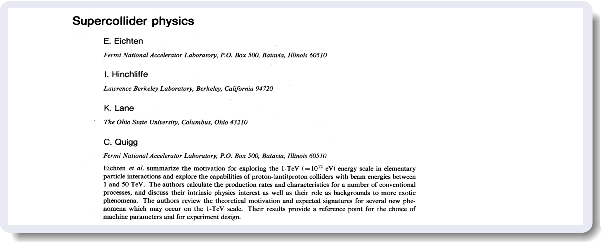

Supercollider physics

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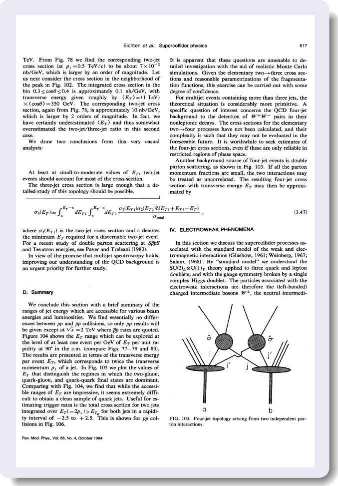

Eichten *et al.*: Supercollider physics 617

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D. Summary

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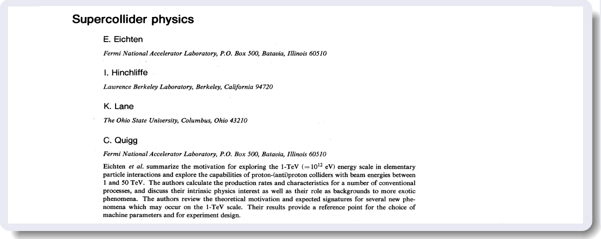

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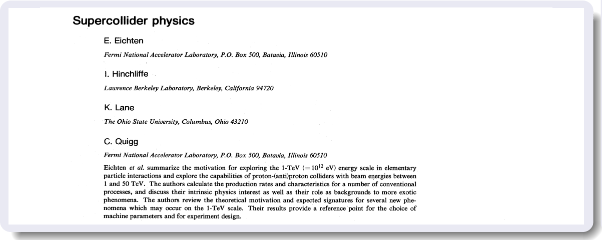

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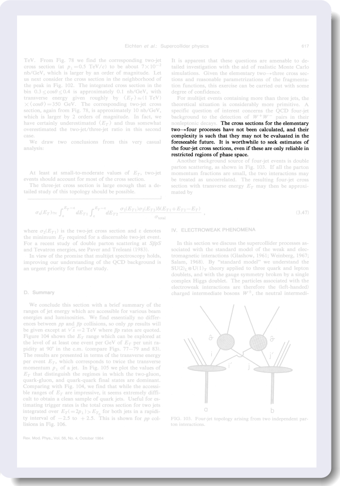

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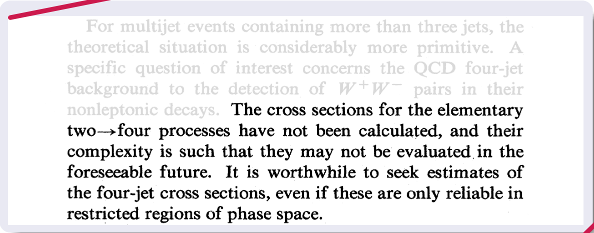

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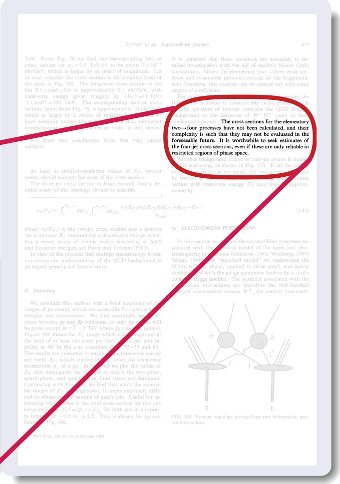

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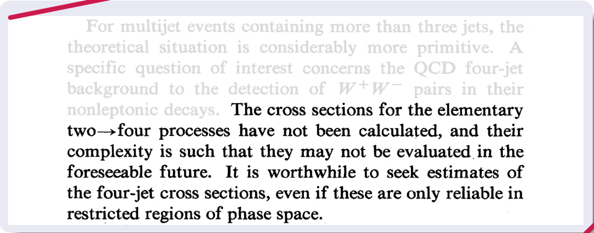
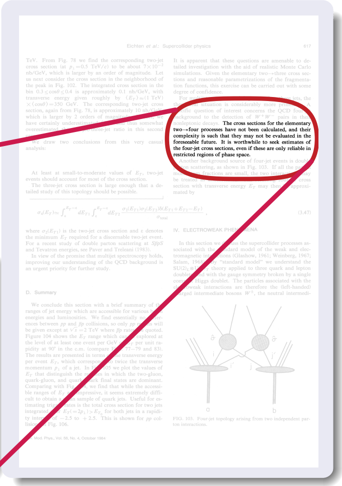


For multijet events containing more than three jets, the theoretical situation is considerably more primitive. A specific question of interest concerns the QCD four-jet background to the detection of W^+W^- pairs in their nonleptonic decays. The cross sections for the elementary two→four processes have not been calculated, and their complexity is such that they may not be evaluated in the foreseeable future. It is worthwhile to seek estimates of the four-jet cross sections, even if these are only reliable in restricted regions of phase space.



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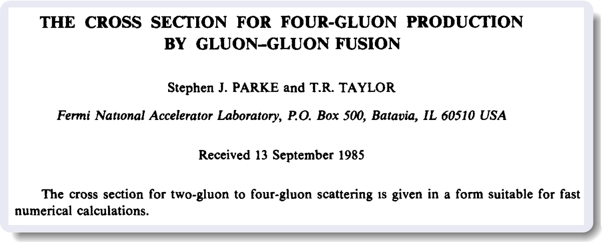

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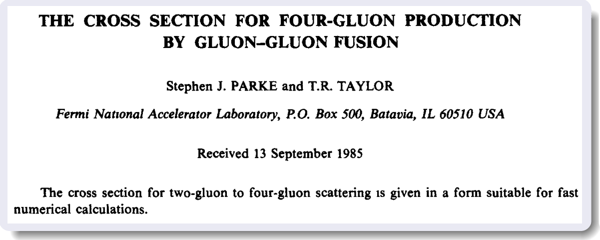

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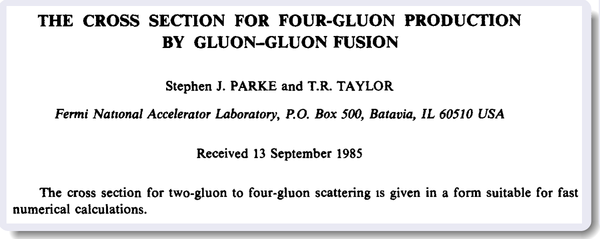

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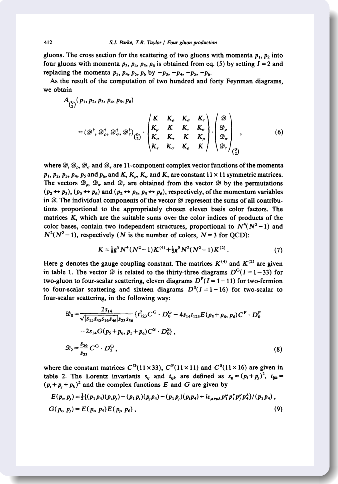

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gluons. The cross section for the scattering of two gluons with momenta p_1, p_2 into four gluons with momenta p_3, p_4, p_5, p_6 is obtained from eq. (5) by setting $I=2$ and replacing the momenta p_3, p_4, p_5, p_6 by $-p_3, -p_4, -p_5, -p_6$.

As the result of the computation of two hundred and forty Feynman diagrams, we obtain

$$A_{\binom{0}{2}}(p_1, p_2, p_3, p_4, p_5, p_6)$$

$$=(\mathscr{D}^\dagger, \mathscr{D}^\dagger_\rho, \mathscr{D}^\dagger_\sigma, \mathscr{D}^\dagger_\tau)_{\binom{0}{2}} \cdot \begin{pmatrix} K & K_\rho & K_\sigma & K_\tau \\ K_\rho & K & K_\tau & K_\sigma \\ K_\sigma & K_\tau & K & K_\rho \\ K_\tau & K_\sigma & K_\rho & K \end{pmatrix} \cdot \begin{pmatrix} \mathscr{D} \\ \mathscr{D}_\rho \\ \mathscr{D}_\sigma \\ \mathscr{D}_\tau \end{pmatrix}_{\binom{0}{2}}, \qquad (6)$$

where $\mathscr{D}$, $\mathscr{D}_\rho$, $\mathscr{D}_\sigma$ and $\mathscr{D}_\tau$ are 11-component complex vector functions of the momenta p_1, p_2, p_3, p_4, p_5 and p_6, and K, K_ρ, K_σ and K_τ are constant 11×11 symmetric matrices. The vectors $\mathscr{D}_\rho$, $\mathscr{D}_\sigma$ and $\mathscr{D}_\tau$ are obtained from the vector $\mathscr{D}$ by the permutations $(p_2 \leftrightarrow p_3)$, $(p_5 \leftrightarrow p_6)$ and $(p_2 \leftrightarrow p_3, p_5 \leftrightarrow p_6)$, respectively, of the momentum variables in $\mathscr{D}$. The individual components of the vector $\mathscr{D}$ represent the sums of all contributions proportional to the appropriately chosen eleven basis color factors. The matrices K, which are the suitable sums over the color indices of products of the color bases, contain two independent structures, proportional to $N^4(N^2-1)$ and $N^2(N^2-1)$, respectively (N is the number of colors, $N=3$ for QCD):

$$K = \tfrac{1}{8}g^8 N^4(N^2-1)K^{(4)} + \tfrac{1}{2}g^8 N^2(N^2-1)K^{(2)}. \qquad (7)$$

Here g denotes the gauge coupling constant. The matrices $K^{(4)}$ and $K^{(2)}$ are given in table 1. The vector $\mathscr{D}$ is related to the thirty-three diagrams $D^{\mathrm{G}}(I=1-33)$ for two-gluon to four-scalar scattering, eleven diagrams $D^{\mathrm{F}}(I=1-11)$ for two-fermion to four-scalar scattering and sixteen diagrams $D^{\mathrm{S}}(I=1-16)$ for two-scalar to four-scalar scattering, in the following way:

$$\mathscr{D}_0 = \frac{2s_{14}}{\sqrt{|s_{15}s_{45}s_{16}s_{46}|}\,s_{23}s_{56}} \{t_{123}^2 C^{\mathrm{G}} \cdot D_0^{\mathrm{G}} - 4s_{14}t_{123}E(p_5+p_6, p_6)C^{\mathrm{F}} \cdot D_0^{\mathrm{F}}$$

$$-2s_{14}G(p_5+p_6, p_5+p_6)C^{\mathrm{S}} \cdot D_0^{\mathrm{S}}\},$$

$$\mathscr{D}_2 = \frac{s_{56}}{s_{23}} C^{\mathrm{G}} \cdot D_2^{\mathrm{G}}, \qquad (8)$$

where the constant matrices $C^{\mathrm{G}}(11\times 33)$, $C^{\mathrm{F}}(11\times 11)$ and $C^{\mathrm{S}}(11\times 16)$ are given in table 2. The Lorentz invariants s_{ij} and t_{ijk} are defined as $s_{ij}=(p_i+p_j)^2$, $t_{ijk}=(p_i+p_j+p_k)^2$ and the complex functions E and G are given by

$$E(p_i, p_j) = \tfrac{1}{2}\{(p_1p_4)(p_ip_j) - (p_1p_i)(p_jp_4) - (p_1p_j)(p_ip_4) + i\varepsilon_{\mu\nu\rho\lambda}p_1^\mu p_i^\nu p_j^\rho p_4^\lambda\}/(p_1p_4),$$

$$G(p_i, p_j) = E(p_i, p_5)E(p_j, p_6), \qquad (9)$$

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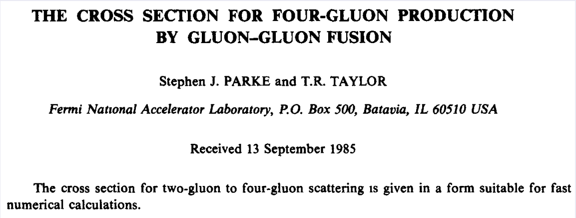

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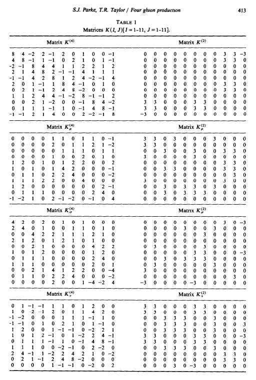

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TABLE 1
Matrices $K(I, J)[I = 1\text{-}11, J = 1\text{-}11]$.

Matrix $K^{(4)}$											Matrix $K^{(2)}$										
8	4	−2	2	−1	2	0	1	0	0	−1	0	0	0	0	0	0	0	0	3	3	−3
4	8	−1	1	−1	0	2	1	0	1	−1	0	0	0	0	0	0	0	0	3	3	0
−2	−1	8	4	4	1	1	2	2	1	2	0	0	0	0	0	0	0	0	0	0	0
2	1	4	8	2	−1	−1	4	1	1	1	0	0	0	0	0	0	0	0	0	0	0
−1	−1	4	2	8	1	2	4	−2	−1	4	0	0	0	0	0	0	0	0	0	0	0
2	0	1	−1	1	8	4	−1	0	1	0	0	0	0	0	0	0	0	0	3	3	0
0	2	1	−1	2	4	8	−2	0	0	0	0	0	0	0	0	0	0	0	3	3	0
1	1	2	4	4	−1	−2	8	−1	−1	2	0	0	0	0	0	0	0	0	0	0	0
0	0	2	1	−2	0	0	−1	8	4	−2	3	3	0	0	0	3	3	0	0	0	0
0	1	1	1	−1	1	0	−1	4	8	−1	3	3	0	0	0	3	3	0	0	0	0
−1	−1	2	1	4	0	0	2	−2	−1	8	−3	0	0	0	0	0	0	0	0	0	0
Matrix $K_{\rho}^{(4)}$											Matrix $K_{\rho}^{(2)}$										
0	0	0	0	1	1	0	1	1	0	−1	3	3	0	3	0	0	0	3	0	0	0
0	0	0	0	2	0	1	1	2	1	−2	3	3	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	1	1	0	1	1	0	0	3	0	0	3	0	0	3	3	0
0	0	0	0	1	0	0	2	0	1	0	3	0	0	0	0	3	0	0	0	0	0
1	2	0	1	0	1	2	2	0	0	2	0	0	0	0	0	0	0	0	3	3	0
1	0	1	0	1	4	2	0	0	0	−1	0	0	3	3	0	0	0	0	3	3	0
0	1	1	0	2	2	4	0	0	0	−2	0	0	0	0	0	0	0	0	0	3	0
1	1	1	2	2	0	0	4	0	0	0	3	0	0	0	0	0	0	0	3	0	0
1	2	0	0	0	0	0	0	0	2	−1	0	0	3	0	3	3	0	3	0	0	0
0	1	1	1	0	0	0	0	2	4	0	0	0	3	0	3	3	3	0	0	0	0
−1	−2	1	0	2	−1	−2	0	−1	0	4	0	0	0	0	0	0	0	0	0	0	0
Matrix $K_{\sigma}^{(4)}$											Matrix $K_{\sigma}^{(2)}$										
4	2	0	2	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	3	0	−3
2	4	0	1	0	0	1	1	0	1	0	0	0	0	0	3	0	0	3	0	0	0
0	0	4	2	2	1	1	1	2	1	0	0	0	0	0	0	0	3	0	0	0	0
2	1	2	0	1	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	2	1	0	0	0	0	4	2	2	0	3	0	0	0	0	3	0	0	0	0
1	0	1	2	0	0	0	0	1	2	0	0	0	0	0	0	3	3	0	0	0	−3
0	1	1	1	0	0	0	0	2	4	0	0	0	3	0	3	3	3	0	0	0	0
1	1	1	0	0	0	0	0	2	0	1	0	3	0	0	0	0	0	3	0	0	0
0	0	2	1	4	1	2	2	0	0	−4	3	0	0	0	0	0	0	0	0	0	0
0	1	1	0	2	2	4	0	0	0	−2	0	0	0	0	0	0	0	0	0	3	0
0	0	0	0	2	0	0	1	−4	−2	4	−3	0	0	0	0	−3	0	0	0	0	0
Matrix $K_{\tau}^{(4)}$											Matrix $K_{\tau}^{(2)}$										
0	1	−1	−1	1	1	0	1	2	0	0	3	3	0	0	0	3	3	0	0	0	0
1	0	−2	−1	2	0	1	1	4	2	0	3	3	0	0	0	3	3	0	0	0	0
−1	−2	0	0	0	1	1	1	−1	1	0	0	0	3	3	3	0	0	3	0	0	0
−1	−1	0	1	0	2	1	0	1	−1	0	0	0	3	3	3	0	0	3	0	0	3
1	2	0	0	1	−1	−1	0	−2	2	1	0	0	3	3	3	0	0	3	0	0	0
1	0	1	2	−1	0	1	−2	2	4	−1	3	3	0	0	0	3	3	0	0	0	−3
0	1	1	1	−1	1	0	−1	4	8	−1	3	3	0	0	0	3	3	0	0	0	0
1	1	1	0	0	−2	−1	0	2	−2	0	0	0	3	3	3	0	0	3	0	0	0
2	4	−1	1	−2	2	4	2	1	0	−2	0	0	0	0	0	0	0	0	3	3	0
0	2	1	−1	2	4	8	−2	0	0	0	0	0	0	0	0	0	0	0	3	3	0
0	0	0	0	1	−1	−1	0	−2	0	2	0	0	0	3	0	−3	0	0	0	0	0

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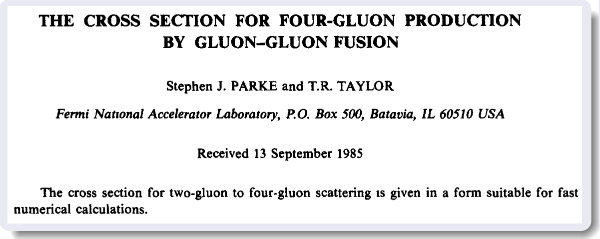

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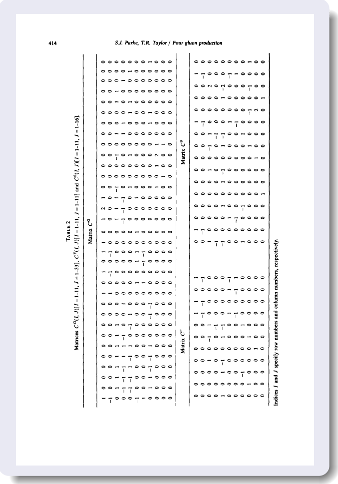

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TABLE 2

Matrices $C^G(I,J)[I=1\text{-}11, J=1\text{-}33]$, $C^F(I,J)[I=1\text{-}11, J=1\text{-}11]$ and $C^S(I,J)[I=1\text{-}11, J=1\text{-}16]$.

Matrix C^G

Matrix C^F

Matrix C^S

Indices I and J specify row numbers and column numbers, respectively.

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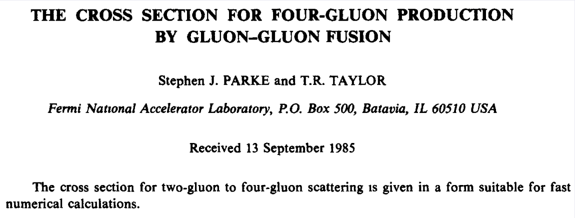

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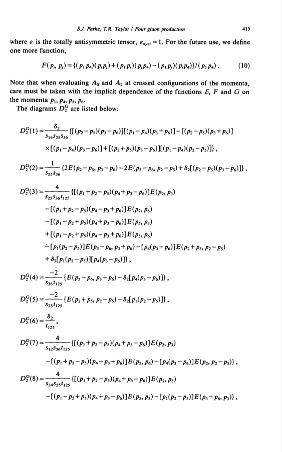

S.J. Parke, T.R. Taylor / Four gluon production 415

where ε is the totally antisymmetric tensor, $\varepsilon_{xyzt}=1$. For the future use, we define one more function,

$$F(p_i, p_j)=\{(p_1p_4)(p_ip_j)+(p_1p_i)(p_jp_4)-(p_1p_j)(p_ip_4)\}/(p_1p_4)\,. \qquad (10)$$

Note that when evaluating A_0 and A_2 at crossed configurations of the momenta, care must be taken with the implicit dependence of the functions E, F and G on the momenta p_1, p_4, p_5, p_6.

The diagrams D_2^{G} are listed below:

$$D_2^{\mathrm{G}}(1)=\frac{\delta_2}{s_{14}s_{25}s_{36}}\{[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)]-[(p_2-p_5)(p_3+p_6)]$$
$$\times[(p_1-p_4)(p_3-p_6)]+[(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)]\}\,,$$

$$D_2^{\mathrm{G}}(2)=\frac{1}{s_{25}s_{36}}\{2E(p_2-p_5, p_3-p_6)-2E(p_3-p_6, p_2-p_5)+\delta_2[(p_2-p_5)(p_3-p_6)]\}\,,$$

$$D_2^{\mathrm{G}}(3)=\frac{4}{s_{25}s_{36}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3)$$
$$-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2, p_6)$$
$$-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5, p_3)$$
$$+[(p_1-p_2+p_5)(p_4-p_3+p_6)]E(p_5, p_6)$$
$$-[p_1(p_2-p_5)]E(p_3-p_6, p_3+p_6)-[p_4(p_3-p_6)]E(p_2+p_5, p_2-p_5)$$
$$+\delta_2[p_1(p_2-p_5)][p_4(p_3-p_6)]\}\,,$$

$$D_2^{\mathrm{G}}(4)=\frac{-2}{s_{36}t_{125}}\{E(p_3-p_6, p_3+p_6)-\delta_2[p_4(p_3-p_6)]\}\,,$$

$$D_2^{\mathrm{G}}(5)=\frac{-2}{s_{25}t_{125}}\{E(p_2+p_5, p_2-p_5)-\delta_2[p_1(p_2-p_5)]\}\,,$$

$$D_2^{\mathrm{G}}(6)=\frac{\delta_2}{t_{125}}\,,$$

$$D_2^{\mathrm{G}}(7)=\frac{4}{s_{12}s_{36}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3)$$
$$-[(p_1+p_2-p_5)(p_4-p_3+p_6)]E(p_2, p_6)-[p_4(p_3-p_6)]E(p_2, p_2-p_5)\}\,,$$

$$D_2^{\mathrm{G}}(8)=\frac{4}{s_{34}s_{25}t_{125}}\{[(p_1+p_2-p_5)(p_4+p_3-p_6)]E(p_2, p_3)$$
$$-[(p_1-p_2+p_5)(p_4+p_3-p_6)]E(p_5, p_3)-[p_1(p_2-p_5)]E(p_3-p_6, p_3)\}\,,$$

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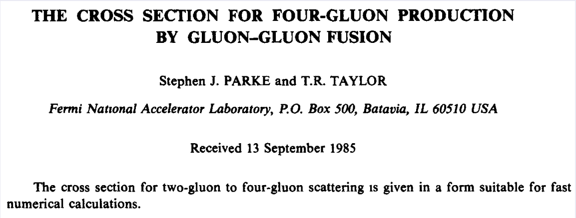

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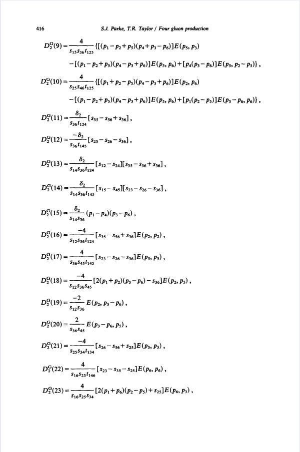

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$$D_2^G(9) = \frac{4}{s_{15}s_{36}t_{125}}\{[(p_1 - p_2 + p_5)(p_4 + p_3 - p_6)]E(p_5, p_3)$$
$$-[(p_1 - p_2 + p_5)(p_4 - p_3 + p_6)]E(p_5, p_6) + [p_4(p_3 - p_6)]E(p_5, p_2 - p_5)\},$$

$$D_2^G(10) = \frac{4}{s_{25}s_{46}t_{125}}\{[(p_1 + p_2 - p_5)(p_4 - p_3 + p_6)]E(p_2, p_6)$$
$$-[(p_1 - p_2 + p_5)(p_4 - p_3 + p_6)]E(p_5, p_6) + [p_1(p_2 - p_5)]E(p_3 - p_6, p_6)\},$$

$$D_2^G(11) = \frac{\delta_2}{s_{36}t_{124}}[s_{35} - s_{56} + s_{36}],$$

$$D_2^G(12) = \frac{-\delta_2}{s_{36}t_{145}}[s_{23} - s_{26} - s_{36}],$$

$$D_2^G(13) = \frac{\delta_2}{s_{14}s_{36}t_{124}}[s_{12} - s_{24}][s_{35} - s_{56} + s_{36}],$$

$$D_2^G(14) = \frac{\delta_2}{s_{14}s_{36}t_{145}}[s_{15} - s_{45}][s_{23} - s_{26} - s_{36}],$$

$$D_2^G(15) = \frac{\delta_2}{s_{14}s_{36}}(p_1 - p_4)(p_3 - p_6),$$

$$D_2^G(16) = \frac{-4}{s_{12}s_{36}t_{124}}[s_{35} - s_{56} + s_{36}]E(p_2, p_2),$$

$$D_2^G(17) = \frac{4}{s_{36}s_{45}t_{145}}[s_{23} - s_{26} - s_{36}]E(p_5, p_5),$$

$$D_2^G(18) = \frac{-4}{s_{12}s_{36}s_{45}}[2(p_1 + p_2)(p_3 - p_6) - s_{36}]E(p_2, p_5),$$

$$D_2^G(19) = \frac{-2}{s_{12}s_{36}}E(p_2, p_3 - p_6),$$

$$D_2^G(20) = \frac{2}{s_{36}s_{45}}E(p_3 - p_6, p_5),$$

$$D_2^G(21) = \frac{-4}{s_{25}s_{34}t_{134}}[s_{26} - s_{56} + s_{25}]E(p_3, p_3),$$

$$D_2^G(22) = \frac{4}{s_{16}s_{25}t_{146}}[s_{23} - s_{35} - s_{25}]E(p_6, p_6),$$

$$D_2^G(23) = \frac{4}{s_{16}s_{25}s_{34}}[2(p_1 + p_6)(p_2 - p_5) + s_{25}]E(p_6, p_3),$$

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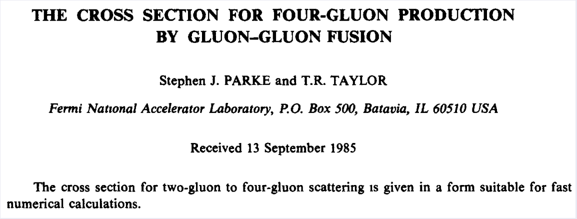

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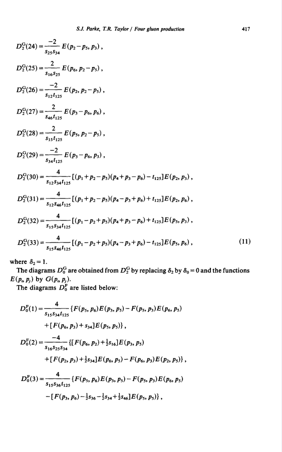

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$$D_2^{\mathrm{G}}(24) = \frac{-2}{s_{25}s_{34}} E(p_2 - p_5, p_3)\,,$$

$$D_2^{\mathrm{G}}(25) = \frac{2}{s_{16}s_{25}} E(p_6, p_2 - p_5)\,,$$

$$D_2^{\mathrm{G}}(26) = \frac{-2}{s_{12}t_{125}} E(p_2, p_2 - p_5)\,,$$

$$D_2^{\mathrm{G}}(27) = \frac{2}{s_{46}t_{125}} E(p_3 - p_6, p_6)\,,$$

$$D_2^{\mathrm{G}}(28) = \frac{2}{s_{15}t_{125}} E(p_5, p_2 - p_5)\,,$$

$$D_2^{\mathrm{G}}(29) = \frac{-2}{s_{34}t_{125}} E(p_3 - p_6, p_3)\,,$$

$$D_2^{\mathrm{G}}(30) = \frac{4}{s_{12}s_{34}t_{125}} [(p_1 + p_2 - p_5)(p_4 + p_3 - p_6) - t_{125}] E(p_2, p_3)\,,$$

$$D_2^{\mathrm{G}}(31) = \frac{4}{s_{12}s_{46}t_{125}} [(p_1 + p_2 - p_5)(p_4 - p_3 + p_6) + t_{125}] E(p_2, p_6)\,,$$

$$D_2^{\mathrm{G}}(32) = \frac{4}{s_{15}s_{34}t_{125}} [(p_1 - p_2 + p_5)(p_4 + p_3 - p_6) + t_{125}] E(p_5, p_3)\,,$$

$$D_2^{\mathrm{G}}(33) = \frac{4}{s_{15}s_{46}t_{125}} [(p_1 - p_2 + p_5)(p_4 - p_3 + p_6) - t_{125}] E(p_5, p_6)\,, \tag{11}$$

where $\delta_2 = 1$.

The diagrams D_0^{G} are obtained from D_2^{G} by replacing δ_2 by $\delta_0 = 0$ and the functions $E(p_i, p_j)$ by $G(p_i, p_j)$.

The diagrams D_0^{F} are listed below:

$$D_0^{\mathrm{F}}(1) = \frac{4}{s_{15}s_{34}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5)$$
$$+ [F(p_6, p_3) + s_{34}]E(p_5, p_5)\}\,,$$

$$D_0^{\mathrm{F}}(2) = \frac{-4}{s_{16}s_{25}s_{34}} \{[F(p_6, p_2) + \tfrac{1}{2}s_{16}]E(p_3, p_5)$$
$$+ [F(p_2, p_3) + \tfrac{1}{2}s_{34}]E(p_6, p_5) - F(p_6, p_3)E(p_2, p_5)\}\,,$$

$$D_0^{\mathrm{F}}(3) = \frac{4}{s_{15}s_{36}t_{125}} \{F(p_5, p_6)E(p_3, p_5) - F(p_5, p_3)E(p_6, p_5)$$
$$- [F(p_3, p_6) - \tfrac{1}{2}s_{36} - \tfrac{1}{2}s_{34} + \tfrac{1}{2}s_{46}]E(p_5, p_5)\}\,,$$

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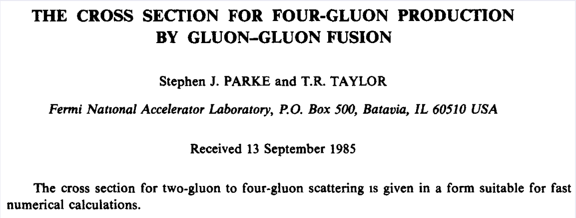

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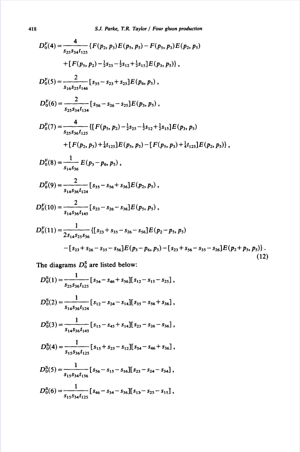

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$$D_0^{\mathrm{F}}(4) = \frac{4}{s_{25}s_{34}t_{125}}\{F(p_2, p_3)E(p_5, p_5) - F(p_5, p_3)E(p_2, p_5)$$
$$+[F(p_5, p_2) - \tfrac{1}{2}s_{25} - \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{15}]E(p_3, p_5)\},$$

$$D_0^{\mathrm{F}}(5) = \frac{2}{s_{16}s_{25}t_{146}}[s_{35} - s_{23} + s_{25}]E(p_6, p_5),$$

$$D_0^{\mathrm{F}}(6) = \frac{2}{s_{25}s_{34}t_{134}}[s_{56} - s_{26} - s_{25}]E(p_3, p_5),$$

$$D_0^{\mathrm{F}}(7) = \frac{4}{s_{25}s_{36}t_{125}}\{[F(p_5, p_2) - \tfrac{1}{2}s_{25} - \tfrac{1}{2}s_{12} + \tfrac{1}{2}s_{15}]E(p_3, p_5)$$
$$+[F(p_2, p_3) + \tfrac{1}{4}t_{125}]E(p_5, p_5) - [F(p_5, p_3) + \tfrac{1}{4}t_{125}]E(p_2, p_5)\},$$

$$D_0^{\mathrm{F}}(8) = \frac{1}{s_{14}s_{36}}E(p_3 - p_6, p_5),$$

$$D_0^{\mathrm{F}}(9) = \frac{2}{s_{14}s_{36}t_{124}}[s_{35} - s_{56} + s_{36}]E(p_2, p_5),$$

$$D_0^{\mathrm{F}}(10) = \frac{2}{s_{14}s_{36}t_{145}}[s_{23} - s_{26} - s_{36}]E(p_5, p_5),$$

$$D_0^{\mathrm{F}}(11) = \frac{1}{2s_{14}s_{25}s_{36}}\{[s_{23} + s_{35} - s_{26} - s_{56}]E(p_2 - p_5, p_5)$$
$$-[s_{23} + s_{26} - s_{35} - s_{56}]E(p_3 - p_6, p_5) - [s_{23} + s_{56} - s_{35} - s_{26}]E(p_2 + p_5, p_5)\}. \qquad (12)$$

The diagrams D_0^{S} are listed below:

$$D_0^{\mathrm{S}}(1) = \frac{1}{s_{25}s_{36}t_{125}}[s_{34} - s_{46} + s_{36}][s_{12} - s_{15} - s_{25}],$$

$$D_0^{\mathrm{S}}(2) = \frac{1}{s_{14}s_{36}t_{124}}[s_{12} - s_{24} - s_{14}][s_{35} - s_{56} + s_{36}],$$

$$D_0^{\mathrm{S}}(3) = \frac{1}{s_{14}s_{36}t_{145}}[s_{15} - s_{45} + s_{14}][s_{23} - s_{26} - s_{36}],$$

$$D_0^{\mathrm{S}}(4) = \frac{1}{s_{15}s_{36}t_{125}}[s_{15} + s_{25} - s_{12}][s_{34} - s_{46} + s_{36}],$$

$$D_0^{\mathrm{S}}(5) = \frac{1}{s_{15}s_{34}t_{156}}[s_{56} - s_{15} - s_{16}][s_{23} - s_{24} - s_{34}],$$

$$D_0^{\mathrm{S}}(6) = \frac{1}{s_{15}s_{34}t_{125}}[s_{46} - s_{34} - s_{36}][s_{12} - s_{25} - s_{15}],$$

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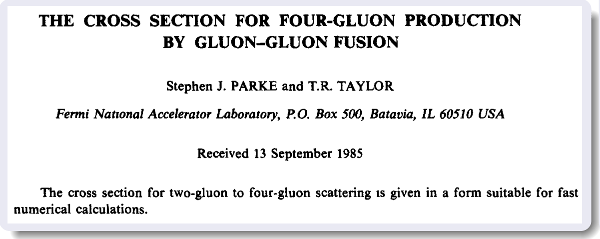

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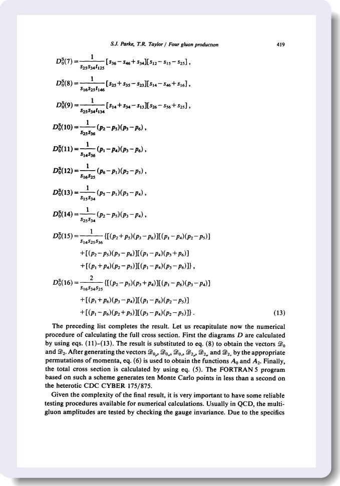

$$D_0^S(7)=\frac{1}{s_{25}s_{34}t_{125}}[s_{36}-s_{46}+s_{34}][s_{12}-s_{15}-s_{25}]\,,$$

$$D_0^S(8)=\frac{1}{s_{16}s_{25}t_{146}}[s_{25}+s_{35}-s_{23}][s_{14}-s_{46}+s_{16}]\,,$$

$$D_0^S(9)=\frac{1}{s_{25}s_{34}t_{134}}[s_{14}+s_{34}-s_{13}][s_{26}-s_{56}+s_{25}]\,,$$

$$D_0^S(10)=\frac{1}{s_{25}s_{36}}(p_2-p_5)(p_3-p_6)\,,$$

$$D_0^S(11)=\frac{1}{s_{14}s_{36}}(p_1-p_4)(p_3-p_6)\,,$$

$$D_0^S(12)=\frac{1}{s_{16}s_{25}}(p_6-p_1)(p_2-p_5)\,,$$

$$D_0^S(13)=\frac{1}{s_{15}s_{34}}(p_5-p_1)(p_3-p_4)\,,$$

$$D_0^S(14)=\frac{1}{s_{25}s_{34}}(p_2-p_5)(p_3-p_4)\,,$$

$$\begin{aligned}D_0^S(15)=\frac{1}{s_{14}s_{25}s_{36}}&\{[(p_2+p_5)(p_3-p_6)][(p_1-p_4)(p_2-p_5)]\\&+[(p_2-p_5)(p_3-p_6)][(p_1-p_4)(p_3+p_6)]\\&+[(p_1+p_4)(p_2-p_5)][(p_1-p_4)(p_3-p_6)]\}\,,\\D_0^S(16)=\frac{2}{s_{16}s_{34}s_{25}}&\{[(p_2-p_5)(p_3+p_4)][(p_1-p_6)(p_3-p_4)]\\&+[(p_1+p_6)(p_3-p_4)][(p_1-p_6)(p_2-p_5)]\\&+[(p_1-p_6)(p_2+p_5)][(p_3-p_4)(p_2-p_5)]\}\,.\end{aligned}\tag{13}$$

The preceding list completes the result. Let us recapitulate now the numerical procedure of calculating the full cross section. First the diagrams D are calculated by using eqs. (11)-(13). The result is substituted to eq. (8) to obtain the vectors $\mathcal{D}_0$ and $\mathcal{D}_2$. After generating the vectors $\mathcal{D}_{0_\rho}$, $\mathcal{D}_{0_\sigma}$, $\mathcal{D}_{0_\tau}$, $\mathcal{D}_{2_\rho}$, $\mathcal{D}_{2_\sigma}$ and $\mathcal{D}_{2_\tau}$ by the appropriate permutations of momenta, eq. (6) is used to obtain the functions A_0 and A_2. Finally, the total cross section is calculated by using eq. (5). The FORTRAN 5 program based on such a scheme generates ten Monte Carlo points in less than a second on the heterotic CDC CYBER 175/875.

