Third-generation ground-based GW Observatory Network, and the future with LISA (1)

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Gravitation, do we still need to study it?

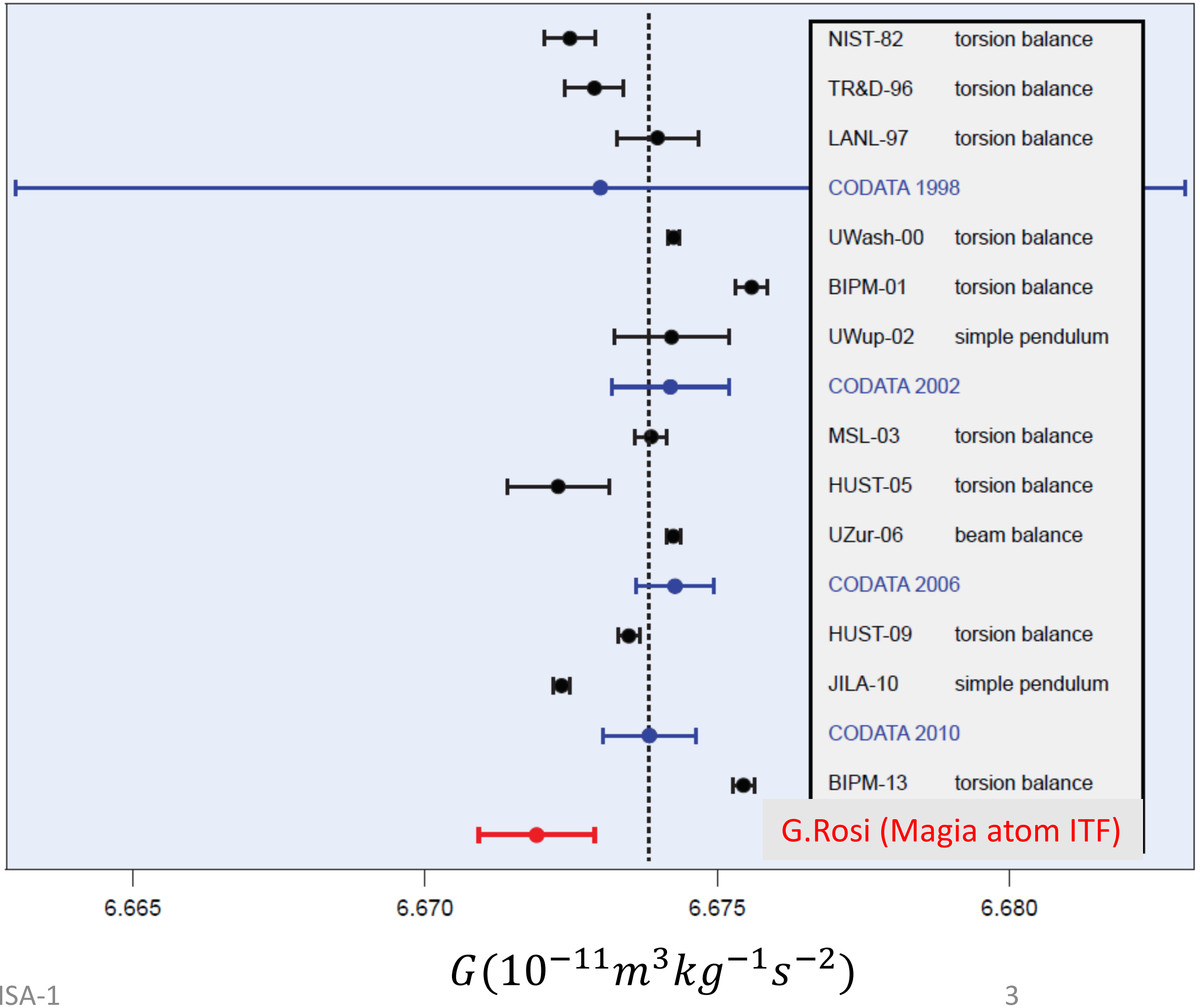
- All the lessons on this schools you had until today are substantially based on the “other 3 fundamental interactions”:
  - Weak, Strong and electromagnetic
- Gravity is the “oldest” interaction described in our books
- Hypothesis, models and theories about the “force” is bonding us to the Earth have been formulated since thousands of years
- Here we focus on the “last two steps”
  - Newton & Einstein

Do we really know gravity?

$$F = -G\frac{m_1 \cdot m_2}{r^2}$$

- Despite the fact that the first measurement of G has been made by Cavendish in 1798, the G value is poorly known
- The comparison with the other "fundamental" constants in physics is impressive

G. Rosi et al., Nature 510, 518-521 (2014)



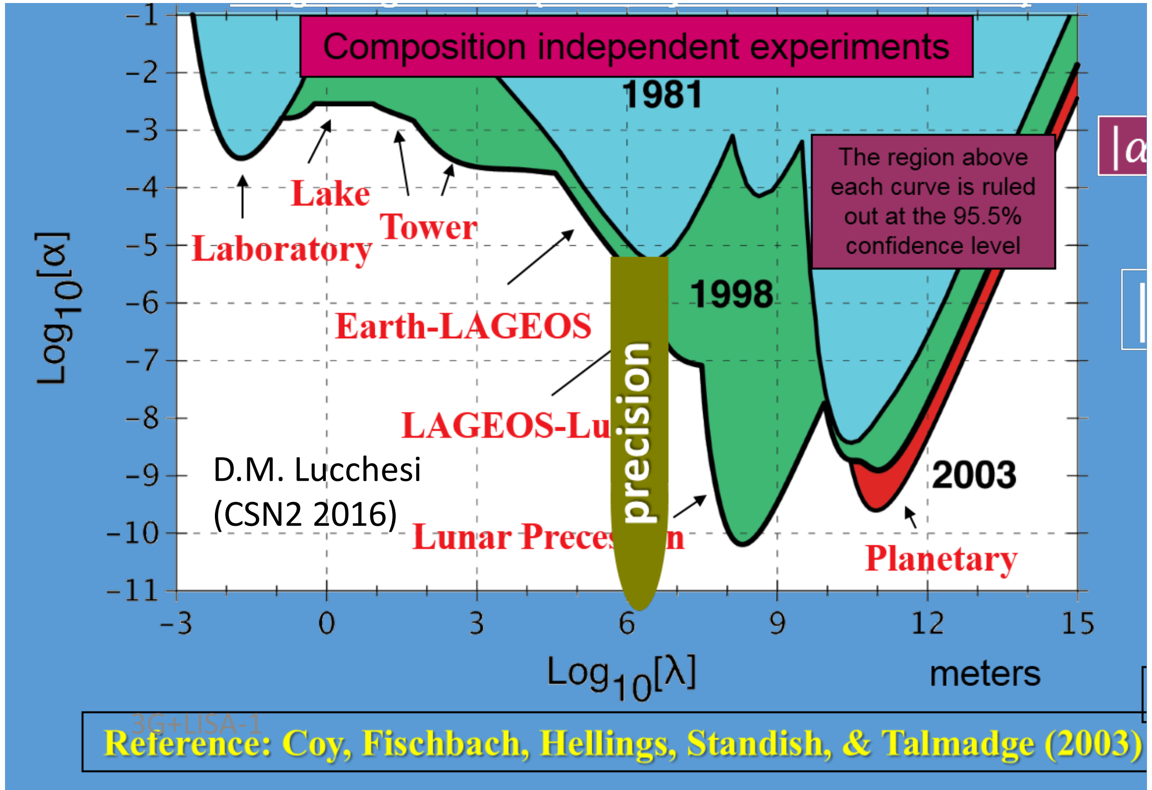
CODATA 2014

TABLE I An abbreviated list of the CODATA recommended values of the fundamental constants of physics and chemistry based on the 2014 adjustment.

Quantity	Symbol	Numerical value	Unit	Relative std. uncert. u_r
speed of light in vacuum	c, c_0	299 792 458	$\mathrm{m\ s^{-1}}$	exact
magnetic constant	μ_0	$4\pi \times 10^{-7}$	$\mathrm{N\ A^{-2}}$	
		$= 12.566\,370\,614... \times 10^{-7}$	$\mathrm{N\ A^{-2}}$	exact
electric constant $1/\mu_0 c^2$	ϵ_0	$8.854\,187\,817... \times 10^{-12}$	$\mathrm{F\ m^{-1}}$	exact
Newtonian constant of gravitation	G	$6.674\,08(31) \times 10^{-11}$	$\mathrm{m^3\ kg^{-1}\ s^{-2}}$	4.7×10^{-5}
Planck constant	h	$6.626\,070\,040(81) \times 10^{-34}$	J s	1.2×10^{-8}
$h/2\pi$	$\hbar$	$1.054\,571\,800(13) \times 10^{-34}$	J s	1.2×10^{-8}
elementary charge	e	$1.602\,176\,6208(98) \times 10^{-19}$	C	6.1×10^{-9}
magnetic flux quantum $h/2e$	Φ_0	$2.067\,833\,831(13) \times 10^{-15}$	Wb	6.1×10^{-9}
conductance quantum $2e^2/h$	G_0	$7.748\,091\,7310(18) \times 10^{-5}$	S	2.3×10^{-10}
electron mass	m_e	$9.109\,383\,56(11) \times 10^{-31}$	kg	1.2×10^{-8}
proton mass	m_p	$1.672\,621\,898(21) \times 10^{-27}$	kg	1.2×10^{-8}
proton-electron mass ratio	m_p/m_e	$1836.152\,673\,89(17)$		9.5×10^{-11}
fine-structure constant $e^2/4\pi\epsilon_0\hbar c$	α	$7.297\,352\,5664(17) \times 10^{-3}$		2.3×10^{-10}
inverse fine-structure constant	α^{-1}	$137.035\,999\,139(31)$		2.3×10^{-10}
Rydberg constant $\alpha^2 m_e c/2h$	R_∞	$10\,973\,731.568\,508(65)$	$\mathrm{m^{-1}}$	5.9×10^{-12}
Avogadro constant	N_A, L	$6.022\,140\,857(74) \times 10^{23}$	$\mathrm{mol^{-1}}$	1.2×10^{-8}
Faraday constant $N_A e$	F	$96\,485.332\,89(59)$	$\mathrm{C\ mol^{-1}}$	6.2×10^{-9}
molar gas constant	R	$8.314\,4598(48)$	$\mathrm{J\ mol^{-1}\ K^{-1}}$	5.7×10^{-7}
Boltzmann constant R/N_A	k	$1.380\,648\,52(79) \times 10^{-23}$	$\mathrm{J\ K^{-1}}$	5.7×10^{-7}
Stefan-Boltzmann constant $(\pi^2/60)k^4/\hbar^3c^2$	σ	$5.670\,367(13) \times 10^{-8}$	$\mathrm{W\ m^{-2}\ K^{-4}}$	2.3×10^{-6}
		Non-SI units accepted for use with the SI		
electron volt (e/C) J	eV	$1.602\,176\,6208(98) \times 10^{-19}$	J	6.1×10^{-9}
(unified) atomic mass unit $\frac{1}{12}m(^{12}\mathrm{C})$	u	$1.660\,539\,040(20) \times 10^{-27}$	kg	1.2×10^{-8}

Is really $F \propto r^{-2}$?

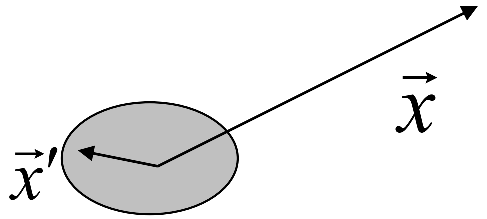
- Let use the Gravitational potential $\longrightarrow$ $$\phi(r) = -G\frac{M}{r}$$
- Let suppose to have a modification according to a Yukawa-like interaction $$\phi(r) = -G\frac{M}{r}\left(1+\alpha e^{-r/\lambda}\right)$$
- λ is the Compton wavelength of the interaction boson ("graviton"): $$\lambda = \frac{\hbar}{m_g c}$$



Gravitational Potential of a mass distribution

- Let consider a continuous distribution of mass having density $\rho(x')$. To evaluate the value of the potential $\phi(\vec{x})$ in a point $\vec{x}$ external to the mass distribution:

$$\phi(\vec{x}) = -\int_V \frac{G \cdot \rho(x')}{|\vec{x} - \vec{x}'|} d^3 x'$$



- Being $\vec{x}$ external to the mass distribution, we can Taylor-expand ${}^{1}/_{|\vec{x}-\vec{x'}|}$ in multi-poles around $\vec{x'} = 0$:

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} = \frac{1}{r} + \sum_k \frac{x^k x'^k}{r^3} + \frac{1}{2} \sum_{k,l} \left(3x'^k x'^l - r'^2 \delta_k^l\right) \frac{x^k x^l}{r^5} + \ldots$$

  - Where $x^{1,2,3}=x,y,z$ and $r = \sqrt{x^2 + y^2 + z^2}$
  - The gravitational potential becomes:

$$\phi(\vec{x}) = -\frac{GM}{r} - \frac{G}{r^3} \sum_k x^k D^k - \frac{G}{2} \sum_{k,l} Q^{kl} \frac{x^k x^l}{r^5} + \ldots$$

Quadrupolar terms of the gravitational potential

- where: $M = \int_V \rho(x') d^3x'$ $D^k = \int_V x'^k \rho(x') d^3x'$ $Q^{kl} = \int_V \left(3x'^k x'^l - r'^2 \delta_l^k\right) \rho(\vec{x}') d^3x'$

- Note: it is possible to find a reference system where the center of mass terms (dipole) D^k vanishes
- If the quadrupolar terms of the mass distribution are $Q^{kl} \neq 0$, a term $\propto r^{-3}$ in $\phi(\vec{x})$ (r^{-4} in force) remains.
- Earth has a difference between the polar and equatorial diameters of 3×10^{-3} and this impacts on the orbits of the satellites (precession of the orbits)

Distance in 3D

- In Newtonian physics, space and time are independent and the distance is defined by Pythagora's formula:

$$\begin{cases} dl^2 = dx^2 + dy^2 + dz^2 & \text{(space interval)} \\ dt^2 & \text{(time interval)} \end{cases}$$

$$dl^2 = dx^2 + dy^2 + dz^2 = \begin{pmatrix} dx & dy & dz \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

Distance in 4D flat space

- In Special Relativity we have a 4 dimensions space-time:

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2$$ (space-time interval)

$$ds^2 = \begin{pmatrix} cdt & dx & dy & dz \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix}$$

$$x^0 \equiv ct, \quad x^1 \equiv x, \quad x^2 \equiv y, \quad x^3 \equiv z$$

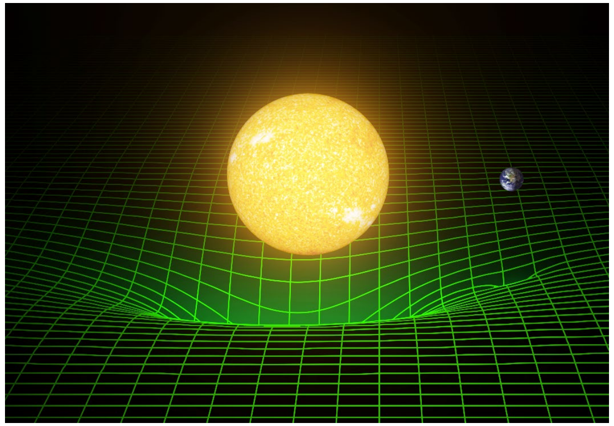
$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Metric tensor or *Minkowski tensor*. It defines the metric of a flat space-time

$$ds^2 = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} \eta_{\mu\nu} dx^\mu dx^\nu \equiv \eta_{\mu\nu} dx^\mu dx^\nu$$

- $ds^2 < 0$ timelike interval: 2 events could lie on the **worldline** (trajectory) of a material particle. In SR (and in Newtonian physics) the worldline of a particle which is not being acted by any external force is a straight line
- $ds^2 > 0$ spacelike interval, 2 events cannot lie on the worldline of a material particle
- $ds^2 = 0$ lightlike; 2 eventsts could be in the worldline of a photon

Curved space-time



- Let generalise the metric of the space-time:

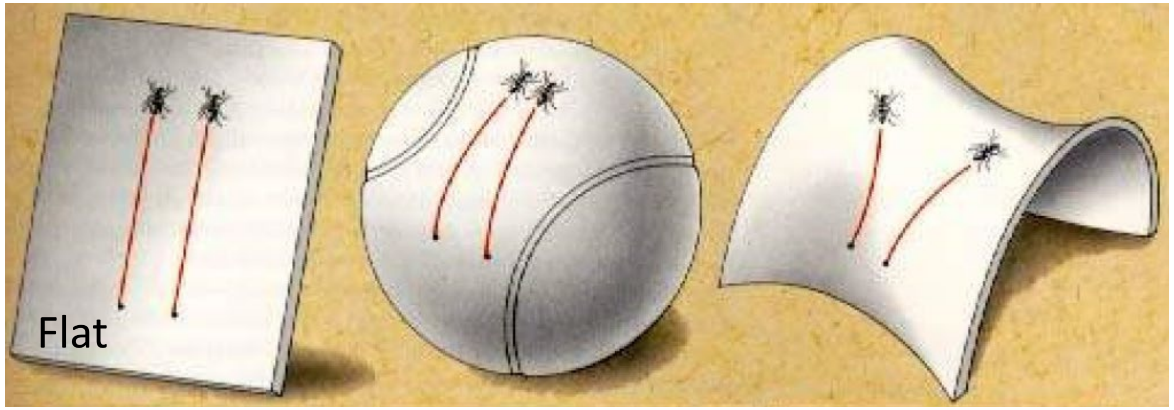
$$ds^2 = \begin{pmatrix} cdt & dx & dy & dz \end{pmatrix} \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix} \begin{pmatrix} cdt \\ dx \\ dy \\ dz \end{pmatrix} = g_{\mu\nu} dx^\mu dx^\nu$$

- In General Relativity (GR) gravity manifests itself as spacetime curvature
  - Trajectories (worldlines) of particle which are not being acted upon any non-gravitational force are generalised to curved path named **geodesics**.
- The correct mathematical definition of a geodesic goes beyond the complexity level of this course (spacetime curves that parallel transport their own tangent vectors), but we can easily define a geodesic as the extremal path:
  - Along a geodesic between two events E_1 and E_2 the elapsed proper time is an extremum:

$$\delta \int_{E_1}^{E_2} d\tau = 0$$

Curvature of the spacetime

- The curvature of the spacetime is revealed through the deviation of neighbouring geodesics



Sphere: positive intrinsic curvature. The geodesics converge

Saddle: negative intrinsic curvature. The geodesics diverge

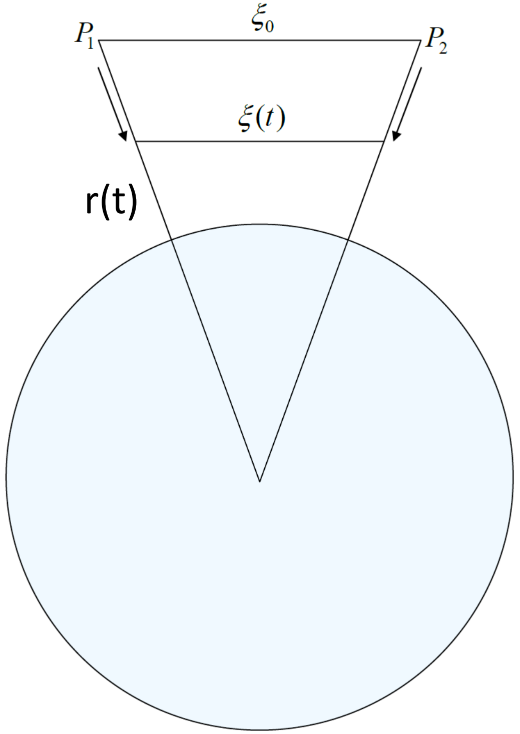
- The **acceleration** of the deviation between neighbouring geodesics is the signature of spacetime curvature due to the presence of a non-uniform (tidal) gravitational fields

Geodesic deviation in Newtonian gravity

- Let jump back for a while to Newtonian Gravity

$\xi(t)$ is the separation between the two free falling particles

Particles in P_1 and in P_2



$$\frac{\xi(t)}{r(t)} = \frac{\xi_0}{r_0} = k$$ Similar triangle – k is a constant

$$\ddot{\xi}(t) = k\,\ddot{r}(t) = -\frac{\xi(t)}{r(t)}\frac{GM}{r^2(t)}$$ ←Using Newton law

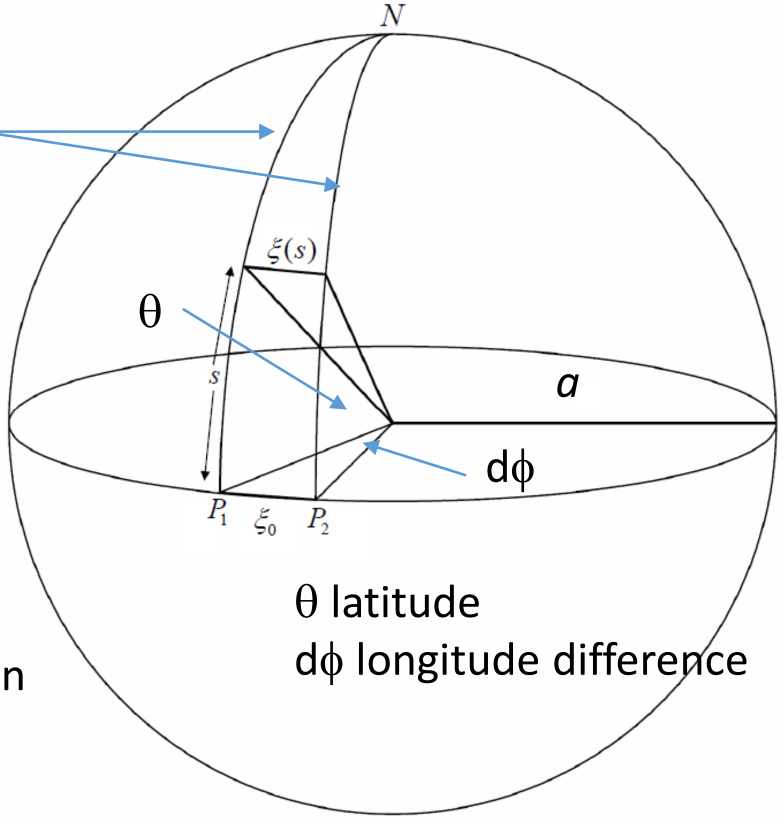
Considering $r(t) \approx R_{Earth} = R$ and making an arbitrary change of variable $t \to ct$:

$$\frac{d^2\xi}{d(ct)^2} = -\frac{GM}{R^3c^2}\xi$$

To understand the meaning of the above equation let consider a 3 dimensional space time (a sphere of radius *a*)

Intrinsic curvature

- Consider the geodesics in a spacetime represented by a sphere of radius a
- A bit of trigonometry:



$$\xi(s) = a\,d\phi\cos\theta = \xi_0\cos\theta = \xi_0\cos\frac{s}{a}$$

$$\frac{d^2\xi}{ds^2} = -\frac{1}{a^2}\xi$$ Comparing with the last formula in the previous page

$$a = \left\{\frac{GM}{R^3c^2}\right\}^{-\frac{1}{2}} \qquad [a] = \left[\frac{GM}{R^3c^2}\right]^{-\frac{1}{2}} = [l]$$

Hence, *a* represents the radius of curvature of the spacetime.

Let compute the radius of curvature of the spacetime deformed by the Earth gravity field:

$R \approx 6.3\times10^6$m

$a \approx 2\times10^{11}$m The spacetime is rather flat around the Earth (weak gravitational field)

Einstein Equation of field

- The correct derivation of the Einstein equation goes beyond this course, but we can try to approach starting from the (differential) Gauss law for the Newtonian gravitational field

$$\vec{\nabla}\cdot\vec{g} = -4\pi G\rho$$

- We stated that in GR the presence of an non-uniform gravitational field is revealed by the acceleration of the divergence of the geodesics:
  - In the left side of a relativistic equation must appears an operator O of the metrics $g_{\mu\nu}$., related to the curvature of the spacetime
  - In the right side must be appear something related to the mass, generating the field. It cannot be the mass itself, but it must have a invariant form: $T_{\mu\nu}$, the energy-momentum tensor of the matter

$$O\left(g_{\mu\nu}\right) = kT_{\mu\nu} \quad \Rightarrow \quad G_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

(A.Einstein 1915)

- $G_{\mu\nu}$ is the curvature tensor, $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu}$
- $R_{\mu\nu}$ is the Ricci tensor, $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$
- R is the ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$
- $\Gamma^{\rho}_{\mu\nu}$ is the Christoffel symbol,

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

- $R^{\mu}_{\nu\rho\sigma}$ is the Rieman tensor, $R^{\mu}_{\nu\rho\sigma} = \partial_\rho\Gamma^{\mu}_{\nu\sigma} - \partial_\sigma\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho}$

$$T^{00} = [\text{energy density}] = \frac{nm}{\sqrt{1-v^2}}$$

$$T^{0k} = T^{k0} = [\text{momentum density } k] = [\text{energy flux density}] = \frac{nmv^k}{\sqrt{1-v^2}}$$

$$T^{kl} = T^{lk} = [\text{momentum } k \text{ density flux in the direction } l] = \frac{nmv^k v^l}{\sqrt{1-v^2}}$$

Einstein Equation of Field

Effect of the deformation $\rightarrow G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \leftarrow$ Cause of the deformation

$$\frac{8\pi G}{c^4} \approx 2 \times 10^{-43} N^{-1}$$

- Comparing (naïf interpretation) the Equation of field to an elastic equation, we can see the space-time as a very stiff elastic medium ($k_{el} \sim c^4/_{8\pi G}$):
  - Very energetic phenomena, determine small curvature of the space-time
- To solve the Einstein equation is an hard task
  - They are a set of highly non linear equations:
    - Knowing or imposing the metric $g_{\mu\nu}$ it is possible to compute $\Gamma_{\mu\nu}$, $R_{\mu\nu}$, $G_{\mu\nu}$ and then, obtaining $T_{\mu\nu}$, determine the spatial and temporal dependence of physical parameters like density or pressure of the system
    - But the way back, from $T_{\mu\nu}$ to the metric of the space-time is usually intractable
    - Luckily, far from heavy masses the equation of field can be simplified (linearized) in the **weak field approximation** (A.Einstein 1916).

Weak gravitational fields

- The spacetime is flat in absence of gravitational field:
  - In presence of a weak field we can define a **nearly flat** space time, a spacetime where **we can find a coordinate system** (called **Nearly Lorentz**) in which the metric has components

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

where: $\eta_{\mu\nu} = diag(-1,1,1,1)$ and $|h_{\mu\nu}| << 1$

- The "secret" to solving tensor equations in GR is, often, to choose the right coordinate system where the equations appear relatively simple
- Not all the systems in a weak field approximation have the above simplified expression and the first target is to find the right coordinate system:
  - When one system is found an infinite class is found thanks to the coordinate transformations:
    - Background Lorentz transformations
    - Gauge transformations

Once we have identified a "nearly Lorentz" coordinate system, we can add an arbitrarily small vector ξ^{α} to the coordinates x^{α} without altering the validity of our assumption that the space is nearly flat

Linearized field equations: GW

- The computation of the linearized field equation and the derivation of the gravitational waves (GW) is beyond the level and the time of this course, but you can use
  - [1] Michele Maggiore, Gravitational Waves. Volume 1, Theory and Experiments, Oxford University Press
  - [2] Bernard Schutz, A first Course in General Relativity. Cambridge
  - [3] Hans C. Ohanian, Remo Ruffini, Gravitazione e Spazio-Tempo, Zanichelli
- Here we highlight just the results and some important intermediate step
  - In case of weak field the Rieman tensor ("the curvature") becomes (at the first order in $h_{\mu\nu}$):

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_\nu \partial_\rho h_{\mu\sigma} + \partial_\mu \partial_\sigma h_{\nu\rho} - \partial_\mu \partial_\rho h_{\nu\sigma} - \partial_\nu \partial_\sigma h_{\mu\rho}\right)$$

Gravitational Waves

- We select the so-called Lorenz or Hilbert gauge: $\partial^{\nu}\bar{h}_{\mu\nu} = 0$
- Where we defined $trace(h_{\mu\nu}) \equiv h \equiv \eta^{\mu\nu}h_{\mu\nu}$ and $\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$
- The Einstein equation of field becomes:

$$\Box\bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}$$

- Where □ is the flat space d'Alambertian $\Box = \eta_{\mu\nu}\partial^{\mu}\partial^{\nu} = \partial_{\mu}\partial^{\mu} = \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)$

- If we are far from matter $T_{\mu\nu} = 0$: $\Box\bar{h}_{\mu\nu} = 0$ Wave eq. $\boxed{\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)\bar{h}_{\mu\nu} = 0}$

Gravitational waves

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}+\nabla^2\right)\bar{h}_{\mu\nu}=0$$

- This is the equation of a wave propagating at speed c: the metric perturbation propagates at the speed of light. This is an effect of the Einstein theory of the GR. Other gravity theories obtain a different speed propagation of the perturbation (massive graviton).
- Note that $g_{\mu\nu}$and then $h_{\mu\nu}$is a symmetric tensor, that corresponds to 6 conditions $h_{\mu\nu}$=$h_{\nu\mu}$per $\mu \neq \nu$. Hence only 10 of the 16 components of $h_{\mu\nu}$ are independent
- But the Lorenz gauge $\partial^\nu \bar{h}_{\mu\nu} = 0$ imposes 4 conditions: the independent components of $\bar{h}_{\mu\nu}$ are now 6
- The Lorenz gauge doesn’t fix completely the gauge and the condition $\partial^\nu \bar{h}_{\mu\nu} = 0$ is not spoiled by a further coordinate transformation $x^\mu \rightarrow x^\mu + \zeta^\mu$ with $\Box\varsigma_\mu = 0$
- We are building the Transverse Traceless (TT) gauge

$$h^{0i}=0 \quad , \quad h^i{}_i=0 \quad , \quad \partial^j h_{ij}=0$$

Polarisations of the GW

- We were arrived to 6 independent components of $h_{\mu\nu}$, thanks to the Lorenz gauge. The choice of the TT gauge tells us that, thanks to the 4 additional conditions the independent components of $h_{\mu\nu}$ are 2.
  - The GW (in GR) has 2 polarisations!
- The equation $\Box \bar{h}_{\mu\nu} = 0$ has plane wave solutions:

$$\boxed{h_{ij}^{TT}(x) = \mathrm{Re}\left\{\varepsilon_{ij}(\mathbf{k})\, e^{ikx}\right\}}$$

with $k^\mu = (\omega/c\,, \boldsymbol{k})$ and $\omega/c = |\boldsymbol{k}|$

- The tensor $\varepsilon_{\mathrm{ij}}(\mathbf{k})$ is called the polarisation tensor. Thanks to the TT conditions, the only non-null components of h_{ij}^{TT} are orthogonal to the direction of the propagation vector $\hat{n} = \boldsymbol{k}/|\boldsymbol{k}|$

Polarisations of the GW

- Choosing z as direction for the propagation $\hat{n}$ and imposing h_{ij} to be symmetric and traceless:

$$\boxed{h_{ij}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij} \cos\left[\omega\left(t - z/c\right)\right]}$$

- Or more simply $\boxed{h_{ab}^{TT}(t,z) = \begin{pmatrix} h_+ & h_\times \\ h_\times & -h_+ \end{pmatrix}_{ab} \cos\left[\omega\left(t - z/c\right)\right]}$
- Where a,b=1,2 are indices in the transvers plane (x,y)
- h_+ is the *plus* polarisation and h_x is the *cross* polarisation

Effect of GW on free particles

- Let suppose to have a free particle in a wave-free region of spacetime
- Let chose a Lorentz frame where the particle is initially at rest:
  - four-velocity $u^\mu \equiv \frac{dx^\mu}{d\tau} = \delta_t^\mu = (1,0,0,0,0)$
- Let write the geodesic equation: $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho} \frac{d^2x^\nu}{d\tau^2} \frac{d^2x^\rho}{d\tau^2} = 0$$

Where τ is the proper time. This is the classical equation of motion of a test mass in the curved spacetime described by $g_{\mu\nu}$.

- In terms of four-velocity: $$\frac{du^\mu}{d\tau} + \Gamma^\mu_{\nu\rho} u^\nu u^\rho = 0$$
- If the particle is initially at rest ($u^\mu = \delta_t^\mu$):

$$\left(\frac{du^\mu}{d\tau} \right)_0 = -\Gamma^\mu_{\nu\rho}\Big|_{\substack{\nu=0 \\ \rho=0}} = -\frac{1}{2}\eta^{\mu\sigma} \left(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho} \right)_{\substack{\nu=0 \\ \rho=0}}$$

Effect of GW on free particles

- Replacing $\nu,\rho=0$ $$\Gamma^{i}_{00} = \frac{1}{2}\left(2\partial_0 h_{0i} - \partial_i h_{00}\right)$$
- But this term vanishes because for the TT gauge are h_{00} and h_{0i} chosen zero.
- This as a crucial effect:

> *In the TT frame, particles that were at rest before the arrival of the wave remain at rest even after the arrival of the wave*

- That means that the coordinates of the particle don't change:
  - The wave has not effect? Is not possible to detect it?
- No, it is an effect of the (correct) choice of the coordinate system
  - The particle remain attached to the initial coordinate position
- But coordinates are merely frame-dependent labels and they do not directly convey any invariant geometrical information about the spacetime

Effect of GW on free particles

- But, let compute the proper distance between two nearby particle, initially both at rest
- Let suppose one particle at the origin of the coordinate system and the other at $x=a$, $y=z=0$
- The proper distance is

$$\Delta\ell \equiv \int \left| g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} \right|^{1/2} = \int_0^a \left| g_{xx} \right|^{1/2} \approx \sqrt{g_{xx}\left(x=0\right)} \cdot a$$

Being: $g_{xx}\left(x=0\right) = \eta_{xx} + h_{xx}^{TT}\left(x=0\right)$ We obtain:

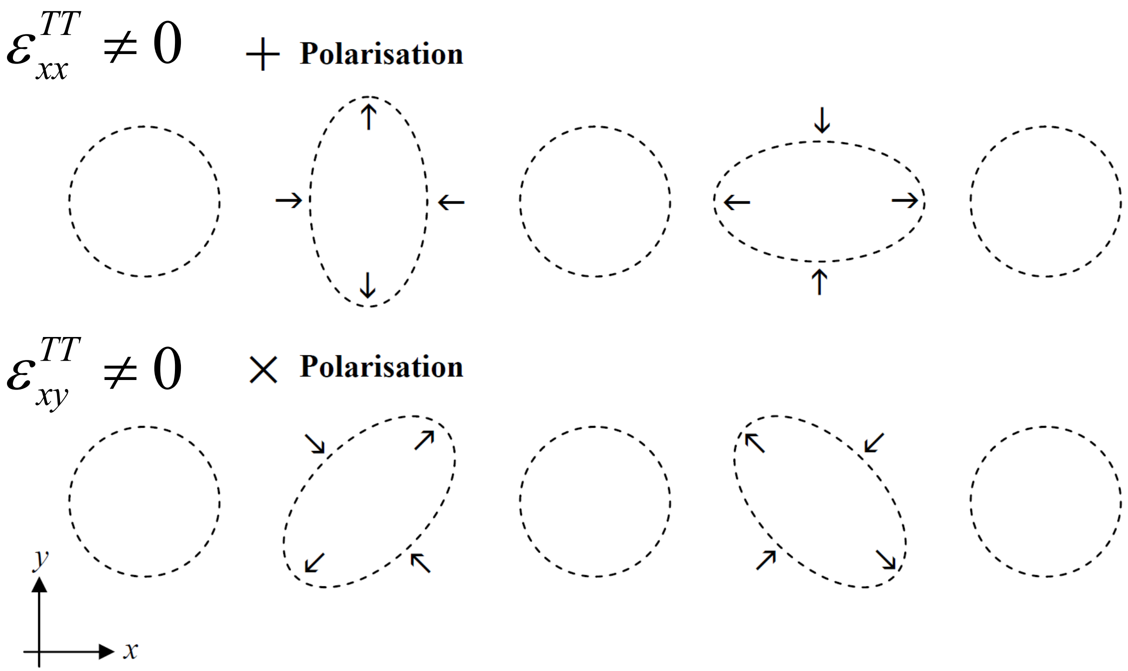
Proper distance: changing

$$\boxed{\Delta\ell \approx \left[1 + \frac{1}{2} h_{xx}^{TT}\left(x=0\right)\right] \cdot a}$$

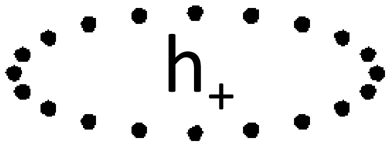
Coordinate distance: unchanged

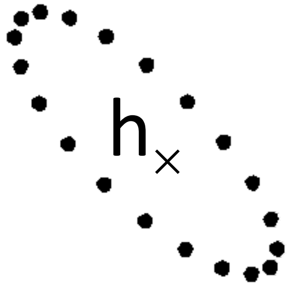
This is a crucial result: the proper distance between two test particle is what the GW detectors are measuring