Given the complexity of the final result, it is very important to have some reliable testing procedures available for numerical calculations. Usually in QCD, the multigluon amplitudes are tested by checking the gauge invariance. Due to the specifics

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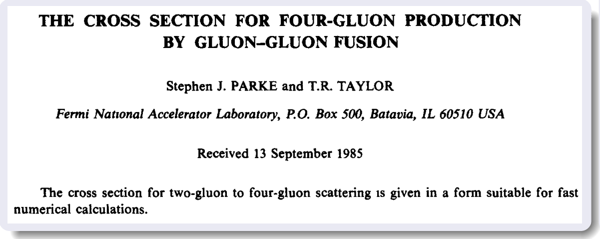

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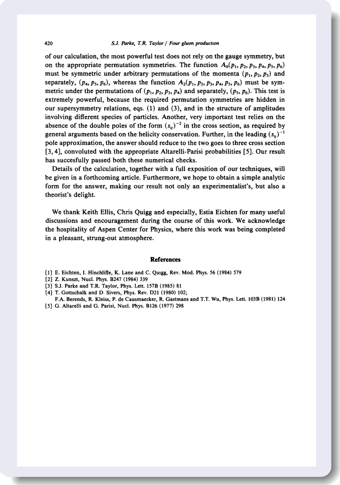

of our calculation, the most powerful test does not rely on the gauge symmetry, but on the appropriate permutation symmetries. The function $A_0(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under arbitrary permutations of the momenta (p_1, p_2, p_3) and separately, (p_4, p_5, p_6), whereas the function $A_2(p_1, p_2, p_3, p_4, p_5, p_6)$ must be symmetric under the permutations of (p_1, p_2, p_3, p_4) and separately, (p_5, p_6). This test is extremely powerful, because the required permutation symmetries are hidden in our supersymmetry relations, eqs. (1) and (3), and in the structure of amplitudes involving different species of particles. Another, very important test relies on the absence of the double poles of the form $(s_{ij})^{-2}$ in the cross section, as required by general arguments based on the helicity conservation. Further, in the leading $(s_{ij})^{-1}$ pole approximation, the answer should reduce to the two goes to three cross section [3, 4], convoluted with the appropriate Altarelli-Parisi probabilities [5]. Our result has succesfully passed both these numerical checks.

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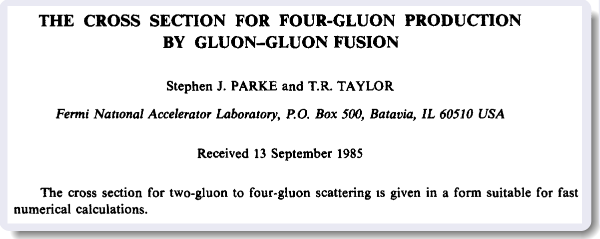

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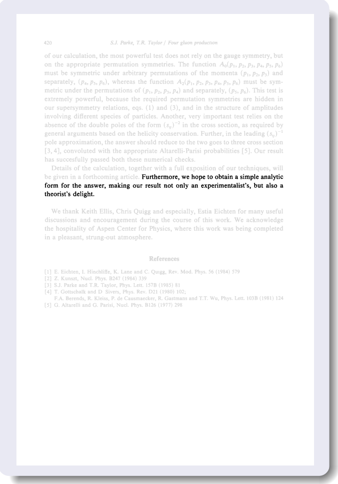

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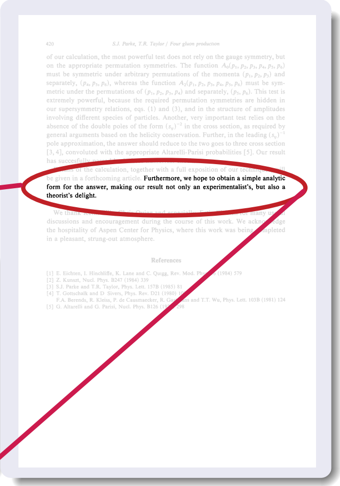

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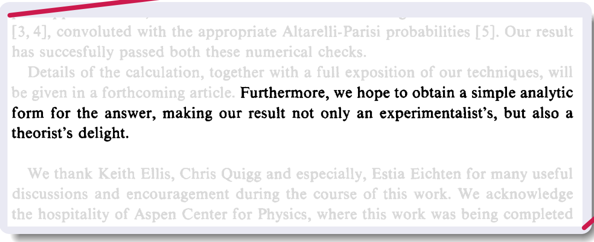

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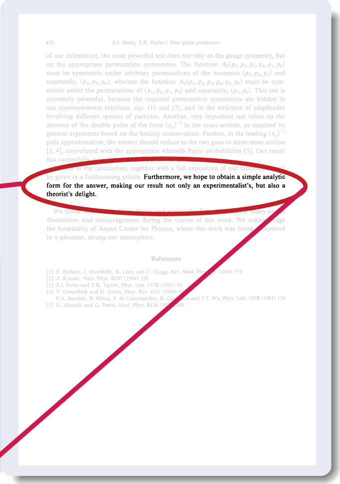
420 *S.J. Parke, T.R. Taylor / Four gluon production*

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References

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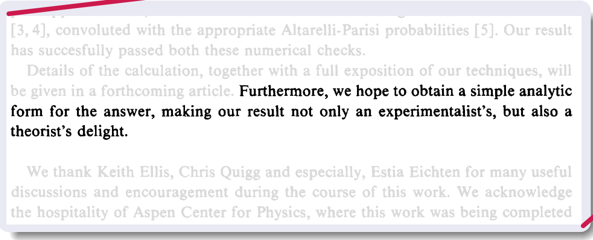

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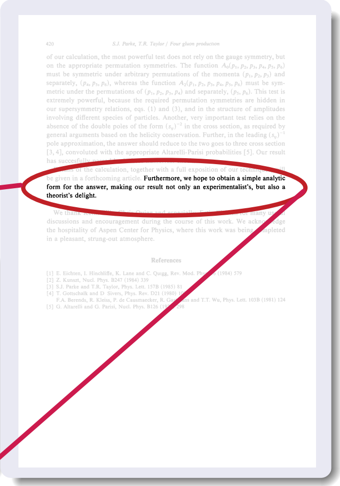

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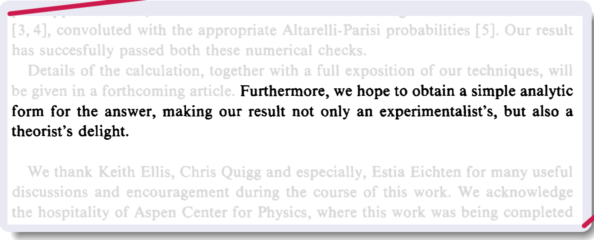

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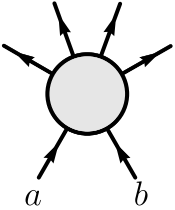

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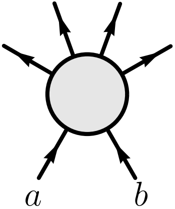

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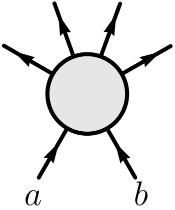

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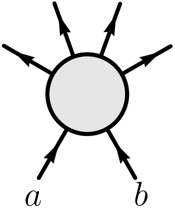

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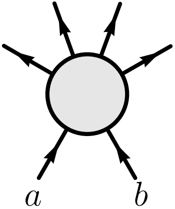

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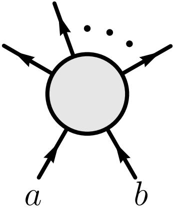

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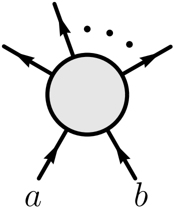

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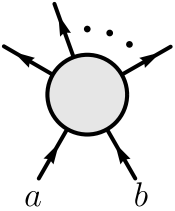

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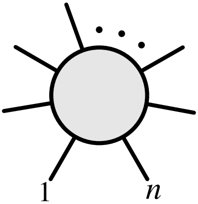

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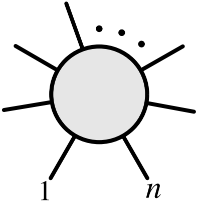

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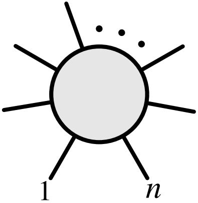

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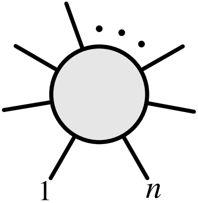

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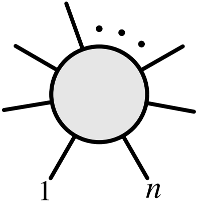

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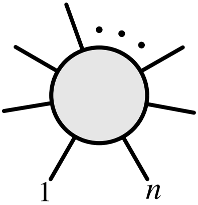
$\mathcal{A}_n \equiv$ 

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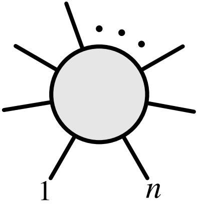

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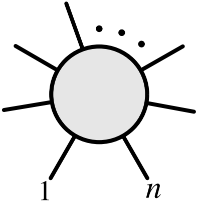

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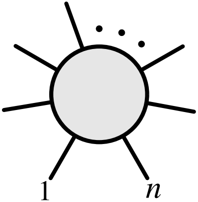

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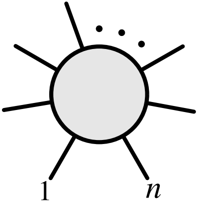
$\mathcal{A}_n \equiv$ 

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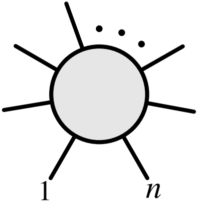
$\mathcal{A}_n \equiv$ 

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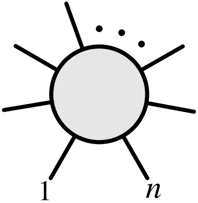

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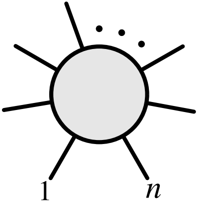
$\mathcal{A}_n \equiv$ 

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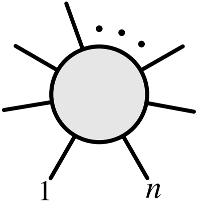
$\mathcal{A}_n \equiv$ 

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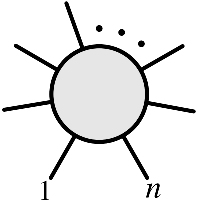
$\mathcal{A}_n \equiv$ 

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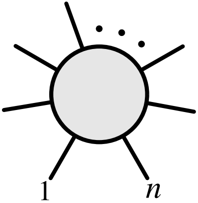
$\mathcal{A}_n \equiv$ 

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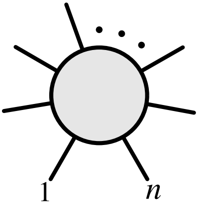

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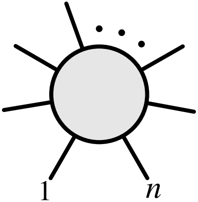

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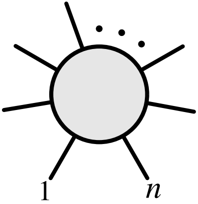
$\mathcal{A}_n \equiv$ 

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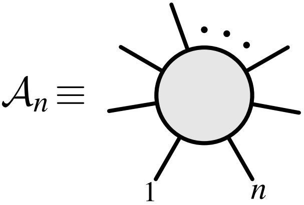

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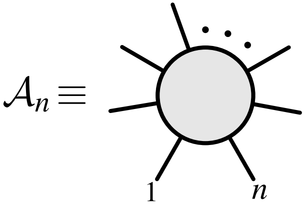

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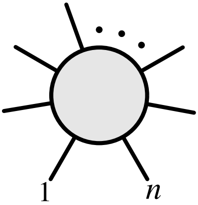
$\mathcal{A}_n \equiv$ 

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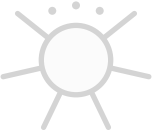

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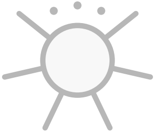

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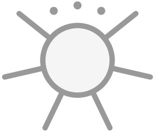

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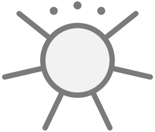

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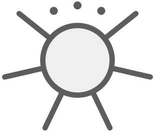

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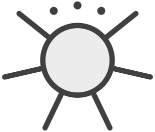

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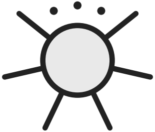

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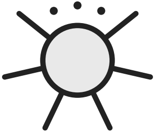

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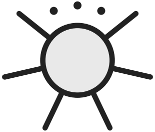

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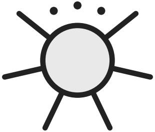

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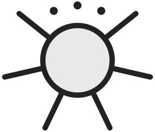

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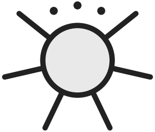

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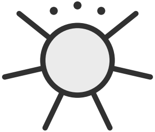

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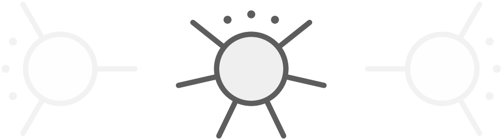

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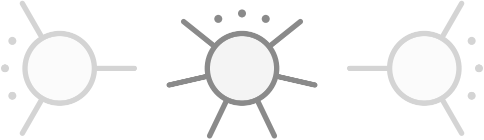

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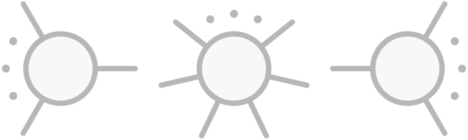

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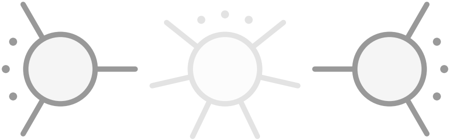

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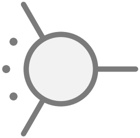

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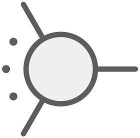

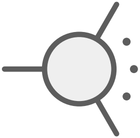

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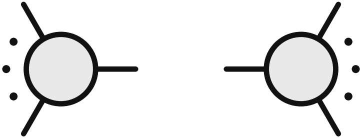

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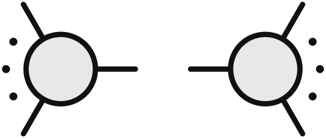

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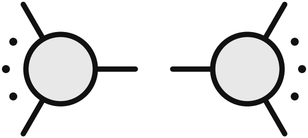

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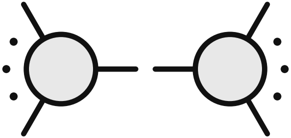

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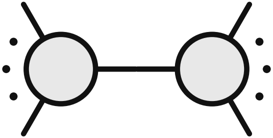

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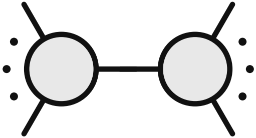

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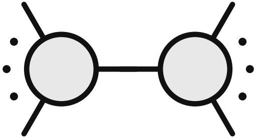

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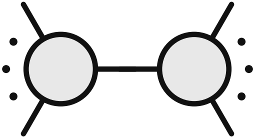

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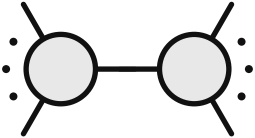

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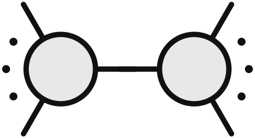

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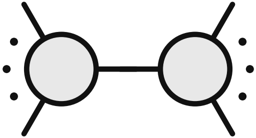

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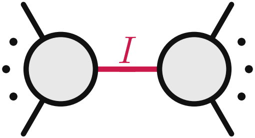

Internal Particles:

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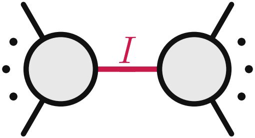

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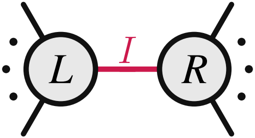

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$$\mathcal{A}_L(\ldots, I)\times\mathcal{A}_R(I,\ldots)$$

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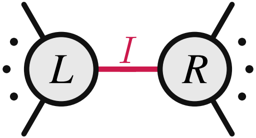

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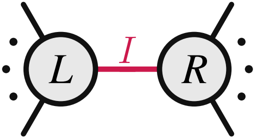

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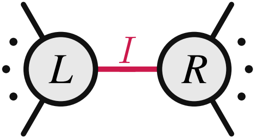

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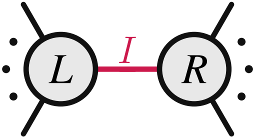

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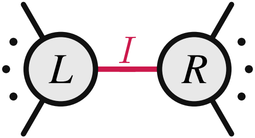

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$$\sum_{\text{states } I} \int d^{d-1}\text{LIPS}_I \; \mathcal{A}_L(\ldots, I) \times \mathcal{A}_R(I, \ldots)$$

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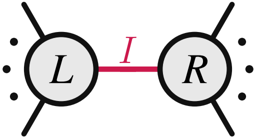

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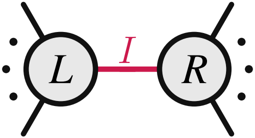

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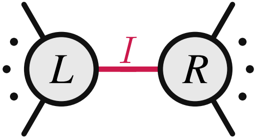

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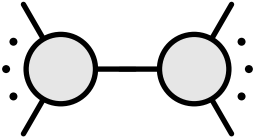

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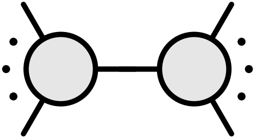

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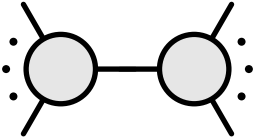

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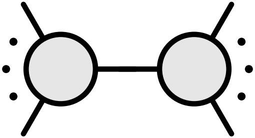

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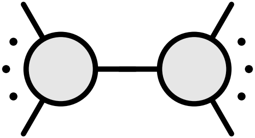

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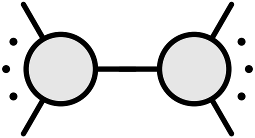

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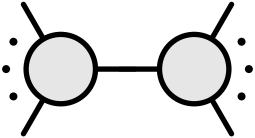

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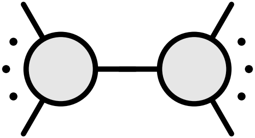

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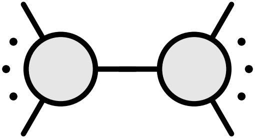

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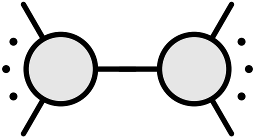

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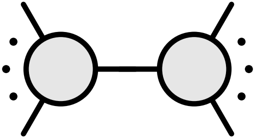

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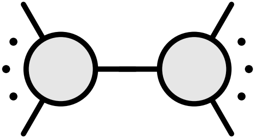

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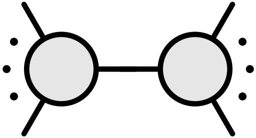

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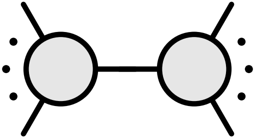

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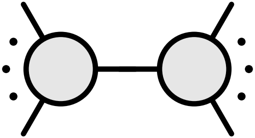

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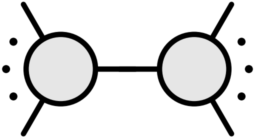

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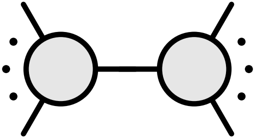

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- When p_a is *real* $\left(p_a \in \mathbb{R}^{3,1}\right)$, $p_a^{\alpha\dot{\alpha}} = (p_a^{\alpha\dot{\alpha}})^\dagger$, which implies that $\left(\lambda_a^\alpha\right)^* = \pm \widetilde{\lambda}_a^{\dot{\alpha}}$. (but allowing for complex momenta, λ_a and $\widetilde{\lambda}_a$ become independent.)

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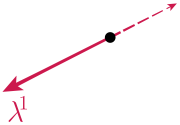

The *Grassmannian* Geometry of Kinematical Constraints

Thus, all the kinematical data can be described by a pair of $(2\times n)$ matrices:

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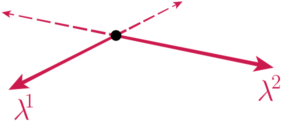

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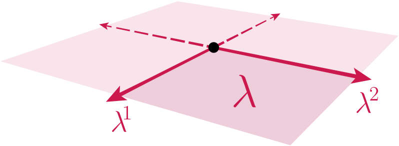

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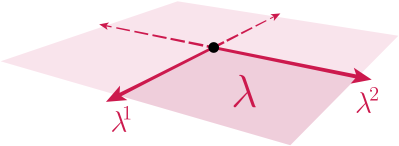

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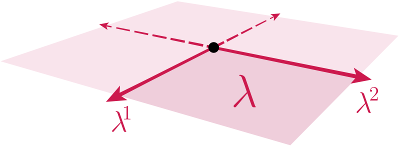

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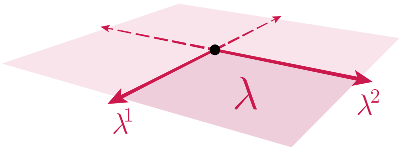

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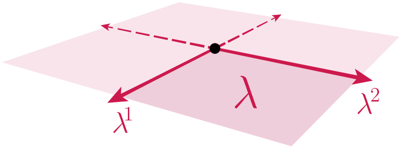

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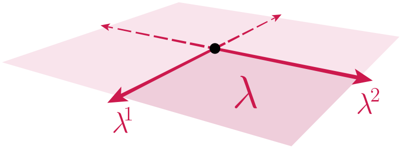

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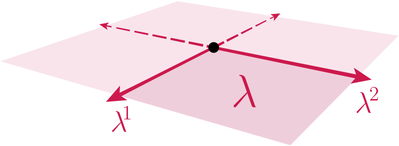

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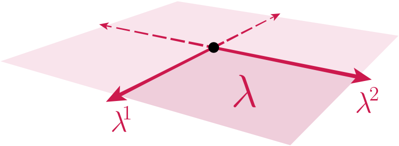

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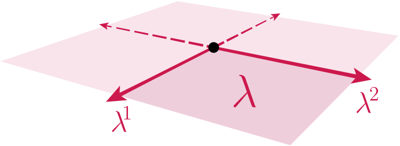

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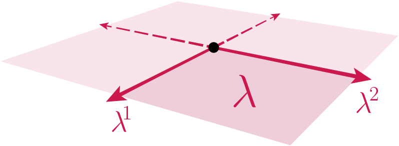

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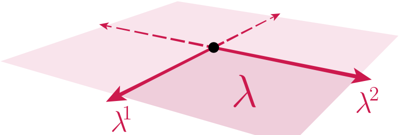

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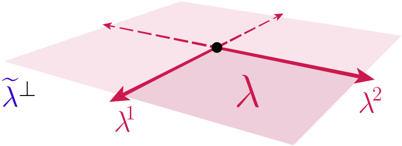

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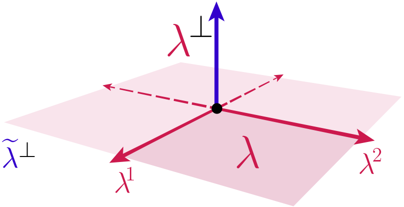

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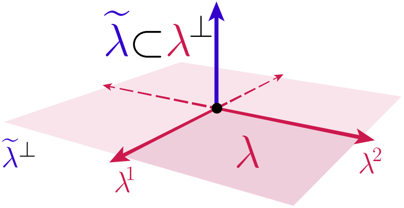

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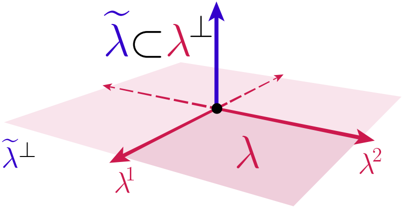

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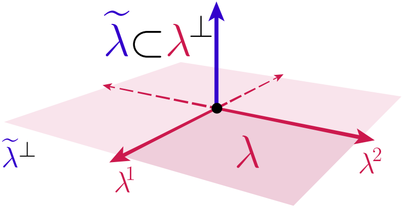

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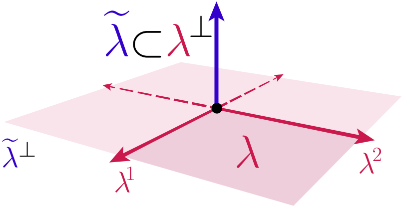

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$$\delta^4\Big(\sum_a p_a^\mu\Big) = \delta^{2\times2}\Big(\sum_a \lambda_a^\alpha \widetilde{\lambda}_a^{\dot{\alpha}}\Big) \equiv \delta^{2\times2}(\lambda\cdot\widetilde{\lambda})$$

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Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).

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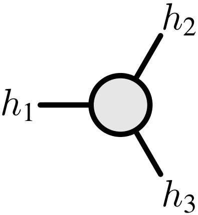

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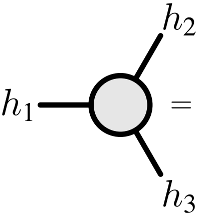

$$= f(\lambda_1\widetilde{\lambda}_1, \lambda_2\widetilde{\lambda}_2, \lambda_3\widetilde{\lambda}_3)\delta^{2\times 2}\big(\lambda\cdot\widetilde{\lambda}\big)$$

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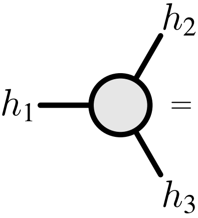

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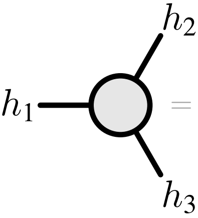

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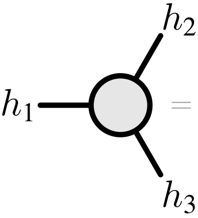

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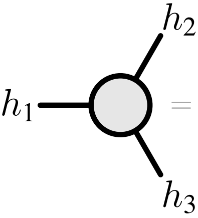

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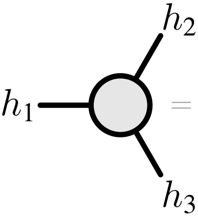

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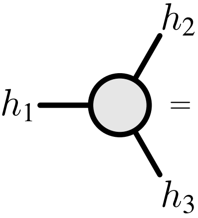

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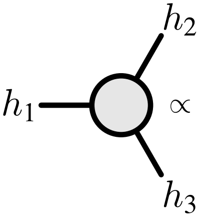

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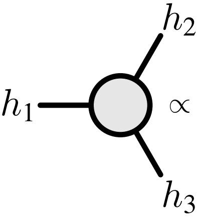

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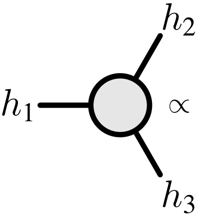

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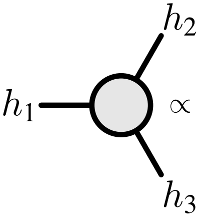

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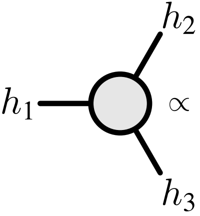

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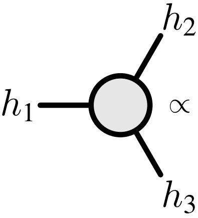

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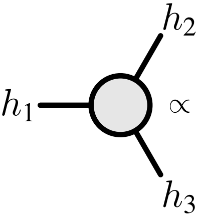

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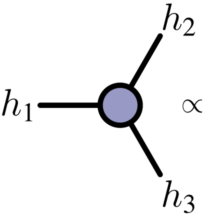

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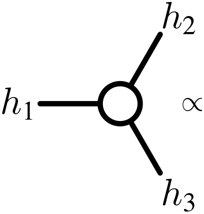

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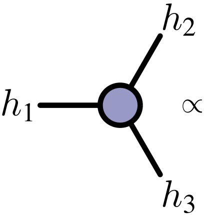

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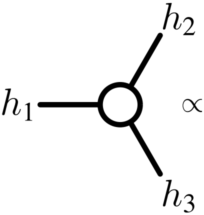

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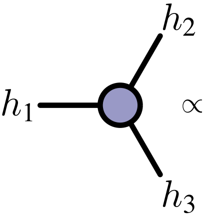

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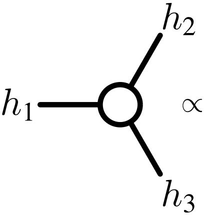

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$$\widetilde{\lambda}\;\equiv\begin{pmatrix}\widetilde{\lambda}_1^{\dot{1}} & \widetilde{\lambda}_2^{\dot{1}} & \widetilde{\lambda}_3^{\dot{1}}\\ \widetilde{\lambda}_1^{\dot{2}} & \widetilde{\lambda}_2^{\dot{2}} & \widetilde{\lambda}_3^{\dot{2}}\end{pmatrix}$$

$$\widetilde{\lambda}^{\perp}\equiv\big([23]\;[31]\;[12]\big)\supset\lambda$$

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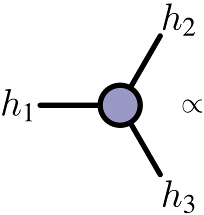
Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).



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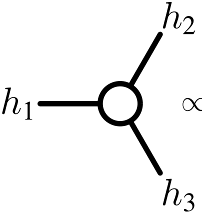

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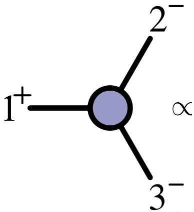

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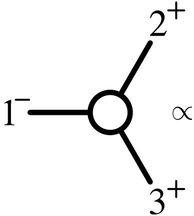

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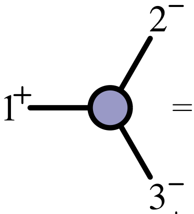
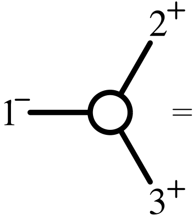


$$= \quad \frac{\langle 2\,3\rangle^4}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\;\delta^{2\times 2}(\lambda\cdot\widetilde{\lambda})$$



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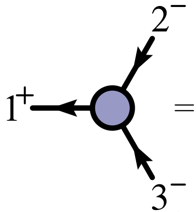

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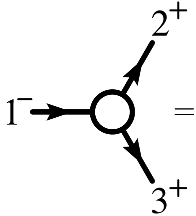

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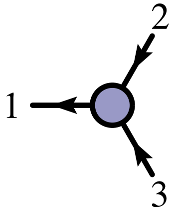

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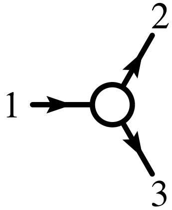

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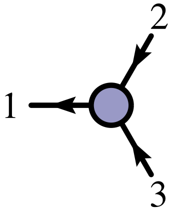

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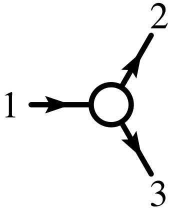

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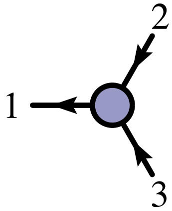

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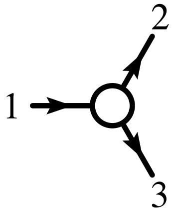

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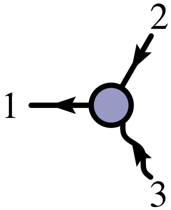

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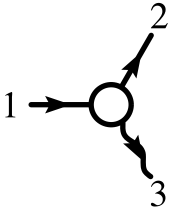

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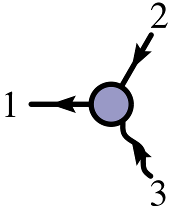

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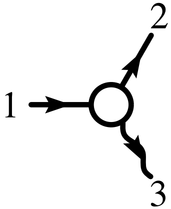

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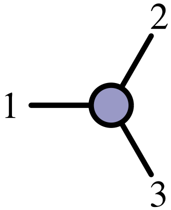
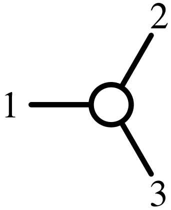


$$= \quad \frac{\delta^{2\times4}\big(\lambda\!\cdot\!\widetilde{\eta}\big)}{\langle1\,2\rangle\langle2\,3\rangle\langle3\,1\rangle}\,\delta^{2\times2}\big(\lambda\!\cdot\!\widetilde{\lambda}\big)\equiv\mathcal{A}_3^{(2)}$$



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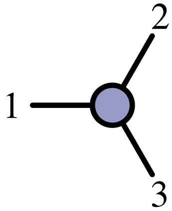

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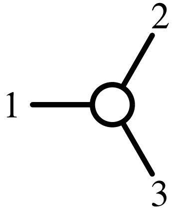

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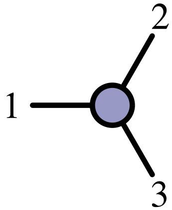

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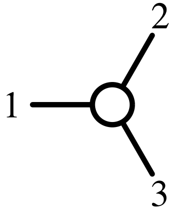

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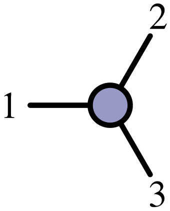

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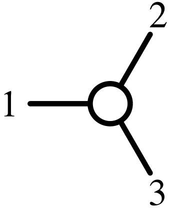

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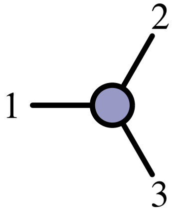

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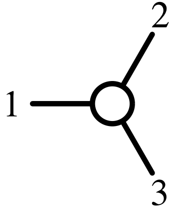

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Extracting Components from Super-Amplitudes

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$$\begin{aligned}
\delta^{1\times 4}\big(\widetilde{\lambda}^{\perp}\cdot\widetilde{\eta}\big) = \quad & \big((1)\,\widetilde{\eta}_1^1\big)\ \big((1)\,\widetilde{\eta}_1^2\big)\ \big((1)\,\widetilde{\eta}_1^3\big)\ \big((1)\,\widetilde{\eta}_1^4\big) + \ldots \quad \text{(3 terms)} \\
+ & \big((1)\,\widetilde{\eta}_1^1\big)\ \big((1)\,\widetilde{\eta}_1^2\big)\ \big((1)\,\widetilde{\eta}_1^3\big)\ \big((2)\,\widetilde{\eta}_2^4\big) + \ldots \text{(24 terms)} \\
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\end{aligned}$$

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$$|a\rangle \equiv |a\rangle^{+1} + \tilde{\eta}_a^I |a\rangle_I^{+1/2} + \frac{1}{2!}\tilde{\eta}_a^I\tilde{\eta}_a^J|a\rangle_{IJ}^{0} + \frac{1}{3!}\tilde{\eta}_a^I\tilde{\eta}_a^J\tilde{\eta}_a^K|a\rangle_{IJK}^{-1/2} + \frac{1}{4!}\tilde{\eta}_a^I\tilde{\eta}_a^J\tilde{\eta}_a^K\tilde{\eta}_a^L|a\rangle_{IJKL}^{-1}$$

$$\mathcal{A}_3^{(1)} \propto \frac{\delta^{1\times4}\big(\tilde{\lambda}^{\perp}\cdot\tilde{\eta}\big)}{[12]\,[23]\,[31]} \quad \text{with} \quad C = \tilde{\lambda}^{\perp} = \left(\begin{array}{ccc} c_1 & c_2 & c_3 \\ \hline [23] & [31] & [12] \end{array}\right)$$

$$\delta^{k\times\mathcal{N}}\big(C\cdot\tilde{\eta}\big) \equiv \prod_{I=1}^{\mathcal{N}}\left(\sum_{1\leq a_1<\cdots<a_k\leq n}(a_1\cdots a_k)\;\tilde{\eta}_{a_1}^I\cdots\tilde{\eta}_{a_k}^I\right)$$

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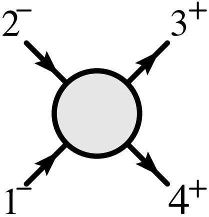

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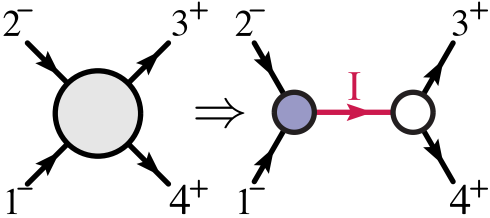

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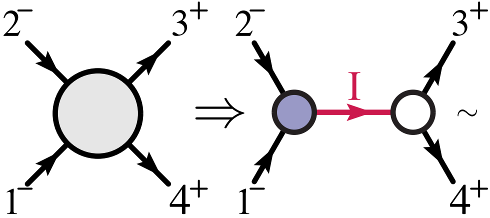

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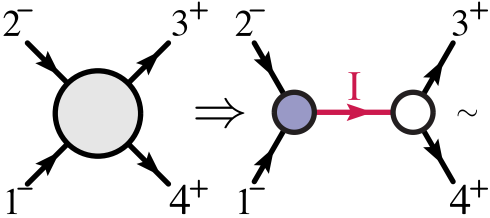

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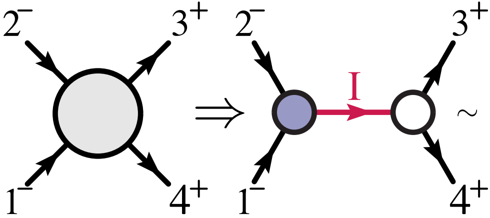

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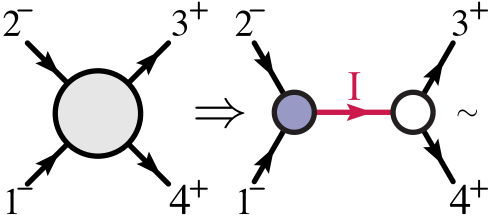

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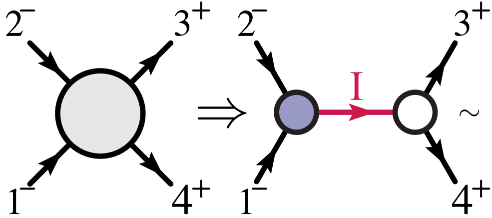