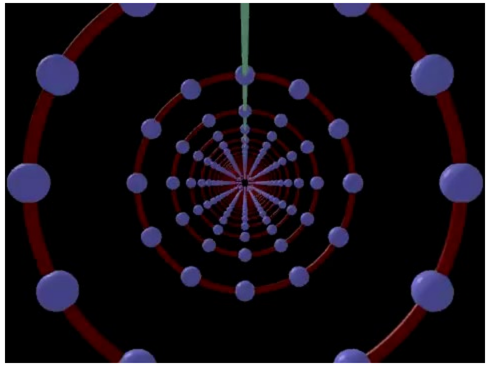
Ring of test particles: passage of a GW



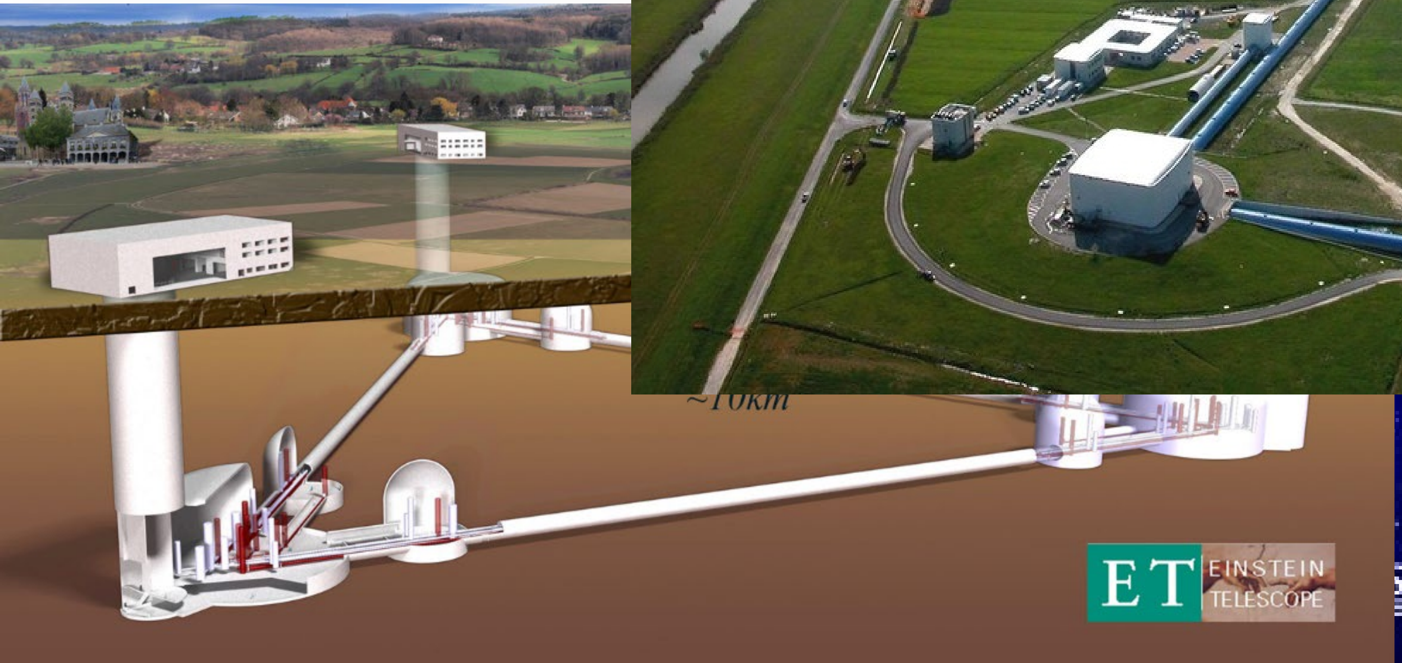
- If we have a pure "+" polarisation wave, propagating along z, the mass distribution undergoes to a deformation described by the upper part of the above panel
- In case of a pure "×" polarisation, the deformation is described by the lower part of that panel





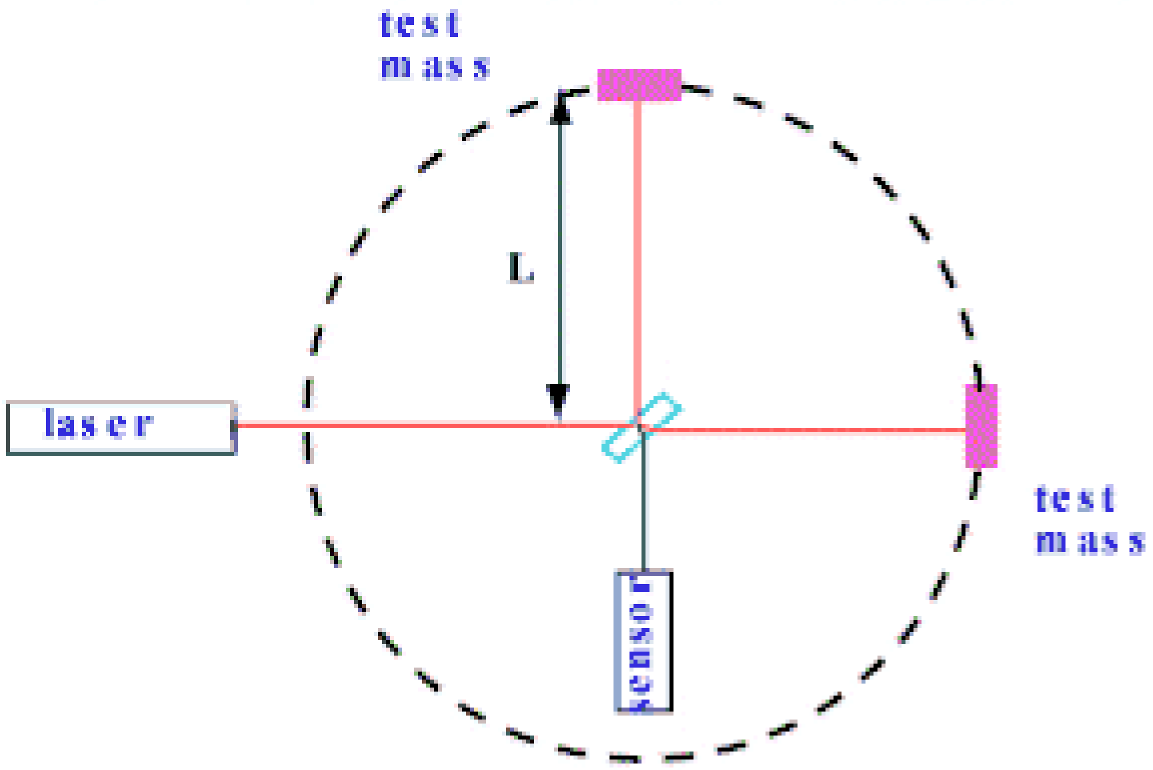


GW Detectors



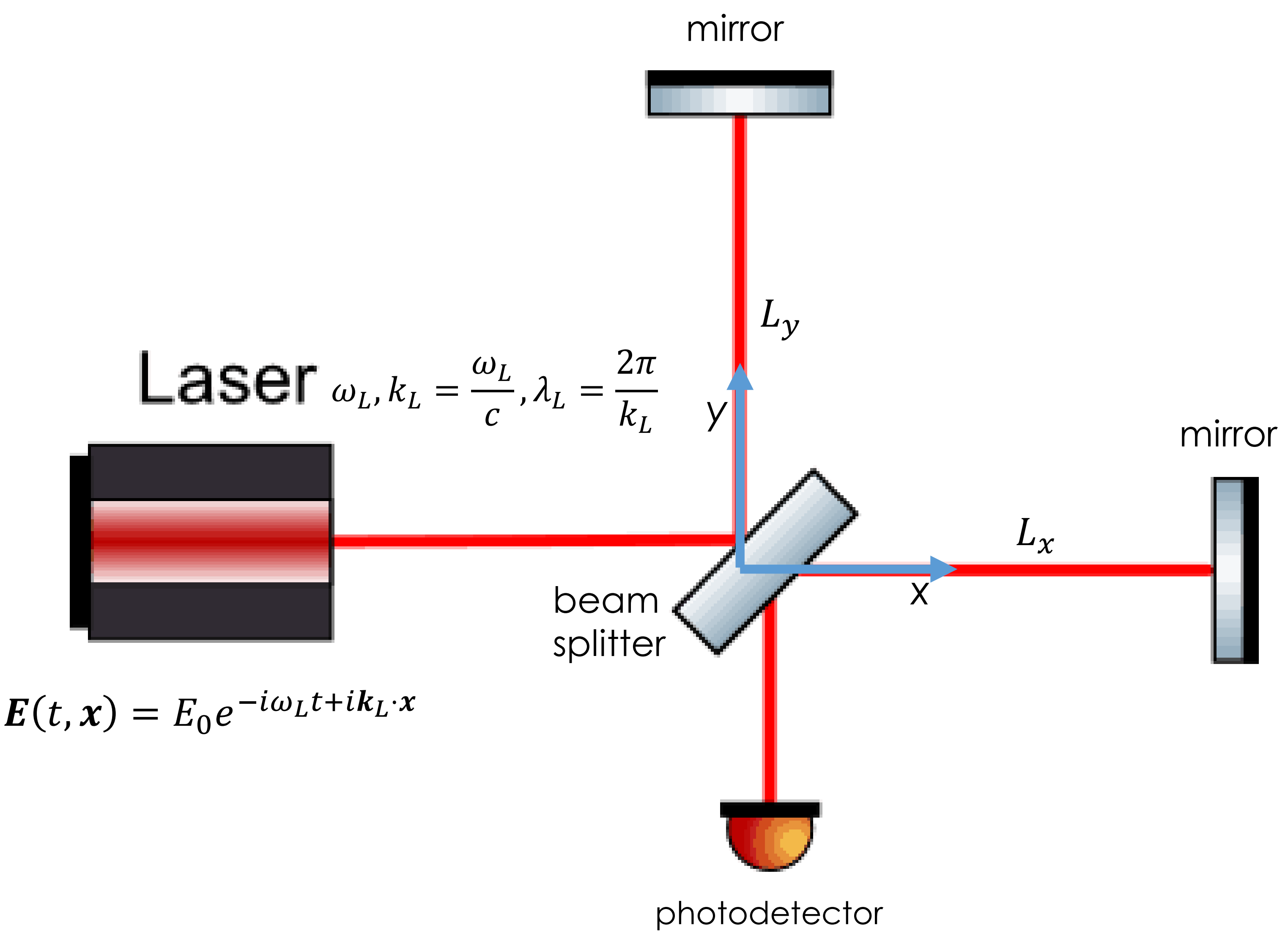
What we have to measure?

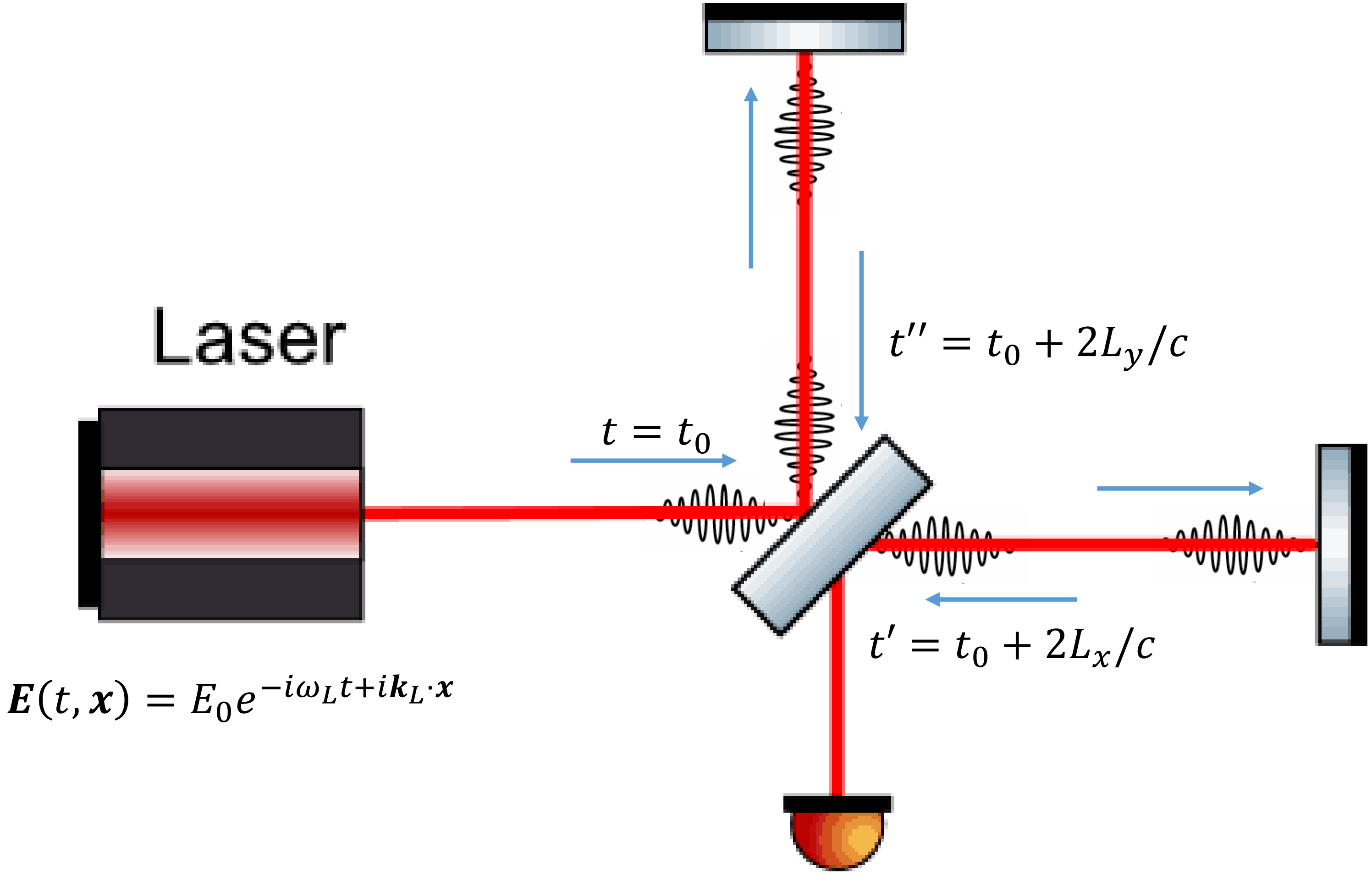
- Let suppose to have a pure “+” wave entering in the screen
- We have to measure a quadrupolar distance modulation:



Credits: G.Gemme

A Michelson interferometer seems a “natural” choice

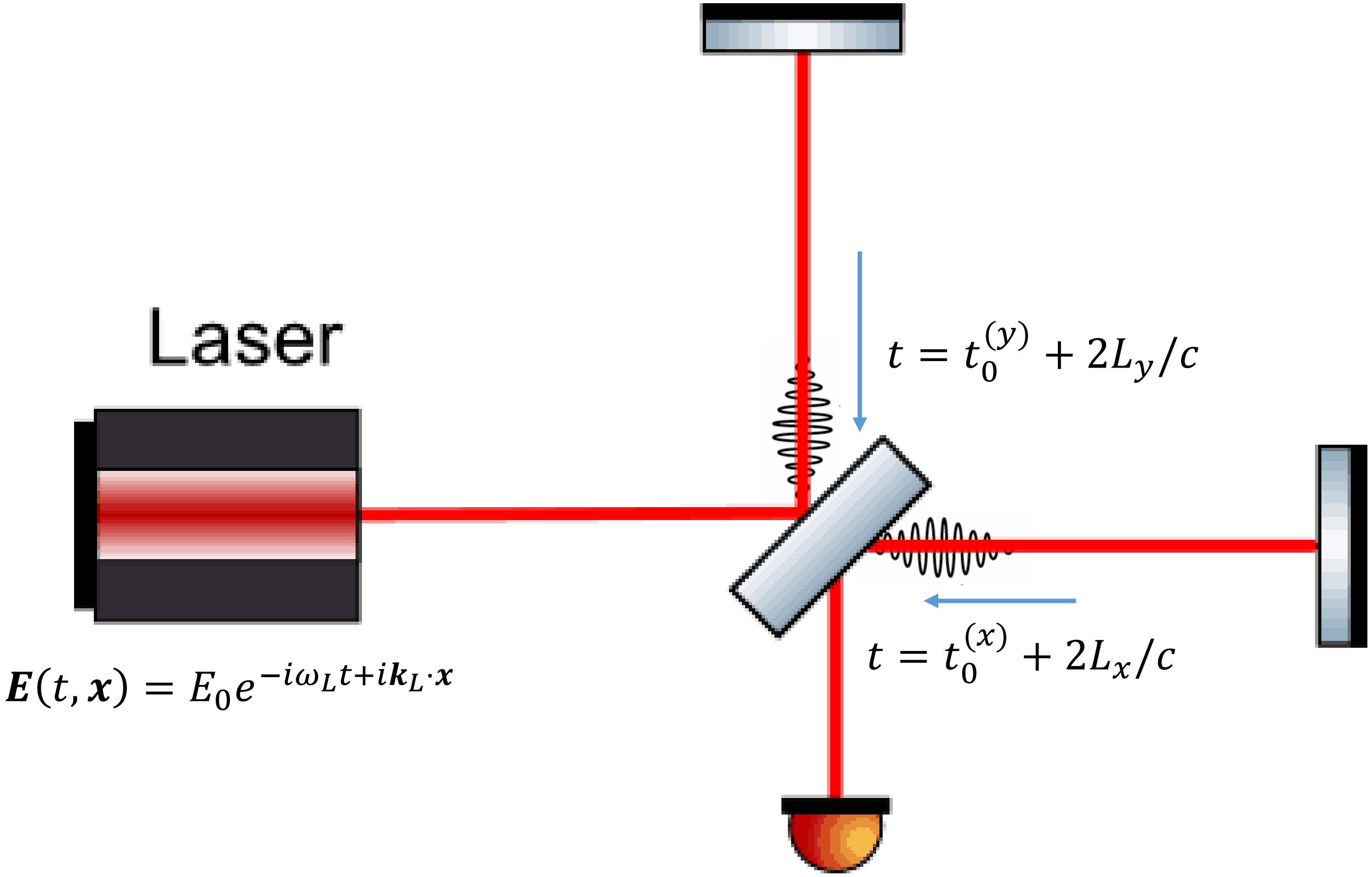

mirror
L_y
Laser $\omega_L, k_L = \frac{\omega_L}{c}, \lambda_L = \frac{2\pi}{k_L}$
y
mirror
L_x
beam
splitter
x
$\boldsymbol{E}(t, \boldsymbol{x}) = E_0 e^{-i\omega_L t + i\boldsymbol{k}_L \cdot \boldsymbol{x}}$
photodetector



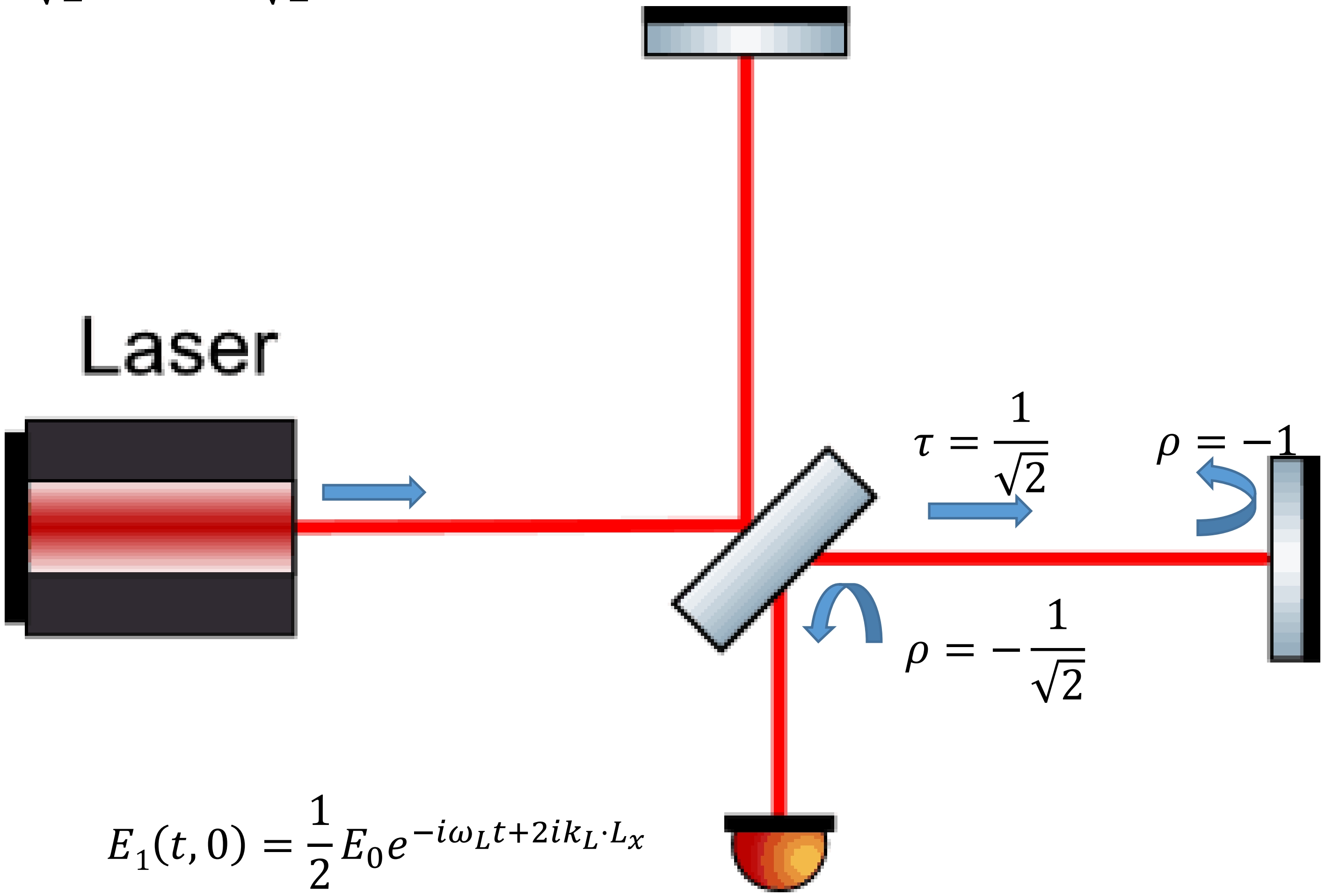
Laser
$t = t_0$
$t'' = t_0 + 2L_y/c$
$t' = t_0 + 2L_x/c$
$\boldsymbol{E}(t, \boldsymbol{x}) = E_0 e^{-i\omega_L t + i\boldsymbol{k}_L \cdot \boldsymbol{x}}$

$$t_0^{(x)} = t - \frac{2L_x}{c} \Longrightarrow \exp\left\{-i\omega_L t_0^{(x)}\right\} \Longrightarrow \exp\left\{-i\omega_L t + 2ik_L L_x\right\}$$

$$t_0^{(y)} = t - \frac{2L_y}{c} \Longrightarrow \exp\left\{-i\omega_L t_0^{(y)}\right\} \Longrightarrow \exp\left\{-i\omega_L t + 2ik_L L_y\right\}$$