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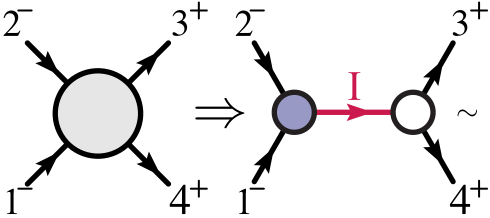

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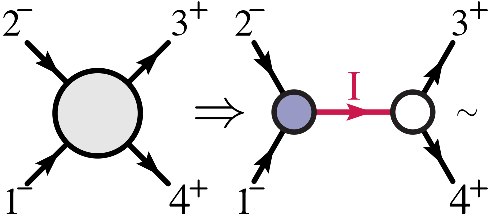

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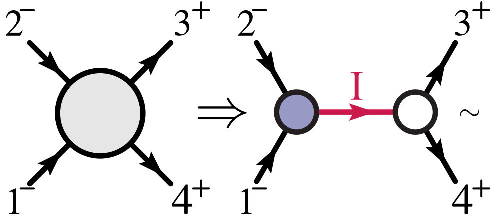

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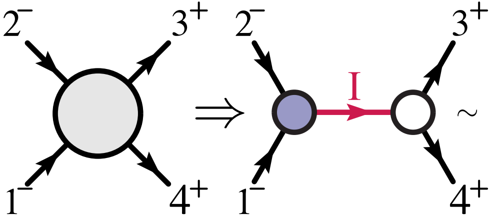

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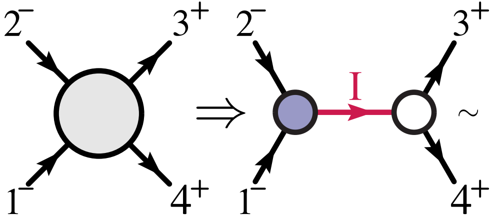

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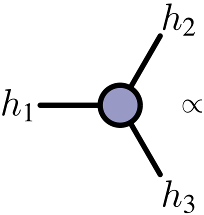
Momentum conservation and Poincaré-invariance **uniquely** fix the kinematical dependence of the amplitude for three massless particles (to all loop orders!).



$$\propto \quad \langle 12\rangle^{h_3-h_1-h_2}\langle 23\rangle^{h_1-h_2-h_3}\langle 31\rangle^{h_2-h_3-h_1}$$

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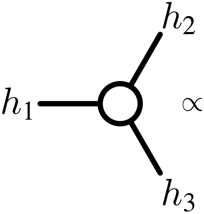

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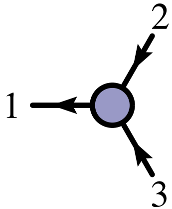

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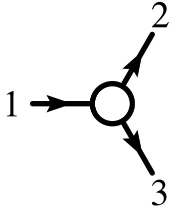

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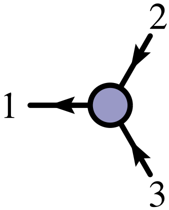

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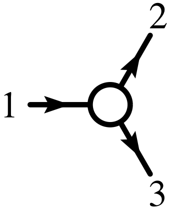

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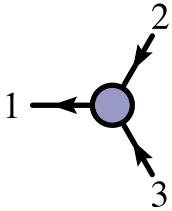

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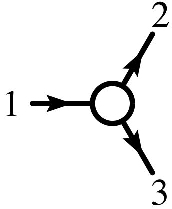

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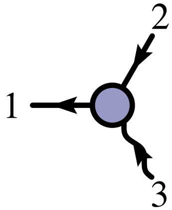

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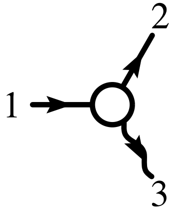

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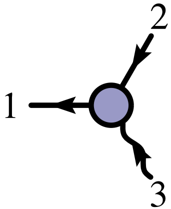

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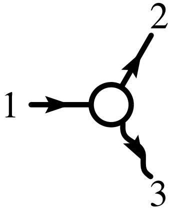

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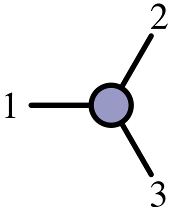

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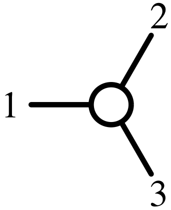

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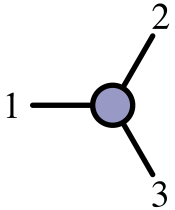

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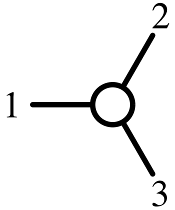

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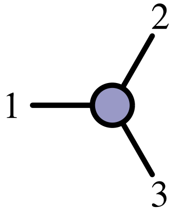

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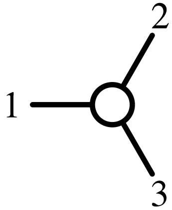

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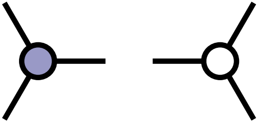

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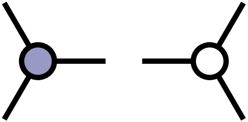

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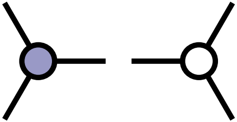

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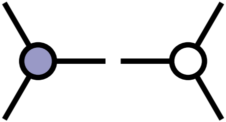

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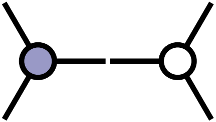

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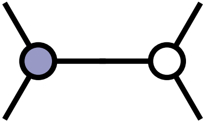

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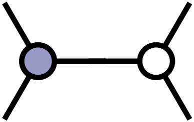

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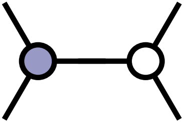

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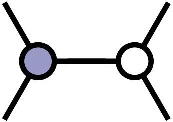

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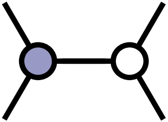

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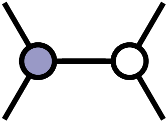

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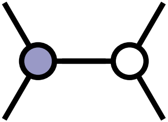

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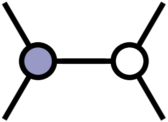

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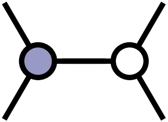

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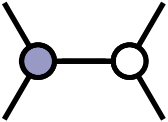

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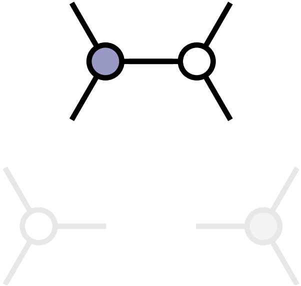

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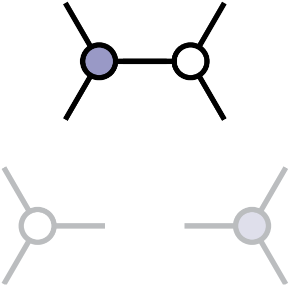

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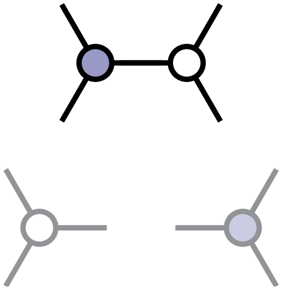

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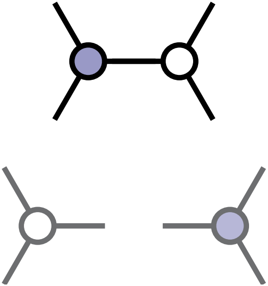

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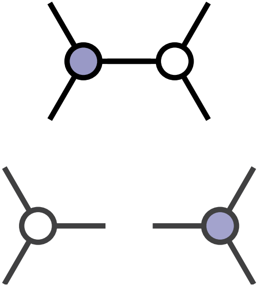

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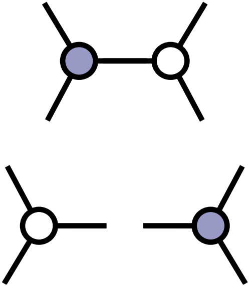

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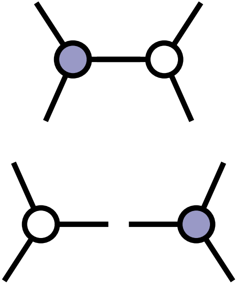

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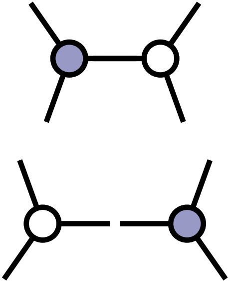

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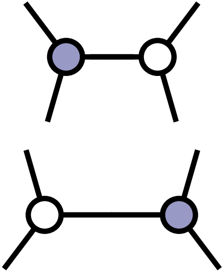

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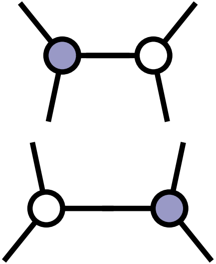

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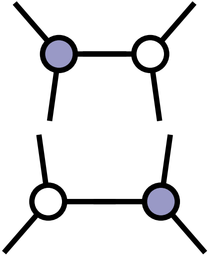

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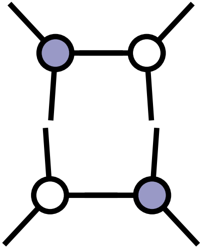

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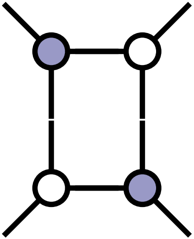

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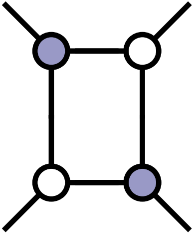

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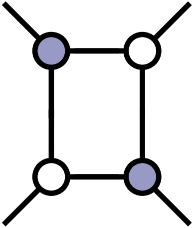

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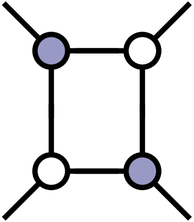

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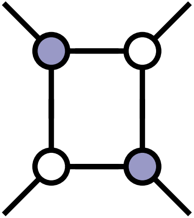

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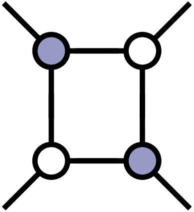

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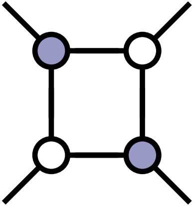

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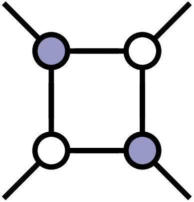

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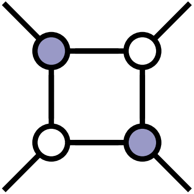

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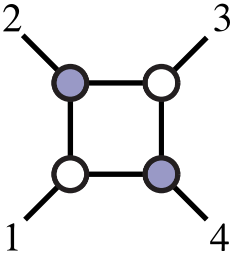

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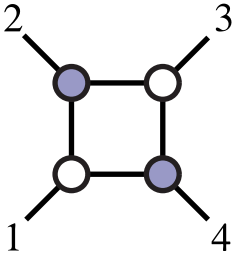

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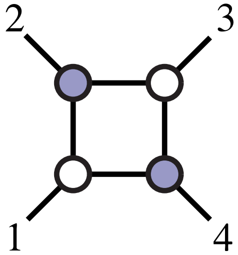

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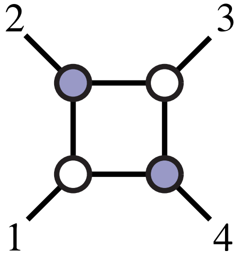

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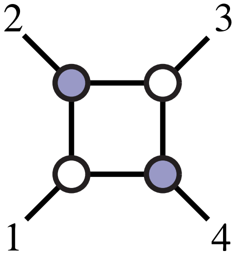

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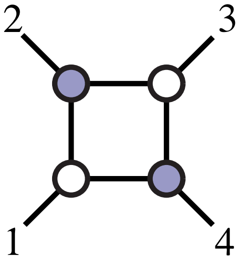

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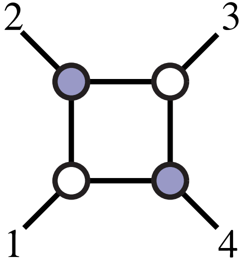

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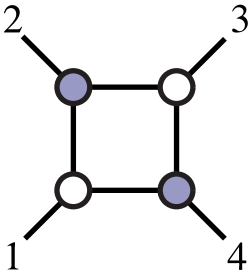

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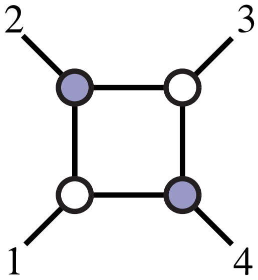

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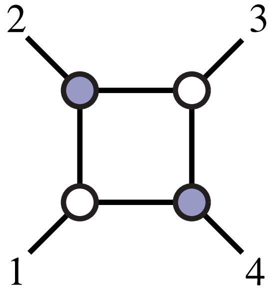

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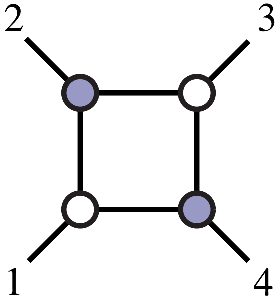

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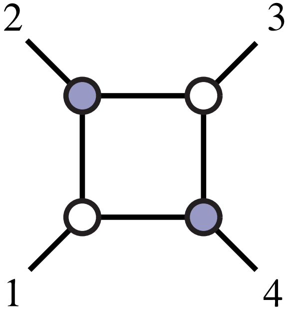

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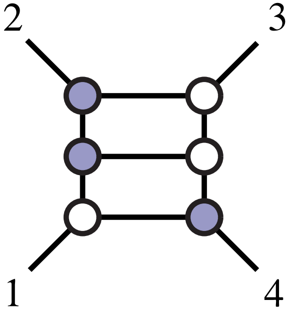

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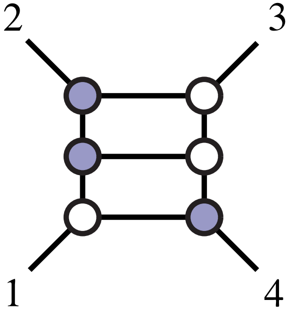

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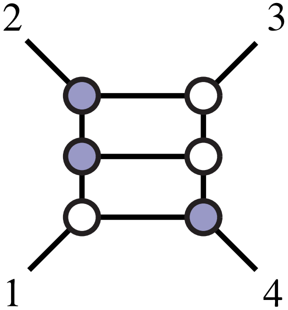

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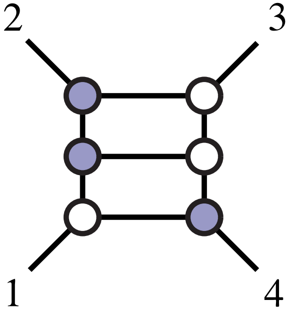

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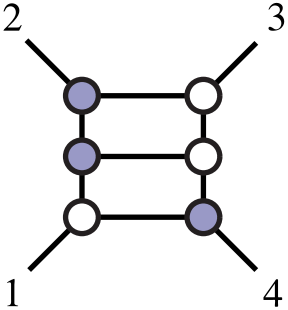

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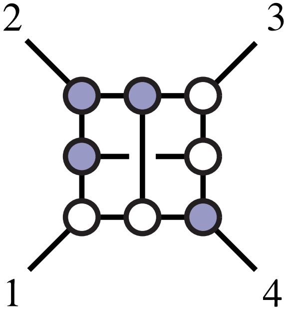

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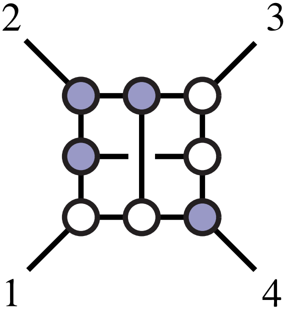

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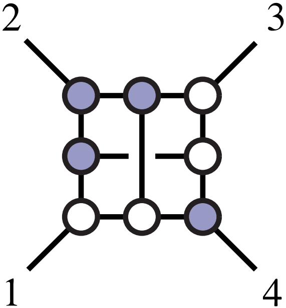

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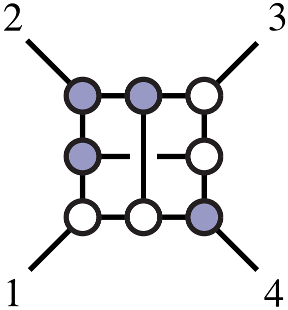

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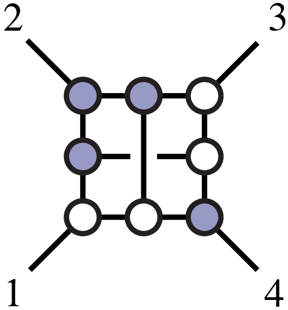

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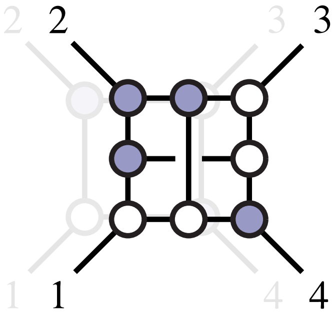

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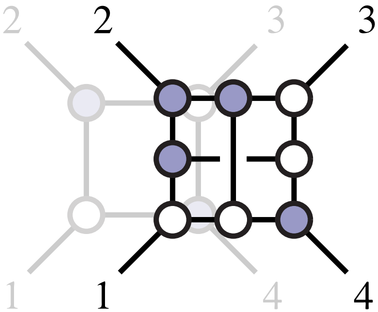

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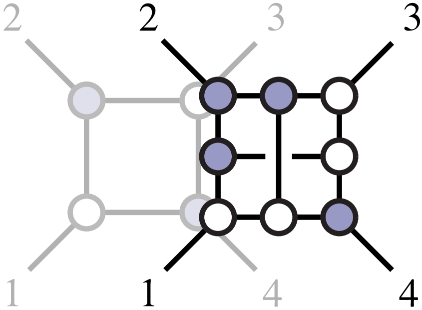

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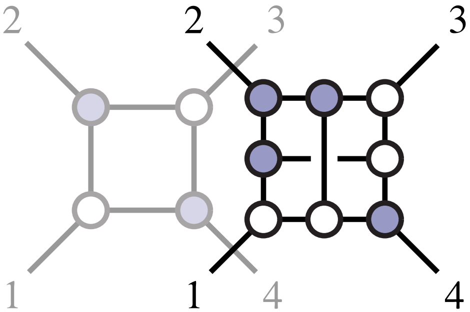

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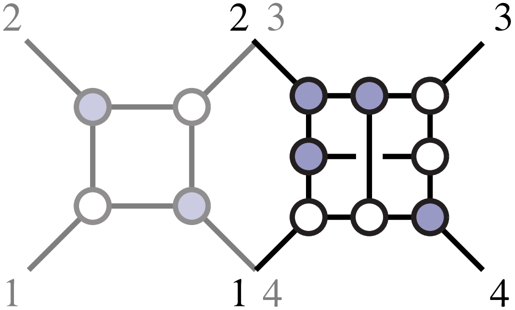

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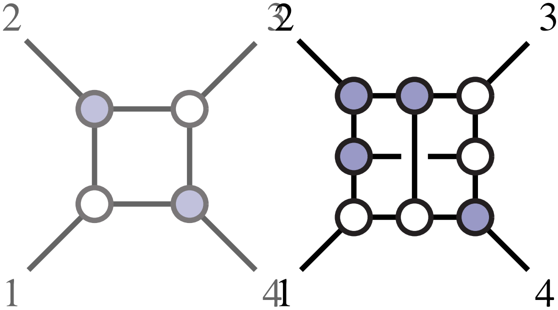

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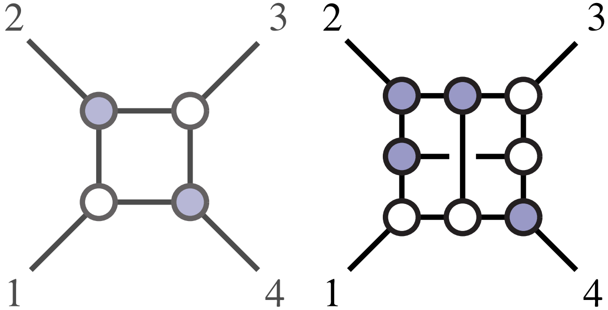

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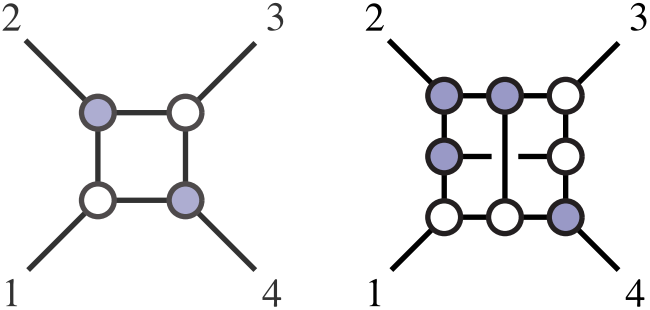

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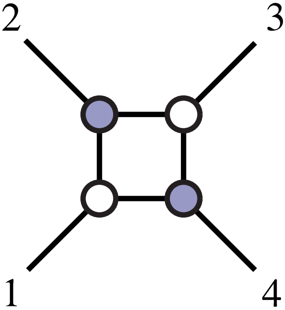

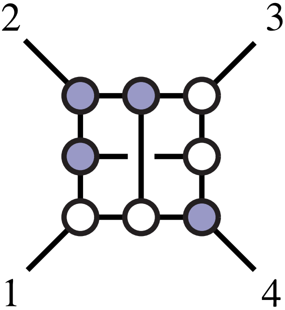

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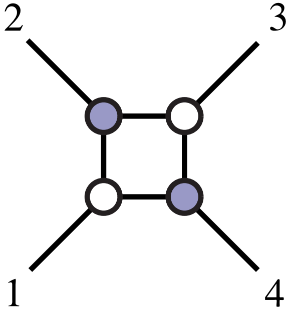

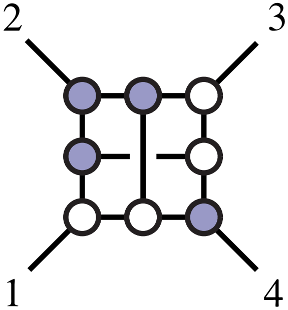

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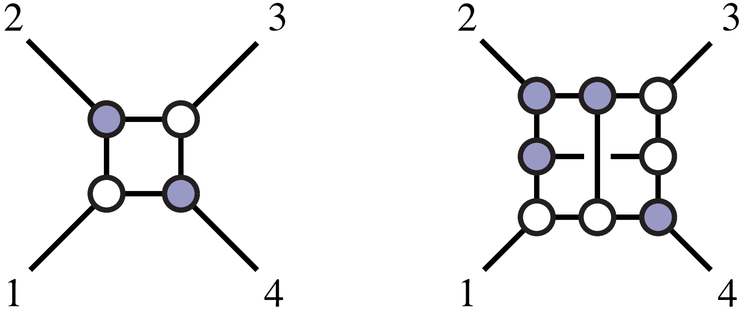

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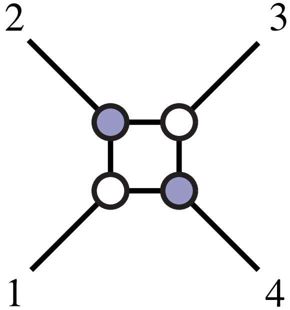

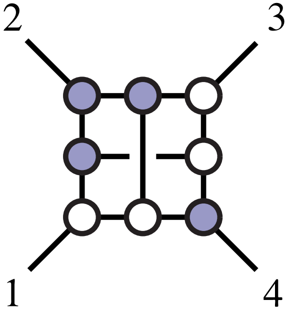

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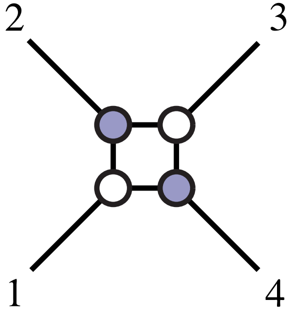

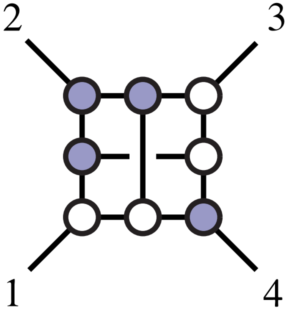

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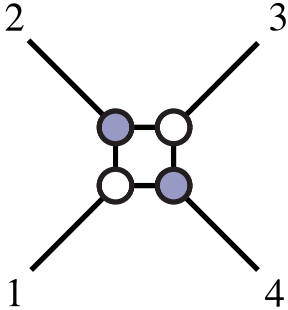

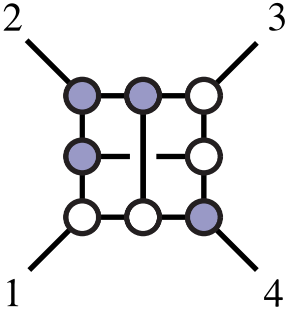

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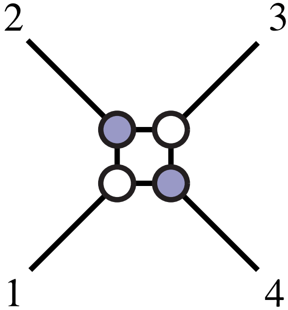

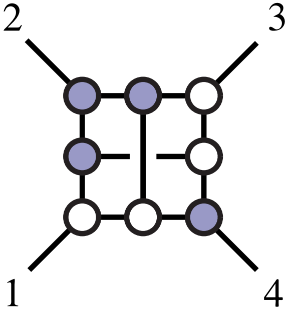

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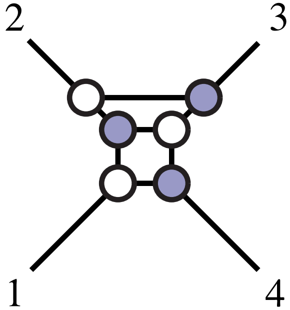

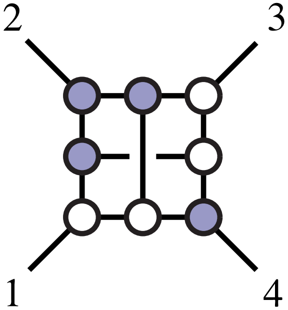

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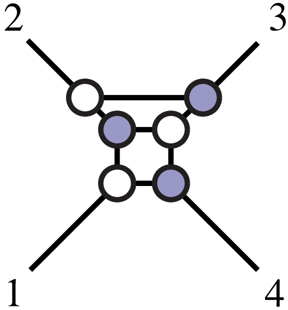

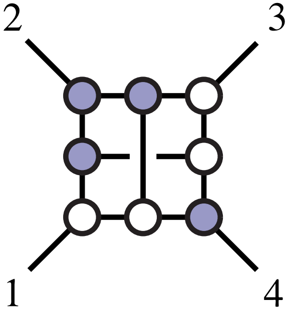

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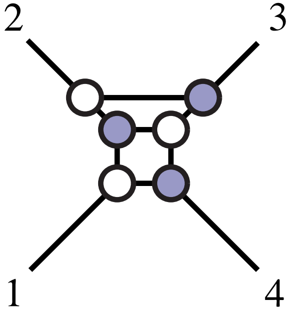

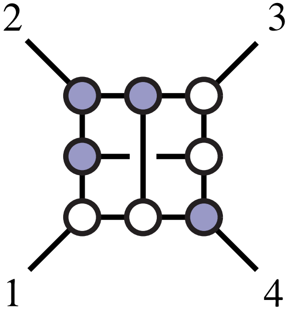

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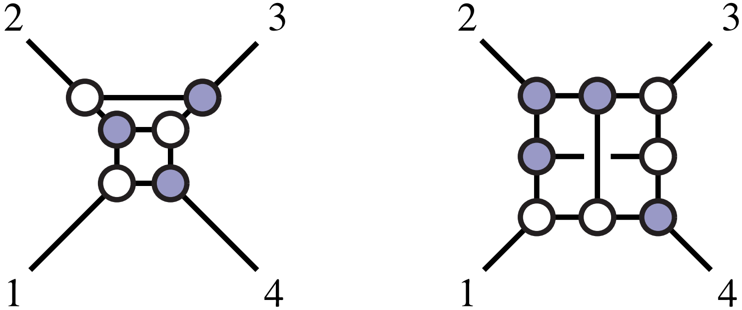

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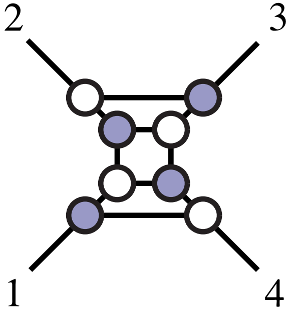

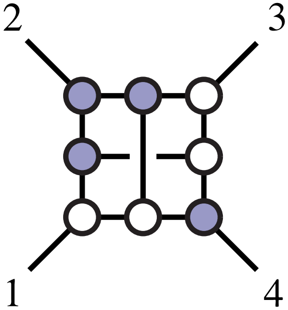

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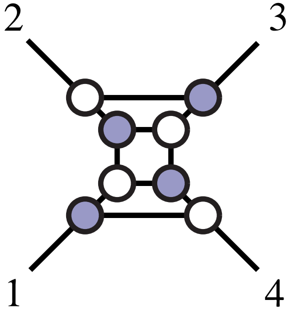

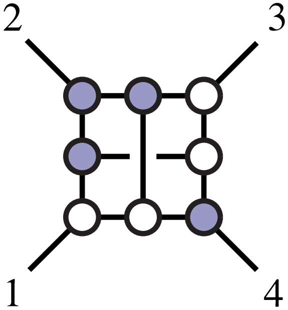

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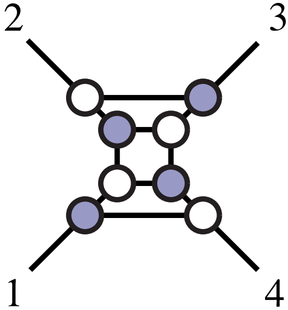

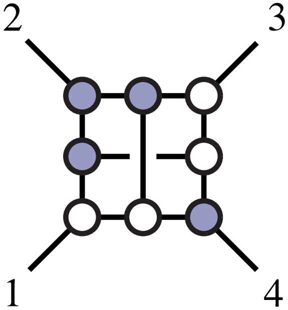

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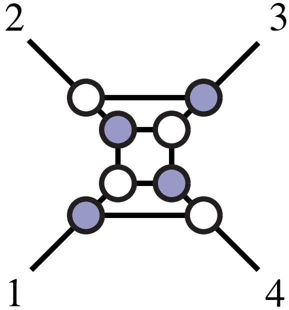

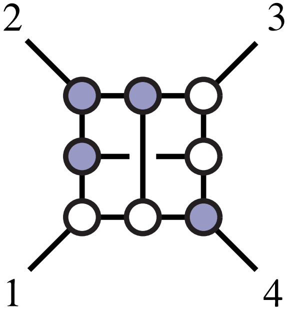

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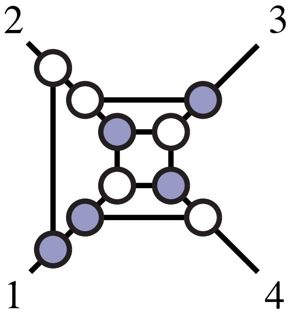

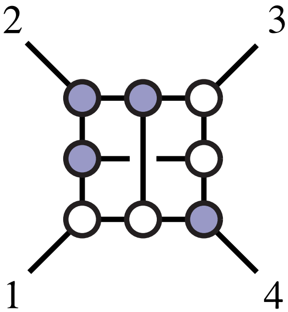

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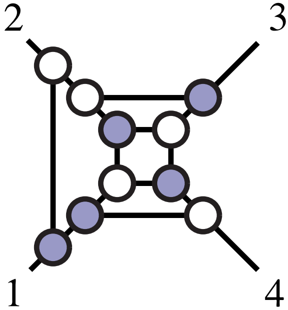

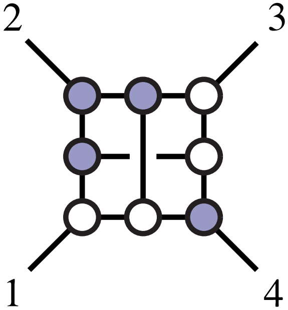

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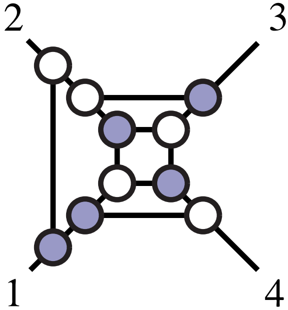

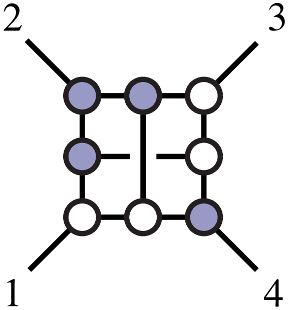

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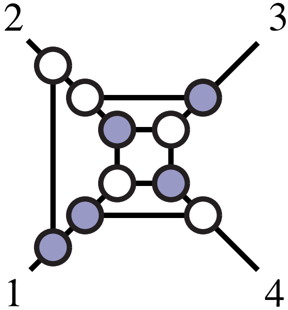

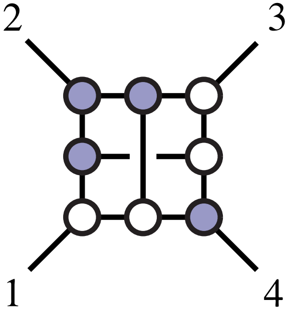

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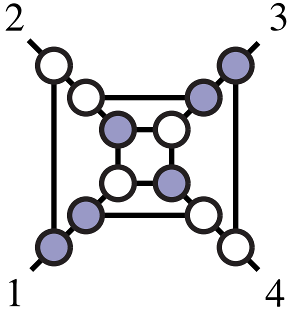

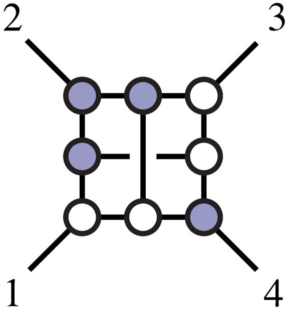

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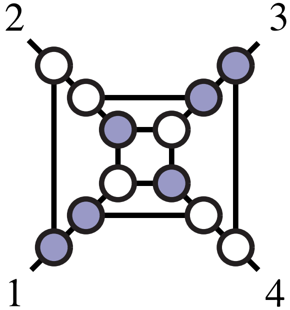

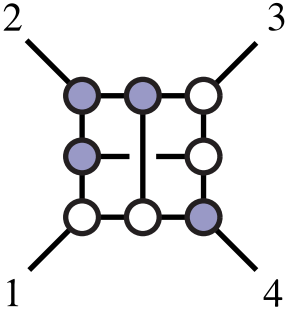

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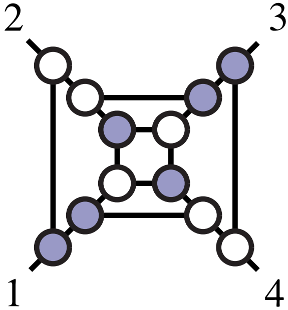

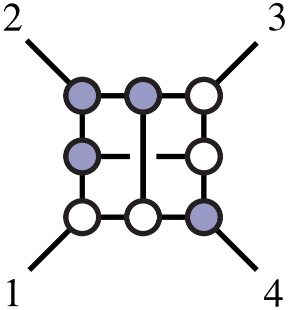

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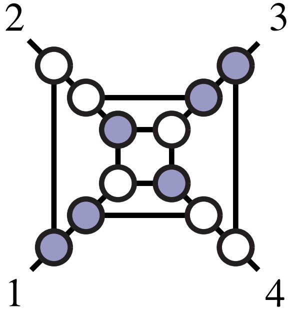

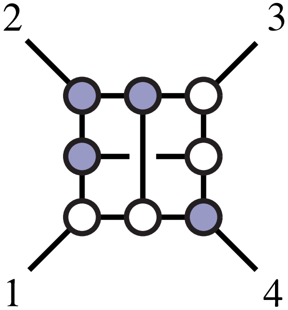

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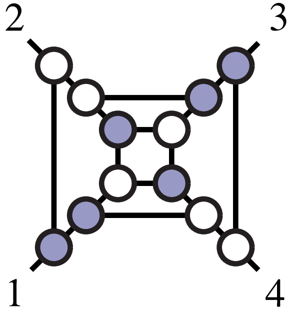

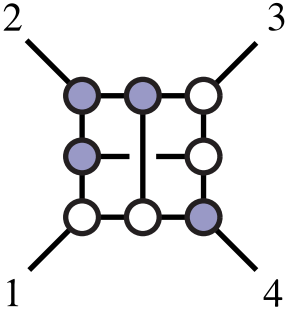

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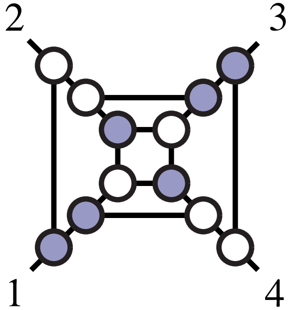

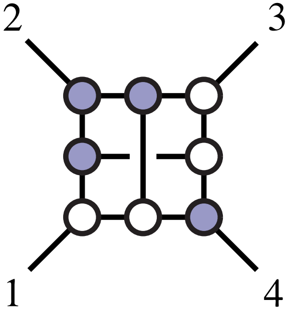

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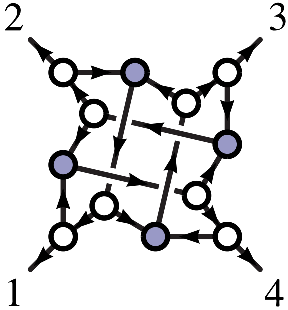

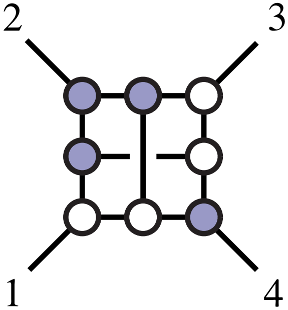

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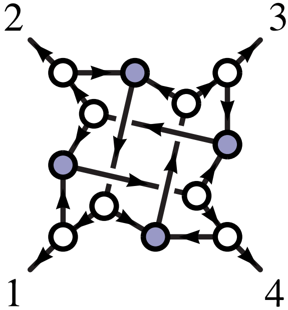