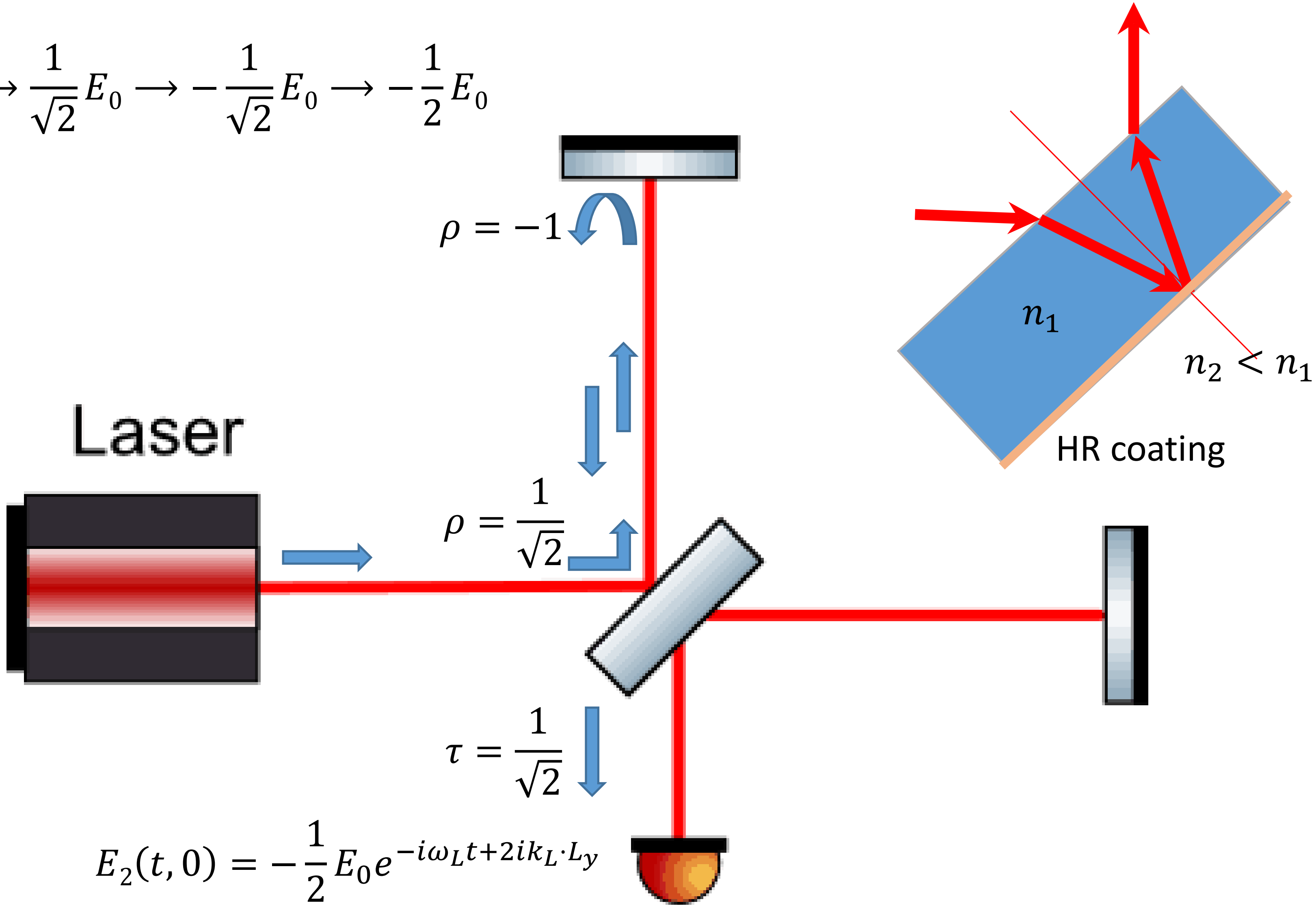
$$E_1 \propto E_0 \longrightarrow \frac{1}{\sqrt{2}} E_0 \longrightarrow -\frac{1}{\sqrt{2}} E_0 \longrightarrow \frac{1}{2} E_0$$



$$E_1(t,0) = \frac{1}{2} E_0 e^{-i\omega_L t + 2ik_L \cdot L_x}$$

$$E_2 \propto E_0 \longrightarrow \frac{1}{\sqrt{2}}E_0 \longrightarrow -\frac{1}{\sqrt{2}}E_0 \longrightarrow -\frac{1}{2}E_0$$

Why here ρ here is positive?



$$E_2(t,0) = -\frac{1}{2}E_0 e^{-i\omega_L t + 2ik_L \cdot L_y}$$

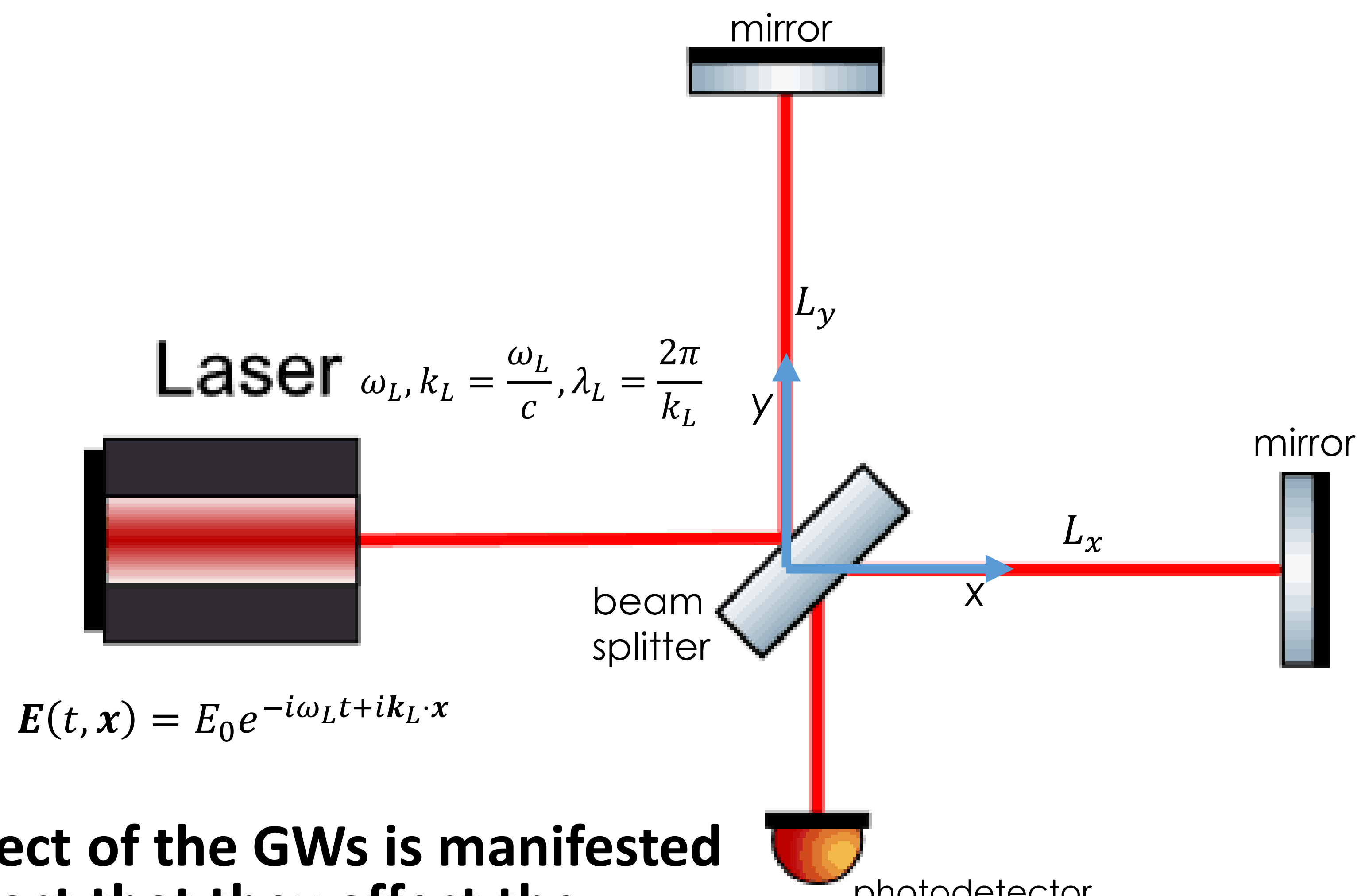
- The total electric field is $E_{out} = E_1 + E_2 \longrightarrow$

$$E_{out} = iE_0\, e^{-i\omega_L t + ik_L(L_x+L_y)} \sin\left[k_L(L_y - L_x)\right]$$

$$\boxed{\left|E_{out}\right|^2 = E_0^2 \sin^2\left[k_L(L_y - L_x)\right]}$$

$$\phi_0 = \ k_L(L_y - L_x) = \frac{2\pi}{\lambda_L} L \frac{L_y - L_x}{L} = 2\pi\, \frac{L}{\lambda_L}\, \frac{\Delta L}{L}$$

In the TT gauge the coordinates of the mirrors are not affected by the passage of the wave



The effect of the GWs is manifested in the fact that they affect the propagation of light

- Let’s take a GW with ‘+’ polarization, propagating along *z*
- In the plane of the IFO (*z* = 0) we have $h_+(t) = h_0 \cos(\omega_{gw} t)$

  and the space-time interval is

$$ds^2 = -c^2 dt^2 + [1 + h_+(t)]\, dx^2 + [1 - h_+(t)]\, dy^2 + dz^2$$

- **Photons travel along null geodesics *ds*² = 0**
- In the *x*-arm we have

$$ds^2 = -c^2 dt^2 + [1 + h_+(t)]\, dx^2 = 0$$

$$c^2 dt^2 = [1 + h_+(t)]\, dx^2$$

$$dx = \pm \frac{c}{\sqrt{1 + h_+(t)}}\, dt \simeq \pm c\, dt \left[1 - \frac{1}{2} h_+(t)\right]$$

- A photon leaves the beam splitter at time t_0
  It reaches the mirror at the fixed coordinate $x = L_x$ at time t_1

$$\int_0^{L_x} dx = +c \int_{t_0}^{t_1} \left[1 - \frac{1}{2} h_+(t')\right] dt'$$

$$L_x = c(t_1 - t_0) - \frac{c}{2} \int_{t_0}^{t_1} h_+(t')\, dt'$$

- Then the photon is reflected back and reaches again the beam splitter at time t_2

$$\int_{L_x}^{0} dx = -c \int_{t_1}^{t_2} \left[1 - \frac{1}{2} h_+(t')\right] dt'$$

$$L_x = c(t_2 - t_1) - \frac{c}{2} \int_{t_1}^{t_2} h_+(t')\, dt'$$

- Summing we get

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{1}{2}\int_{t_0}^{t_2} h_+(t')\,dt'$$

- In the integral we can approximate $t_2 \simeq t_0 + \frac{2L_x}{c} + O(h_0)$

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{1}{2}\int_{t_0}^{t_0+2L_x/c} h_0\cos(\omega_{gw}t')\,dt' =$$

$$\frac{2L_x}{c} + \frac{h_0}{2\omega_{gw}}\left\{\sin\left[\omega_{gw}(t_0 + 2L_x/c)\right] - \sin(\omega_{gw}t_0)\right\}$$

- Since $\sin(\alpha + 2\beta) - \sin\alpha = 2\sin\beta\cos(\alpha + \beta)$ we can write

$$t_2 - t_0 = \frac{2L_x}{c} + \frac{h_0 L_x}{c}\frac{\sin(\omega_{gw}L_x/c)}{(\omega_{gw}L_x/c)}\cos\left[\omega_{gw}(t_0 + L_x/c)\right]$$

$$\boxed{t_2 - t_0 = \frac{2L_x}{c} + \frac{L_x}{c}\,h(t_0 + L_x/c)\,\frac{\sin(\omega_{gw}L_x/c)}{(\omega_{gw}L_x/c)}}$$

- In the *y*-arm the analysis is similar and we get

$$t_2 - t_0 = \frac{2L_y}{c} - \frac{L_y}{c} h(t_0 + L_y/c)\,\mathrm{sinc}(\omega_{gw} L_y/c)$$

- We are interested in the light that comes out of the IFO at a given time t
- We fix $t_2 \equiv t$ and compute the corresponding values of t_0 for the two arms: $t_0^{(x)}$ and $t_0^{(y)}$

$$t_0^{(x)} = t - \frac{2L_x}{c} - \frac{L_x}{c} h(t - L_x/c)\,\mathrm{sinc}(\omega_{gw} L_x/c)$$

$$t_0^{(y)} = t - \frac{2L_y}{c} + \frac{L_y}{c} h(t - L_y/c)\,\mathrm{sinc}(\omega_{gw} L_y/c)$$

where, at first order in h, we have set ($a = x, y$)

$$h(t_0 + L_a/c) = h(t - 2L_a/c + L_a/c) = h(t - L_a/c)$$

- The light that is at the beam splitter (x = 0) at $t = t_0^{(x)}$ has phase $\exp\left\{-i\omega_L t_0^{(x)}\right\}$ so

$$E^{(x)}(t) = \frac{1}{2}E_0 e^{-i\omega_L t_0^{(x)}} = \frac{1}{2}E_0 e^{-i\omega_L (t-2L_x/c) + i\Delta\phi_x(t)}$$

with

$$\Delta\phi_x(t) = h_0 \frac{\omega_L L_x}{c} \operatorname{sinc}(\omega_{gw} L_x/c) \cos\left[\omega_{gw}(t - L_x/c)\right]$$

- Similarly for the y-arm we have

$$E^{(y)}(t) = -\frac{1}{2}E_0 e^{-i\omega_L t_0^{(y)}} = -\frac{1}{2}E_0 e^{-i\omega_L (t-2L_y/c) + i\Delta\phi_y(t)}$$

and

$$\Delta\phi_y(t) = -h_0 \frac{\omega_L L_y}{c} \operatorname{sinc}(\omega_{gw} L_y/c) \cos\left[\omega_{gw}(t - L_y/c)\right]$$

- In general we build our detector to have $L_x \cong L_y$ in order to cancel common noise in the two arms

- Since we have $$L_x = \frac{L_x + L_y}{2} + \frac{L_x - L_y}{2} \simeq L + O(h_0)$$

  in $\Delta\phi_x$ and $\Delta\phi_y$, which are already $O(h_0)$, we can replace L_x and L_y by L

- In the terms $t - \frac{2L_x}{c}$ and $t - \frac{2L_y}{c}$ we need to take into account the difference between L_x and L_y.

  We write $$2L_x = 2L + (L_x - L_y)$$ $$2L_y = 2L - (L_x - L_y)$$

- Then

$$E^{(x)}(t) = \frac{1}{2}E_0 e^{-i\omega_L(t-2L/c)+i\phi_0+i\Delta\phi(t)}$$
$$E^{(y)}(t) = -\frac{1}{2}E_0 e^{-i\omega_L(t-2L/c)-i\phi_0-i\Delta\phi(t)}$$

- with $\phi_0 = k_L\left(L_x - L_y\right)$ and

$$\boxed{\begin{aligned}\Delta\phi &= h_0\, k_L L\,\mathrm{sinc}\left(\omega_{gw}L/c\right)\cos\left[\omega_{gw}\left(t-L/c\right)\right]= \\ &= 2\pi\frac{L}{\lambda_L}\mathrm{sinc}\left(2\pi\frac{L}{\lambda_{gw}}\right)\cdot h_0\,\cos\left[\omega_{gw}\left(t-L/c\right)\right]\end{aligned}}$$

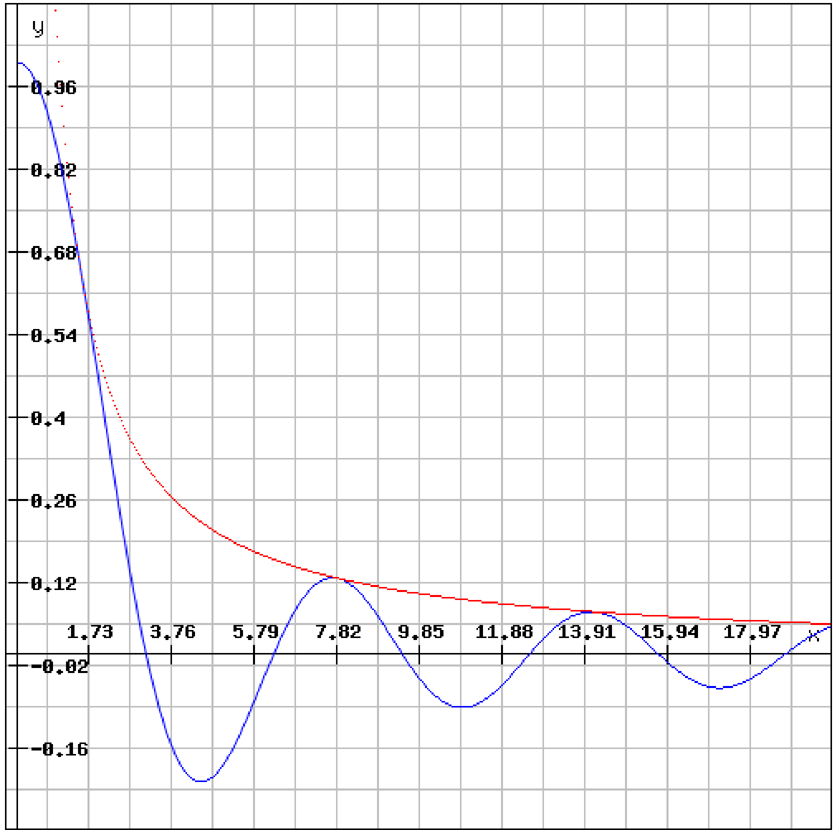
- Note the function «sine cardinal»:

$$\boxed{\mathrm{sinc}\left(2\pi\frac{L}{\lambda_{gw}}\right)=\frac{\sin\left(2\pi\frac{L}{\lambda_{gw}}\right)}{\left(2\pi\frac{L}{\lambda_{gw}}\right)}}$$