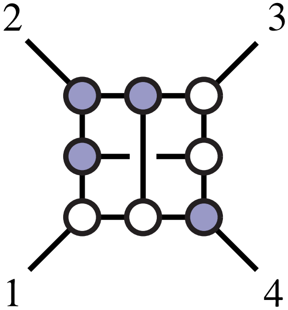

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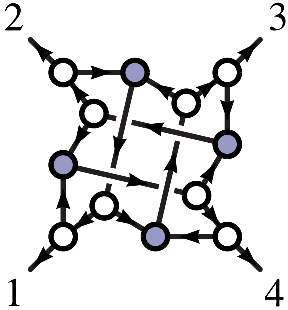

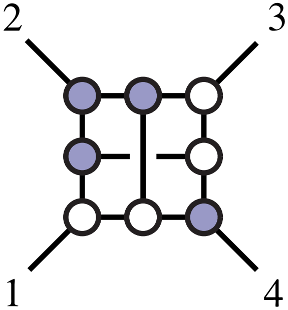

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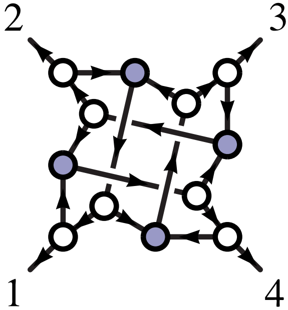

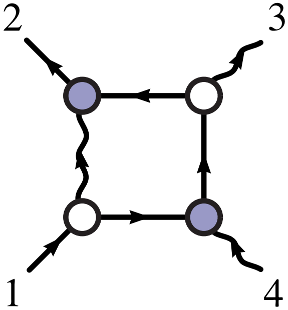

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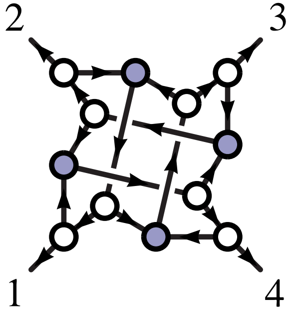

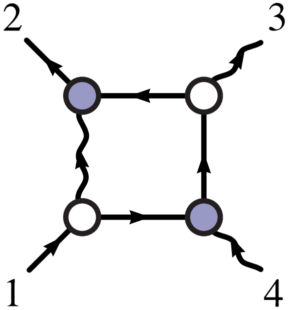

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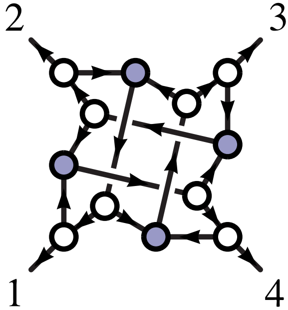

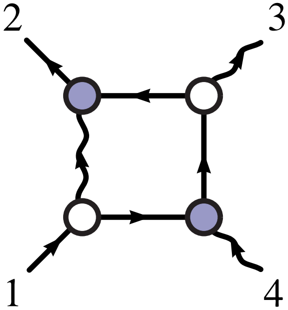

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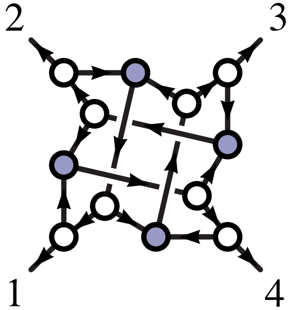

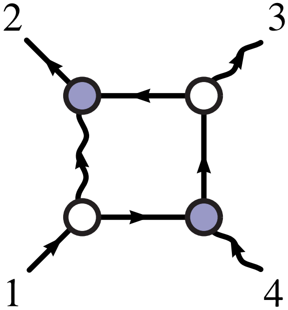

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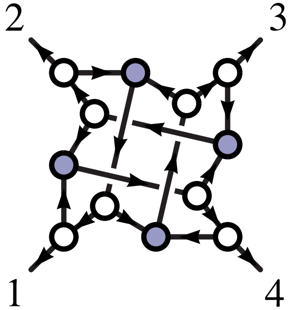

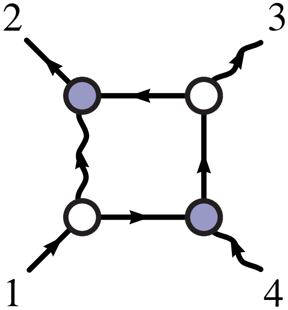

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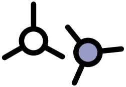

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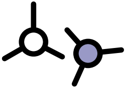

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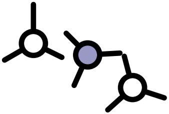

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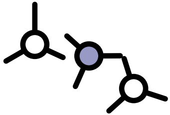

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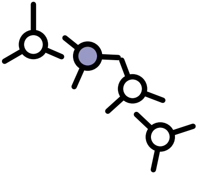

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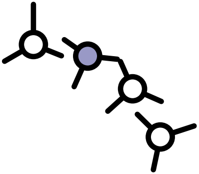

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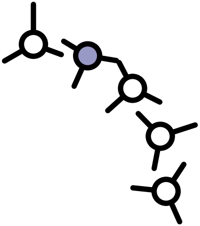

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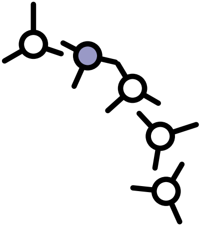

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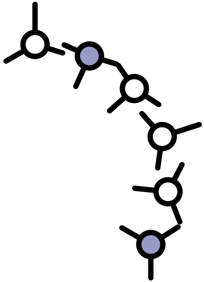

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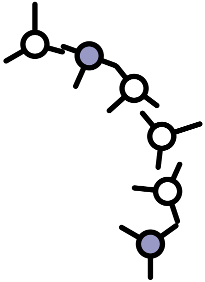

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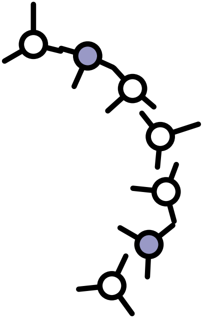

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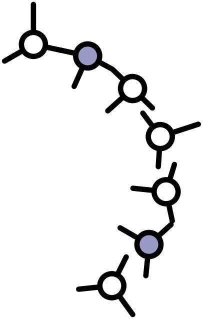

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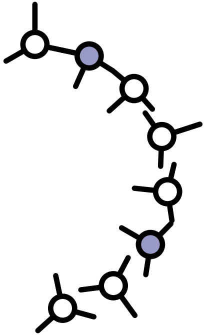

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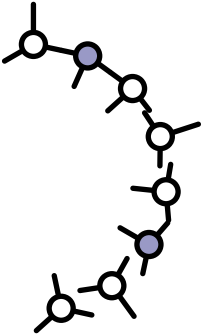

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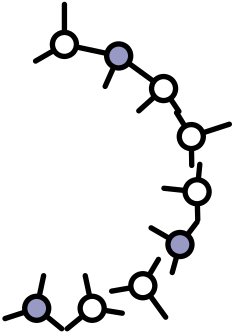

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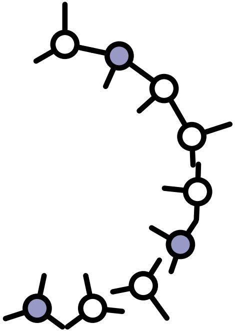

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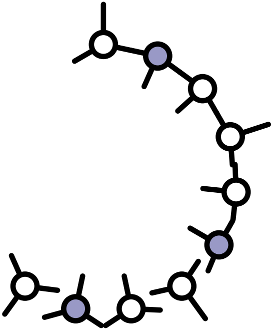

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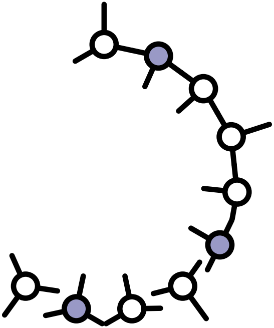

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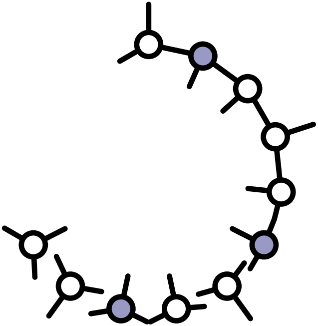

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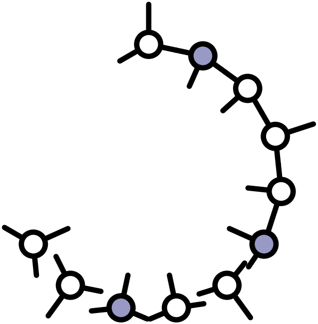

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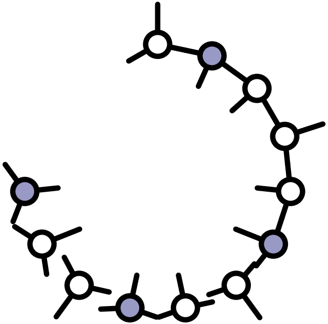

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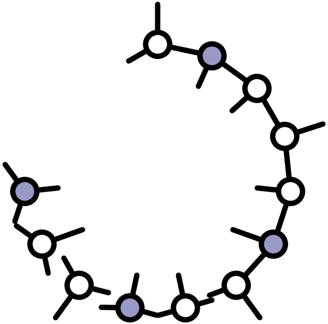

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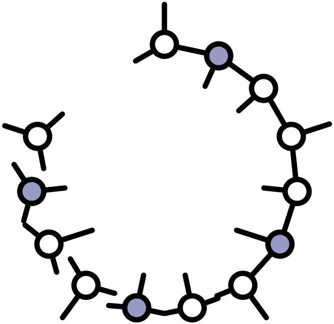

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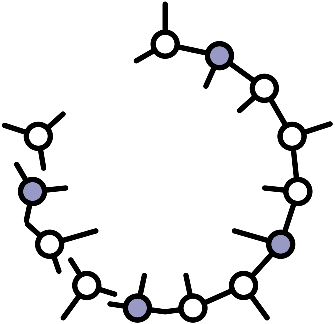

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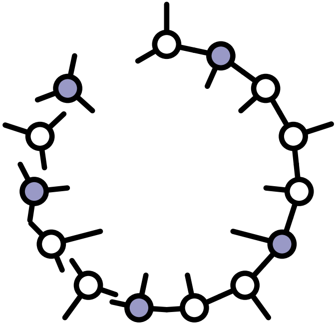

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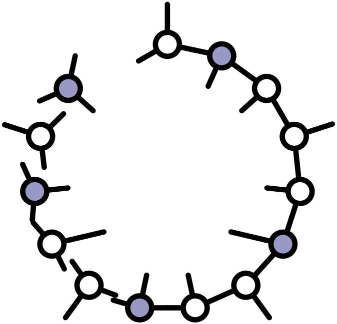

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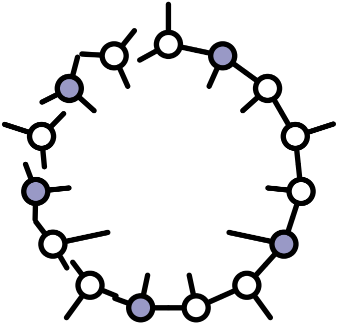

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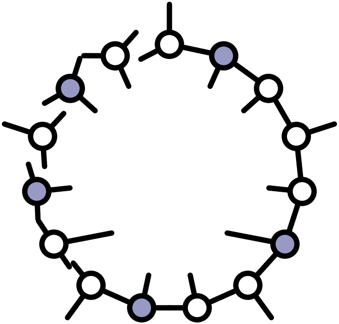

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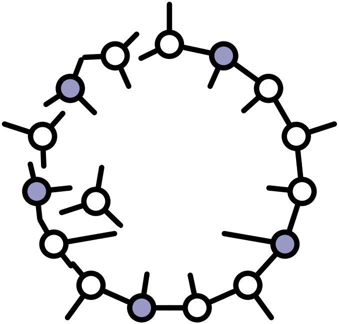

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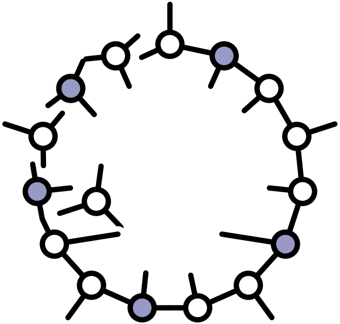

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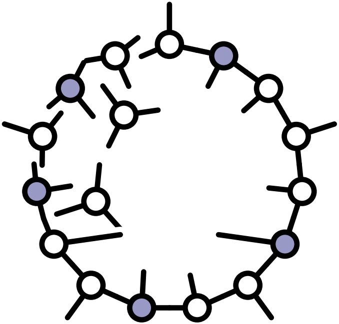

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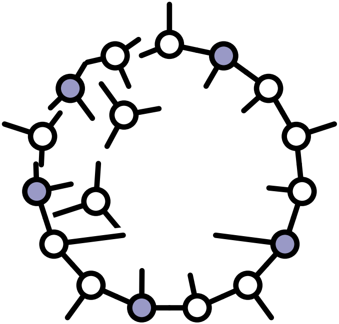

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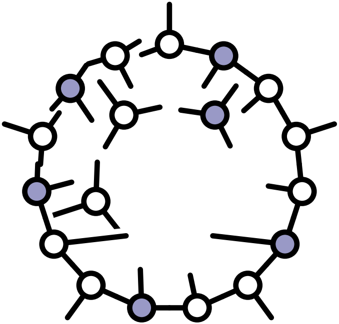

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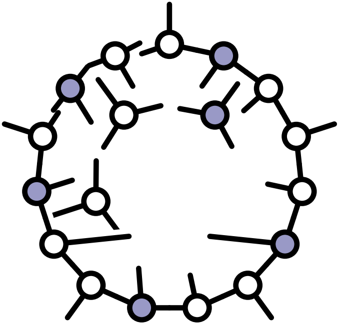

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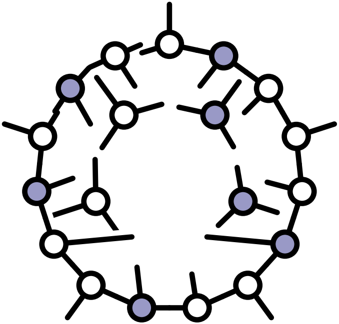

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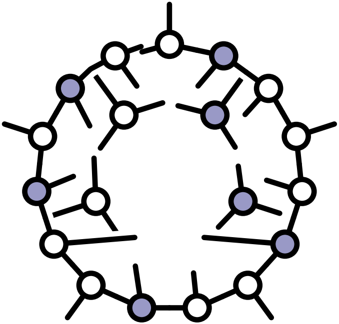

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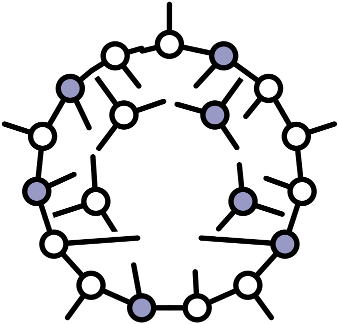

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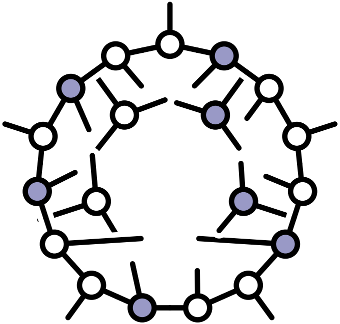

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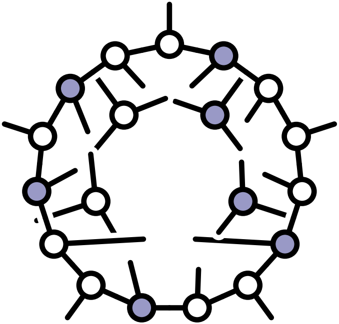

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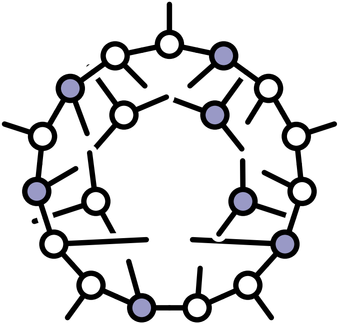

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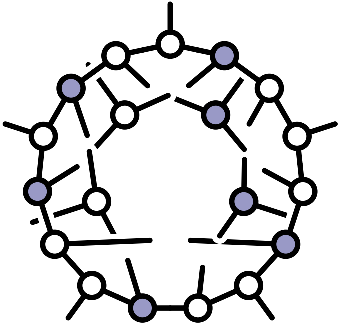

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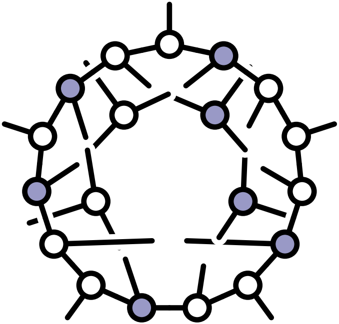

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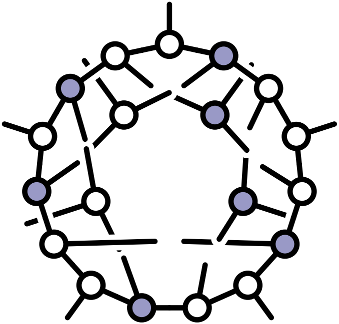

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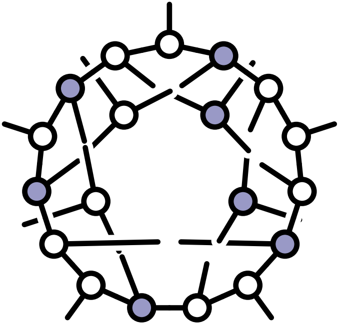

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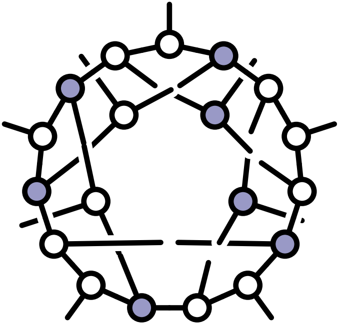

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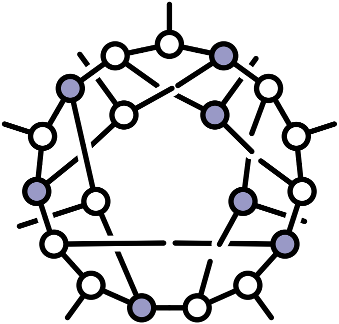

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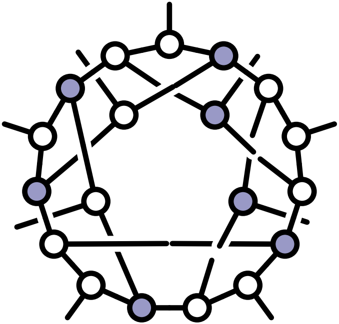

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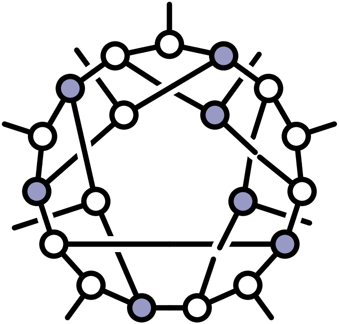

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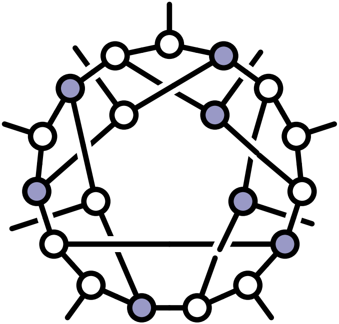

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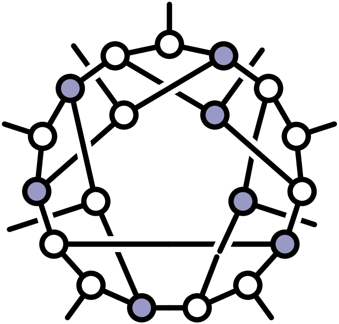

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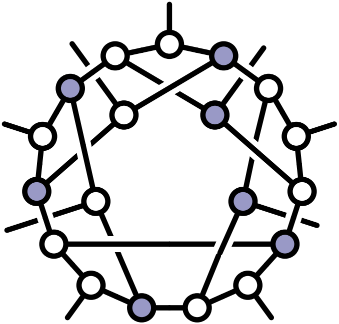

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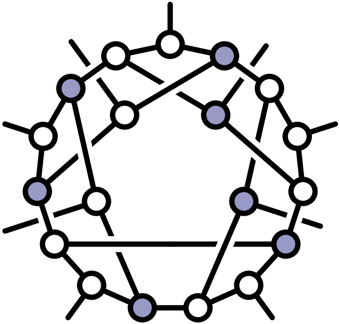

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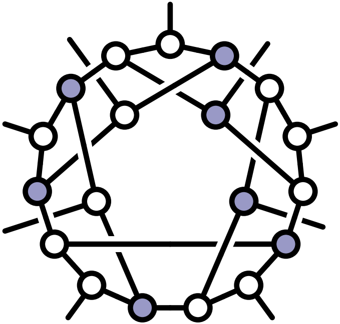

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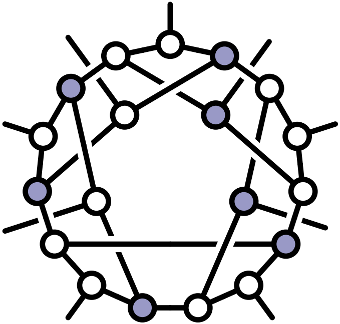

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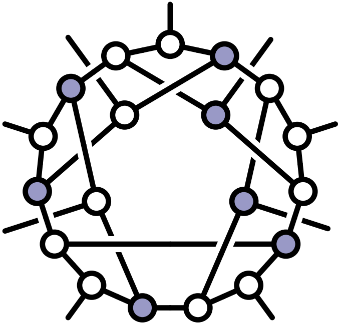

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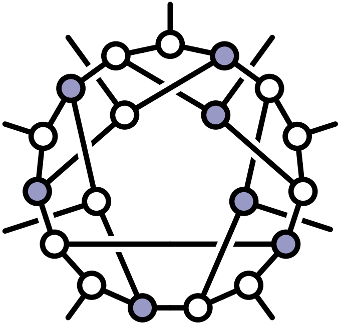

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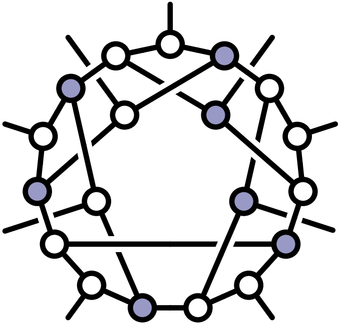

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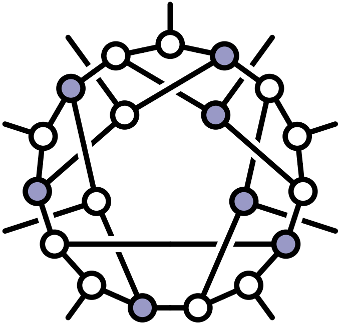

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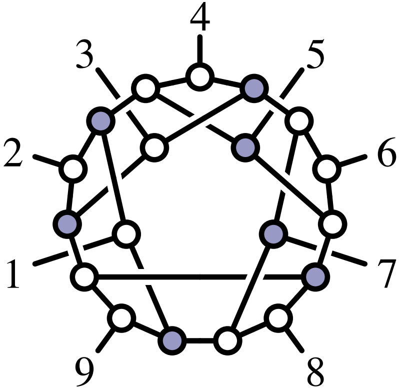

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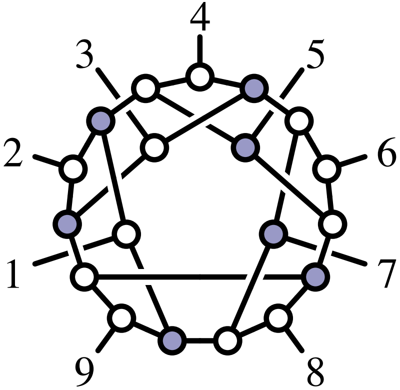

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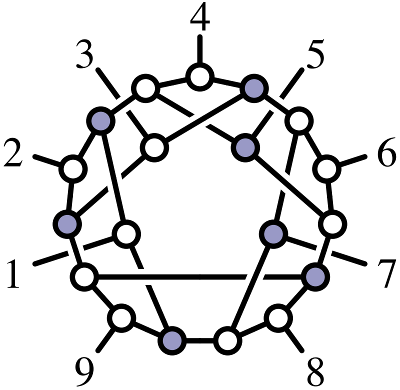

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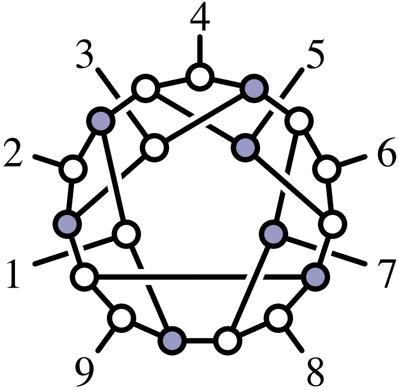

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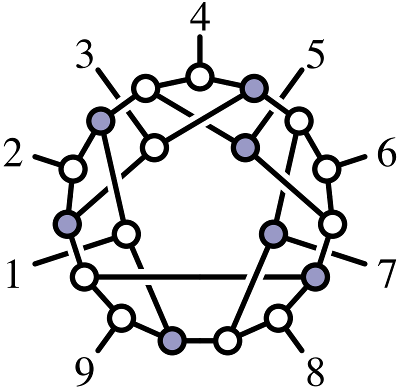

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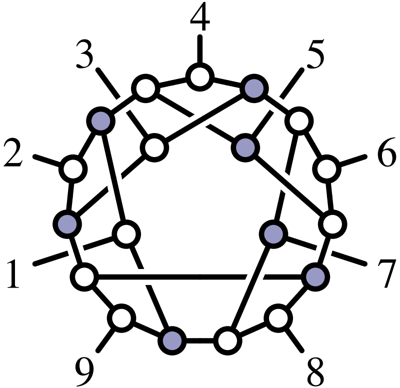

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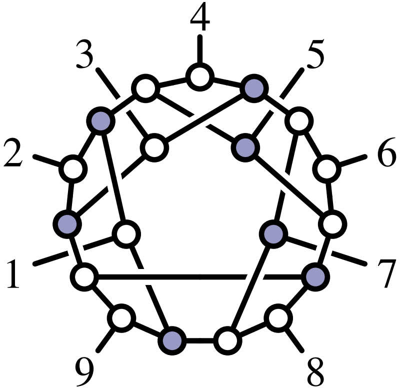

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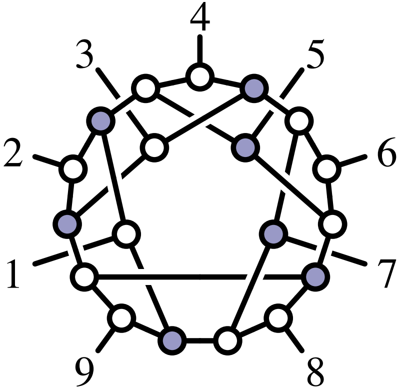

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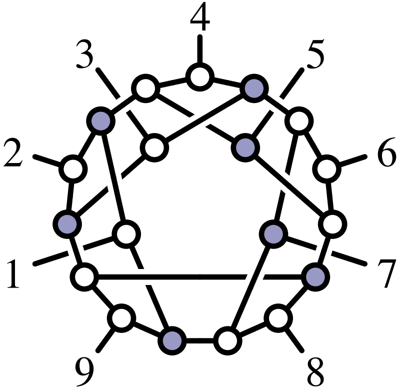

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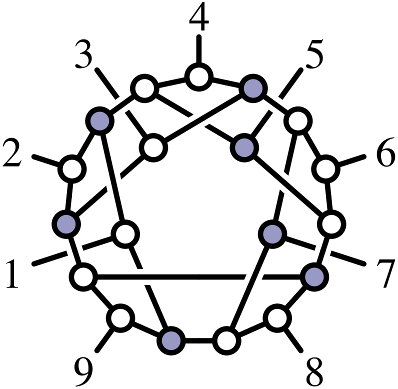

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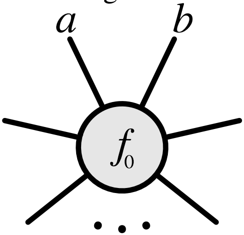

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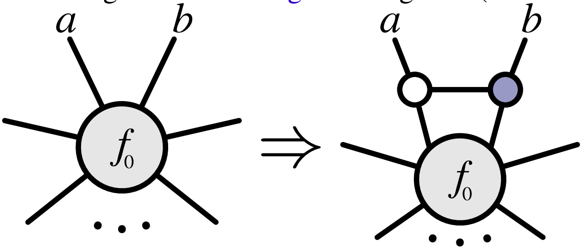

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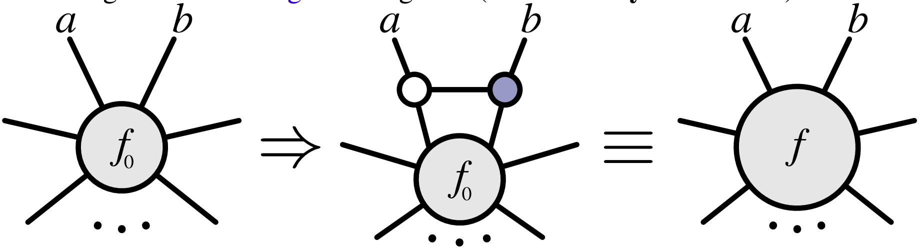

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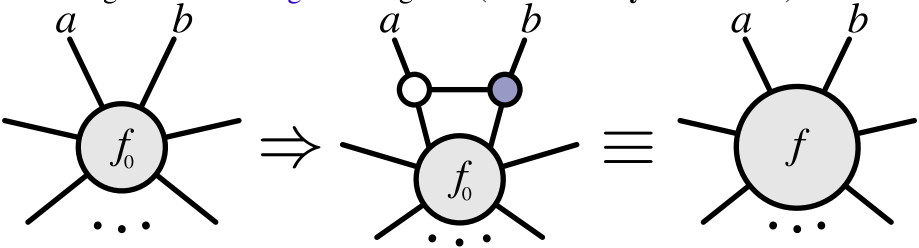

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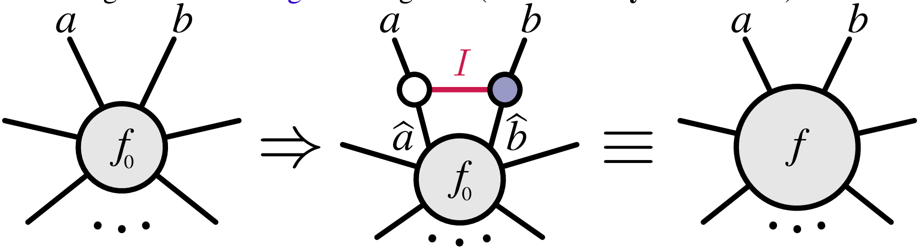

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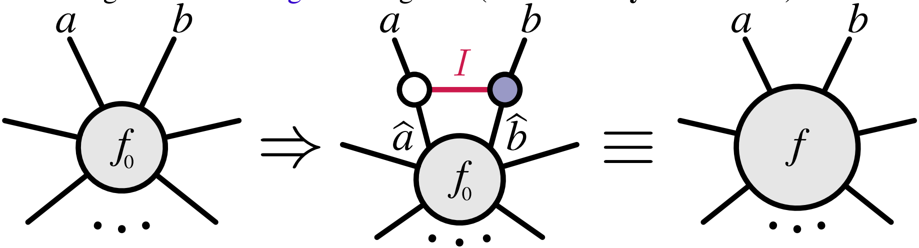

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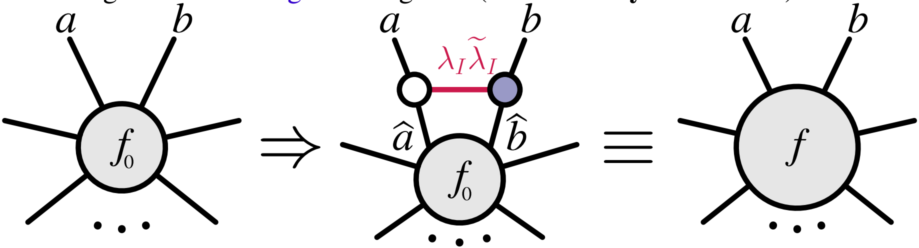

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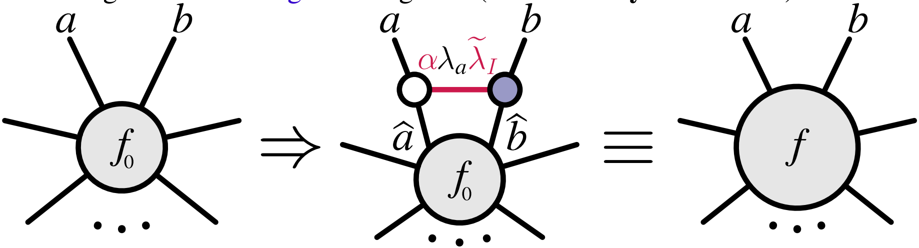

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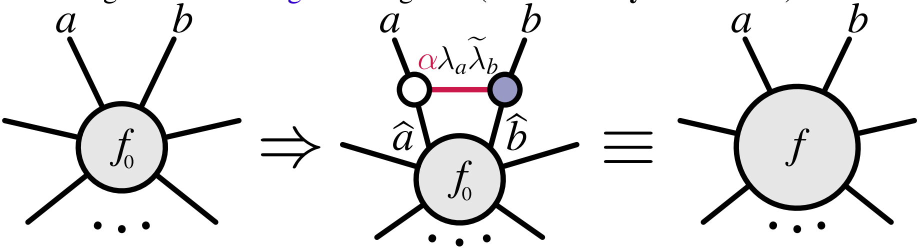

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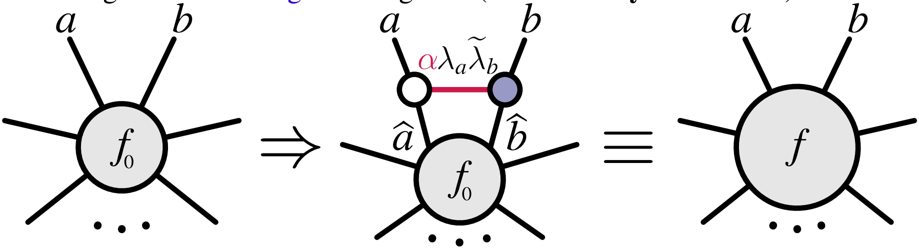

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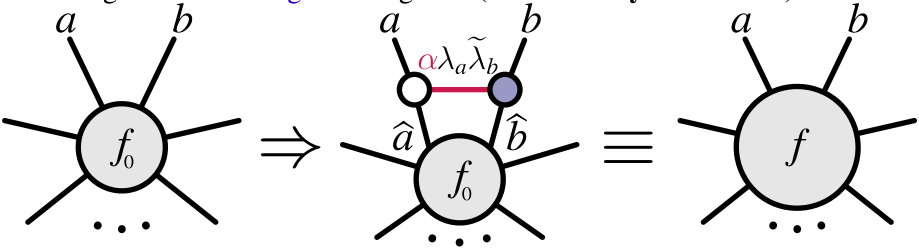

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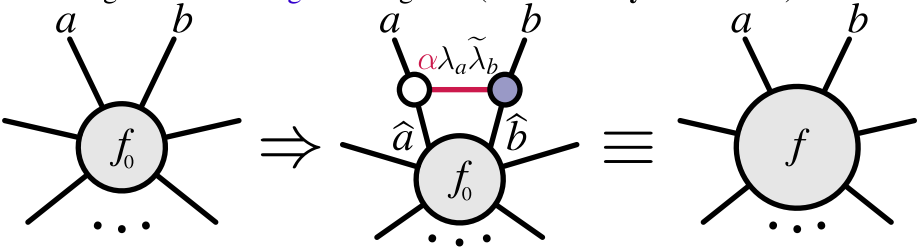

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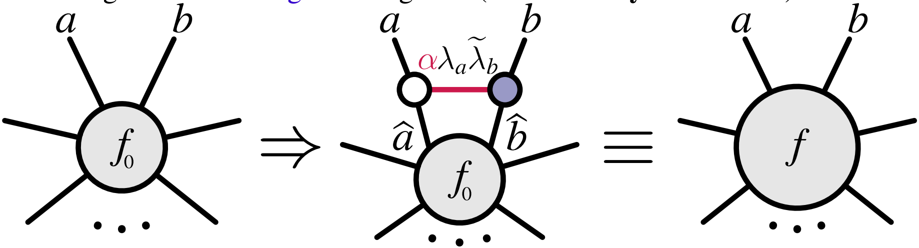

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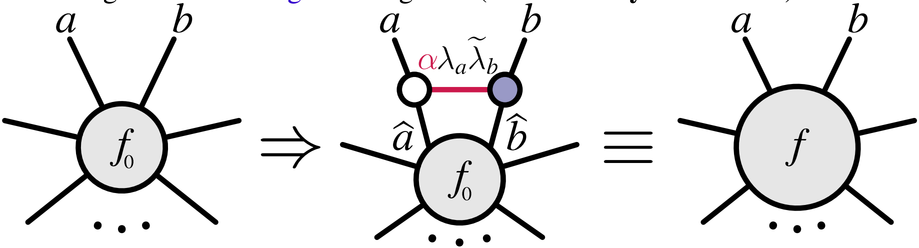

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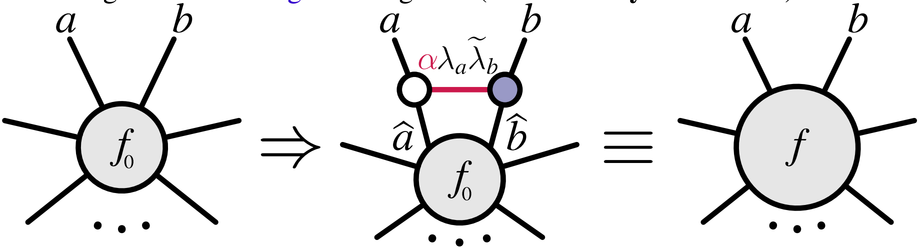

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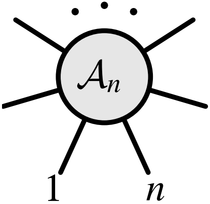