Blind at some frequencies

$$\operatorname{sinc}\left(2\pi\frac{L}{\lambda_{gw}}\right)=\frac{\sin\left(2\pi\frac{L}{\lambda_{gw}}\right)}{\left(2\pi\frac{L}{\lambda_{gw}}\right)}=0 \qquad \begin{array}{l}\lambda_{gw}=2L/n \\ f_{gw}=n\left(c/2L\right)\end{array}, \qquad n=1,2,\ldots$$

- Virgo L=3000m
- First zero at 50kHz



Going back to the difference of phase:

$$\Delta\phi = 2\pi \frac{L}{\lambda_L} \operatorname{sinc}\left(2\pi \frac{L}{\lambda_{gw}}\right) \cdot h_0 \cos\left[\omega_{gw}\left(t - L/c\right)\right]$$

Going back to the difference of phase, considering the quasi static approximation $\lambda_{gw} >> L$:

$$\Delta\phi \approx 2\pi \frac{L}{\lambda_L} \cdot h\left(t - L/c\right) = 1.9 \times 10^{10} \cdot h\left(t - L/c\right)$$

Comparing this expression with the difference of phase for arm length difference (slide 11) we have:

$$\left.\begin{aligned} \Delta\phi &\approx 2\pi \frac{L}{\lambda_L} \cdot h\left(t - L/c\right) \\ \phi_0 &= 2\pi \frac{L}{\lambda_L} \cdot \frac{L_x - L_y}{L} \end{aligned}\right\} \Rightarrow h \approx \frac{\Delta\left(L_x - L_y\right)}{L} \quad \Rightarrow \quad \Delta\left(L_x - L_y\right) \approx h \cdot L$$

We need giant detectors

- The total electric field at the output is

$$E_{tot}(t) = E^{(x)}(t) + E^{(y)}(t) = iE_0 e^{-i\omega_L(t-2L/c)} \sin\left[\phi_0 + \Delta\phi(t)\right]$$

adjustable parameter(dark fringe)

GWs

- The total power $P \sim |E_0|^2$ at the photodetector is

$$P = P_0 \sin^2\left[\phi_0 + \Delta\phi(t)\right] = \frac{P_0}{2}\left\{1 - \cos\left[2\phi_0 + 2\Delta\phi(t)\right]\right\}$$

- We need to maximize $\Delta\phi$. The amplitude of $\Delta\phi$ depends on *L* trough

$$|\Delta\phi| \sim \frac{L}{\lambda_L}\frac{\lambda_{gw}}{L}\sin\left(2\pi\frac{L}{\lambda_{gw}}\right) = \frac{\lambda_{gw}}{\lambda_L}\sin\left(2\pi\frac{L}{\lambda_{gw}}\right)$$

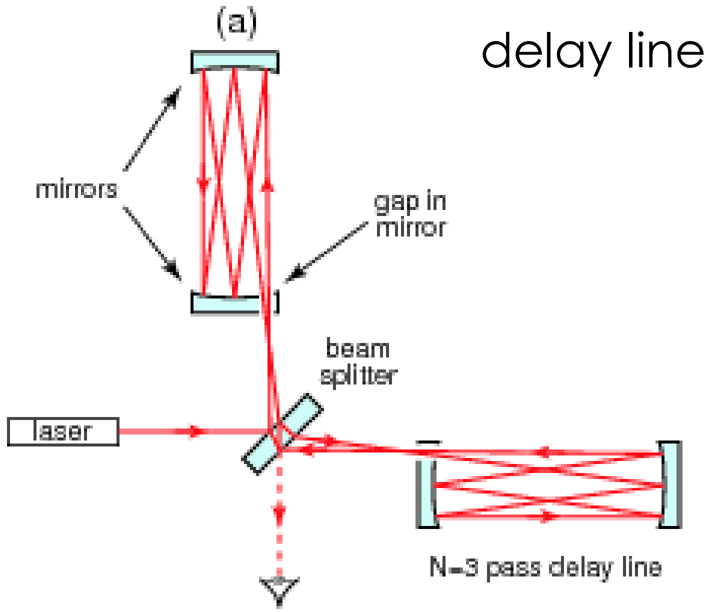
- The optimal arm length is given by

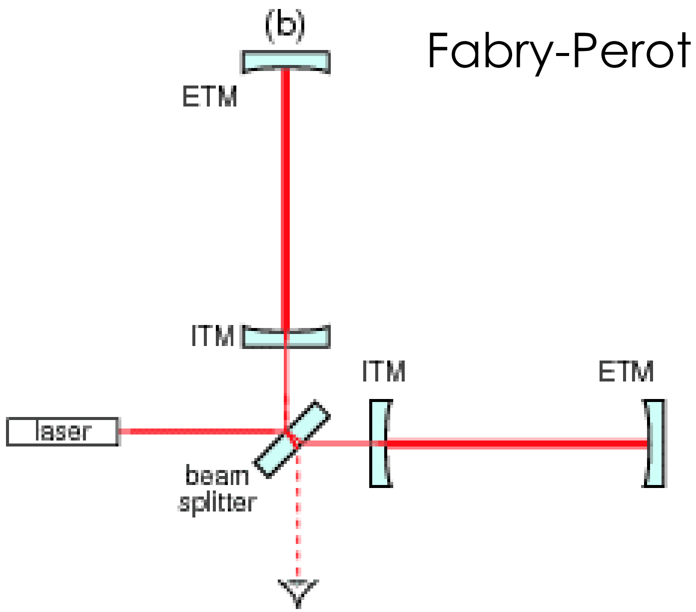
$$2\pi\frac{L}{\lambda_{gw}} = \frac{\pi}{2} \rightarrow L = \frac{\lambda_{gw}}{4}$$

- In terms of $f_{gw} = \omega_{gw}/(2\pi)$ this gives

$$\boxed{L \simeq 750\,\mathrm{km}\left(\frac{100\,\mathrm{Hz}}{f_{gw}}\right)}$$

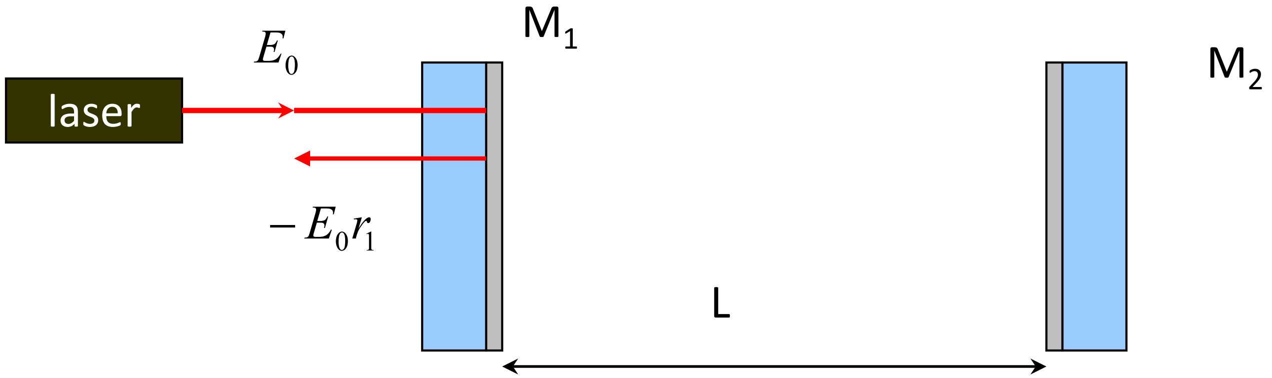
- Quite difficult to realise on Earth … we need a trick! (on space we need different solutions)





Fabry-Perot

- A Fabry-Perot cavity is realized with (at least) two mirrors:



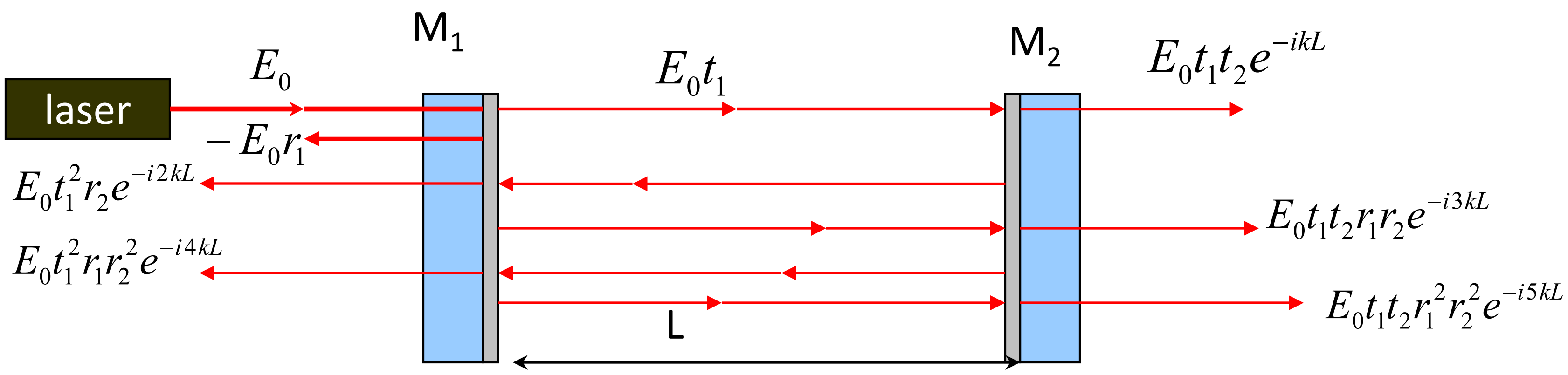
- Each mirror is has deposited an high reflectivity coating (HR) having an amplitude reflectivity coefficient $r_{1,2}$.
- For each of the two substrates (plus coatings) is defined an amplitude transmission coefficient $t_{1,2}$ and absorption $a_{1,2}$. We have:

$$r_{1,2}^2 + t_{1,2}^2 + a_{1,2}^2 = 1$$

Neglecting losses: $$r_{1,2}^2 + t_{1,2}^2 = 1$$

- Let suppose to imping a laser light (plane wave $E_0 e^{i(kx-\omega t)}$)
- The first mirror reflects a fraction $-E_0 r_1$ of the impinging electric field

… Fabry-Perot (2)



- A fraction $E_0 t_1$ of the impinging beam enters in the cavity
- Let follow the beam …

- Summing up all the components, we compute the cavity amplitude transmission and reflection coefficients:

$$t_c = \frac{E_t}{E_0} = t_1 t_2 e^{-ikL} \sum_{n=0}^{\infty} (r_1 r_2)^n e^{-2inkL} = \frac{t_1 t_2 e^{-ikL}}{1 - r_1 r_2 e^{-2ikL}}$$

$$r_c = \frac{E_r}{E_0} = -r1 + t_1^2 r_2 e^{-i2kL} \sum_{n=0}^{\infty} (r_1 r_2)^n e^{-2inkL} = -r_1 + \frac{t_1 t_2 e^{-i2kL}}{1 - r_1 r_2 e^{-2ikL}}$$

FP - Resonance condition

- The field in the cavity (at x=0) is

$$E_{cav}\left(0\right) = E_0 e^{-i\omega t} t_1 \sum_{n=0}^{\infty} \left(r_1 r_2\right)^n e^{-2inkL} = \frac{t_1}{1 - r_1 r_2 e^{-2ikL}} E_0 e^{-i\omega t}$$

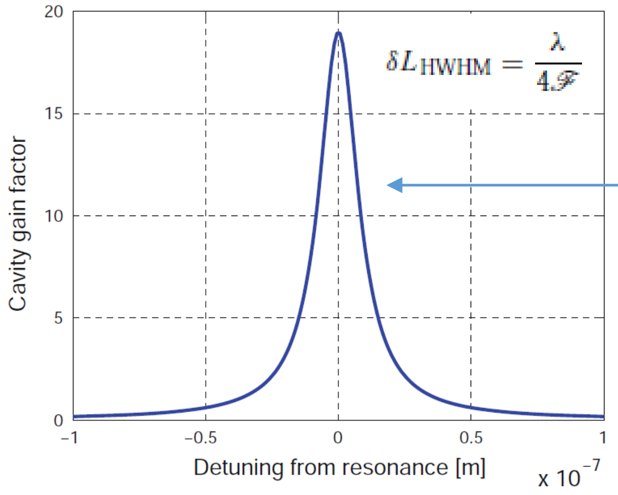
- The field in the cavity is maximum when

$$2kL \equiv 2\frac{2\pi}{\lambda} L = 2n\pi \Rightarrow L = n\frac{\lambda}{2}$$

Resonance condition

- At the resonance, also the other fields are quite interesting:

Fig. 3.4 Example of how the power stored inside a resonant Fabry-Perot cavity depends on the detuning from the resonance position. In this example the input mirror reflectivity is 0.9, corresponding to a relatively low finesse of about 30



The power is stored in the cavity near resonant condition: photons are bouncing from one mirror to the other

Cavity outside the resonance is essentially reflecting

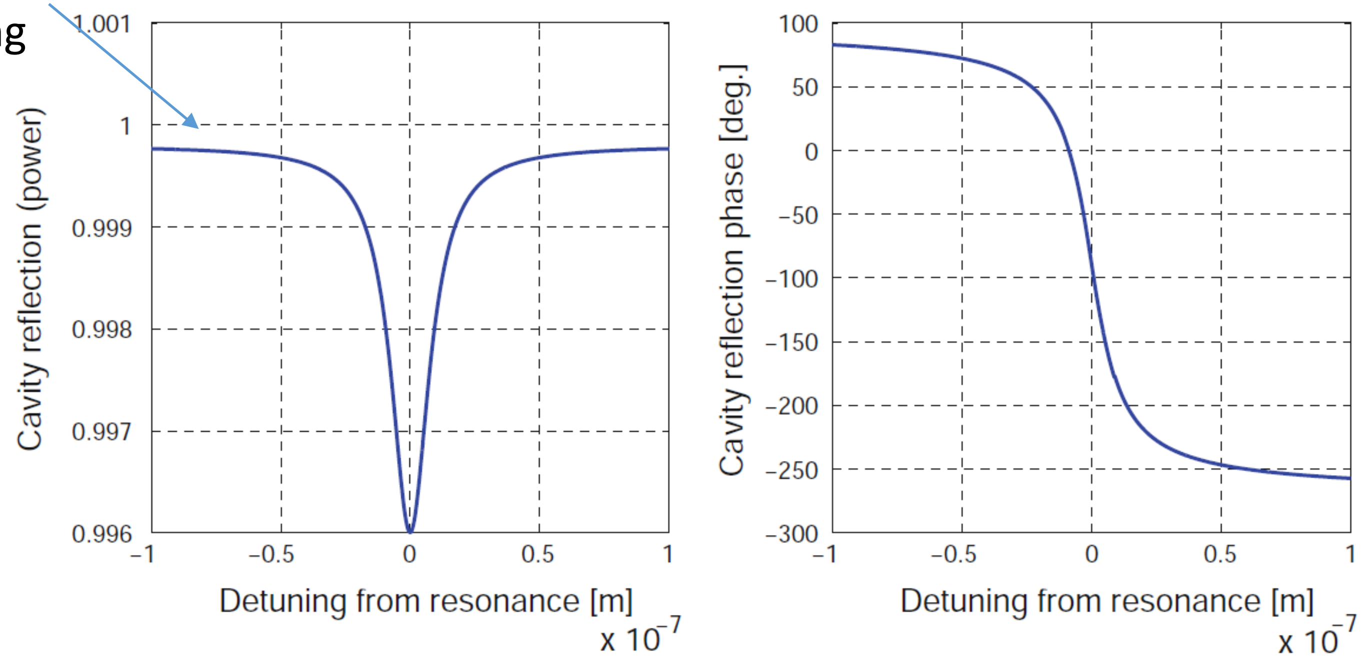
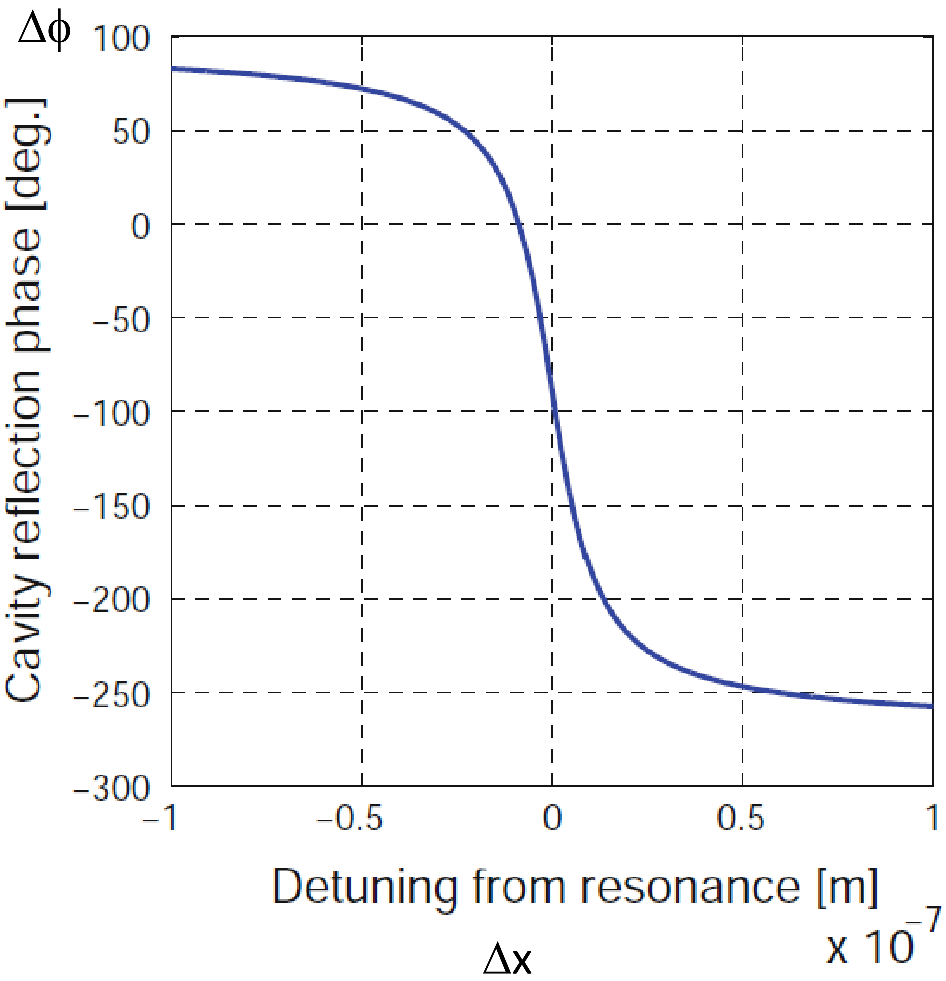


Fig. 3.5 Example of the reflection of a Fabry-Perot cavity as a function of the detuning from resonance. The cavity considered here has the same parameters used for Fig. 3.4, plus additional (large) round-trip losses of 200 ppm

We are interested to $\frac{\partial\phi}{\partial x}$: we are blind far the resonance, but extremely more sensitive (with respect to a Michelson) around the resonance