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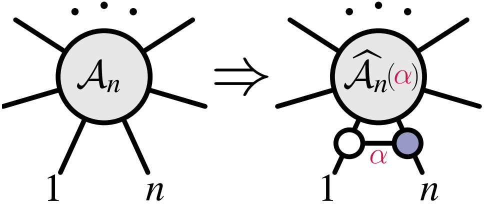

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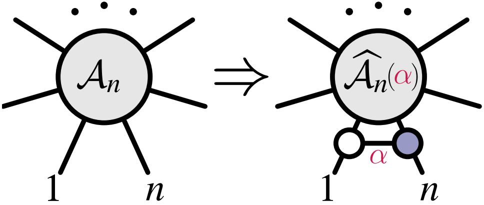

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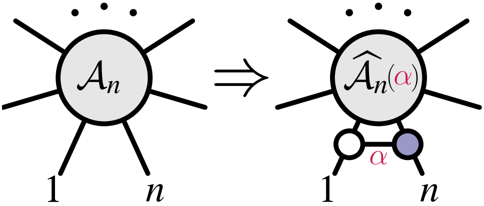

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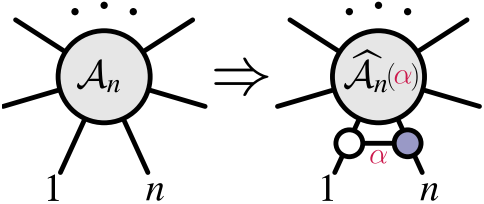

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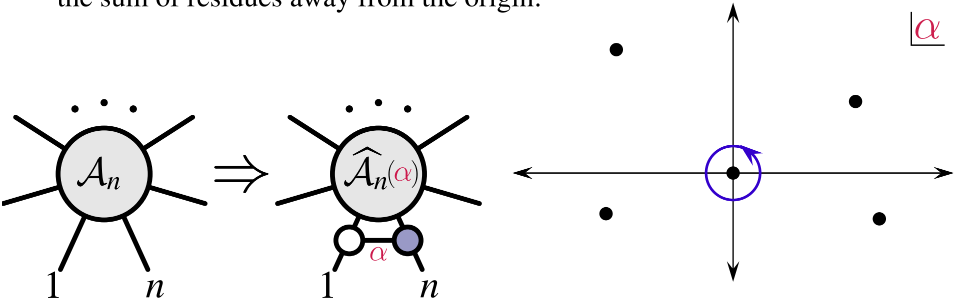

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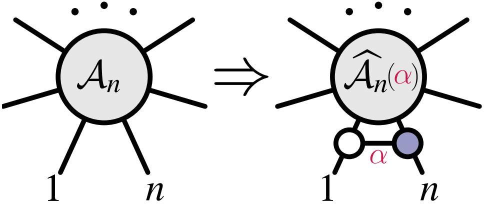

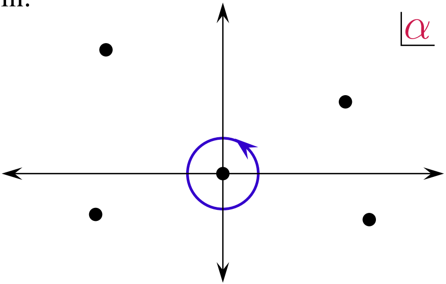

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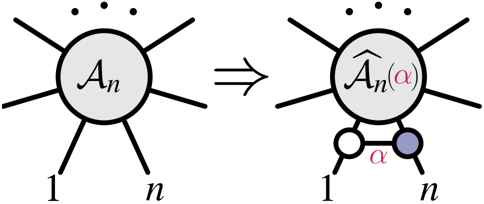

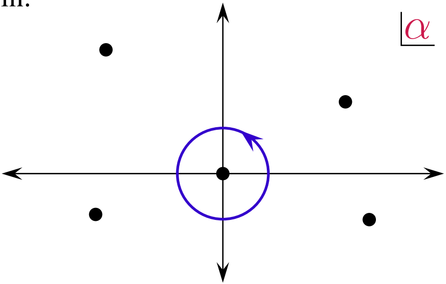

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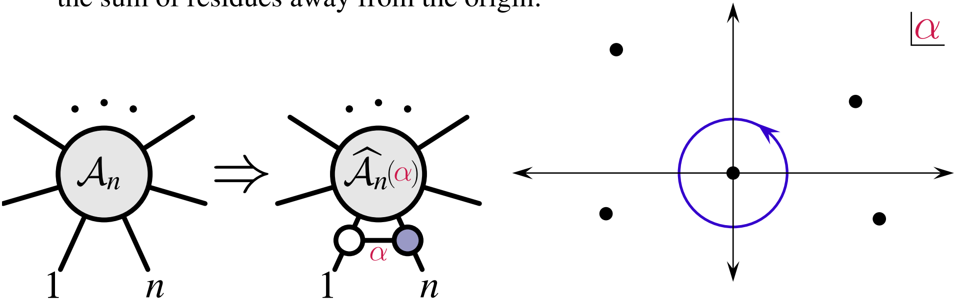

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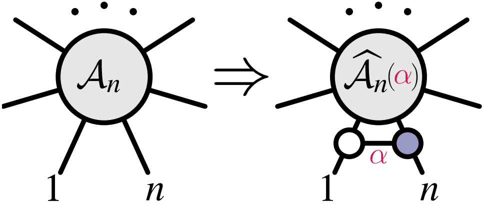

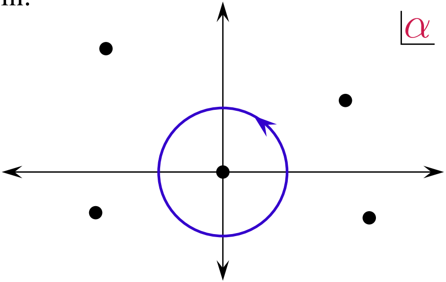

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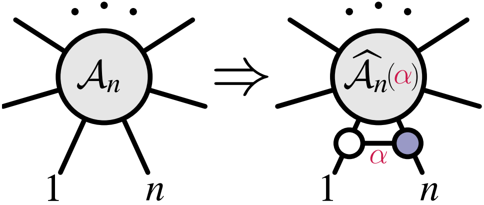

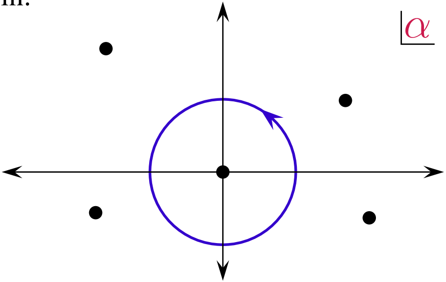

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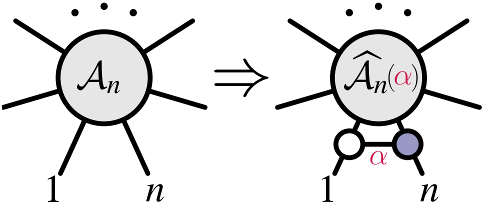

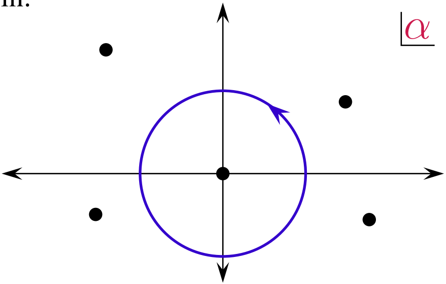

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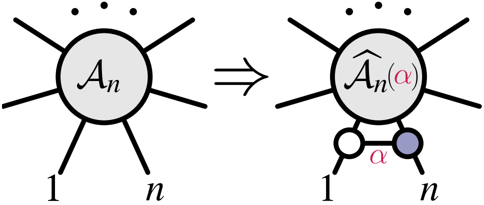

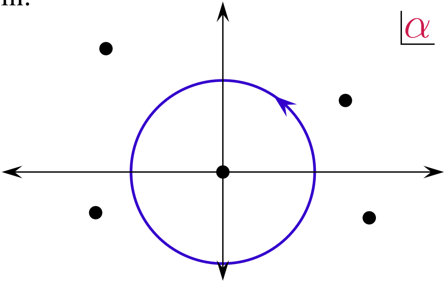

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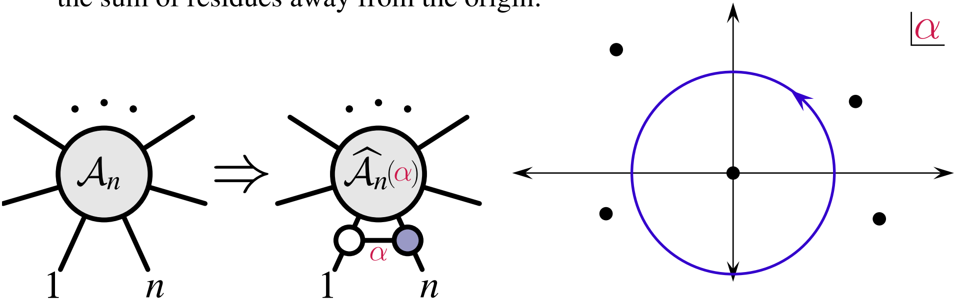

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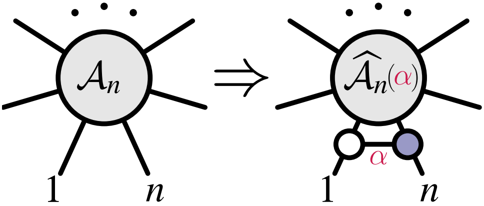

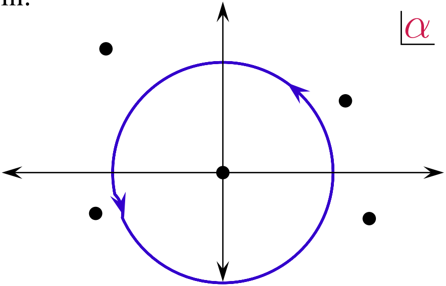

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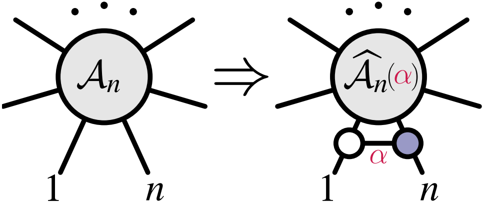

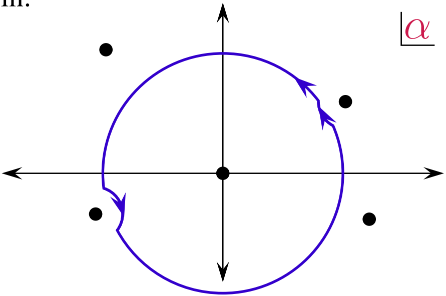

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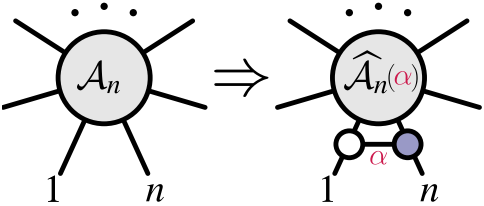

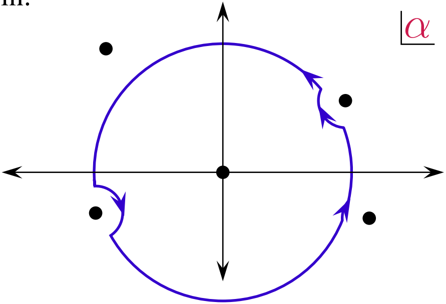

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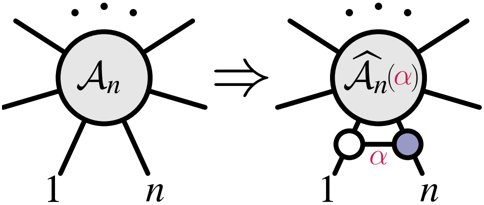

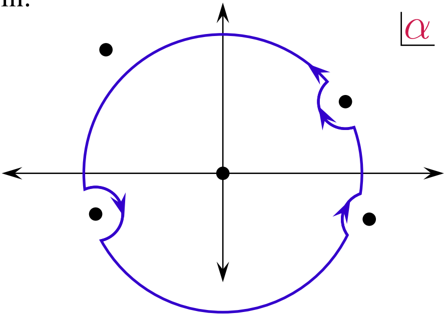

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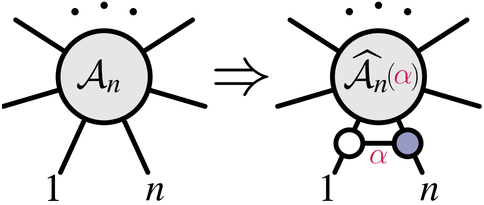

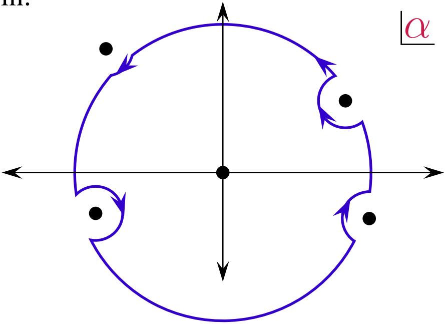

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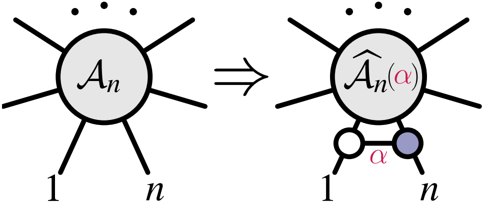

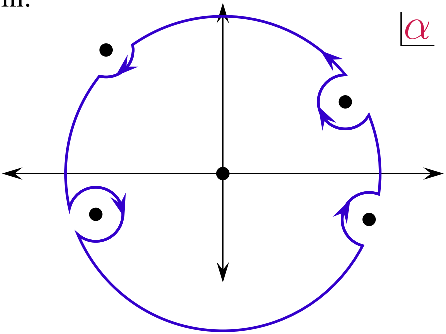

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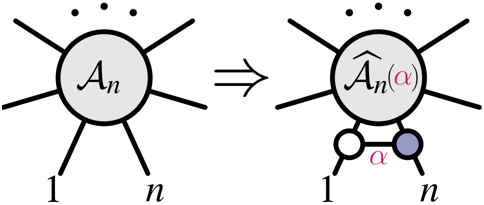

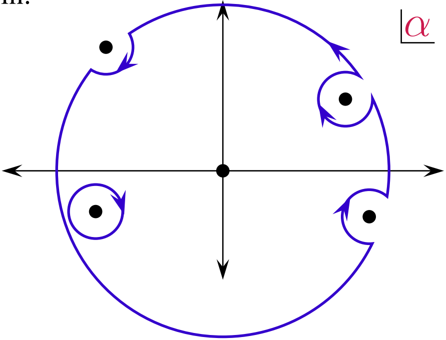

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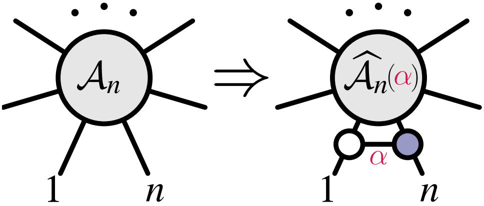

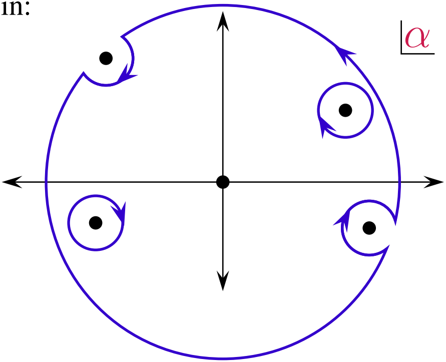

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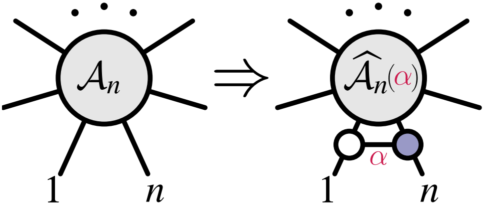

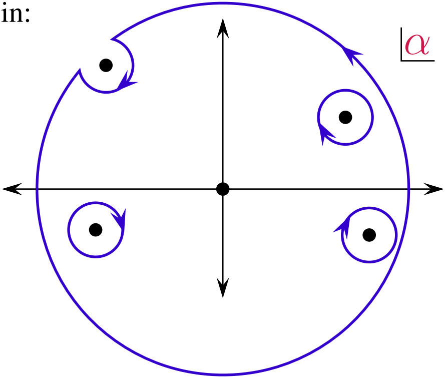

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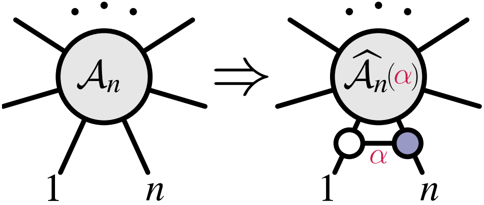

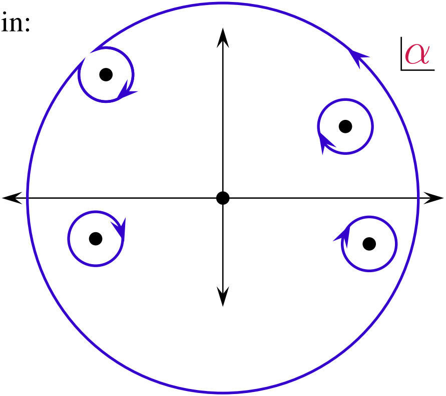

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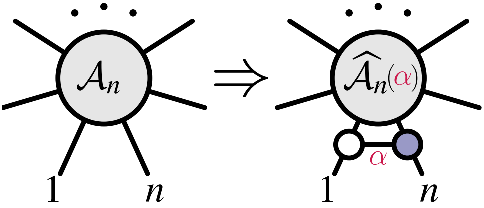

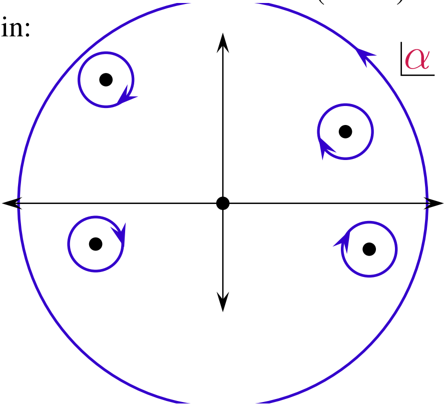

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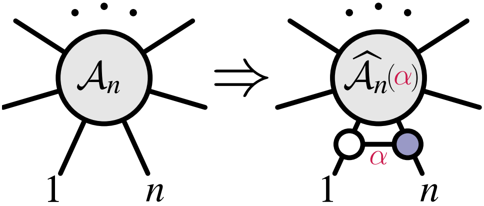

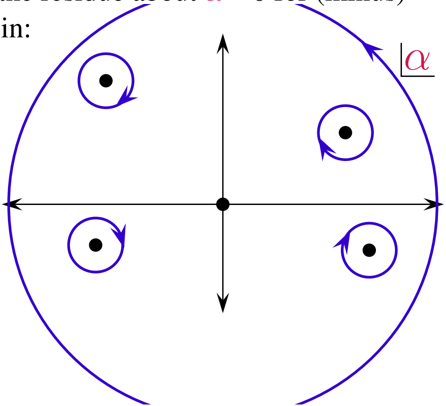

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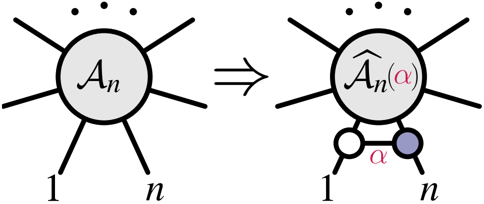

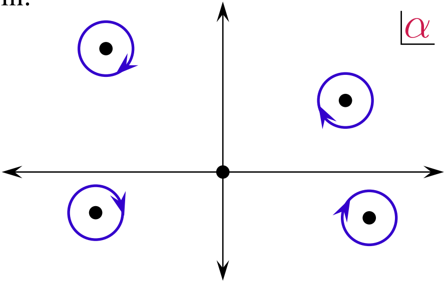

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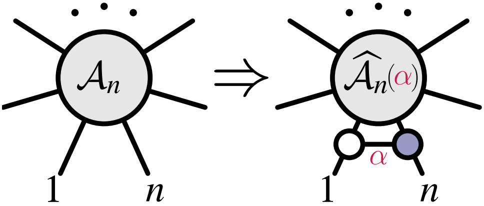

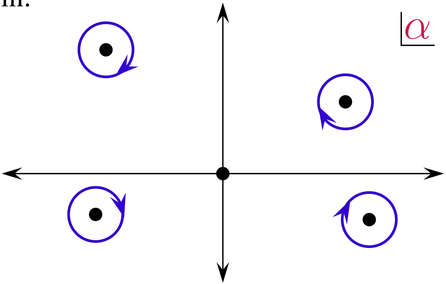

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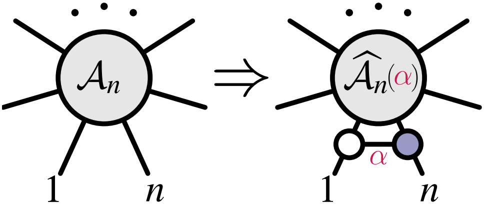

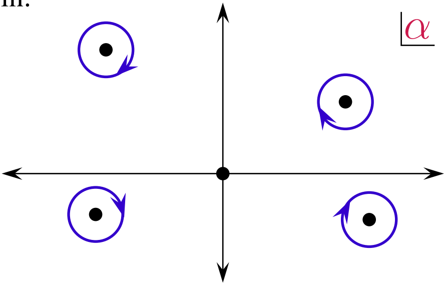

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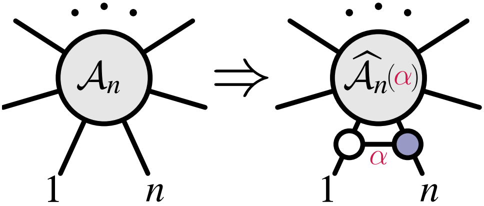

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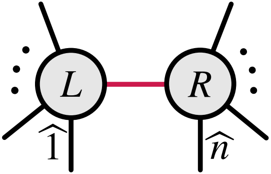

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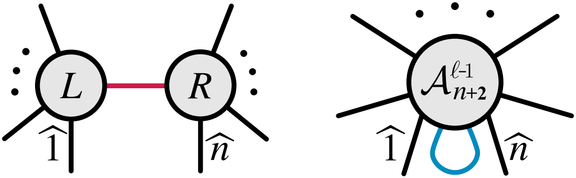

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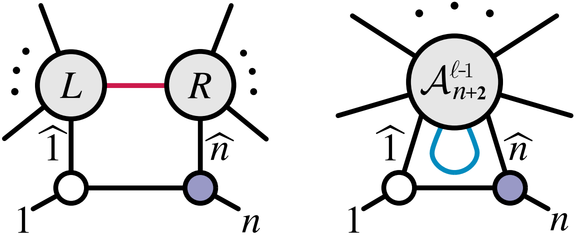

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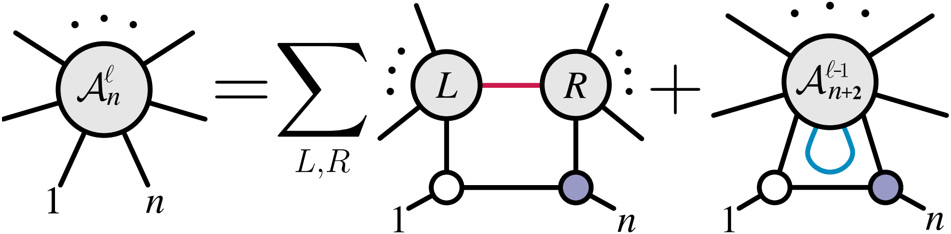

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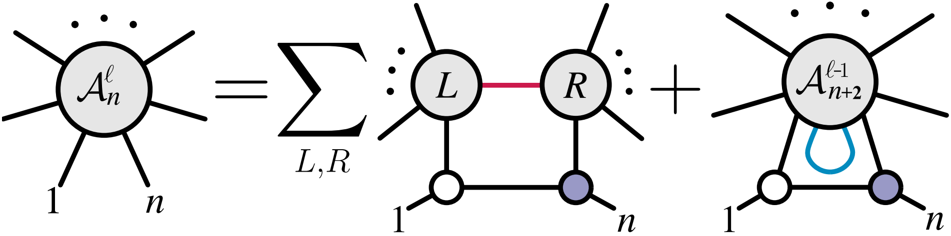

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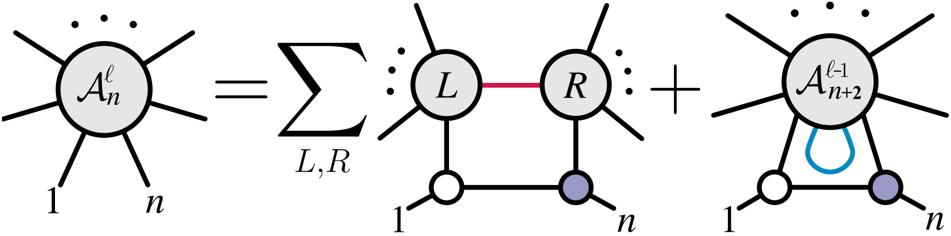

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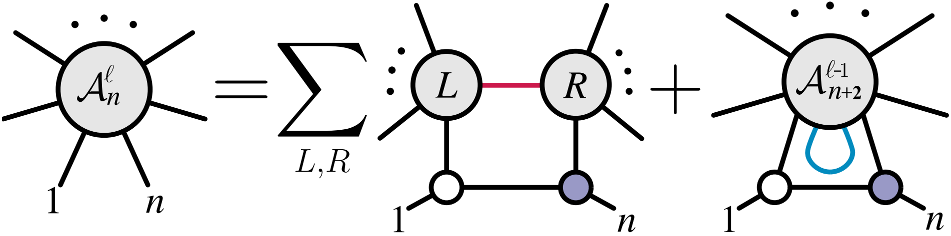

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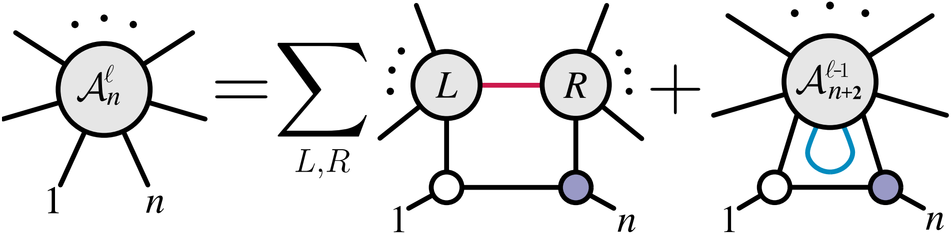

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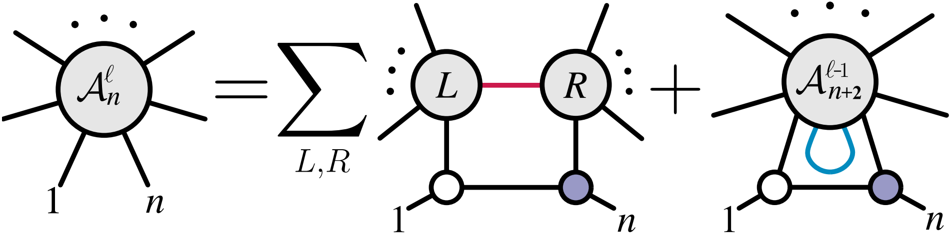

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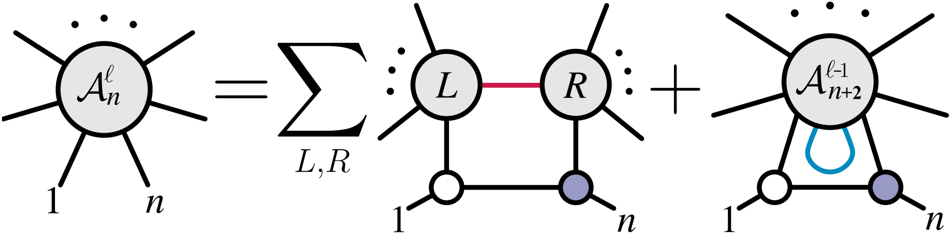

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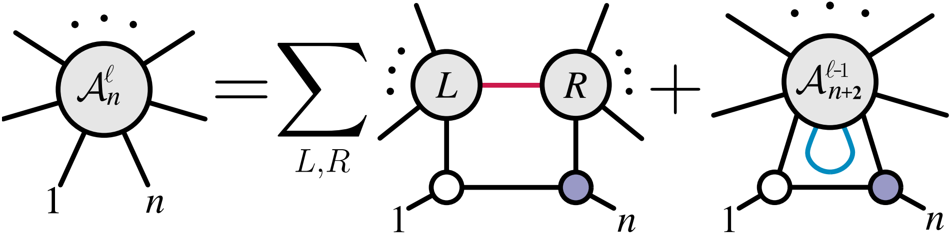

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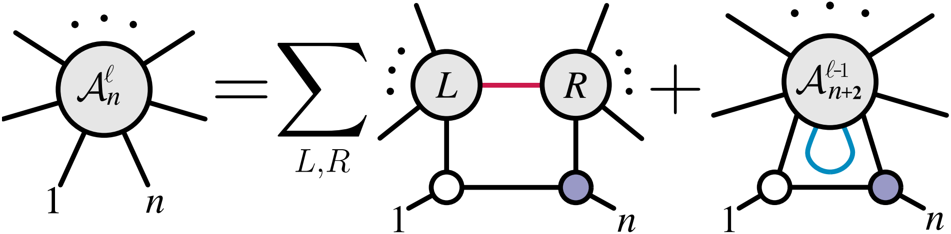

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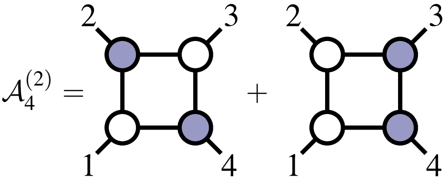

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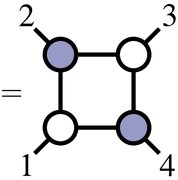

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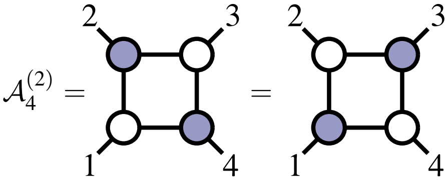

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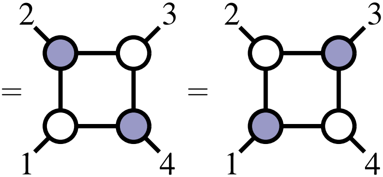

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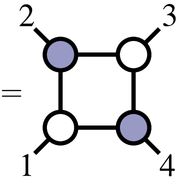

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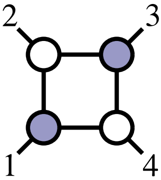

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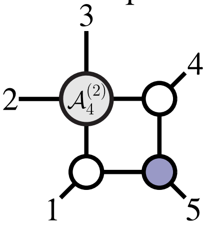

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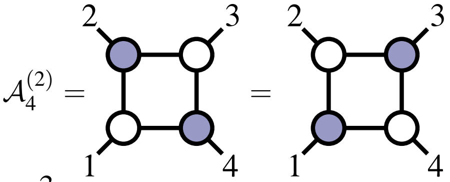

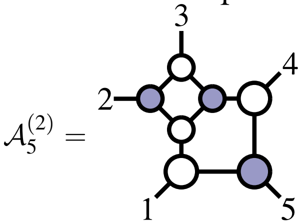

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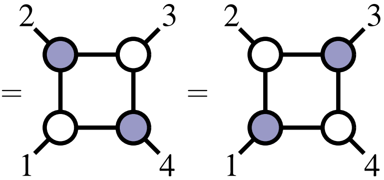

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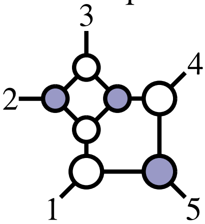

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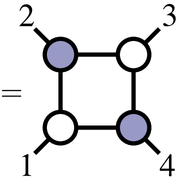

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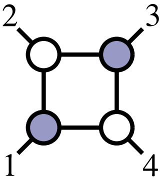

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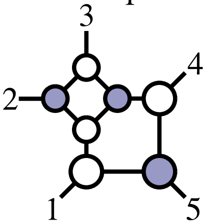

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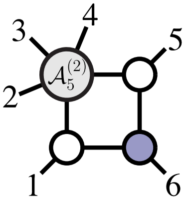

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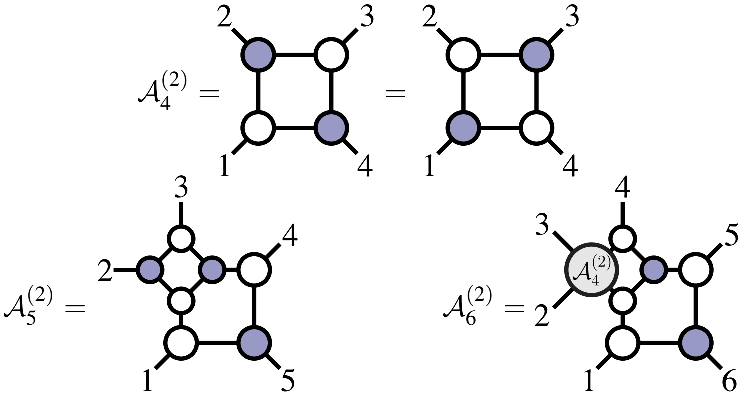

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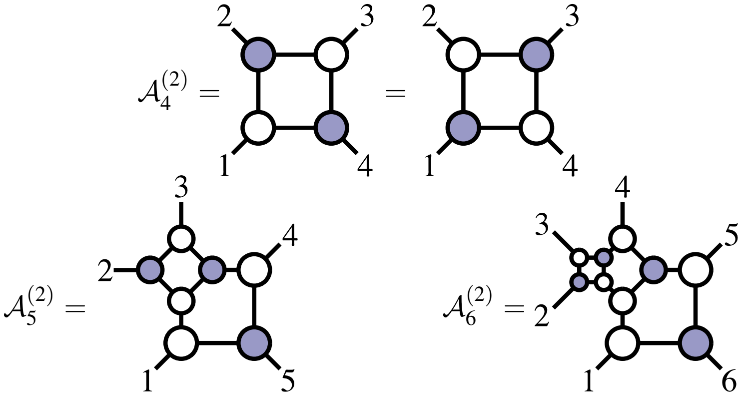

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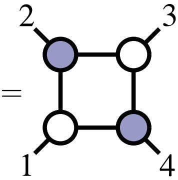

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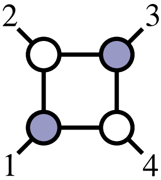

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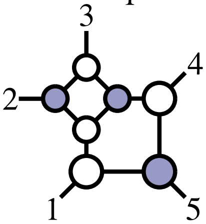

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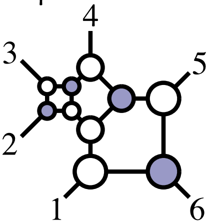

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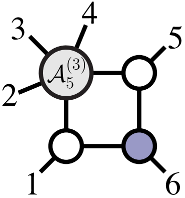

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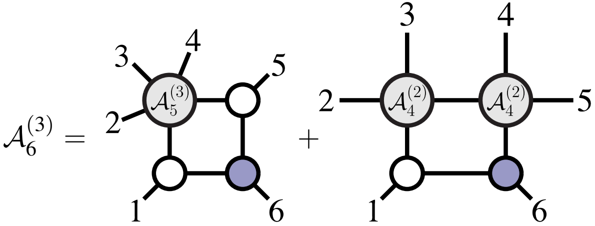

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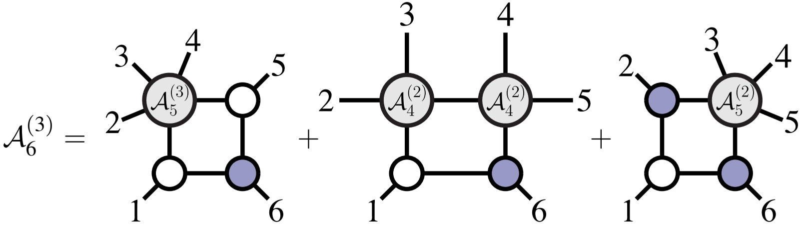

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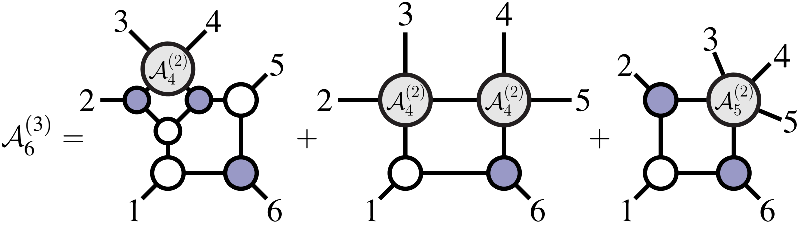

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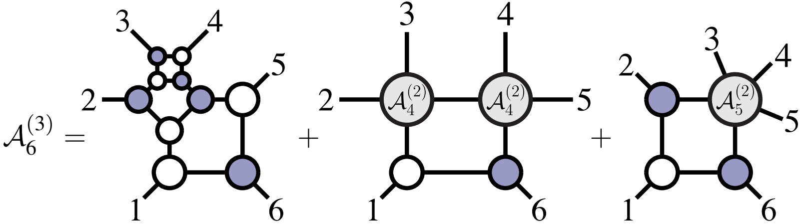

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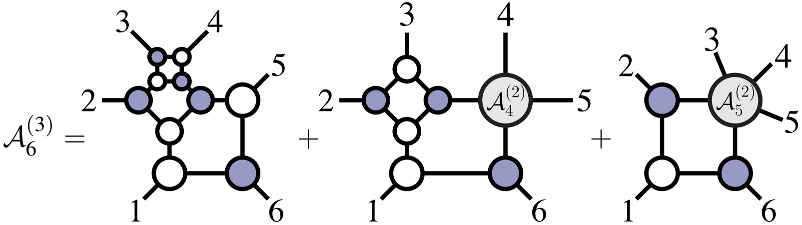

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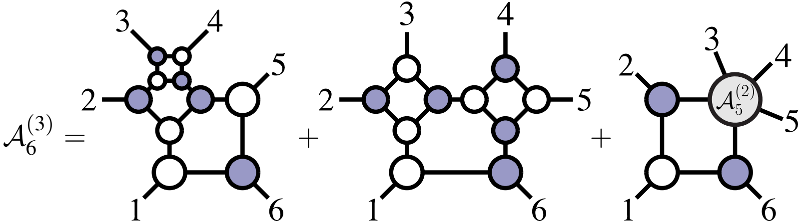