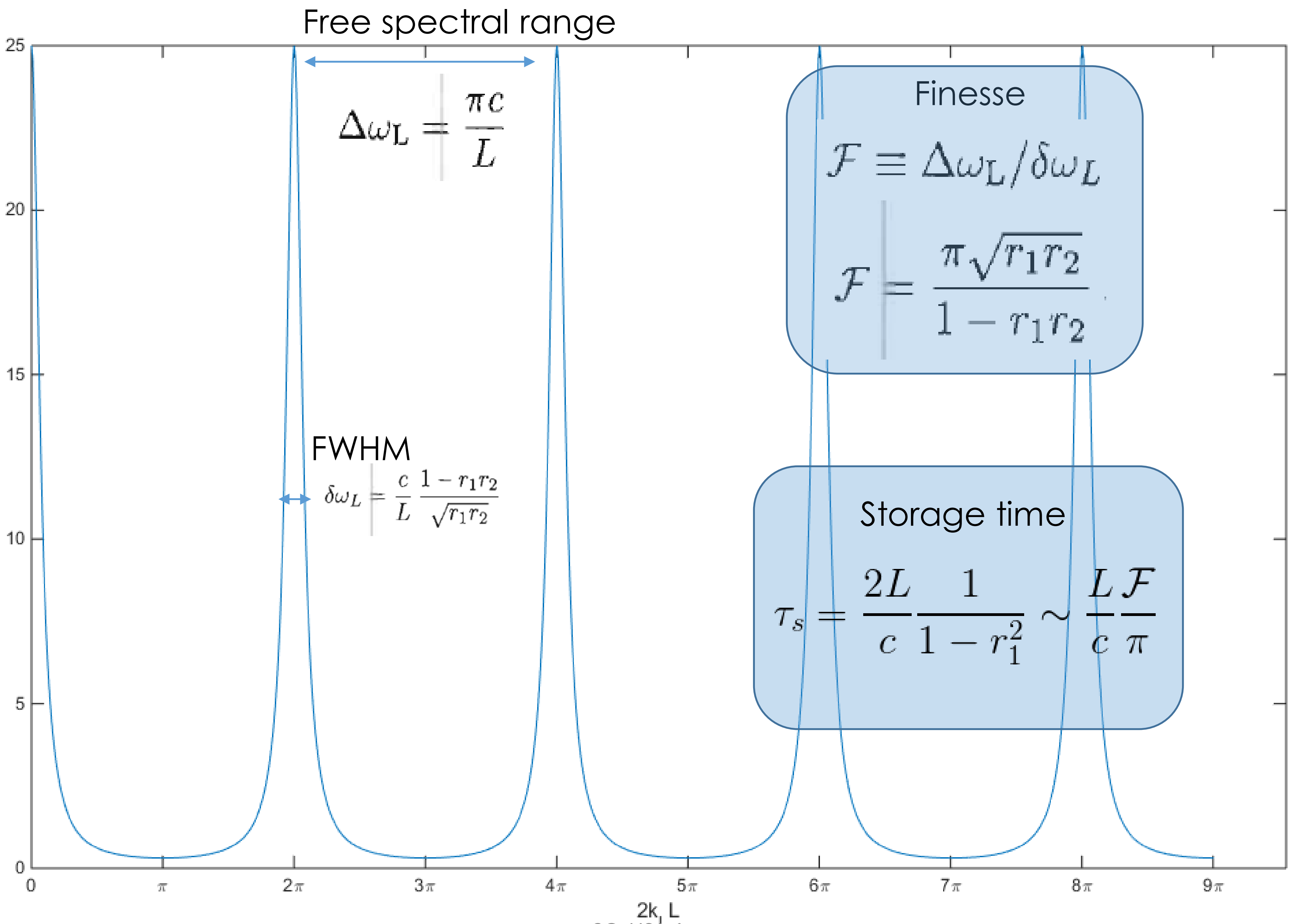


$$\frac{\partial\phi_{Mic}}{\partial x} = \frac{2\pi}{\lambda}$$

$$\left.\frac{\partial\phi_{FP}}{\partial x}\right|_{reson} = \frac{2\mathcal{F}}{\pi}\frac{2\pi}{\lambda}$$

Where $\mathcal{F}$ is named Finesse and its is defined through the transmitted power

$$|E_t|^2 = E_0^2 \frac{(t_1\, t_2)^2}{1 + (r_1\, r_2)^2 - 2\, r_1 r_2 \cos 2k_L L}$$

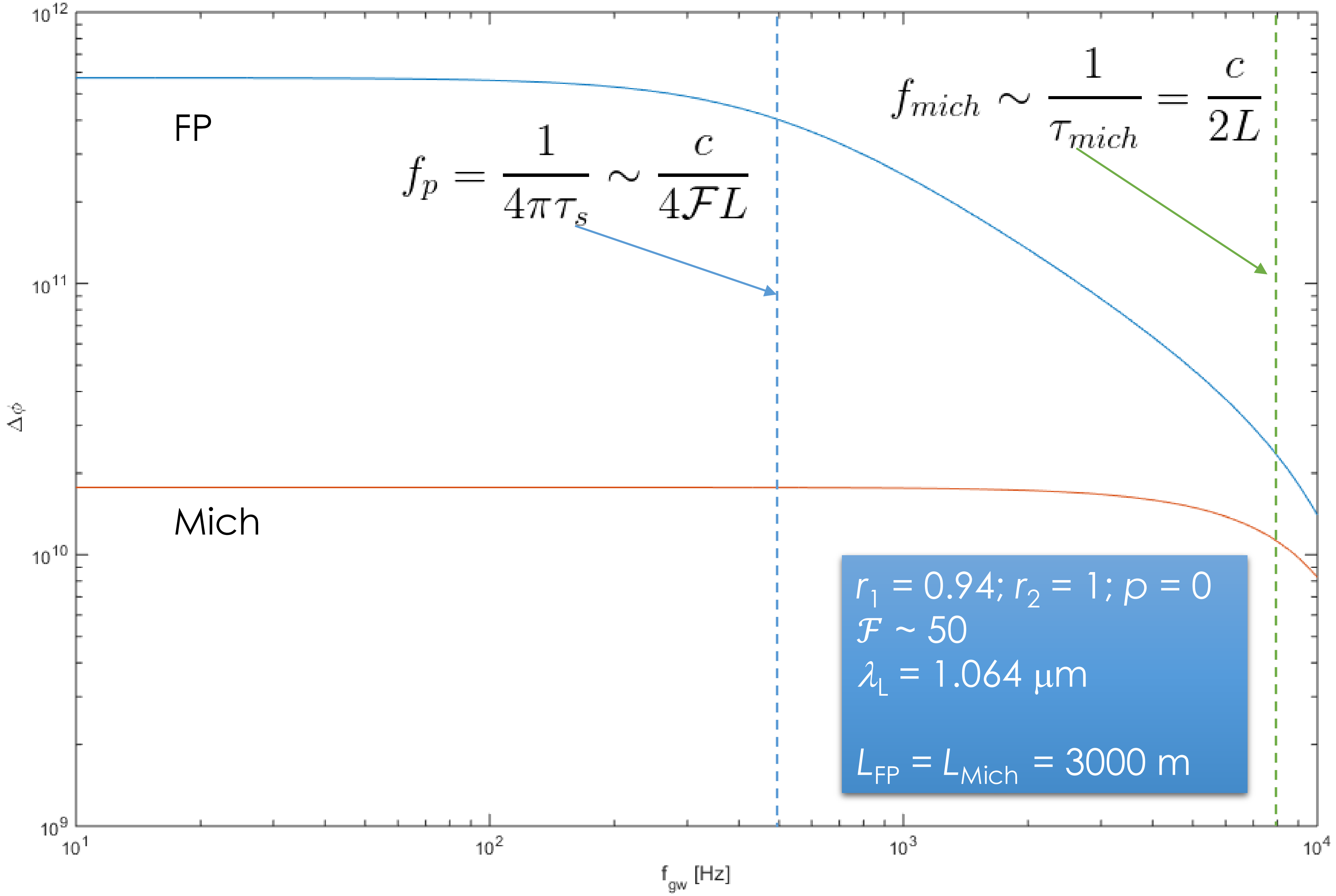

Free spectral range
$\Delta\omega_\mathrm{L} = \frac{\pi c}{L}$
FWHM
$\delta\omega_L = \frac{c}{L}\frac{1 - r_1 r_2}{\sqrt{r_1 r_2}}$
Finesse
$\mathcal{F} \equiv \Delta\omega_\mathrm{L}/\delta\omega_L$
$\mathcal{F} = \frac{\pi\sqrt{r_1 r_2}}{1 - r_1 r_2}$
Storage time
$\tau_s = \frac{2L}{c}\frac{1}{1 - r_1^2} \sim \frac{L}{c}\frac{\mathcal{F}}{\pi}$
25
20
15
10
5
0
0
π
2π
3π
4π
5π
6π
7π
8π
9π
$2k_L L$

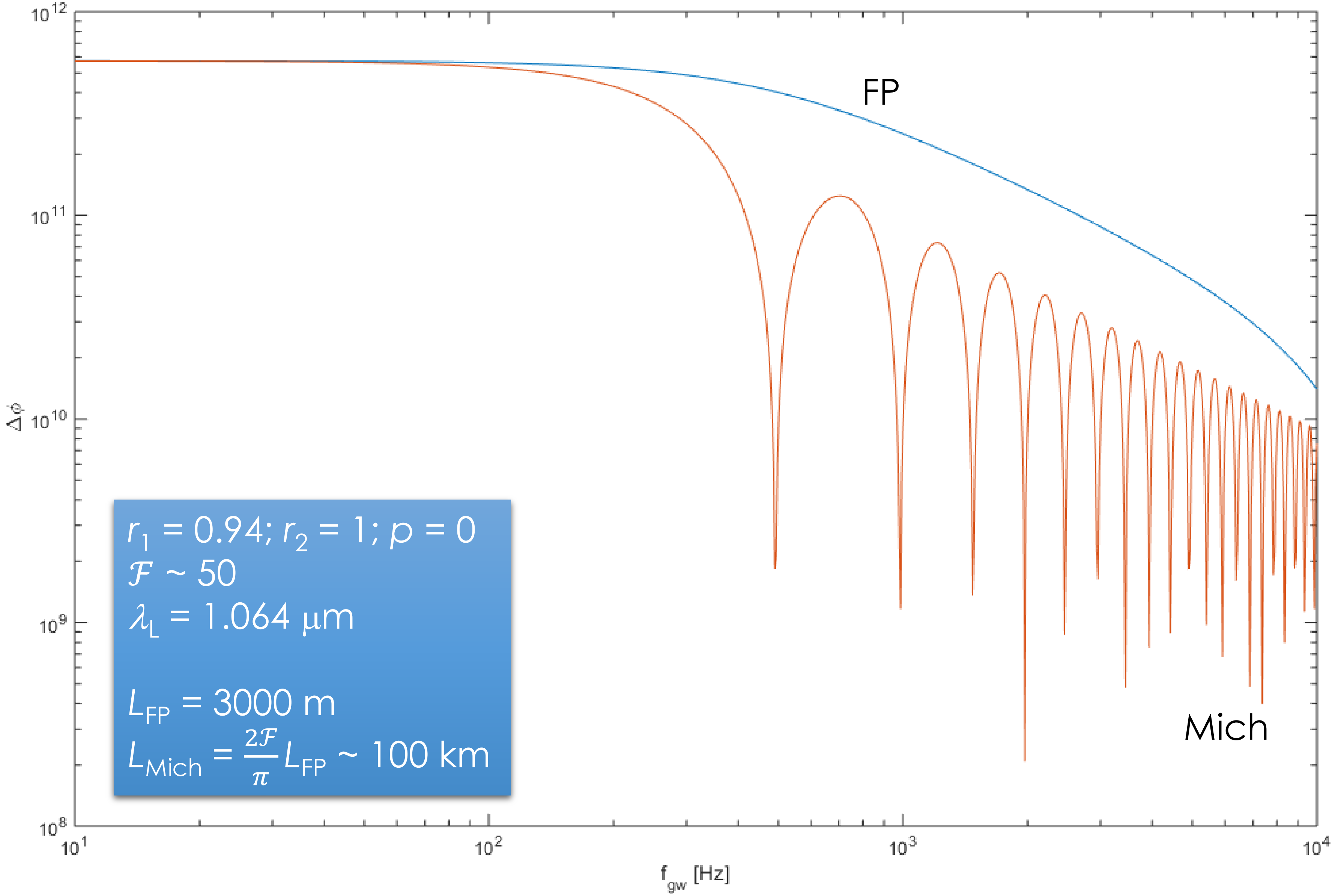
Response to the GW passage

- We seen that the FP amplifies the response in $\Delta\phi$ with respect to a Michelson:
  - Its response to a GW is roughly corresponding to the response of a Michelson having the arms $2\mathcal{F}/\pi$ longer
  - This is true at the first approximation, but we need to consider the storage time and the low pass filtering behaviour of the cavity:

$$\left|\Delta\phi_x\right|_{reson} \approx h_0 2k_L L \frac{\mathcal{F}}{\pi} \frac{1}{\sqrt{1+ {f_{gw}^2}/{f_p^2}}}$$

where $$f_p = \frac{1}{4\pi\tau_s} \approx \frac{c}{4\mathcal{F}L}$$


10^{12}
10^{11}
10^{10}
10^{9}
$\Delta\phi$
10^{1}
10^{2}
10^{3}
10^{4}
f_{gw} [Hz]
FP
Mich
$f_p = \frac{1}{4\pi\tau_s} \sim \frac{c}{4\mathcal{F}L}$
$f_{mich} \sim \frac{1}{\tau_{mich}} = \frac{c}{2L}$
$r_1 = 0.94$; $r_2 = 1$; $p = 0$
$\mathcal{F} \sim 50$
$\lambda_L = 1.064$ μm
$L_{FP} = L_{Mich} = 3000$ m

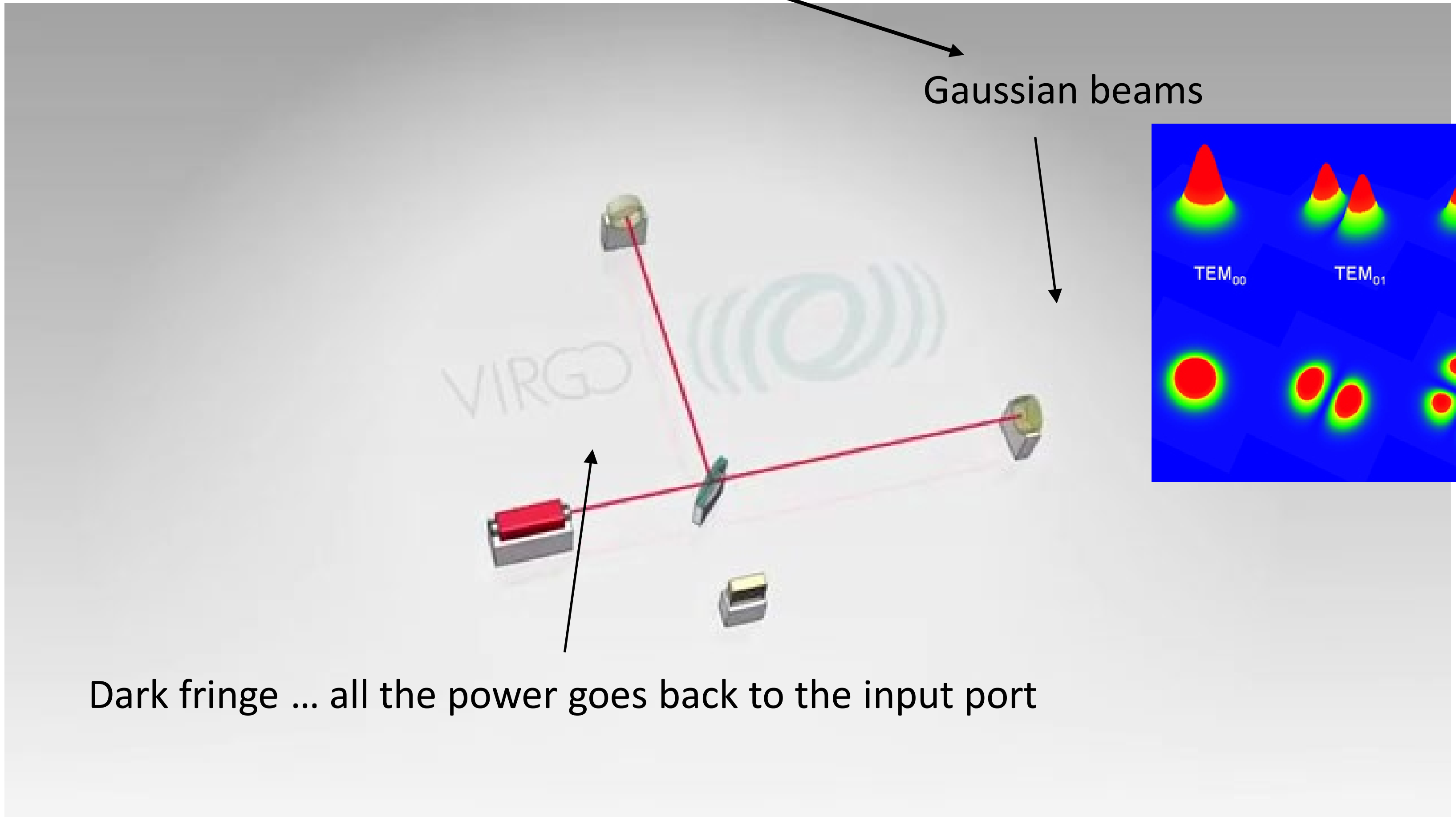


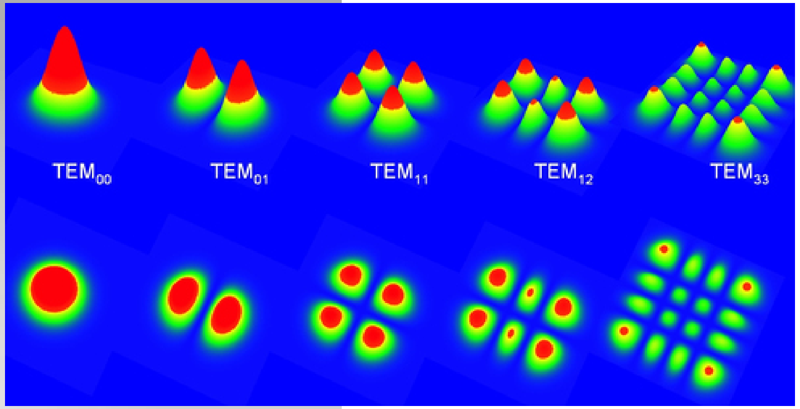
10^{12}
10^{11}
10^{10}
10^{9}
10^{8}
$\Delta\phi$
10^{1}
10^{2}
10^{3}
10^{4}
f_{gw} [Hz]
FP
Mich
$r_1 = 0.94; r_2 = 1; p = 0$
$\mathcal{F} \sim 50$
$\lambda_L = 1.064\ \mu m$
$L_{FP} = 3000$ m
$L_{Mich} = \frac{2\mathcal{F}}{\pi} L_{FP} \sim 100$ km

Building Advanced Virgo

$$E_{mn}(x,y,z,t) = E_0 \frac{w_0}{w(x)} \psi_m\left(\frac{\sqrt{2}y}{w(x)}\right) \psi_n\left(\frac{\sqrt{2}z}{w(x)}\right) \cdot e^{i(m+n+1)\varphi} \, e^{-ik\frac{y^2+z^2}{2R(x)}} \, e^{-i(kx-\omega t)}$$

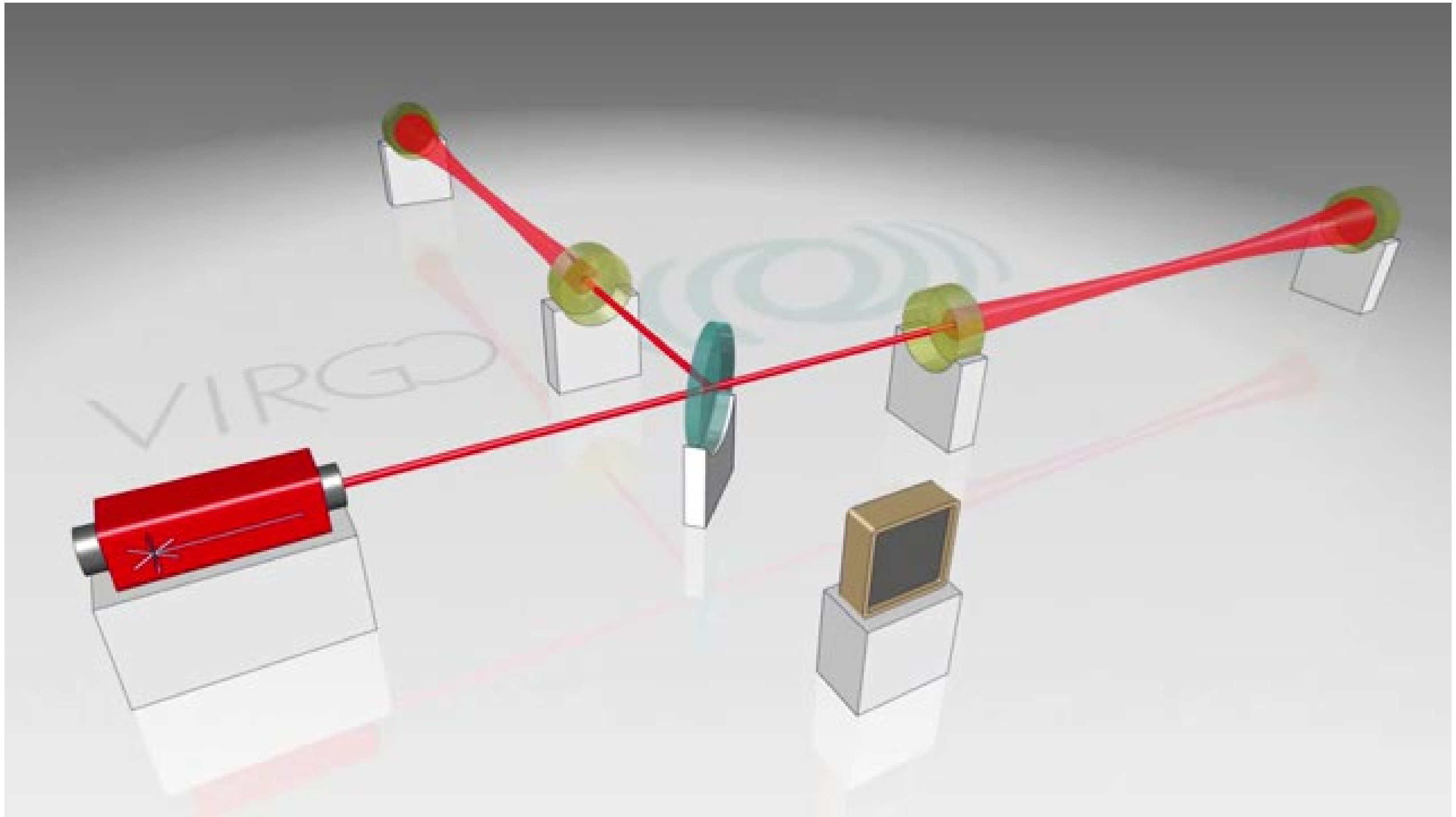
Gaussian beams





Dark fringe … all the power goes back to the input port

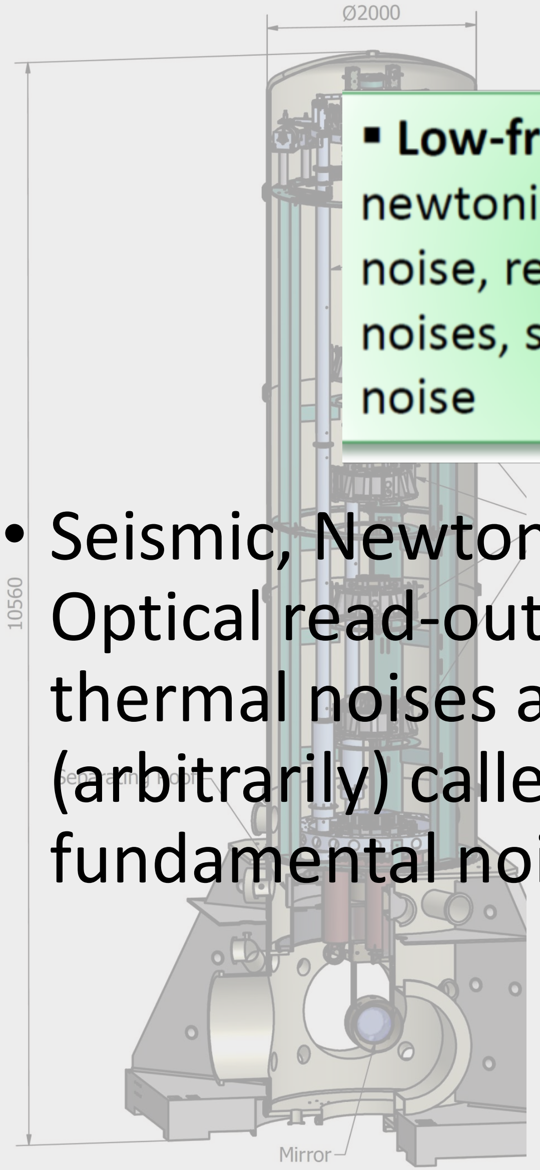
Building Advanced Virgo



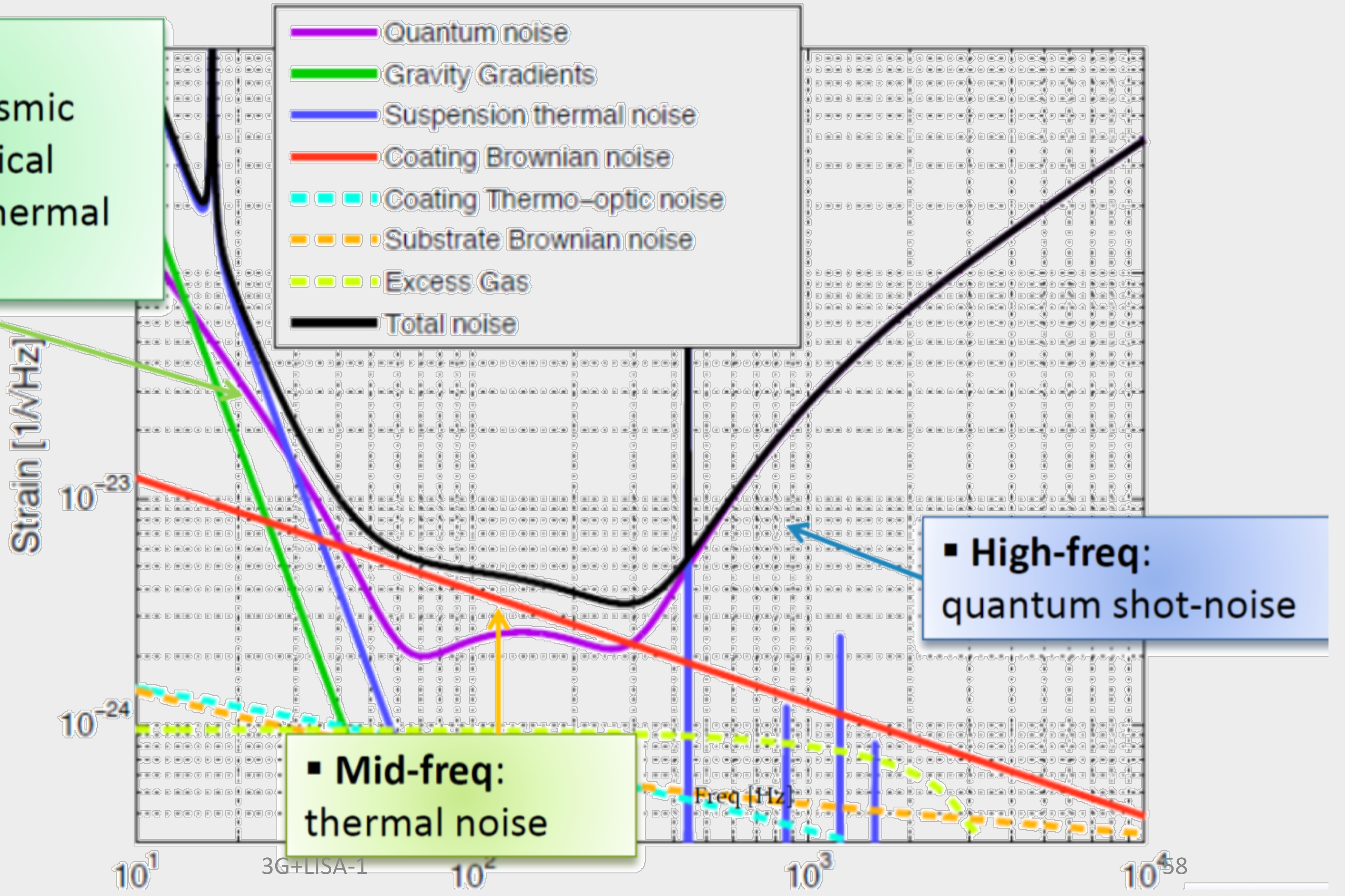
Free falling masses

- We worked until now in the hypothesis to have free falling masses
  - The mirrors should be subjected only to gravitational forces, in order to move on geodesics …. But is it true?
- How is it possible to realise free falling mirror on the Earth?
  - Mirror suspended through a pendulum:
    - For small oscillations, in the horizontal direction, the mirror is (in principle) free
    - Obviously vertically is constrained by the suspension to “avoid a true free falling” damage
- But we need to suppress (in the detection frequency range) all the spurious forces acting on the mirrors

“Fundamental” noises

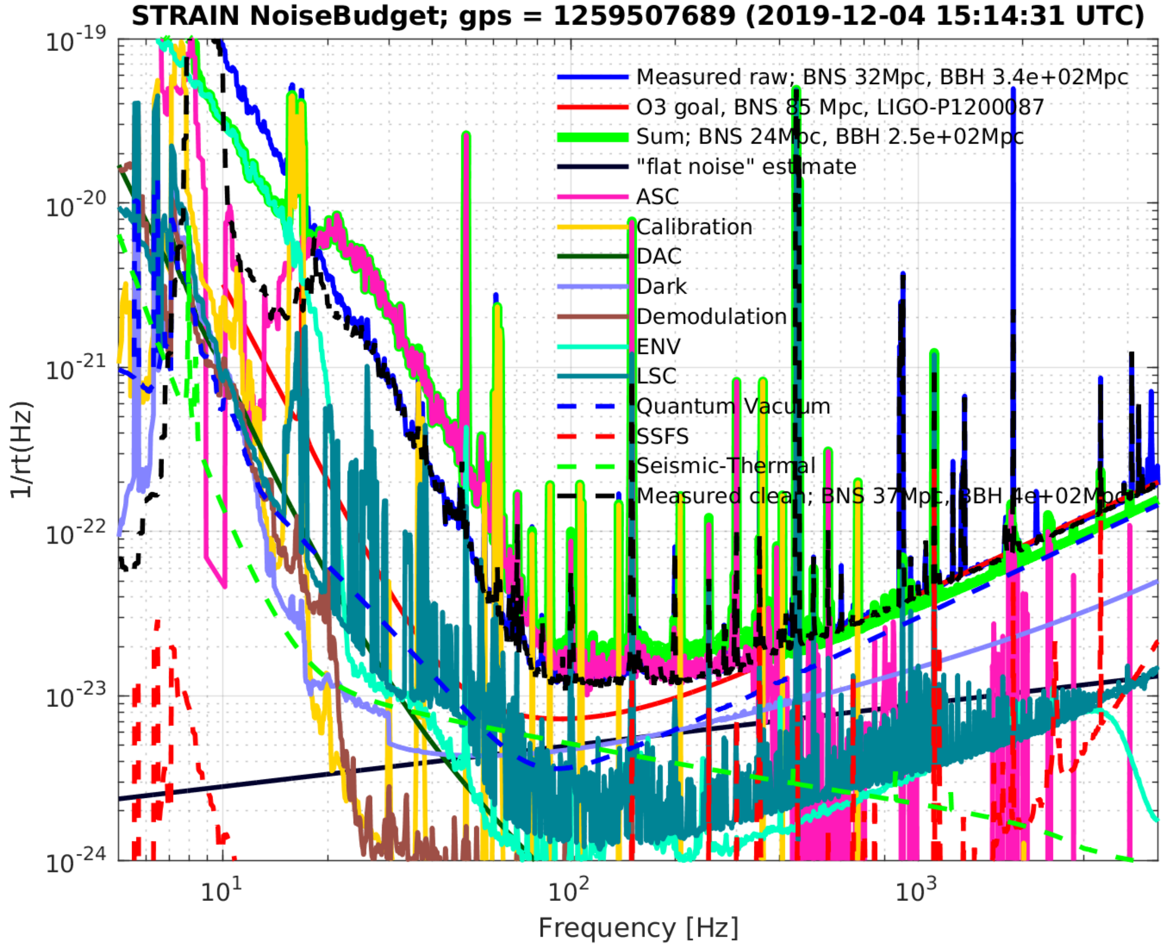


- Seismic, Newtonian, Optical read-out, thermal noises are (arbitrarily) called fundamental noises

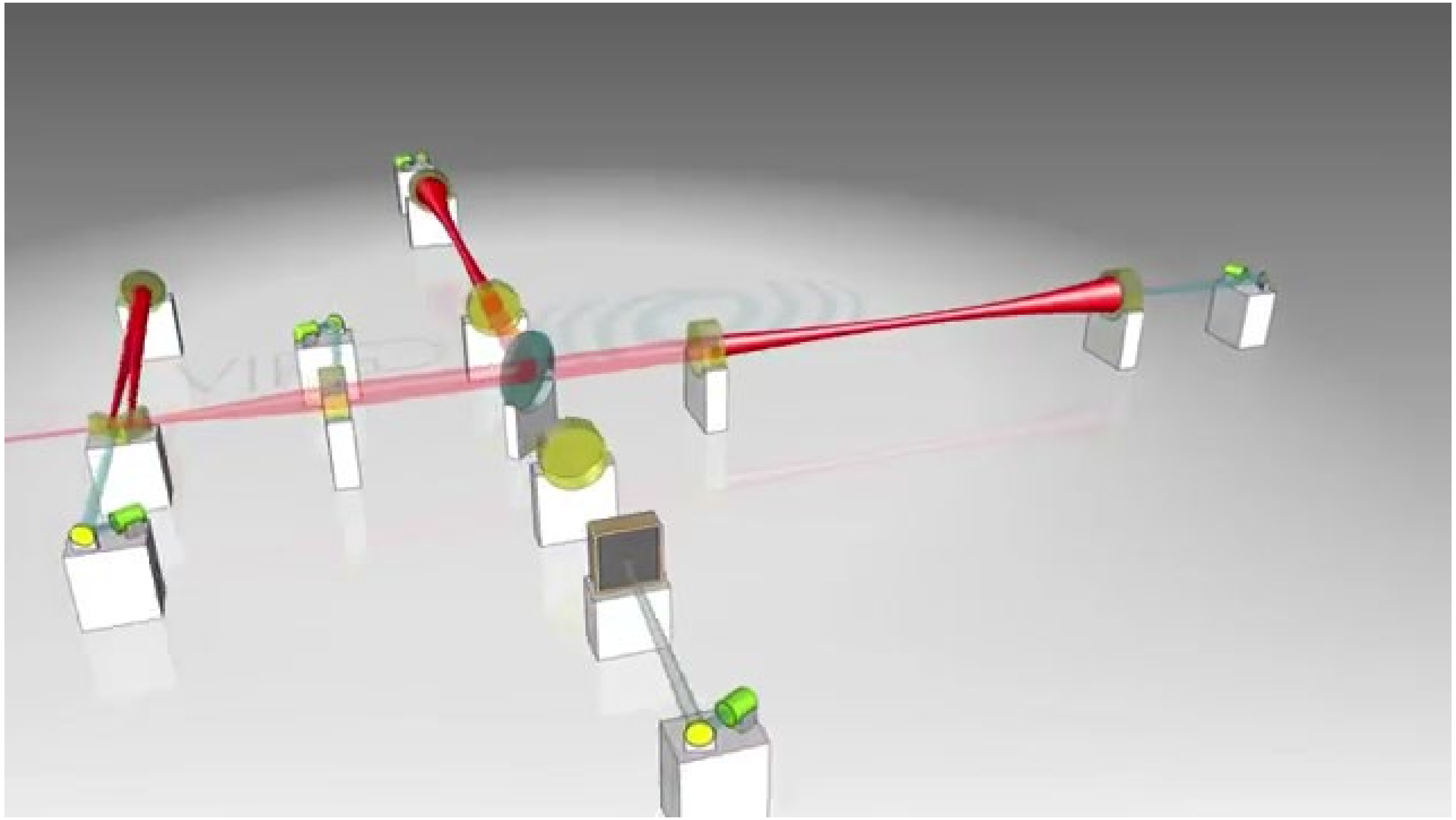


Noise budget

- But, in effect, we are fighting against a plethora of noises of “technical” origin and the long periods of commissioning in each detector are mainly addressed to the reduction of the noises limiting the observation sensitivity