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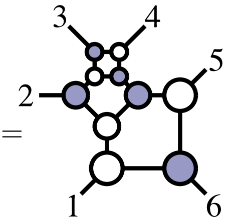

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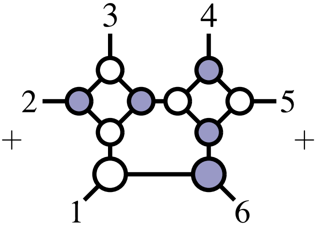

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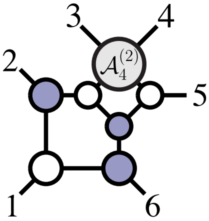

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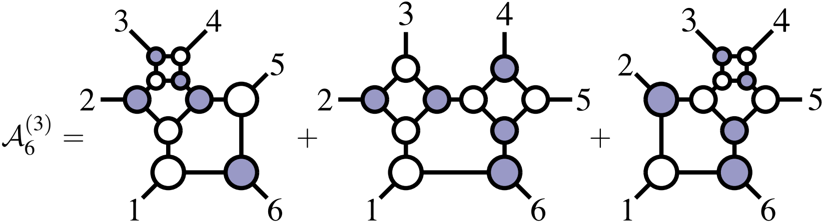

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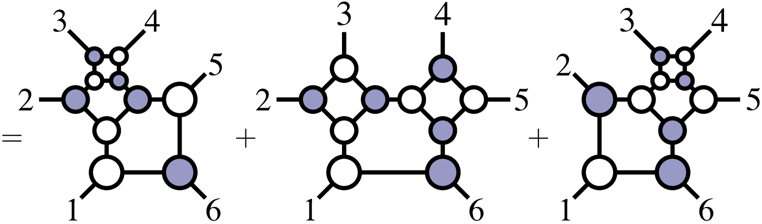

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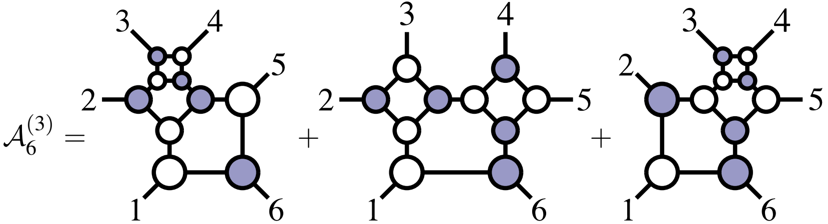

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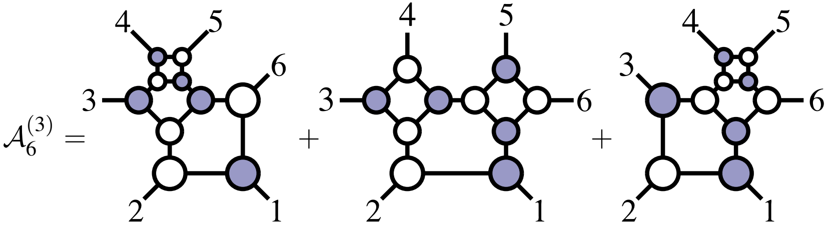

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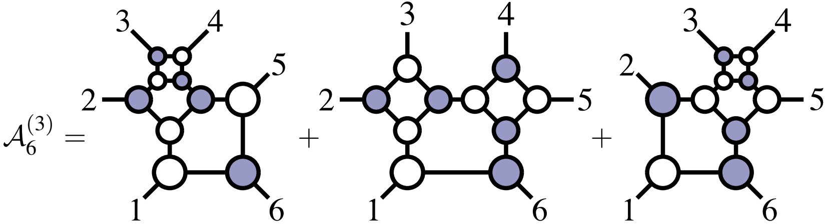

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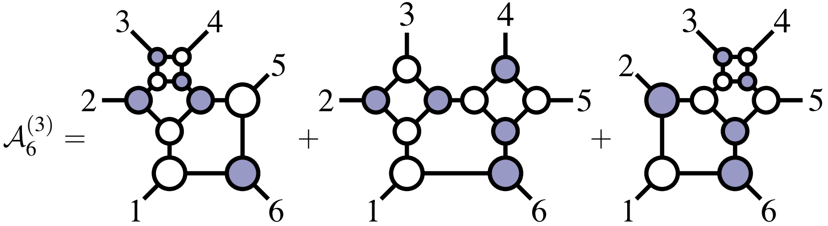

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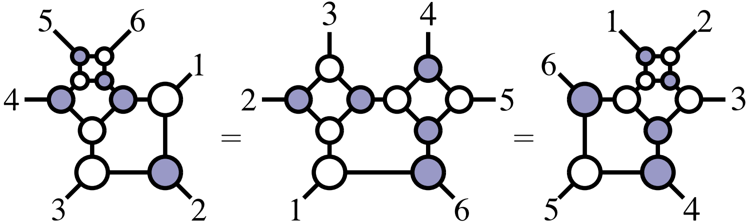

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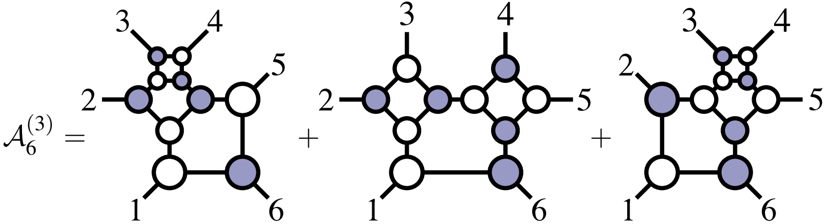

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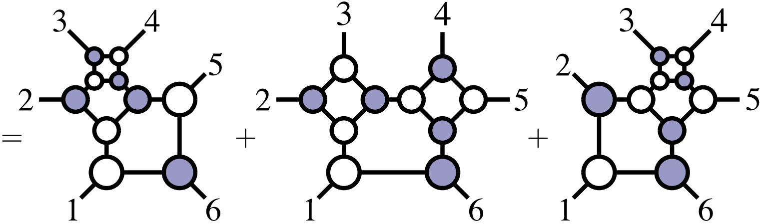

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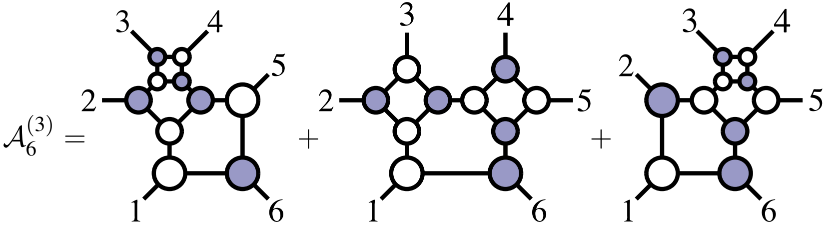

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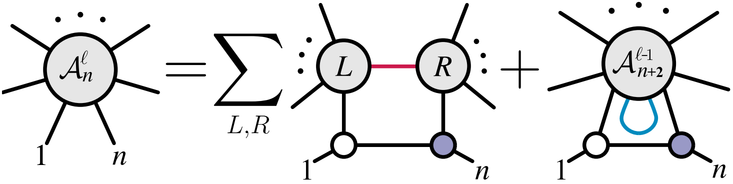

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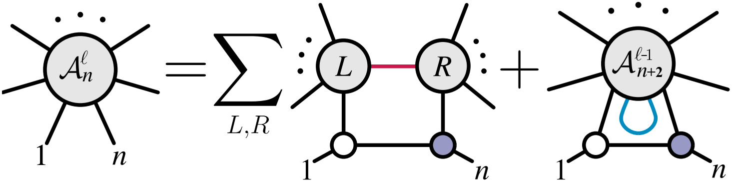

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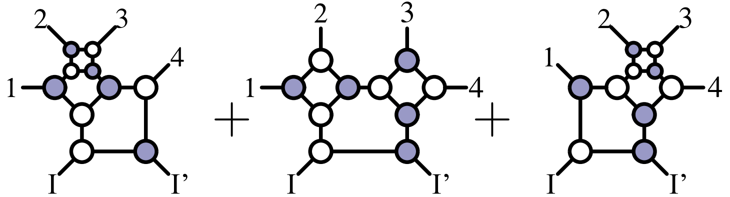

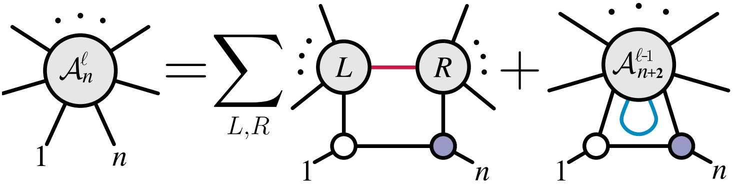

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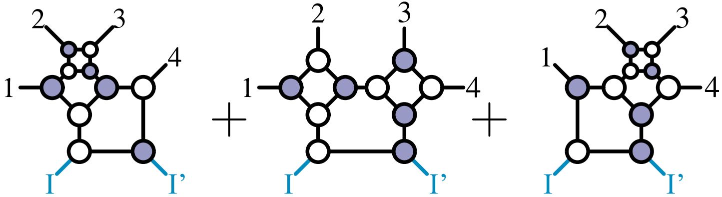

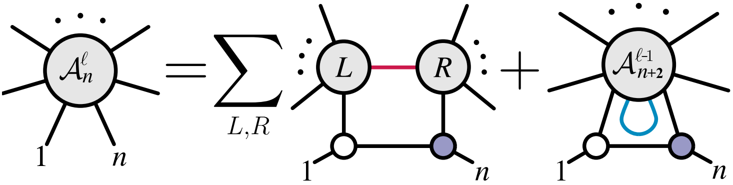

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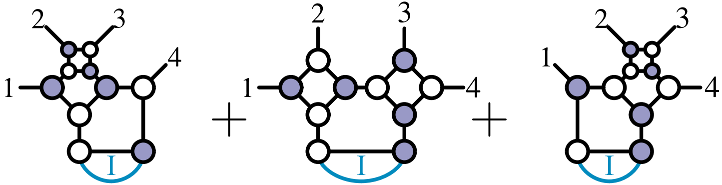

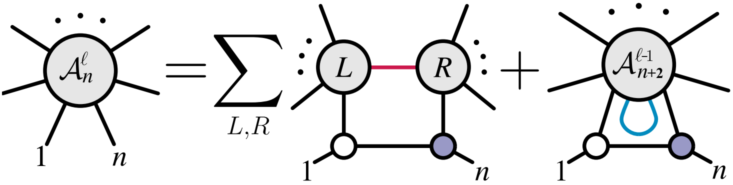

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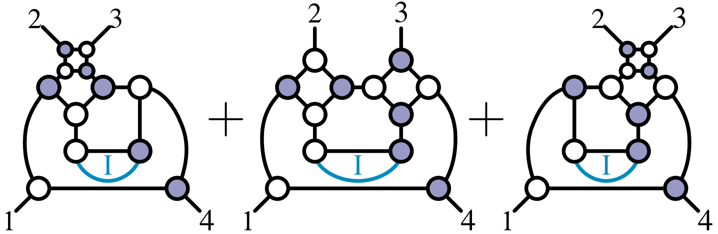

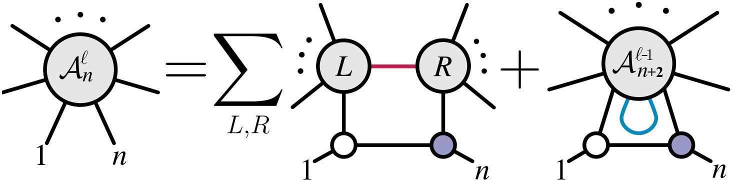

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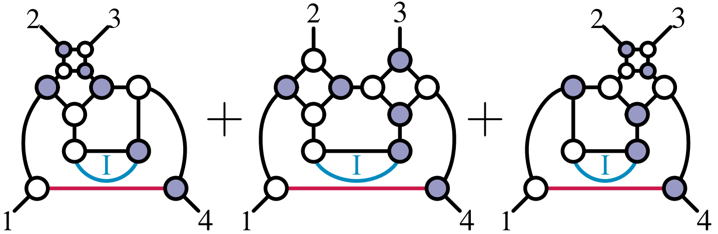

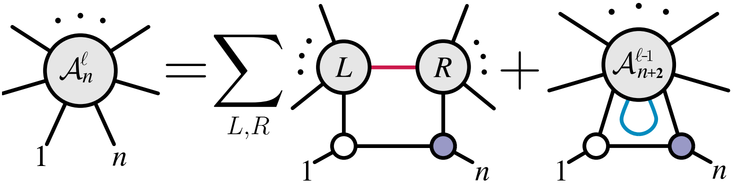

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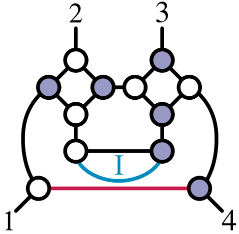

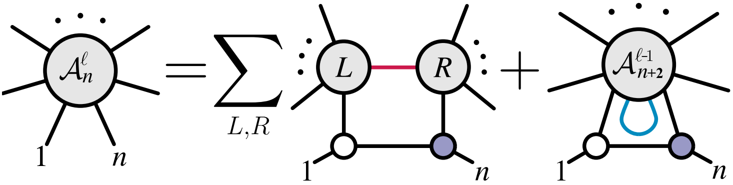

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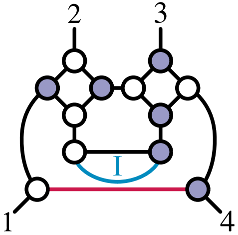

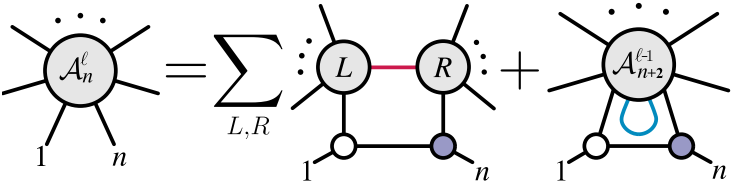

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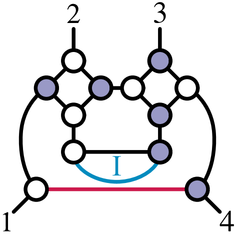

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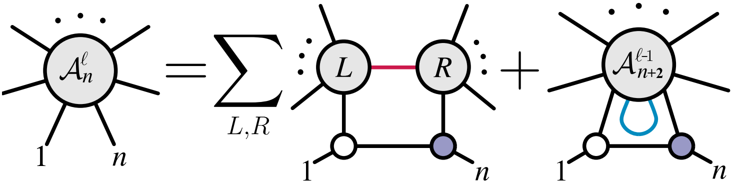

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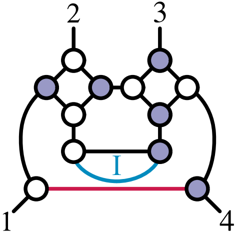

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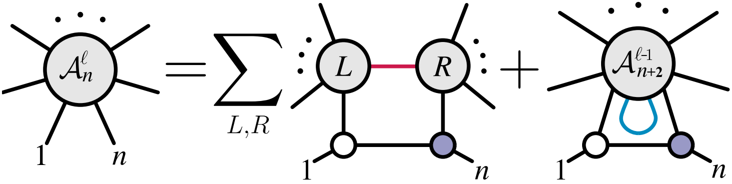

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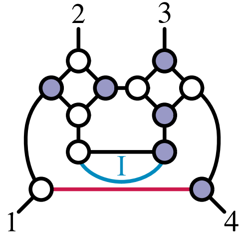

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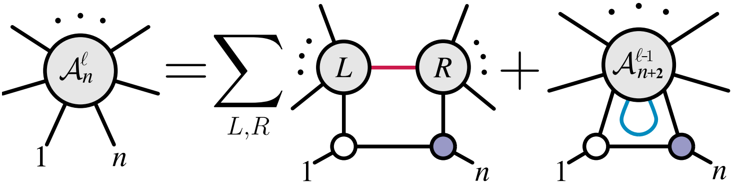

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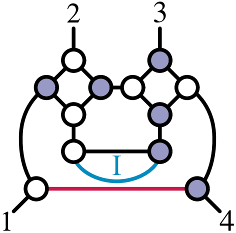

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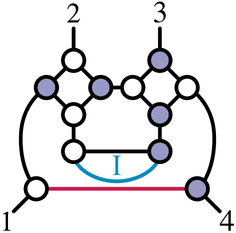

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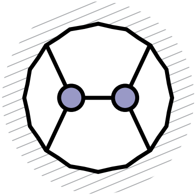

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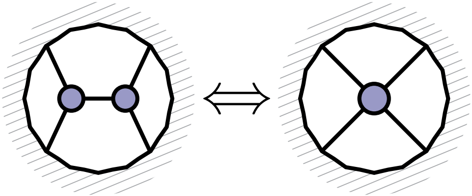

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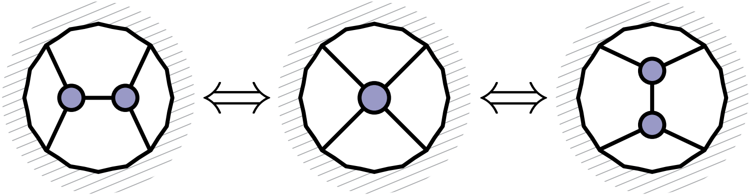

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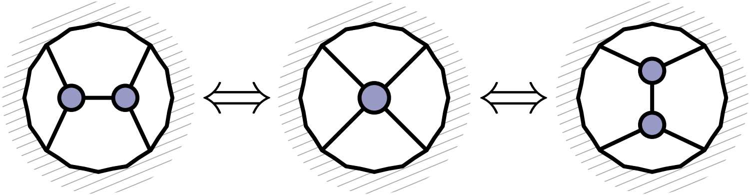

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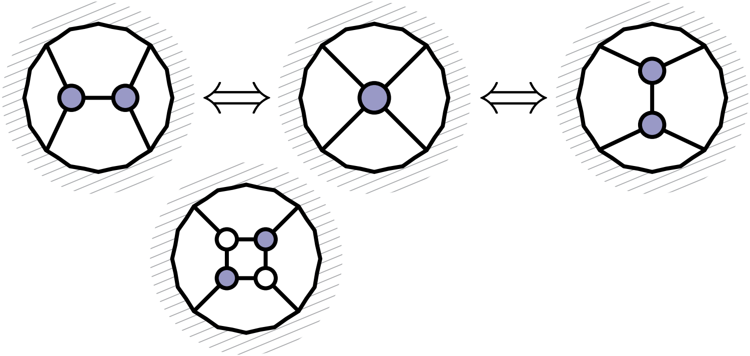

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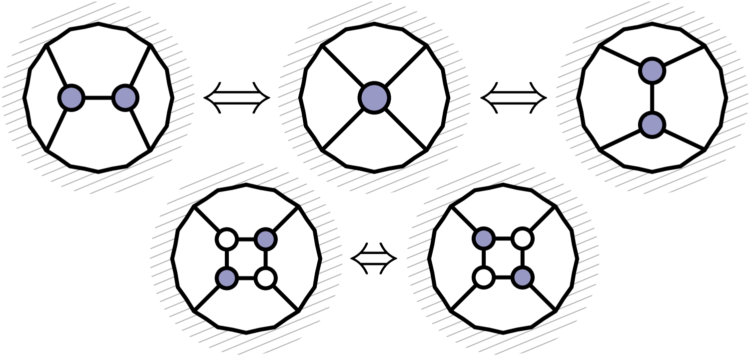

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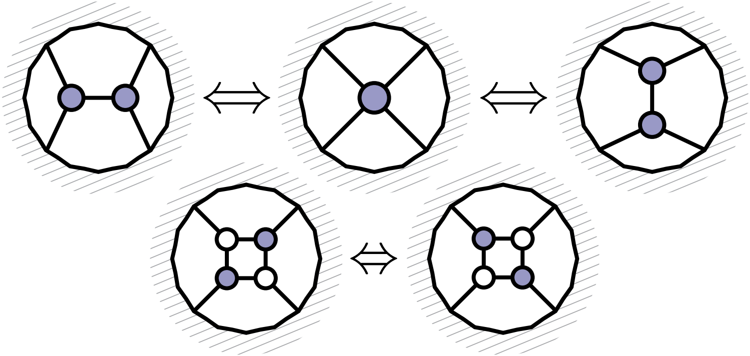

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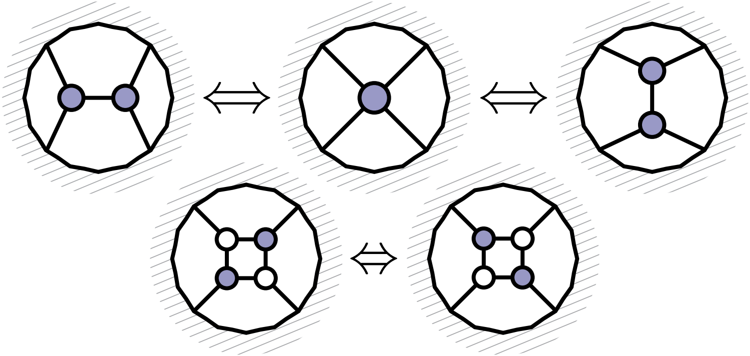

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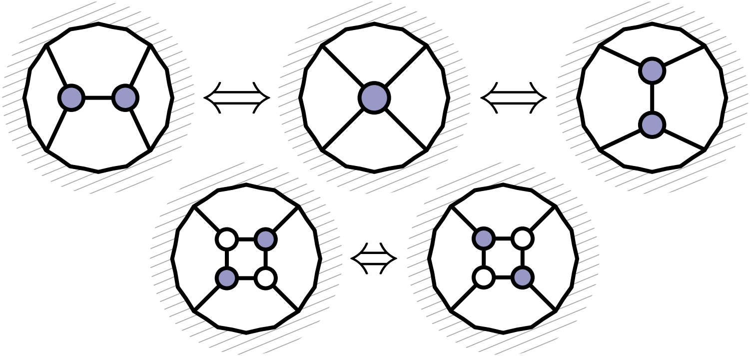

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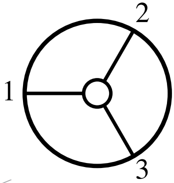

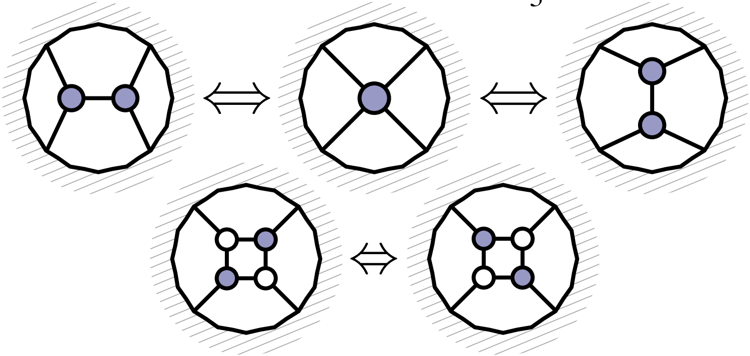

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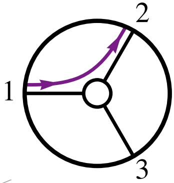

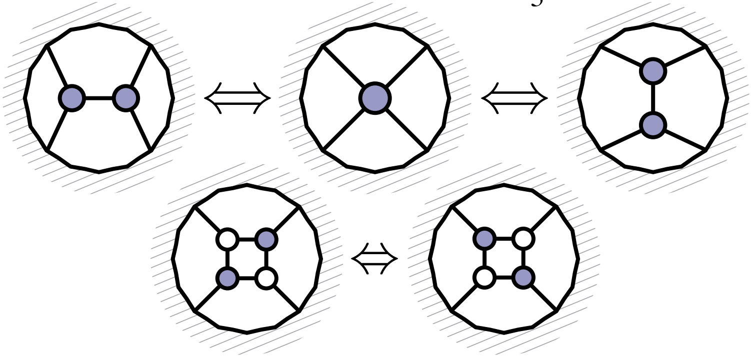

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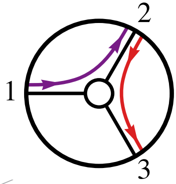

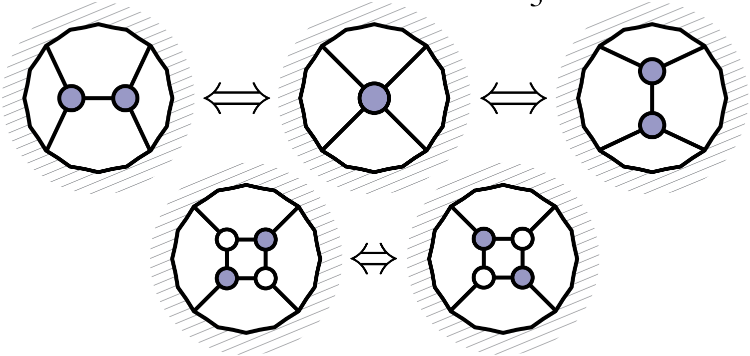

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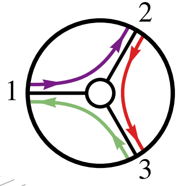

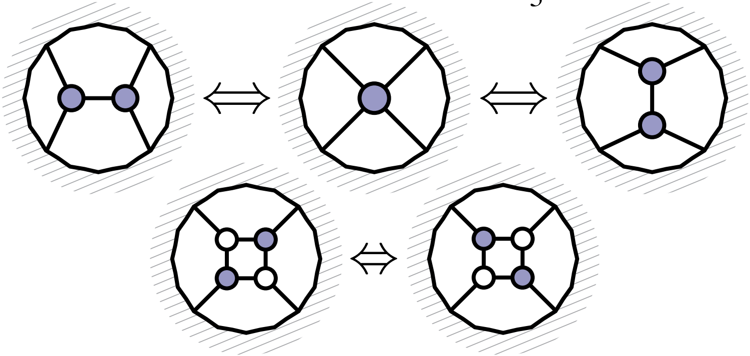

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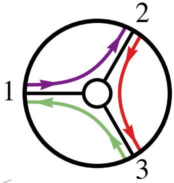

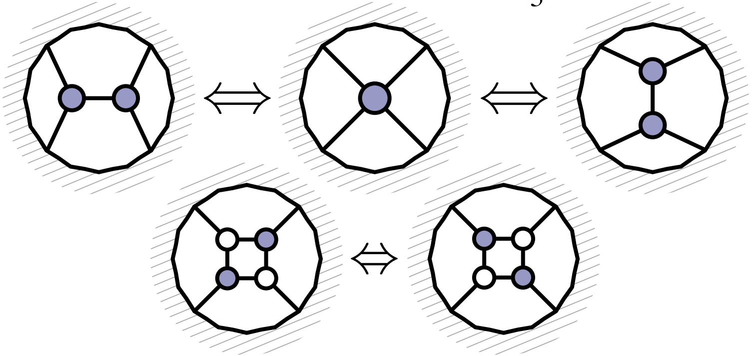

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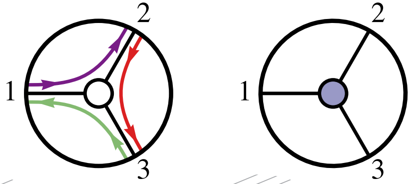

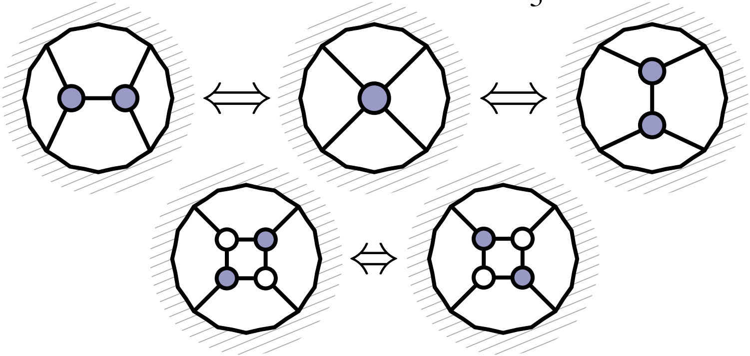

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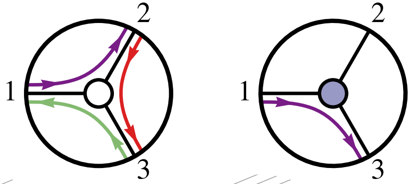

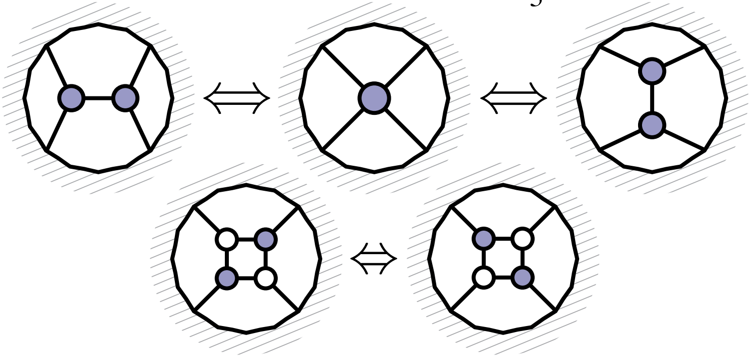

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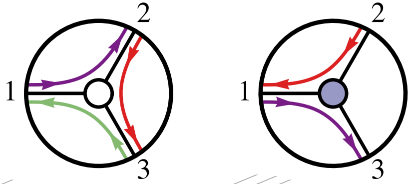

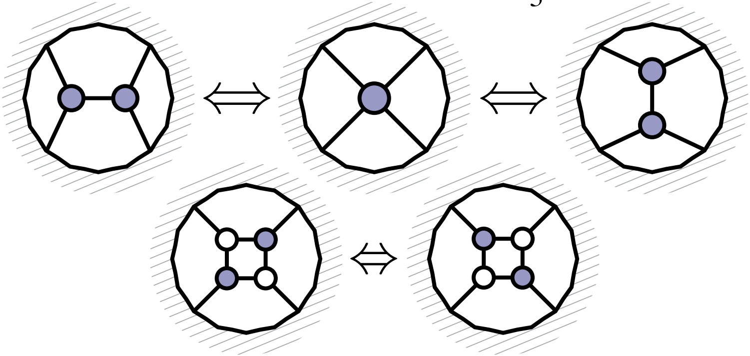

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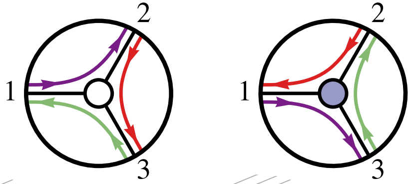

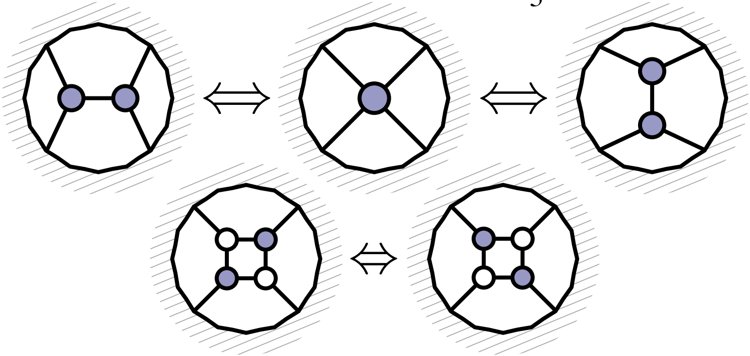

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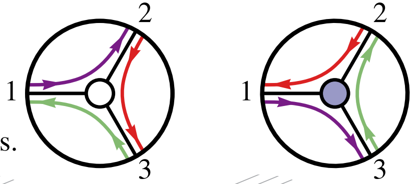

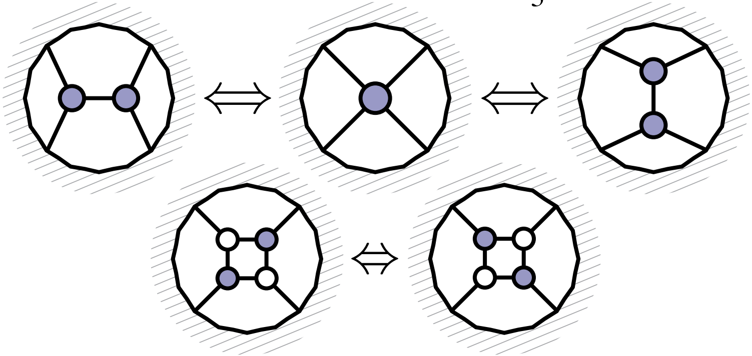

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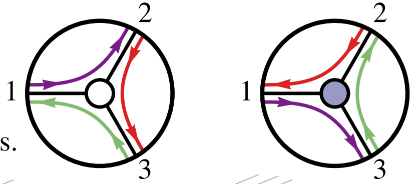

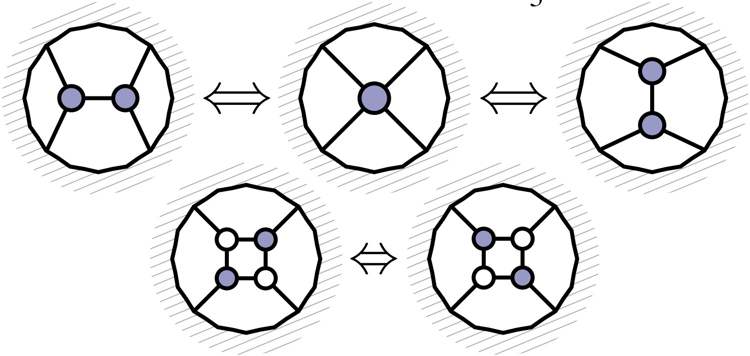

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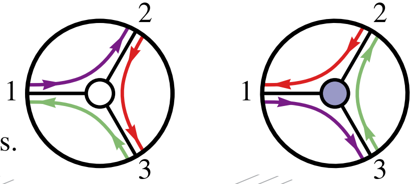

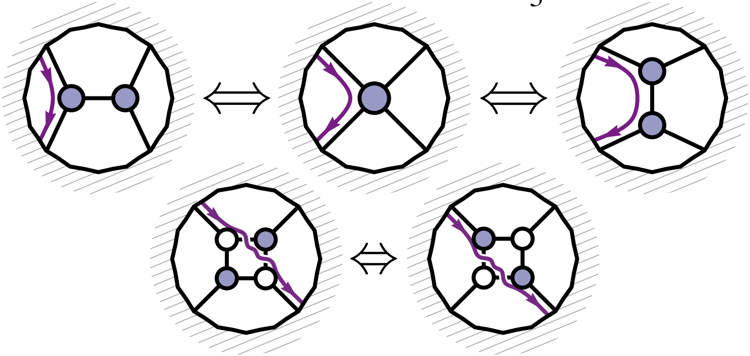

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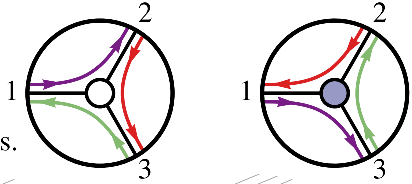

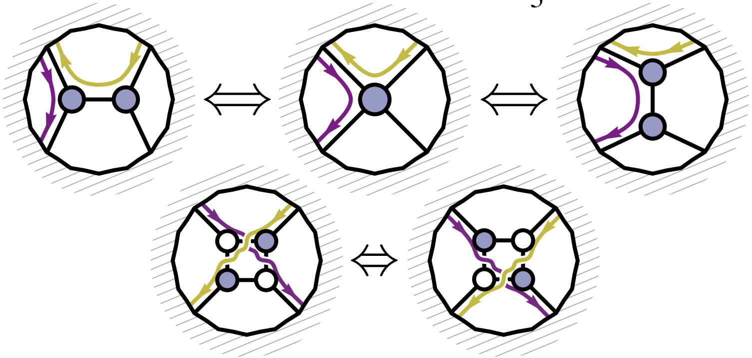

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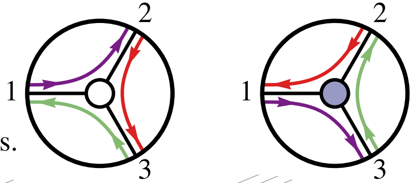

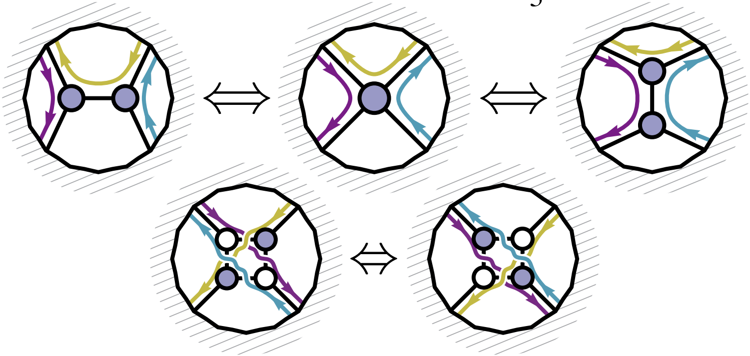

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Starting from any leg a, turn:

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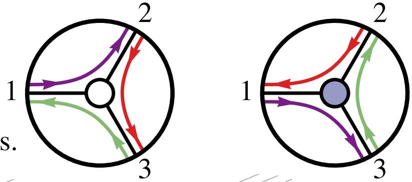

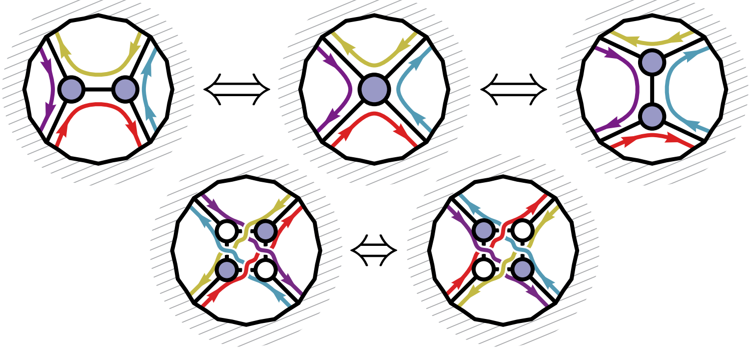

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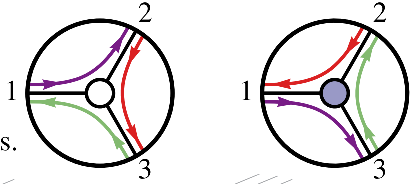

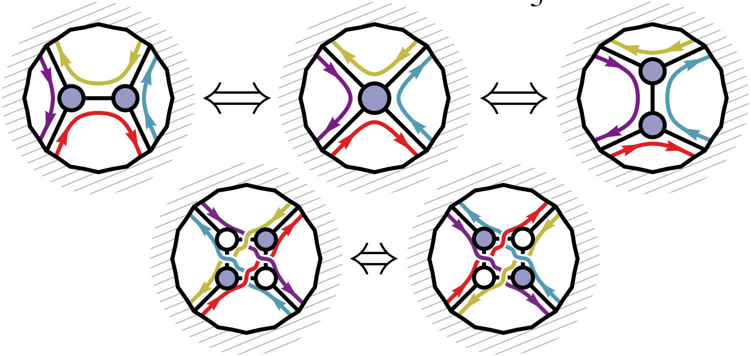

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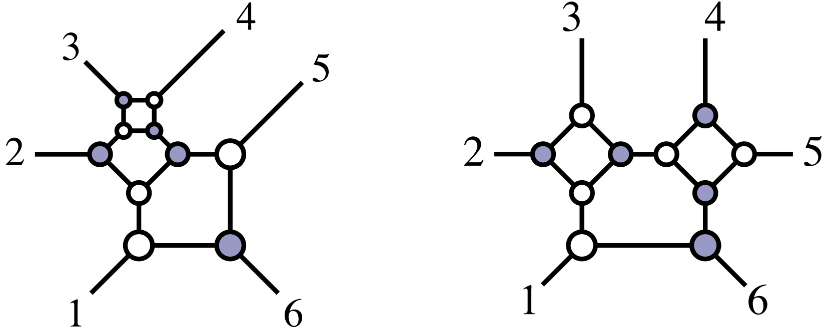

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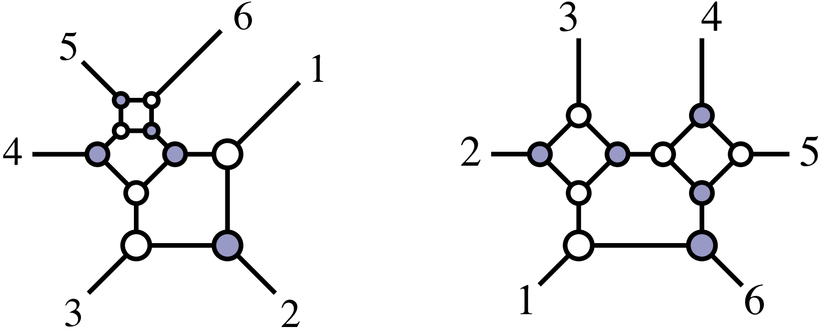

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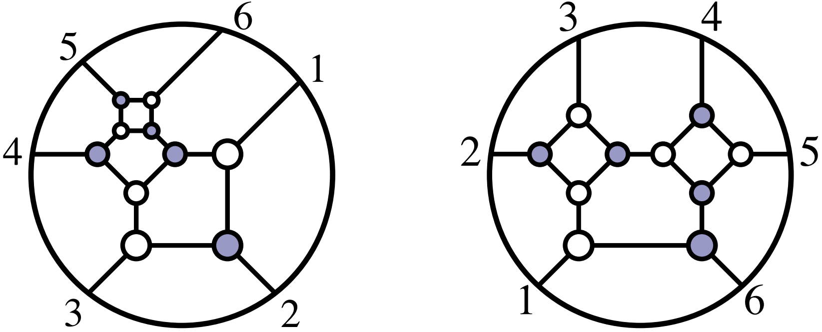

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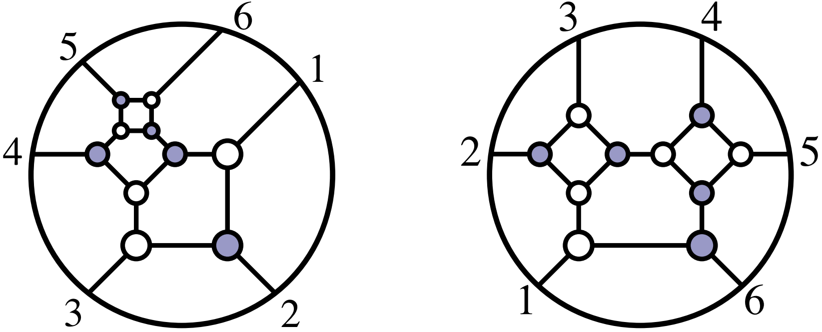

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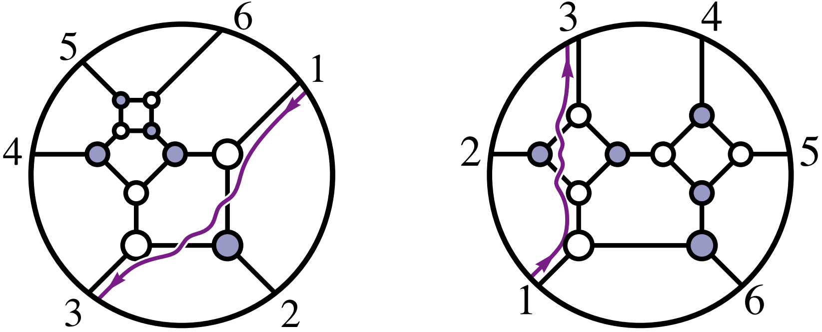

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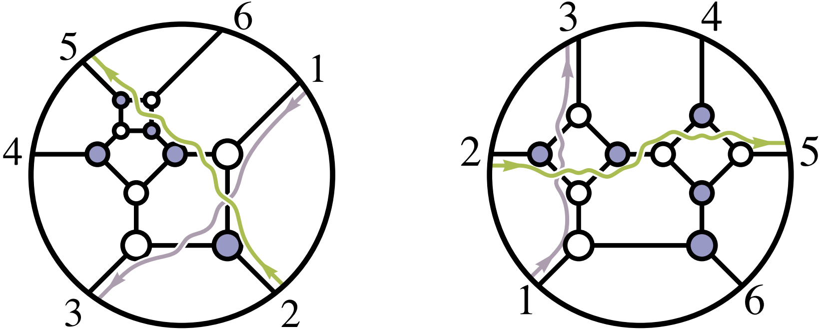

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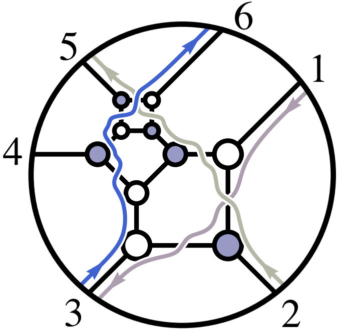

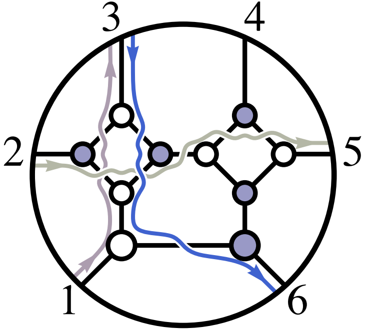

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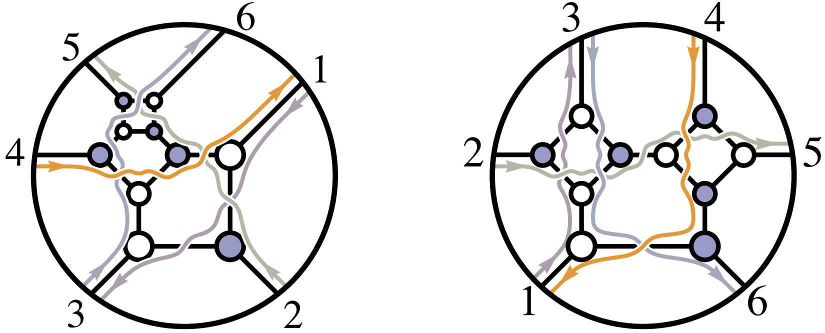

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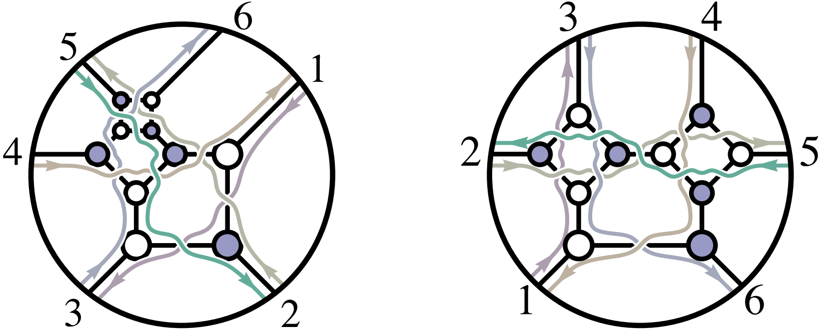

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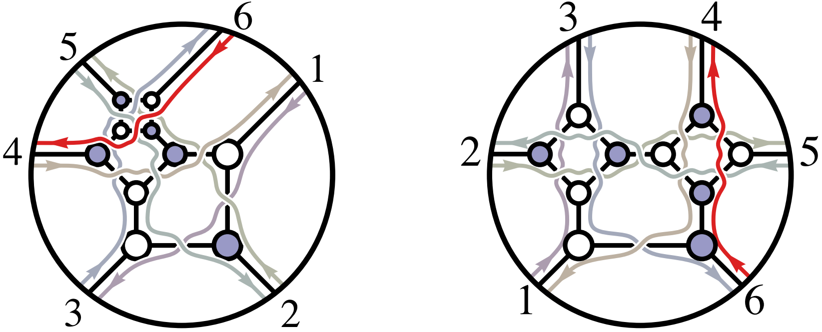

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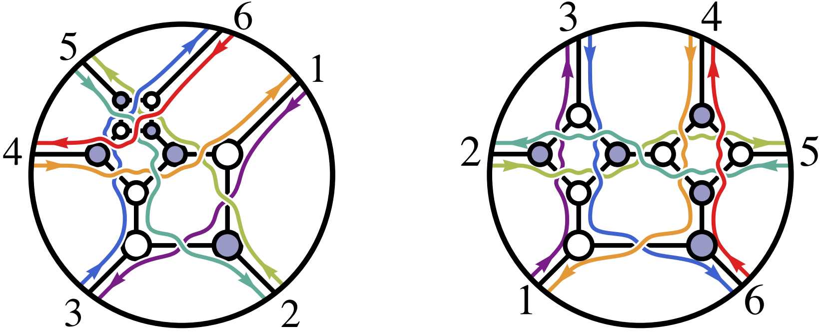

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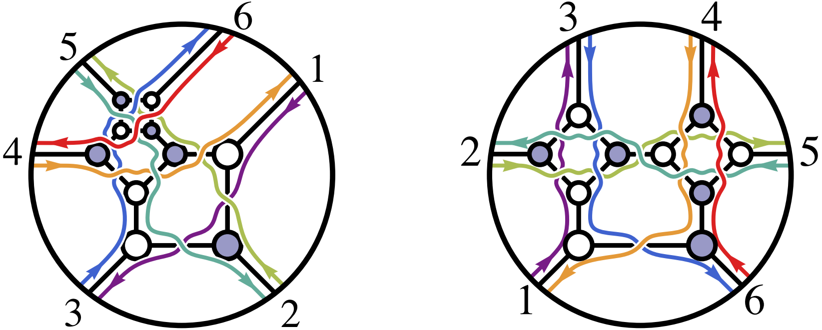

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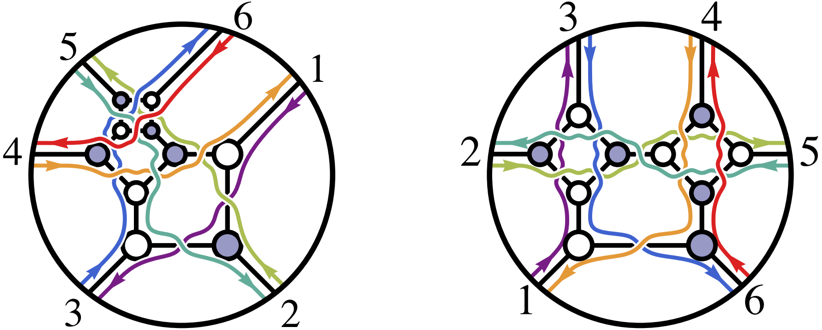

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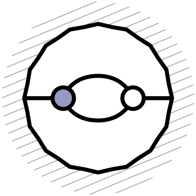

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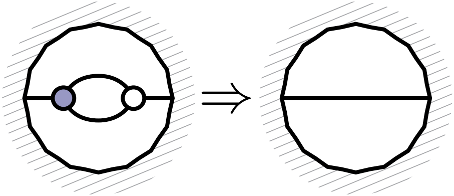

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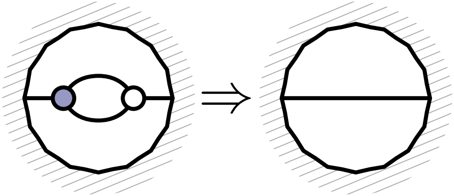

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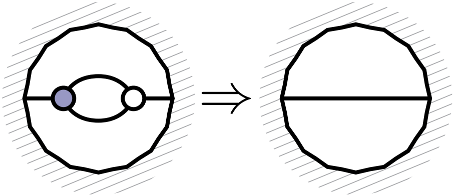

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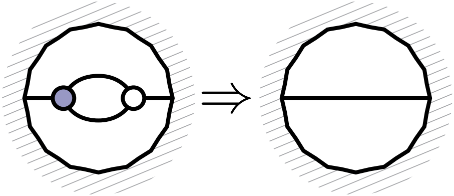

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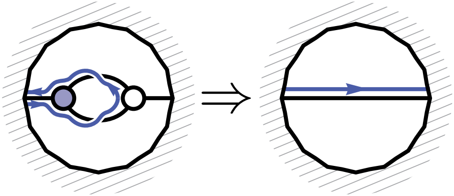

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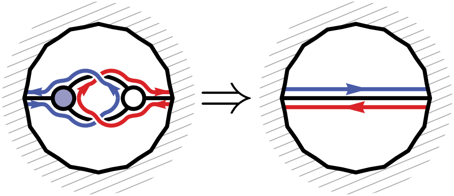

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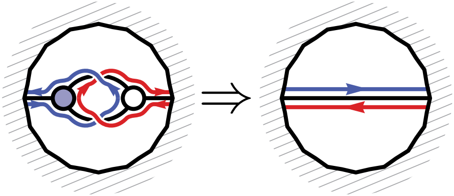
Such factors of $d\alpha/\alpha$ arising from bubble deletion encode loop integrands!



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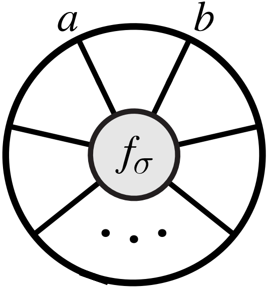

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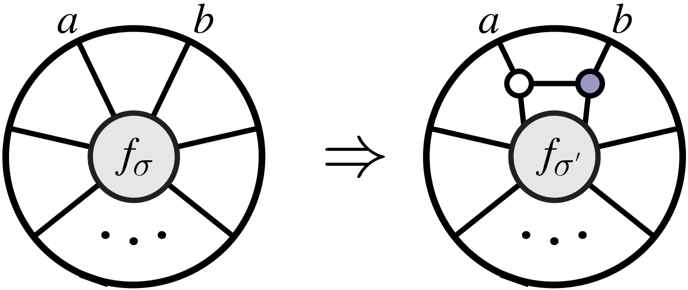

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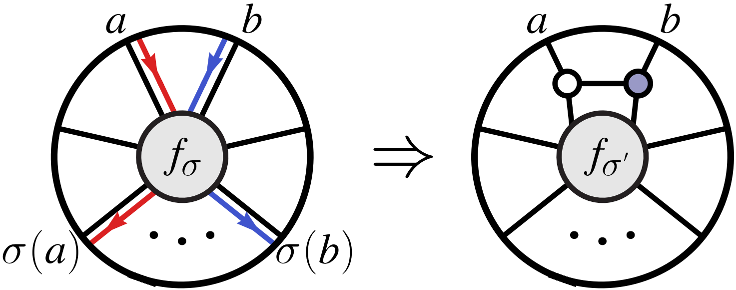

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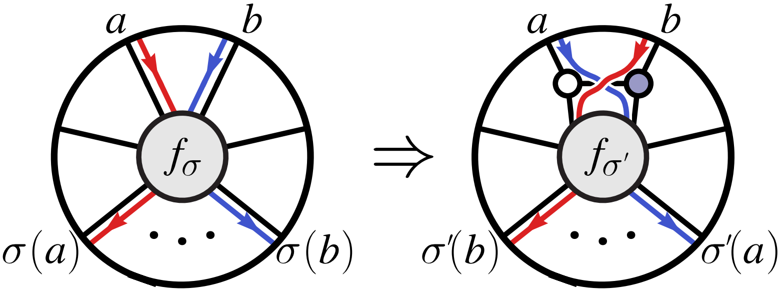

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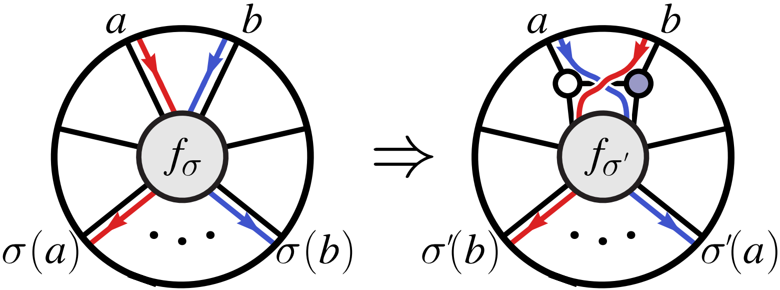

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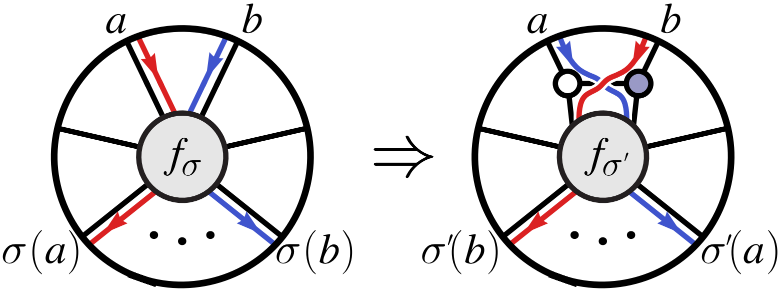

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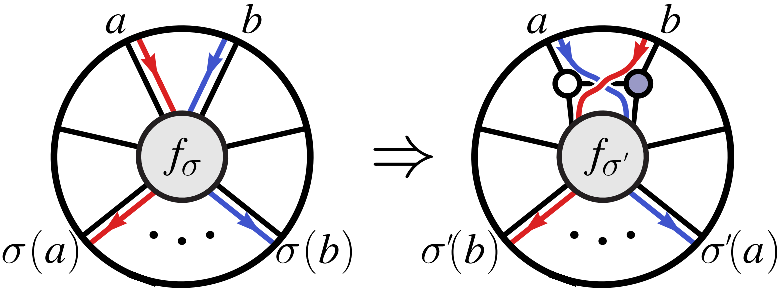

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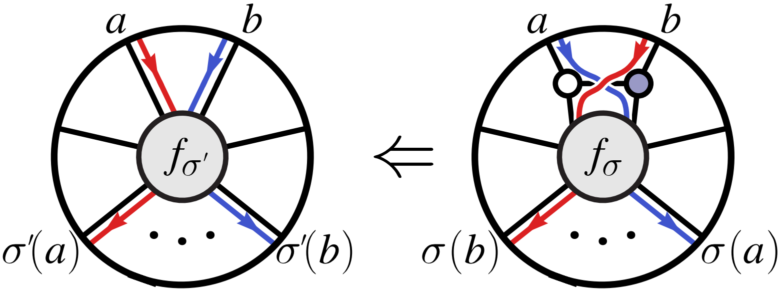

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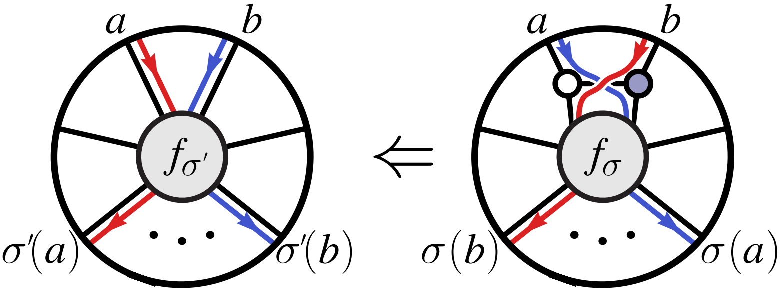
Recall that attaching 'BCFW bridges' can lead to very rich on-shell diagrams.

Read the other way, we can 'peel-off' bridges and thereby decompose a permutation into transpositions according to $\sigma = (a\, b) \circ \sigma'$



Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions

'Bridge' Decomposition

$$\sigma:\begin{pmatrix}1&2&3&4&5&6\\\downarrow&\downarrow&\downarrow&\downarrow&\downarrow&\downarrow\\3&5&6&7&8&10\end{pmatrix}$$

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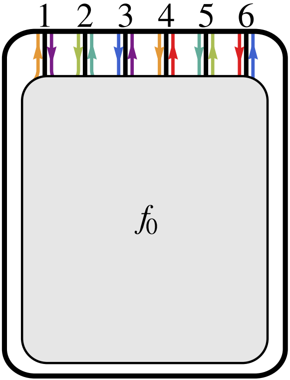

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$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & \end{array}$$

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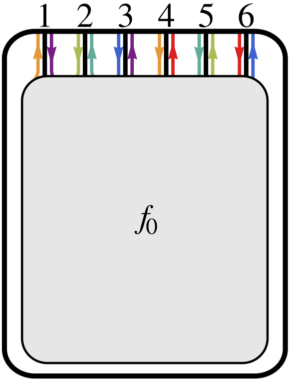

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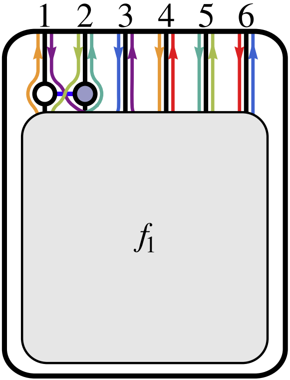

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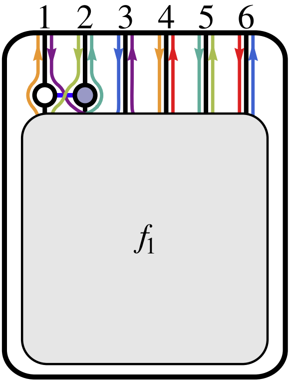


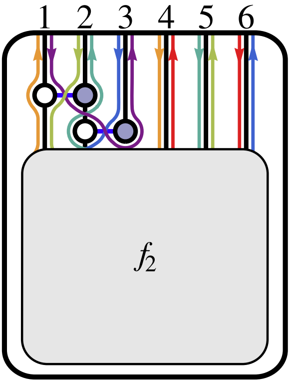
‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2} f_2$$



‘Bridge’ Decomposition

$$\begin{array}{lccccccl}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\
f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & \\
 & & & & & & & (1\,2) \\
f_1 & \{5 & 3 & 6 & 7 & 8 & 10\} & \\
 & & & & & & & (2\,3) \\
f_2 & \{5 & 6 & 3 & 7 & 8 & 10\} &
\end{array}$$

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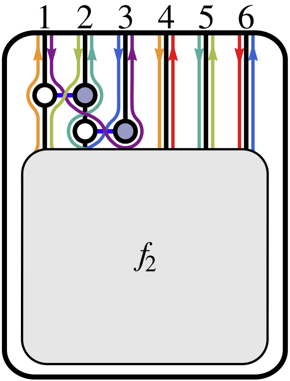

‘Bridge’ Decomposition

$$\begin{array}{lcccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \\
f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & & (1\,2) \\
f_1 & \{5 & 3 & 6 & 7 & 8 & 10\} & & (2\,3) \\
f_2 & \{5 & 6 & 3 & 7 & 8 & 10\} & & (1\,2)
\end{array}$$

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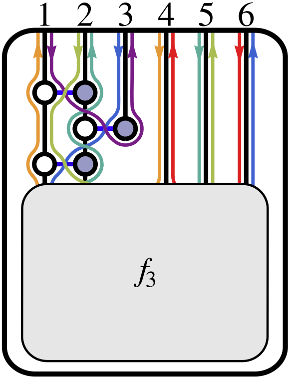

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	

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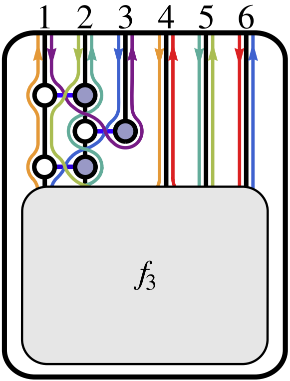

‘Bridge’ Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$

Canonical Coordinates for Computing On-Shell Functions

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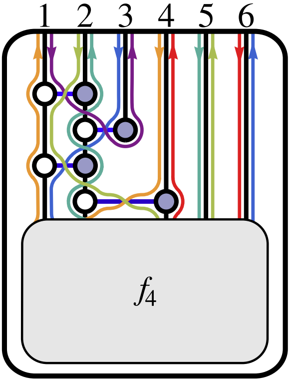

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	

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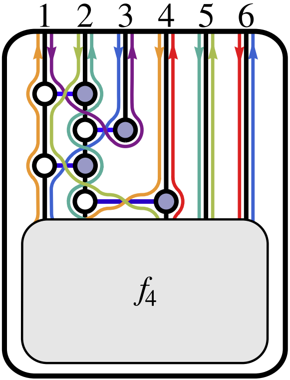

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)

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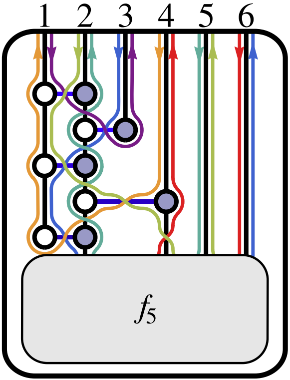
$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}f_5$$



'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	

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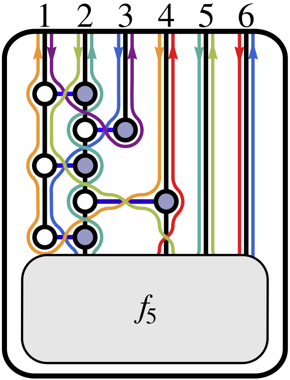

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$

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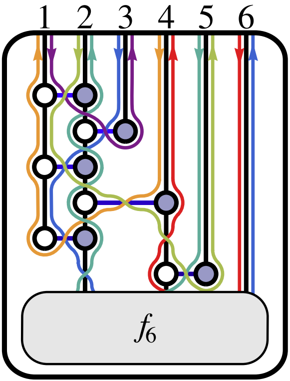
$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}f_6$$



'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	

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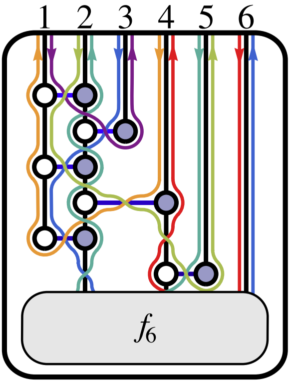

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$

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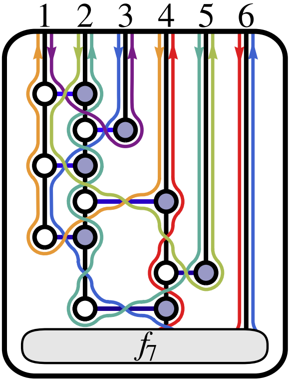

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	

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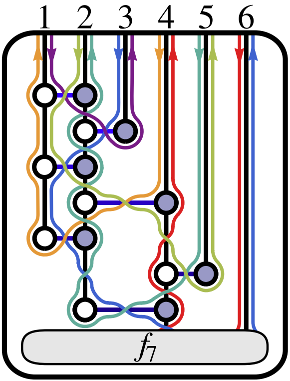

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
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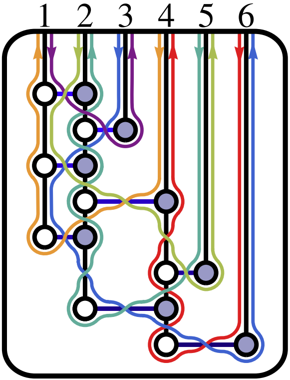

‘Bridge’ Decomposition

	$1 \downarrow$	$2 \downarrow$	$3 \downarrow$	$4 \downarrow$	$5 \downarrow$	$6 \downarrow$	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

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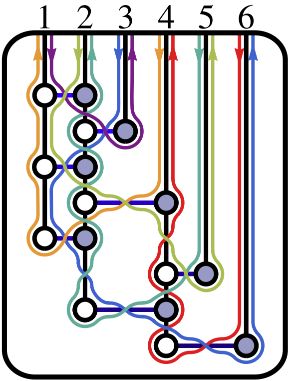

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

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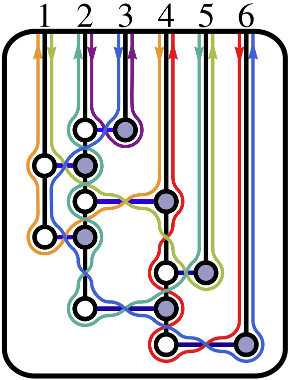

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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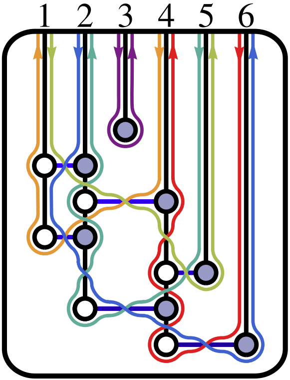

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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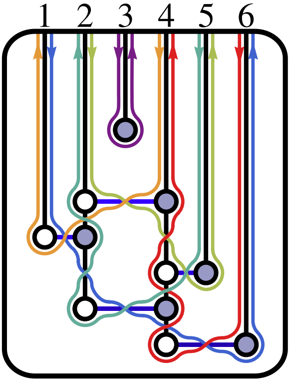

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

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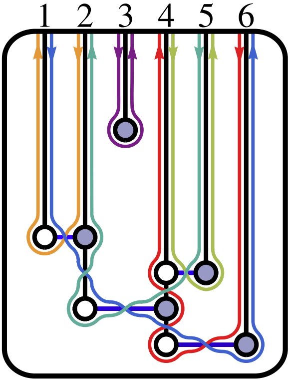

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

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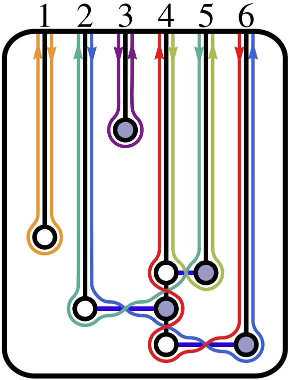
$$f_0 = \frac{d\alpha_1}{\alpha_1}\frac{d\alpha_2}{\alpha_2}\frac{d\alpha_3}{\alpha_3}\frac{d\alpha_4}{\alpha_4}\frac{d\alpha_5}{\alpha_5}\frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8} f_8$$



‘Bridge’ Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6 }	

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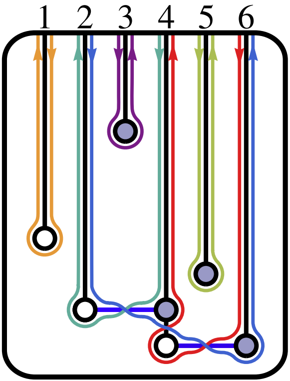

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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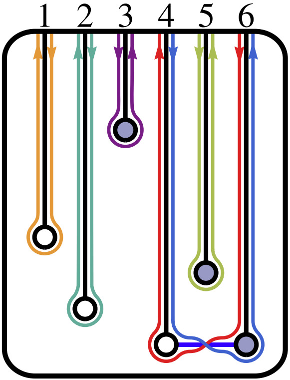

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_7	$\{7$	8	3	6	5	$10\}$	$(4\,6)$
f_8	$\{7$	8	3	10	5	$6\}$	

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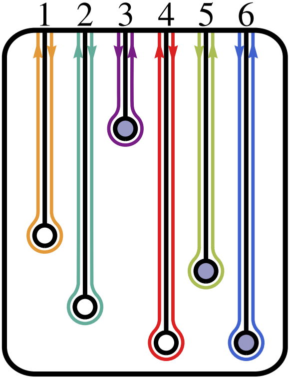

'Bridge' Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

f_8 $\{7\ \ 8\ \ 3\ \ 10\ \ 5\ \ 6\ \}$

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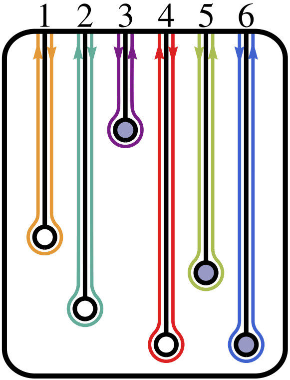

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$$f_8 = \prod_{a=\sigma(a)+n}\Big(\delta^4(\widetilde{\eta}_a)\,\delta^2(\widetilde{\lambda}_a)\Big)\prod_{b=\sigma(b)}\Big(\delta^2(\lambda_b)\Big)$$

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$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

‘Bridge’ Decomposition

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \end{array}$$

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‘Bridge’ Decomposition

$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\,\} \end{array}$$

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$$f_7 = \frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

$$(4\,6):\quad c_6 \mapsto c_6 + \alpha_8\, c_4$$

'Bridge' Decomposition

$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & (4\,6) \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\,\} & \end{array}$$

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$$f_6 = \frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

$$(2\,4):\quad c_4 \mapsto c_4 + \alpha_7\, c_2$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_6	{7	6	3	8	5	10}	$(2\,4)$
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$$f_5 = \frac{d\alpha_6}{\alpha_6}\frac{d\alpha_7}{\alpha_7}\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(4\,5):\quad c_5 \mapsto c_5 + \alpha_6\,c_4$$

'Bridge' Decomposition

$$\begin{array}{ccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \tau \\ f_5 & \{7 & 6 & 3 & 5 & 8 & 10\} & & (4\,5) \\ f_6 & \{7 & 6 & 3 & 8 & 5 & 10\} & & (2\,4) \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & & (4\,6) \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\} & & \end{array}$$

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$$f_4=\frac{d\alpha_5}{\alpha_5}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C\equiv\left(\begin{array}{cccccc}1&2&3&4&5&6\\\hline 1&\alpha_5&0&0&0&0\\0&1&0&\alpha_7&\alpha_6\alpha_7&0\\0&0&0&1&\alpha_6&\alpha_8\end{array}\right)$$

$$(1\,2):\quad c_2\mapsto c_2+\alpha_5\,c_1$$

‘Bridge’ Decomposition

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_4 & \{6 & 7 & 3 & 5 & 8 & 10\} & \\ & & & & & & & (1\,2) \\ f_5 & \{7 & 6 & 3 & 5 & 8 & 10\} & \\ & & & & & & & (4\,5) \\ f_6 & \{7 & 6 & 3 & 8 & 5 & 10\} & \\ & & & & & & & (2\,4) \\ f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & \\ & & & & & & & (4\,6) \\ f_8 & \{7 & 8 & 3 & 10 & 5 & 6\,\} & \end{array}$$

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$$f_3=\frac{d\alpha_4}{\alpha_4}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C\equiv\left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \alpha_5 & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(2\,4):\quad c_4\mapsto c_4+\alpha_4\,c_2$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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$$f_2=\frac{d\alpha_3}{\alpha_3}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\delta^{3\times2}(C\cdot\widetilde{\lambda})\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C\equiv\left(\begin{array}{cccccc} 1&2&3&4&5&6\\ \hline 1&(\alpha_3+\alpha_5)&0&\alpha_4\alpha_5&0&0\\ 0&1&0&(\alpha_4+\alpha_7)&\alpha_6\alpha_7&0\\ 0&0&0&1&\alpha_6&\alpha_8\end{array}\right)$$

$$(1\,2):\quad c_2\mapsto c_2+\alpha_3\,c_1$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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$$f_1=\frac{d\alpha_2}{\alpha_2}\;\cdots\;\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

$$C\equiv\left(\begin{array}{cccccc}1&2&3&4&5&6\\ \hline 1&(\alpha_3+\alpha_5)&\alpha_2(\alpha_3+\alpha_5)&\alpha_4\alpha_5&0&0\\ 0&1&\alpha_2&(\alpha_4+\alpha_7)&\alpha_6\alpha_7&0\\ 0&0&0&1&\alpha_6&\alpha_8\end{array}\right)$$

$$(2\,3):\quad c_3\mapsto c_3+\alpha_2\,c_2$$

‘Bridge’ Decomposition

	$1\atop\downarrow$	$2\atop\downarrow$	$3\atop\downarrow$	$4\atop\downarrow$	$5\atop\downarrow$	$6\atop\downarrow$	τ
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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$$f_0 = \frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}(C\cdot\widetilde{\eta})\,\delta^{3\times2}(C\cdot\widetilde{\lambda})\,\delta^{2\times3}(\lambda\cdot C^{\perp})$$

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(1\,2):\quad c_2 \mapsto c_2 + \alpha_1\,c_1$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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$$f_0=\frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_8}{\alpha_8}\delta^{3\times4}\big(C\cdot\widetilde{\eta}\big)\delta^{3\times2}\big(C\cdot\widetilde{\lambda}\big)\delta^{2\times3}\big(\lambda\cdot C^{\perp}\big)$$

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‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:

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'Bridge' Decomposition

$$\begin{array}{lcccccccc} & & 1 & 2 & 3 & 4 & 5 & 6 & \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau \\ f_0 & \{ & 3 & 5 & 6 & 7 & 8 & 10 & \} \\ & & & & & & & & (1\,2) \\ f_1 & \{ & 5 & 3 & 6 & 7 & 8 & 10 & \} \\ & & & & & & & & (2\,3) \\ f_2 & \{ & 5 & 6 & 3 & 7 & 8 & 10 & \} \\ & & & & & & & & (1\,2) \\ f_3 & \{ & 6 & 5 & 3 & 7 & 8 & 10 & \} \\ & & & & & & & & (2\,4) \\ f_4 & \{ & 6 & 7 & 3 & 5 & 8 & 10 & \} \\ & & & & & & & & (1\,2) \\ f_5 & \{ & 7 & 6 & 3 & 5 & 8 & 10 & \} \\ & & & & & & & & (4\,5) \\ f_6 & \{ & 7 & 6 & 3 & 8 & 5 & 10 & \} \\ & & & & & & & & (2\,4) \\ f_7 & \{ & 7 & 8 & 3 & 6 & 5 & 10 & \} \\ & & & & & & & & (4\,6) \\ f_8 & \{ & 7 & 8 & 3 & 10 & 5 & 6 & \} \end{array}$$

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$$(6\,1):\quad c_1 \mapsto c_1 + \alpha_0\, c_6$$

'Bridge' Decomposition

	1 $\downarrow$	2 $\downarrow$	3 $\downarrow$	4 $\downarrow$	5 $\downarrow$	6 $\downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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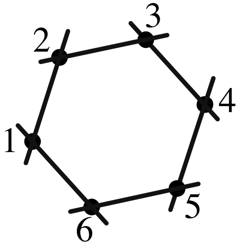

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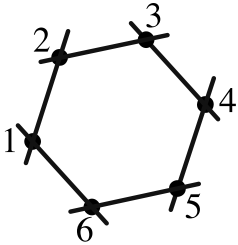

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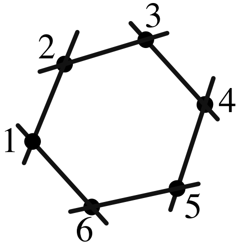

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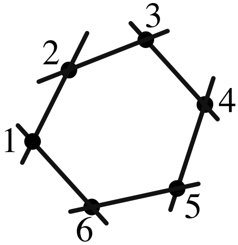

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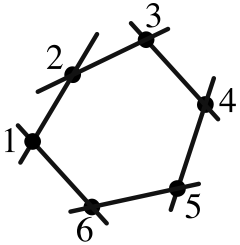

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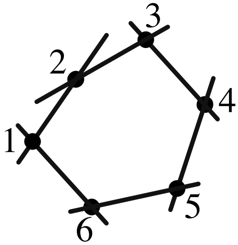

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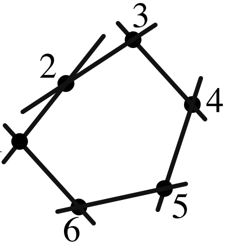

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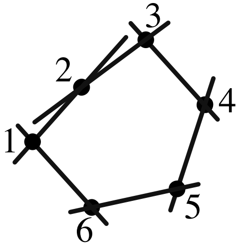

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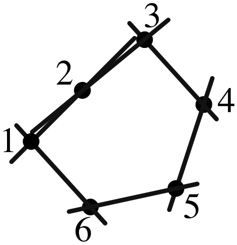

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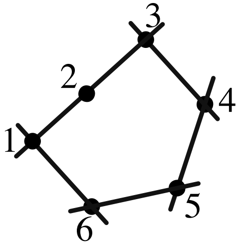

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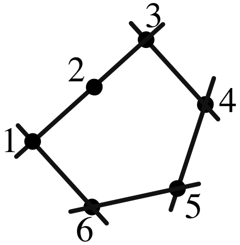

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

‘Bridge’ Decomposition

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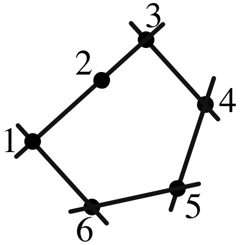

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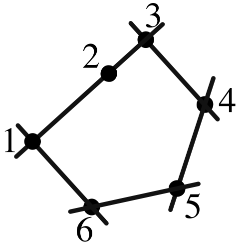

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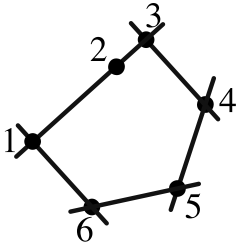

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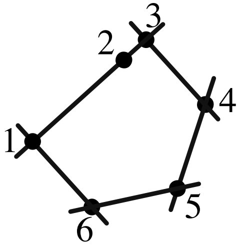

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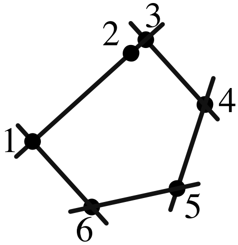

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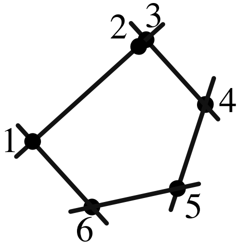

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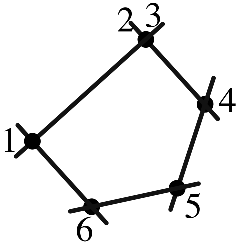

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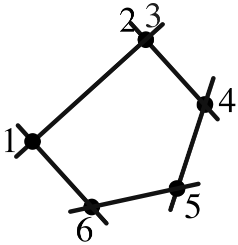

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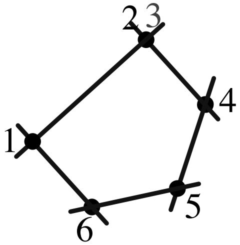

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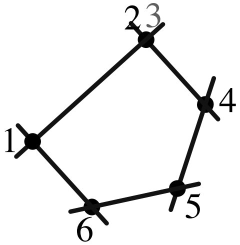

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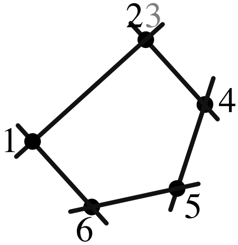

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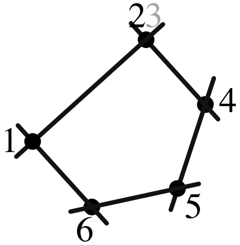

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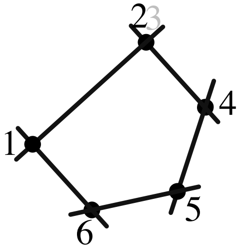

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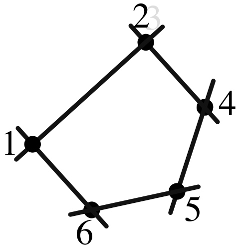

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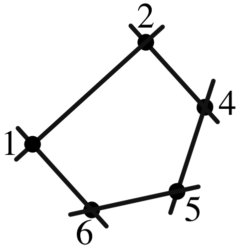

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

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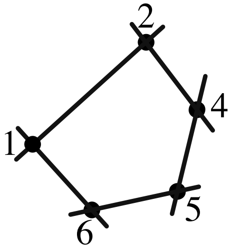

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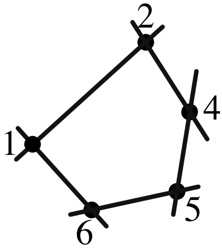

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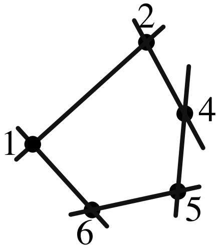

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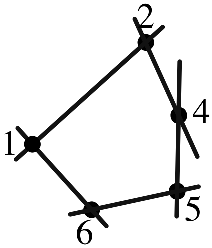

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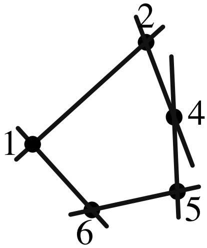

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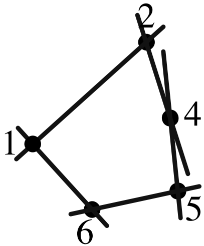

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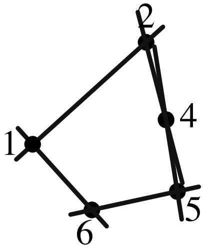

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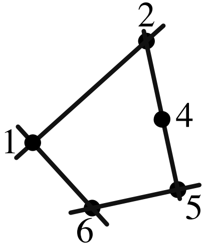

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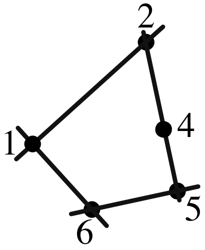

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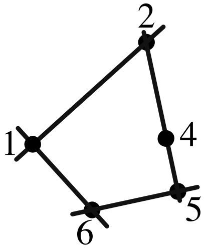

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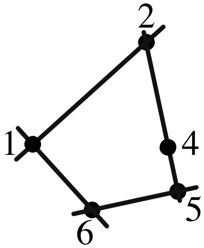

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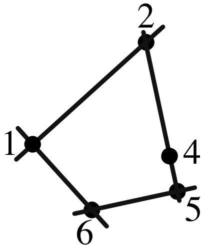

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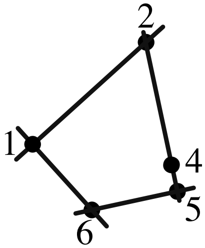

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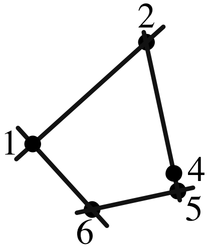

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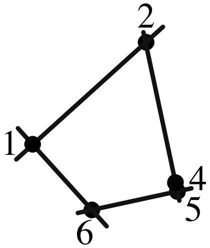

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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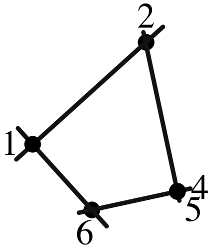

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‘Bridge’ Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
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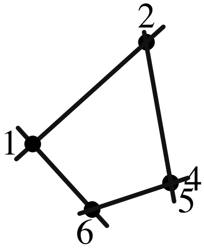

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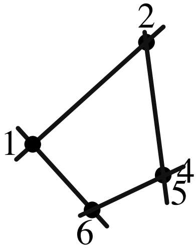

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	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
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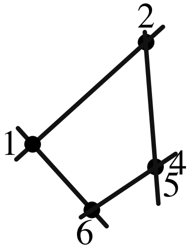

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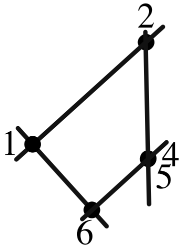

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	↓	↓	↓	↓	↓	↓	
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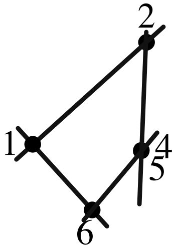

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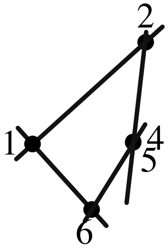

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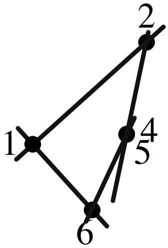

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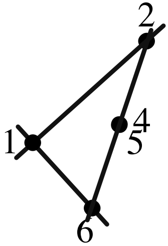

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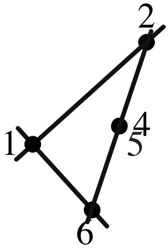

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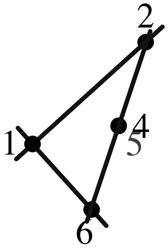

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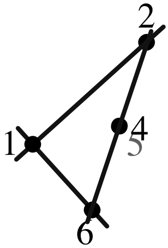

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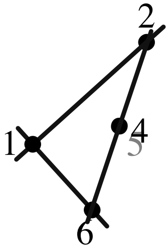

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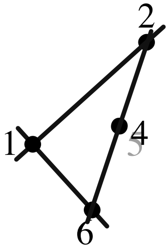

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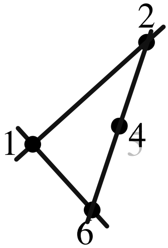

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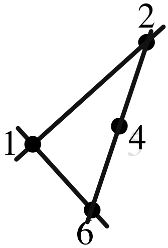

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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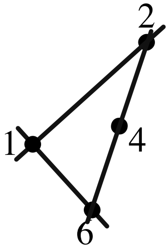

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‘Bridge’ Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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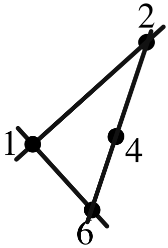

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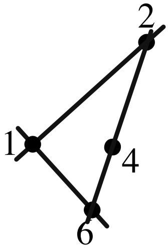

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‘Bridge’ Decomposition

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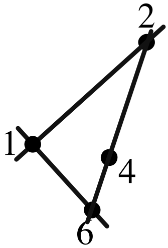

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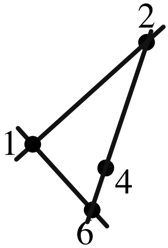

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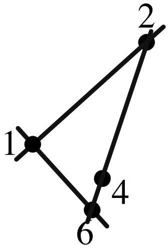

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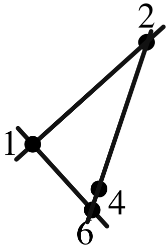

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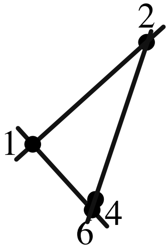

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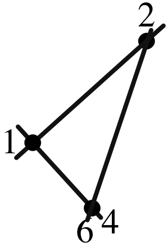

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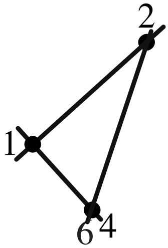

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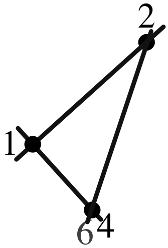

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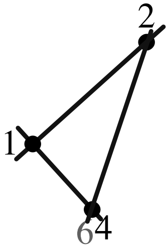

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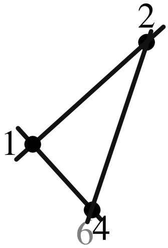

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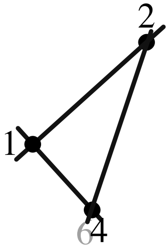

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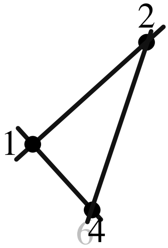

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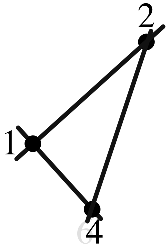

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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Canonical Coordinates for Computing On-Shell Functions

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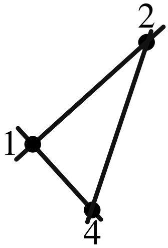

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
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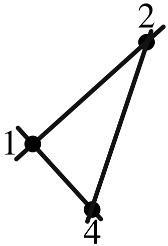

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
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f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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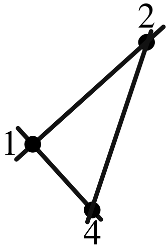

$$C\equiv\left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6\\ \hline 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right)$$

'Bridge' Decomposition

	1 $\downarrow$	2 $\downarrow$	3 $\downarrow$	4 $\downarrow$	5 $\downarrow$	6 $\downarrow$	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
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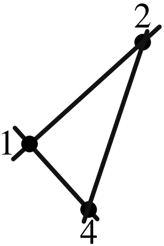

$$C\equiv\left(\begin{array}{cccccc}1&2&3&4&5&6\\\hline 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\end{array}\right)$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
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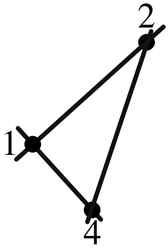

$$C\equiv\left(\begin{array}{cccccc}1&2&3&4&5&6\\\hline 1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\end{array}\right)$$

'Bridge' Decomposition

$$\begin{array}{c|cccccc|c}
 & 1 & 2 & 3 & 4 & 5 & 6 & \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \tau\\
f_0 & \{3 & 5 & 6 & 7 & 8 & 10\} & (1\,2)\\
f_1 & \{5 & 3 & 6 & 7 & 8 & 10\} & (2\,3)\\
f_2 & \{5 & 6 & 3 & 7 & 8 & 10\} & (1\,2)\\
f_3 & \{6 & 5 & 3 & 7 & 8 & 10\} & (2\,4)\\
f_4 & \{6 & 7 & 3 & 5 & 8 & 10\} & (1\,2)\\
f_5 & \{7 & 6 & 3 & 5 & 8 & 10\} & (4\,5)\\
f_6 & \{7 & 6 & 3 & 8 & 5 & 10\} & (2\,4)\\
f_7 & \{7 & 8 & 3 & 6 & 5 & 10\} & (4\,6)\\
f_8 & \{7 & 8 & 3 & 10 & 5 & 6\} &
\end{array}$$

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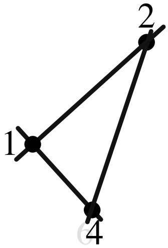

$$C\equiv\left(\begin{array}{cccccc}1&2&3&4&5&6\\ \hline 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&1&0&\alpha_8\end{array}\right)$$

$$(4\,6):\quad c_6\mapsto c_6+\alpha_8\,c_4$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
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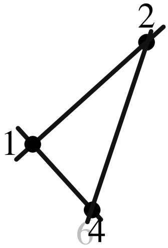

$$C\equiv\left(\begin{array}{cccccc}1 & 2 & 3 & 4 & 5 & 6\\ \hline 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & \alpha_8\end{array}\right)$$

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	1	2	3	4	5	6	τ
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f_0	{3	5	6	7	8	10}	$(1\,2)$
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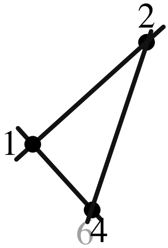

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	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
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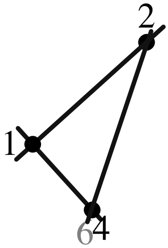

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

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'Bridge' Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	$\{3$	5	6	7	8	$10\}$	$(1\,2)$
f_1	$\{5$	3	6	7	8	$10\}$	$(2\,3)$
f_2	$\{5$	6	3	7	8	$10\}$	$(1\,2)$
f_3	$\{6$	5	3	7	8	$10\}$	$(2\,4)$
f_4	$\{6$	7	3	5	8	$10\}$	$(1\,2)$
f_5	$\{7$	6	3	5	8	$10\}$	$(4\,5)$
f_6	$\{7$	6	3	8	5	$10\}$	$(2\,4)$
f_7	$\{7$	8	3	6	5	$10\}$	$(4\,6)$
f_8	$\{7$	8	3	10	5	$6\}$	

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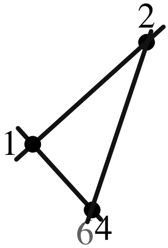

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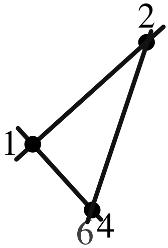

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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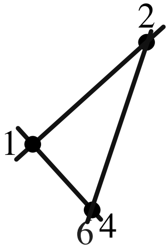

$$C \equiv \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}$$

$$(4\,6): \quad c_6 \mapsto c_6 + \alpha_8\, c_4$$

'Bridge' Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	(1 2)
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f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
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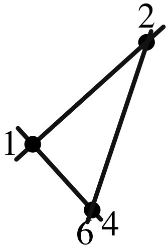

$$C\equiv\left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6\\ \hline 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

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‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
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f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
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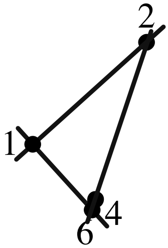

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

$$(2\,4):\quad c_4 \mapsto c_4 + \alpha_7\, c_2$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

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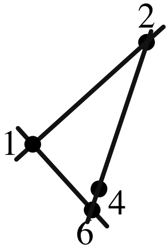

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

$(2\,4): \quad c_4 \mapsto c_4 + \alpha_7\, c_2$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
f_5	{7	6	3	5	8	10}	$(4\,5)$
f_6	{7	6	3	8	5	10}	$(2\,4)$
f_7	{7	8	3	6	5	10}	$(4\,6)$
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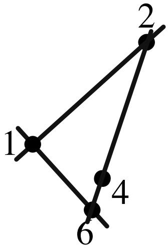

$$C\equiv\left(\begin{array}{cccccc}1&2&3&4&5&6\\\hline 1&0&0&0&0&0\\0&1&0&\alpha_7&0&0\\0&0&0&1&0&\alpha_8\end{array}\right)$$

$(2\,4):\quad c_4\mapsto c_4+\alpha_7\,c_2$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
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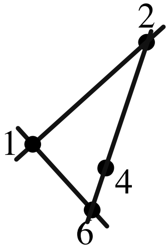

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \alpha_8 \end{array}\right)$$

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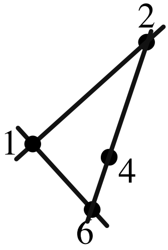

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‘Bridge’ Decomposition

	1	2	3	4	5	6	
	↓	↓	↓	↓	↓	↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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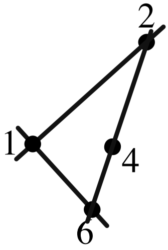

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	1	2	3	4	5	6	τ
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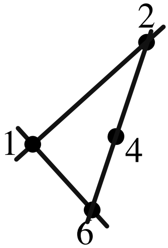

$$C \equiv \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline \end{array}$$

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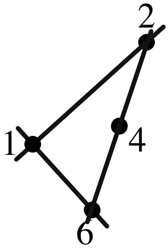

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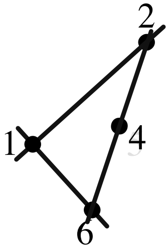

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	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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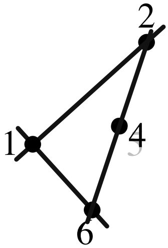

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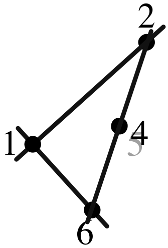

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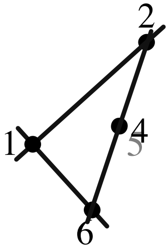

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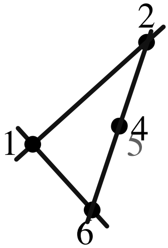

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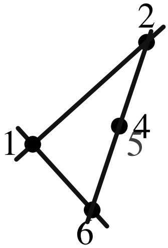

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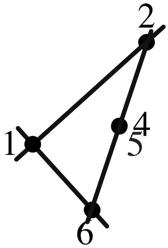

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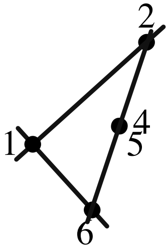

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	1	2	3	4	5	6	τ
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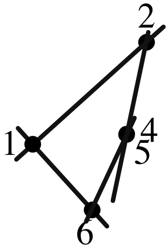

$$C\equiv\begin{array}{c}\begin{array}{cccccc}1&2&3&4&5&6\end{array}\\\left(\begin{array}{cccccc}1&\alpha_5&0&0&0&0\\0&1&0&\alpha_7&\alpha_6\alpha_7&0\\0&0&0&1&\alpha_6&\alpha_8\end{array}\right)\end{array}$$

$$(1\,2):\quad c_2\mapsto c_2+\alpha_5\,c_1$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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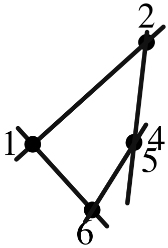
f_8	{7	8	3	10	5	6 }	

Canonical Coordinates for Computing On-Shell Functions

There are many ways to decompose a permutation into transpositions—*e.g.*, always choose the first transposition $\tau \equiv (a\,b)$ such that $\sigma(a) < \sigma(b)$:



$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \alpha_5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \alpha_7 & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(1\,2): \quad c_2 \mapsto c_2 + \alpha_5\, c_1$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
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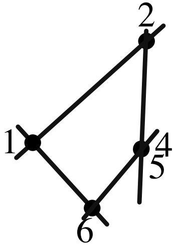

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‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
f_2	{5	6	3	7	8	10}	$(1\,2)$
f_3	{6	5	3	7	8	10}	$(2\,4)$
f_4	{6	7	3	5	8	10}	$(1\,2)$
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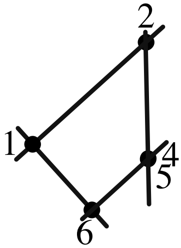

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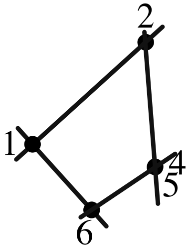

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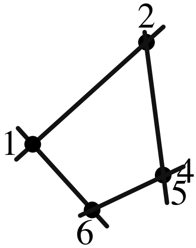

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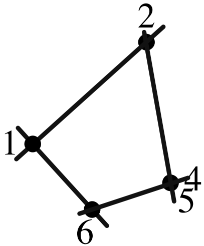

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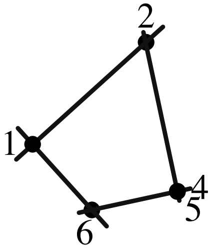

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'Bridge' Decomposition

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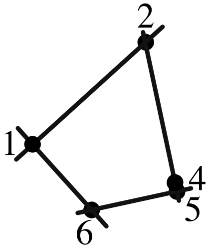

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & \alpha_5 & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(2\,4): \quad c_4 \mapsto c_4 + \alpha_4\, c_2$$

'Bridge' Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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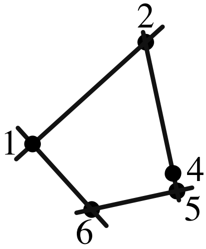

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‘Bridge’ Decomposition

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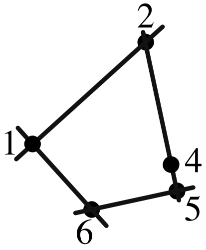

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'Bridge' Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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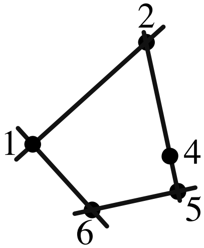

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‘Bridge’ Decomposition

	$1 \downarrow$	$2 \downarrow$	$3 \downarrow$	$4 \downarrow$	$5 \downarrow$	$6 \downarrow$	τ
f_0	$\{3$	5	6	7	8	$10\}$	$(1\,2)$
f_1	$\{5$	3	6	7	8	$10\}$	$(2\,3)$
f_2	$\{5$	6	3	7	8	$10\}$	$(1\,2)$
f_3	$\{6$	5	3	7	8	$10\}$	$(2\,4)$
f_4	$\{6$	7	3	5	8	$10\}$	$(1\,2)$
f_5	$\{7$	6	3	5	8	$10\}$	$(4\,5)$
f_6	$\{7$	6	3	8	5	$10\}$	$(2\,4)$
f_7	$\{7$	8	3	6	5	$10\}$	$(4\,6)$
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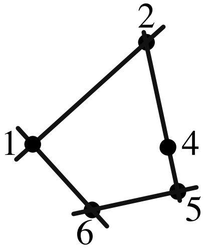

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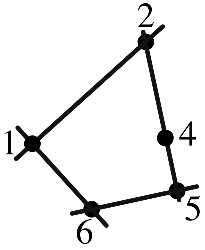

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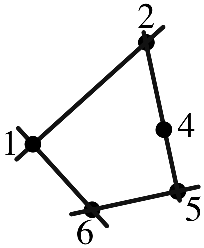

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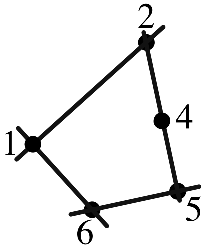

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‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
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f_1	{5	3	6	7	8	10}	(2 3)
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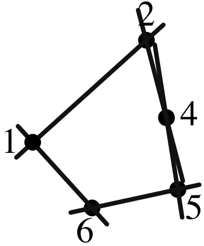

$$C \equiv \begin{array}{c} \\ \\ \end{array}\left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_3+\alpha_5) & 0 & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & 0 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(1\,2): \quad c_2 \mapsto c_2 + \alpha_3\, c_1$$

'Bridge' Decomposition

	$1 \downarrow$	$2 \downarrow$	$3 \downarrow$	$4 \downarrow$	$5 \downarrow$	$6 \downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
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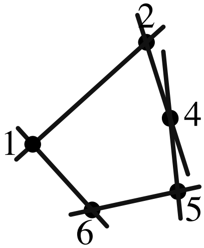

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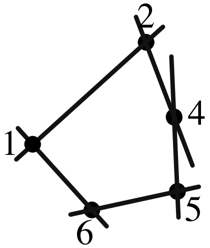

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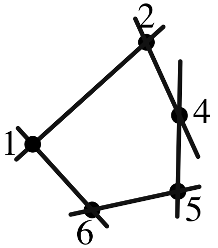

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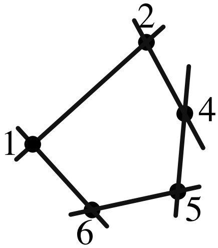

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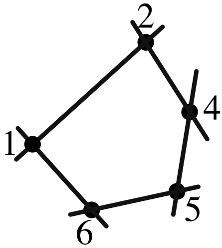

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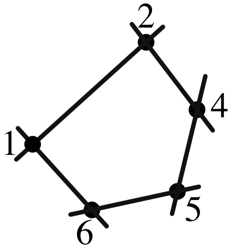

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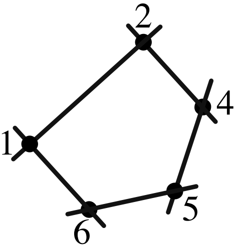

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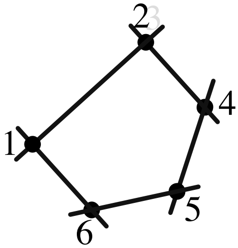

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‘Bridge’ Decomposition

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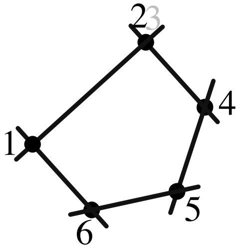

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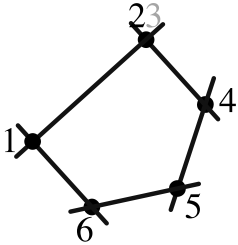

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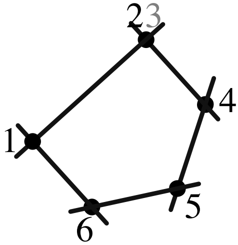

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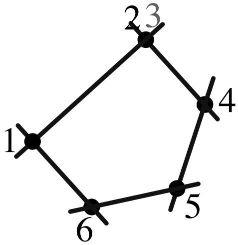

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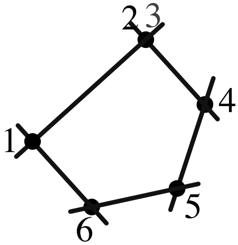

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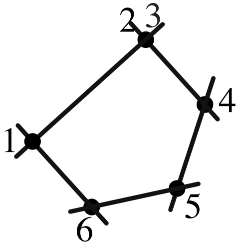

$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ 0 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

$$(2\,3): \quad c_3 \mapsto c_3 + \alpha_2\, c_2$$

‘Bridge’ Decomposition

	1	2	3	4	5	6	τ
	↓	↓	↓	↓	↓	↓	
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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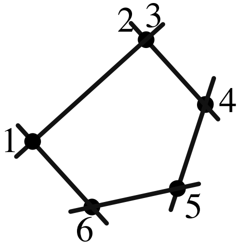

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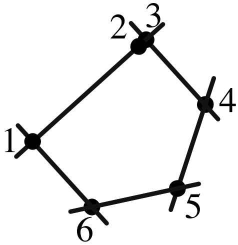

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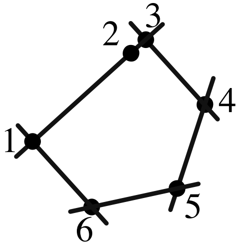

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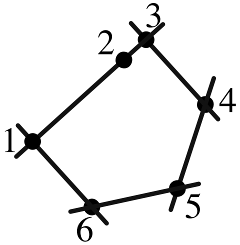

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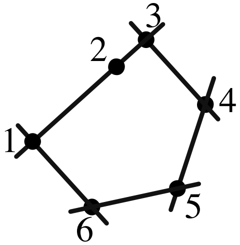

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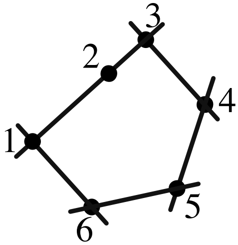

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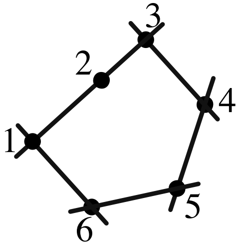

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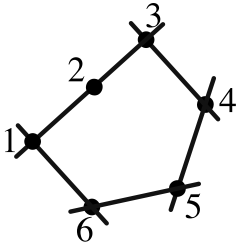

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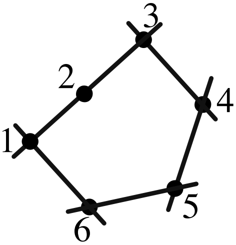

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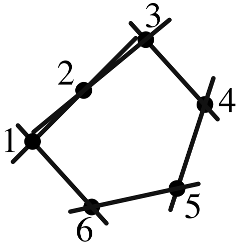

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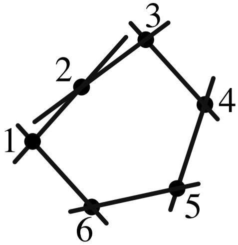

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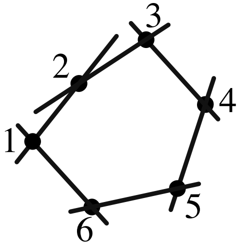

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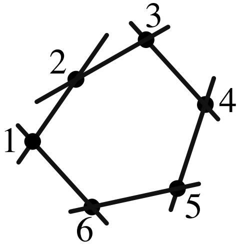

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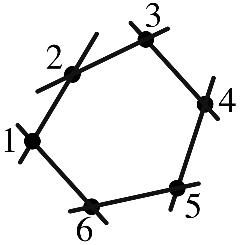

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$$(6\,1):\quad c_1 \mapsto c_1 + \alpha_0\, c_6$$

‘Bridge’ Decomposition

	$1\downarrow$	$2\downarrow$	$3\downarrow$	$4\downarrow$	$5\downarrow$	$6\downarrow$	τ
f_0	{3	5	6	7	8	10}	$(1\,2)$
f_1	{5	3	6	7	8	10}	$(2\,3)$
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Canonical Coordinates for Computing On-Shell Functions

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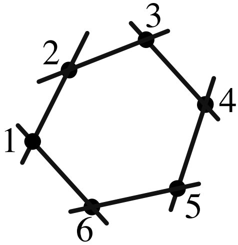

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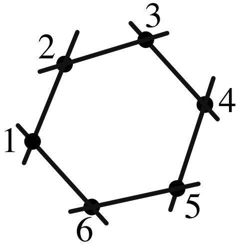

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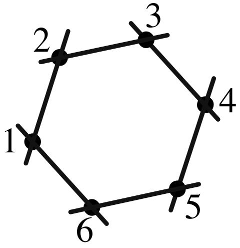

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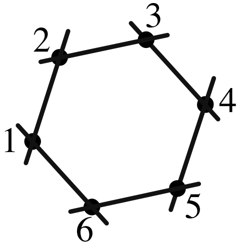

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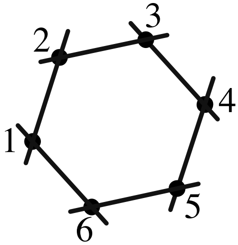

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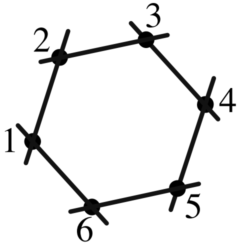

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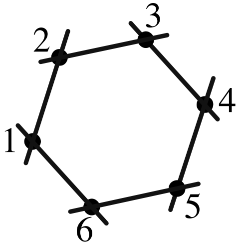
Canonical Coordinates for Computing On-Shell Functions



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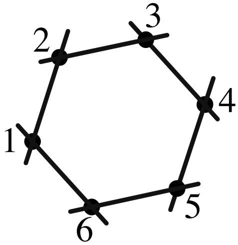
Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{6,3} \equiv \frac{d\alpha_0}{\alpha_0}\cdots\frac{d\alpha_8}{\alpha_8}$$



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‘Bridge’ Decomposition

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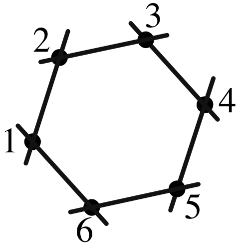

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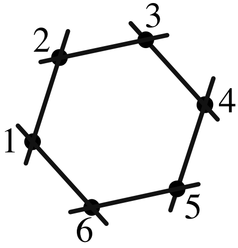
Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{6,3} \equiv \frac{d\alpha_0}{\alpha_0} \cdots \frac{d\alpha_8}{\alpha_8} \quad = \frac{d^{3\times 6}C}{\mathrm{vol}(GL(3))} \frac{1}{(123)(234)(345)(456)(561)(612)}$$



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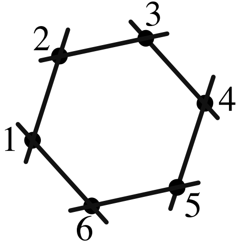

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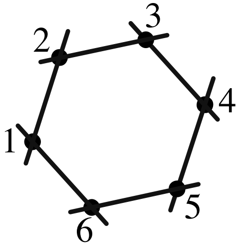

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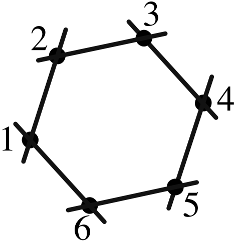

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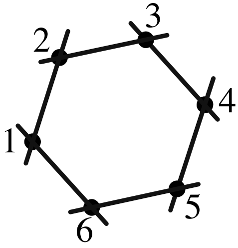
Canonical Coordinates for Computing On-Shell Functions

$$\mathcal{L}_{n,k} \equiv \frac{d\alpha_1}{\alpha_1}\cdots\frac{d\alpha_{k(n-k)}}{\alpha_{k(n-k)}} = \frac{d^{k\times n}C}{\mathrm{vol}(GL(k))}\,\frac{1}{(1\cdots k)\,(2\cdots k{+}1)\cdots(n\cdots k{-}1)}$$



$$C \equiv \left(\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & (\alpha_1+\alpha_3+\alpha_5) & \alpha_2(\alpha_3+\alpha_5) & \alpha_4\alpha_5 & 0 & 0 \\ 0 & 1 & \alpha_2 & (\alpha_4+\alpha_7) & \alpha_6\alpha_7 & 0 \\ \alpha_0\alpha_8 & 0 & 0 & 1 & \alpha_6 & \alpha_8 \end{array}\right)$$

‘Bridge’ Decomposition

	1 ↓	2 ↓	3 ↓	4 ↓	5 ↓	6 ↓	τ
f_0	{3	5	6	7	8	10}	(1 2)
f_1	{5	3	6	7	8	10}	(2 3)
f_2	{5	6	3	7	8	10}	(1 2)
f_3	{6	5	3	7	8	10}	(2 4)
f_4	{6	7	3	5	8	10}	(1 2)
f_5	{7	6	3	5	8	10}	(4 5)
f_6	{7	6	3	8	5	10}	(2 4)
f_7	{7	8	3	6	5	10}	(4 6)
f_8	{7	8	3	10	5	6}	

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