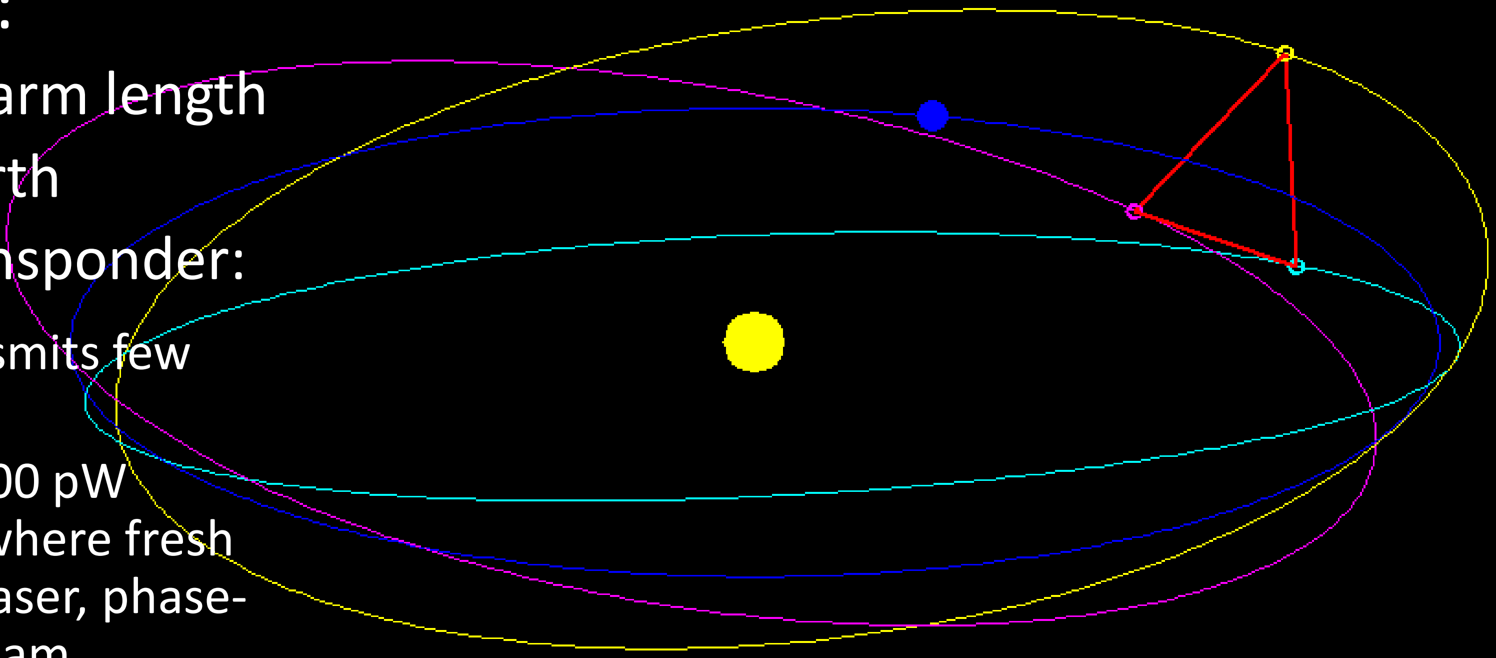


Building Advanced Virgo



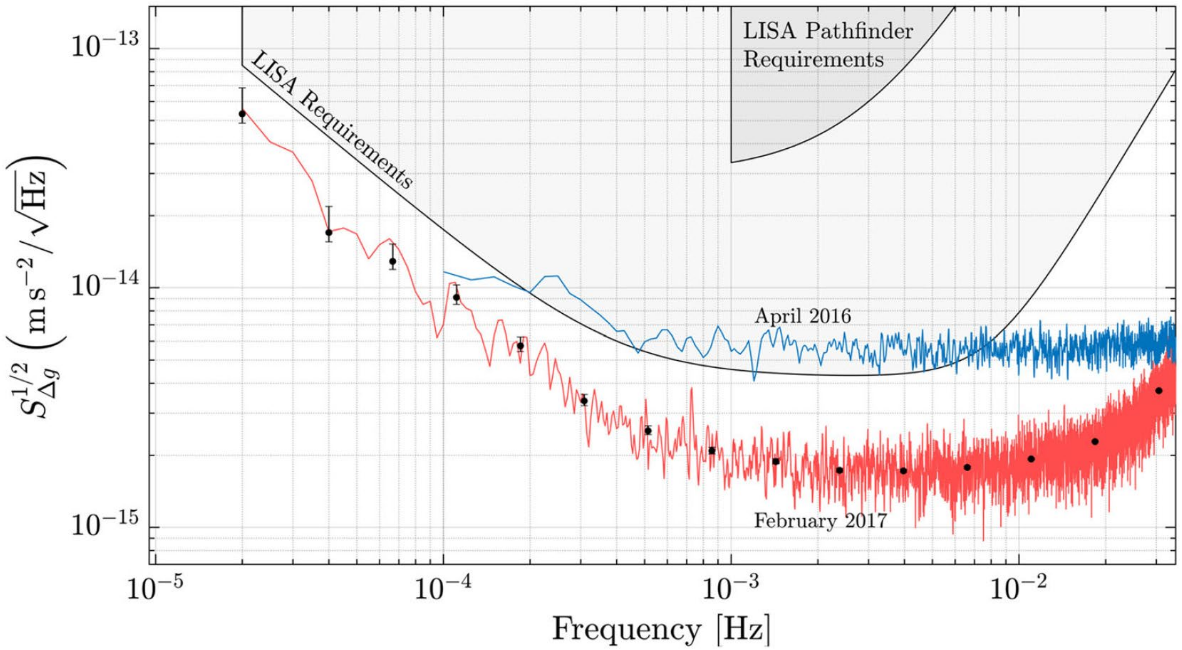
Jumping in Space (LISA) $\Delta L \approx h \cdot L$

- In space, the length limitations (due, for example, to the curvature of the Earth) are naturally solved:
  - LISA project, 2.5×10^6km arm length
  - 50×10^6km away from Earth
  - Each vertex is an active transponder:
    - A 1064 nm stable laser transmits few watt from each vertex
    - Due to beam divergence ~100 pW arrive to the next satellite, where fresh power is emitted by a new laser, phase-locked with the incoming beam

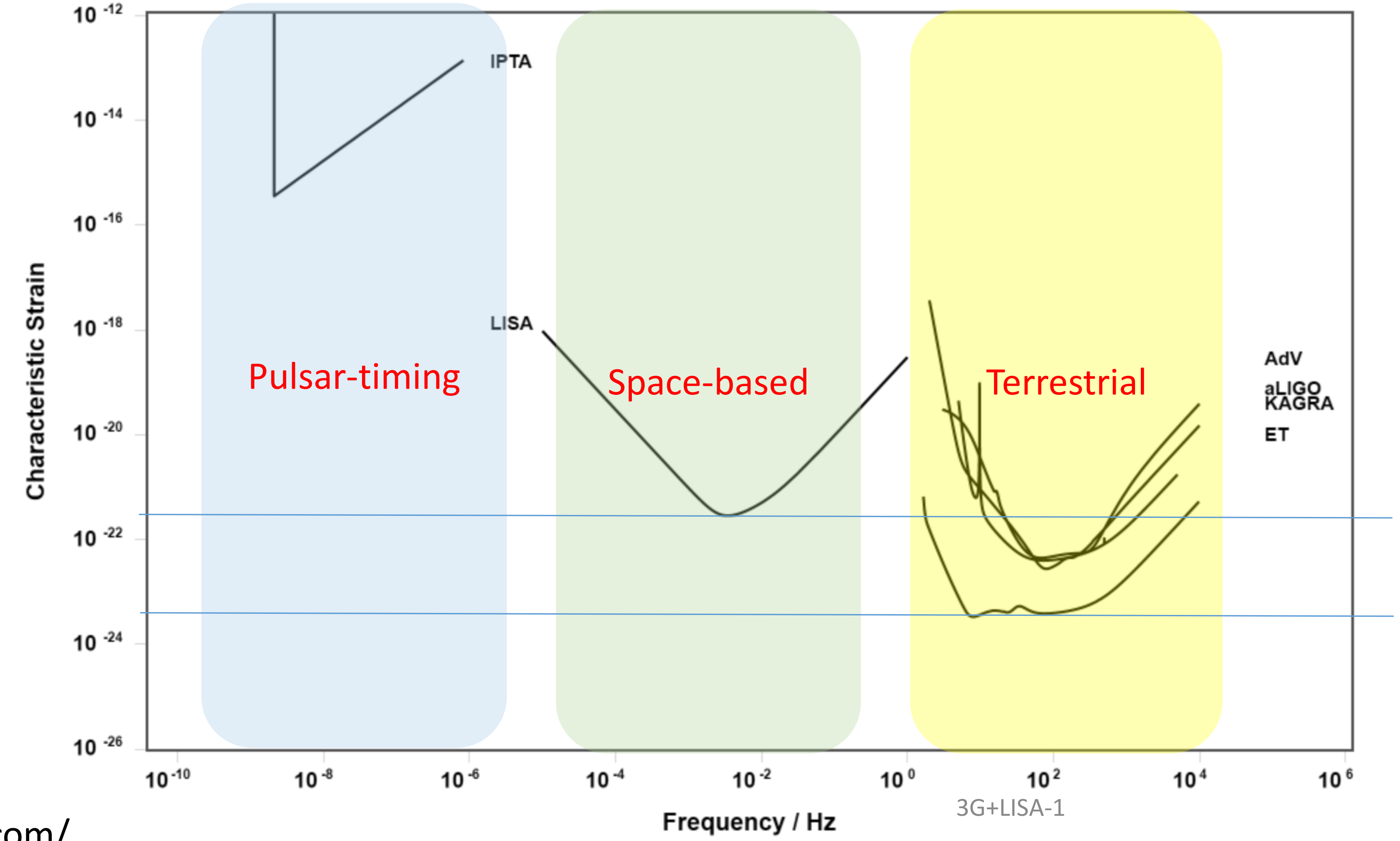


- Combining "offline" the 3 independent arms it is possible to build-up three interferometric signals:
  - 2 Michelson Interferometers (→2 polarisations)
  - 1 Sagnac Interferometer (null stream)

LISA Pathfinder

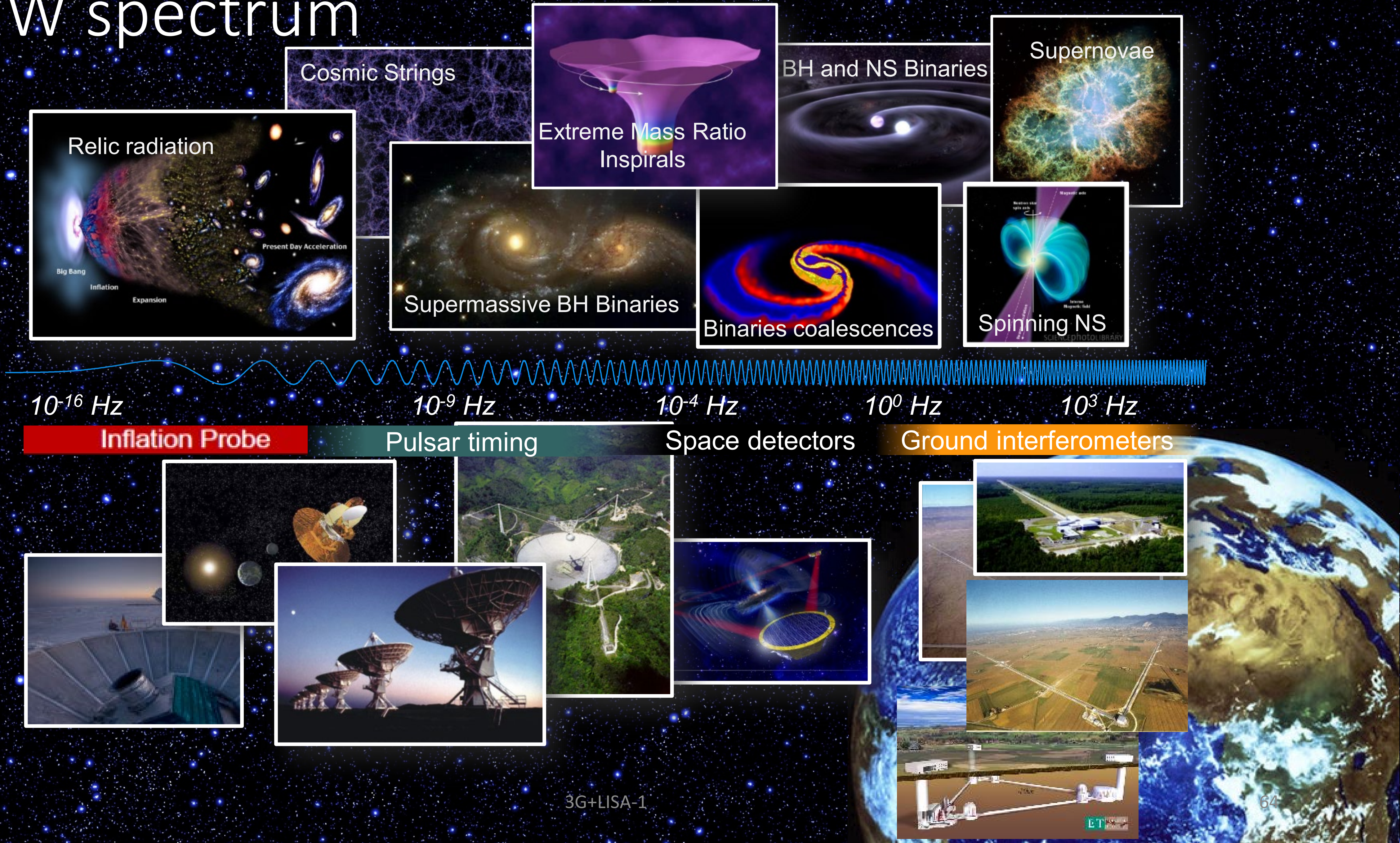


Sensitivity

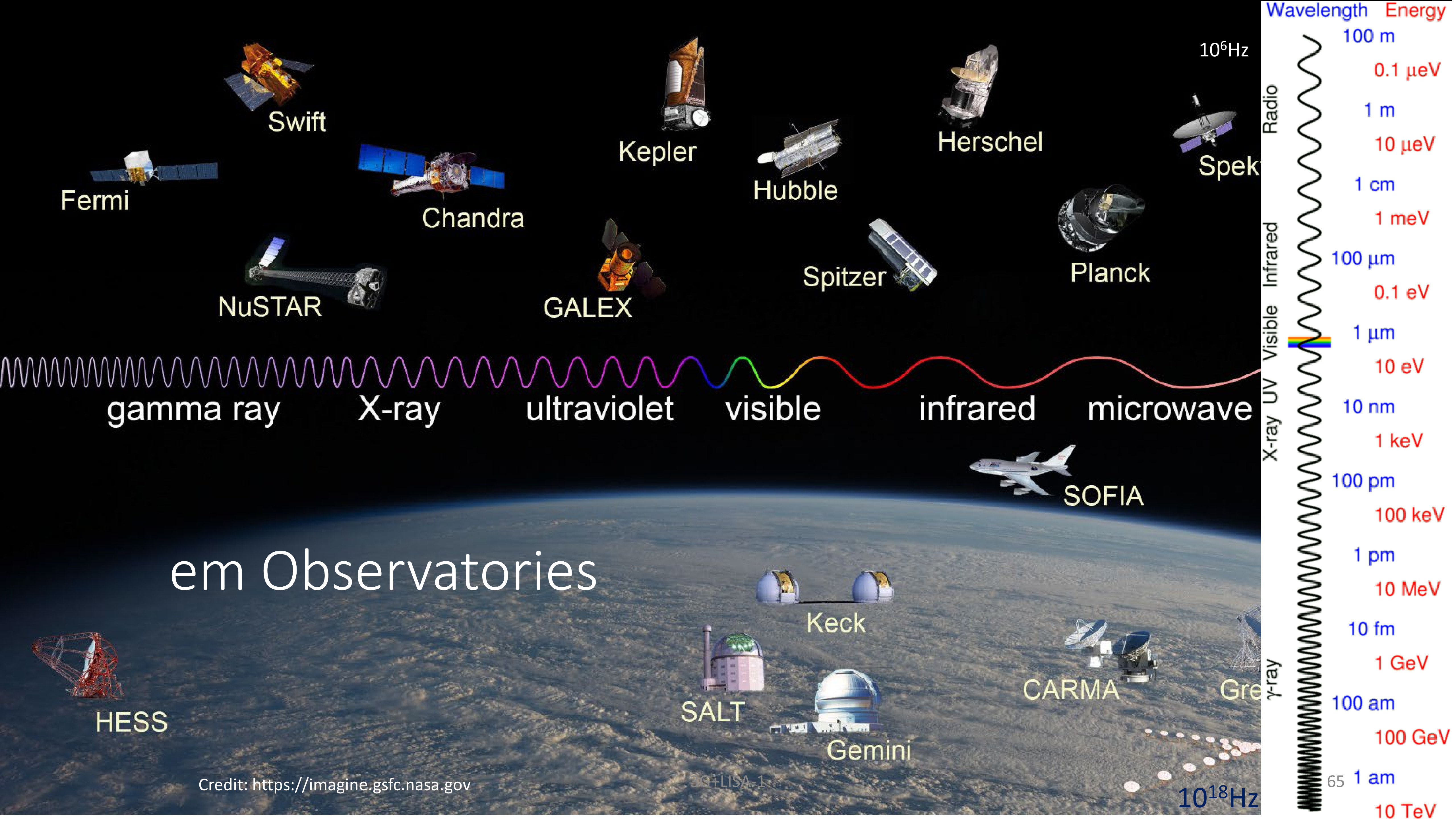


http://gwplotter.com/

The GW spectrum



em Observatories



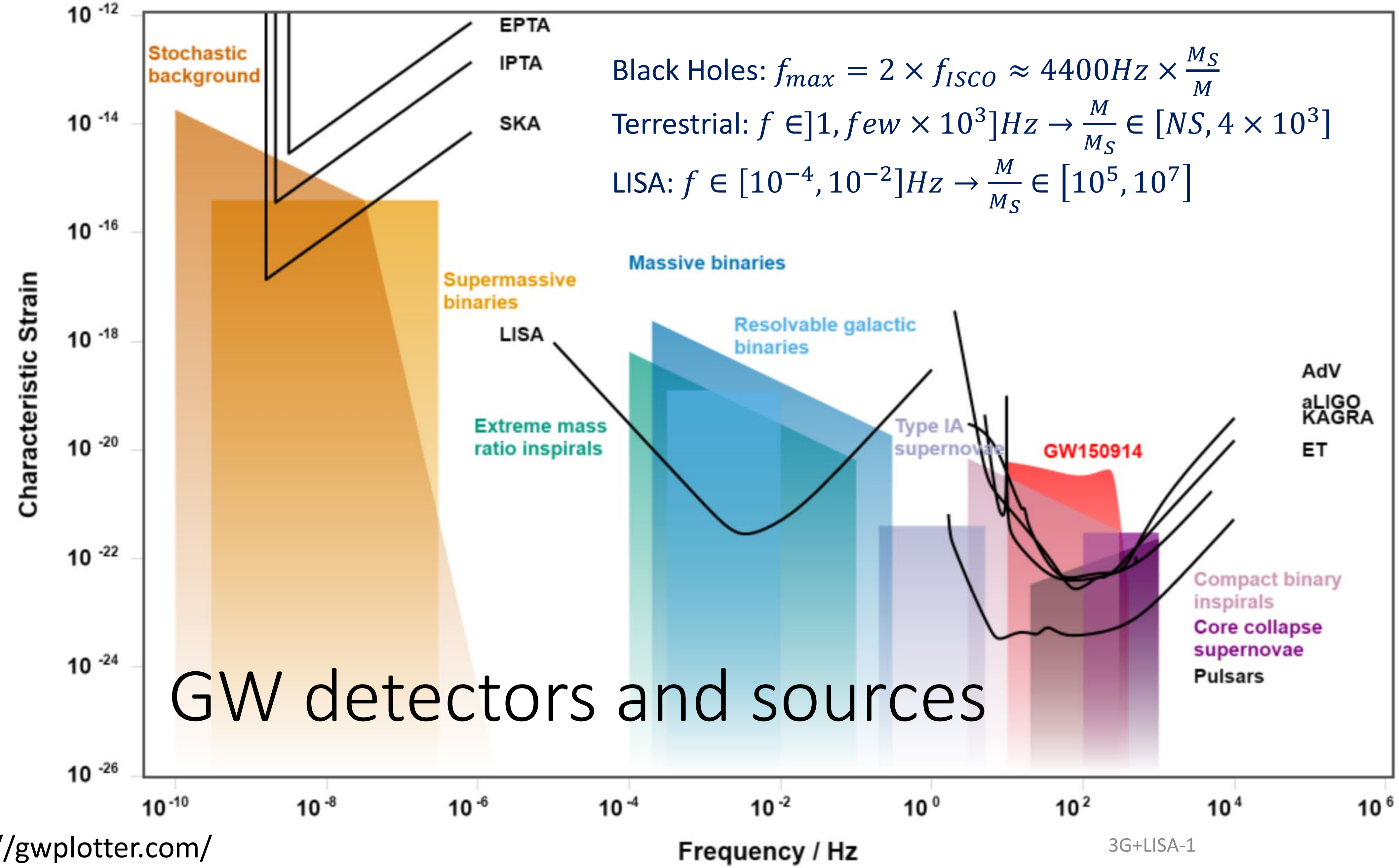
Credit: https://imagine.gsfc.nasa.gov

GW detectors and sources

Black Holes: $f_{max} = 2 \times f_{ISCO} \approx 4400Hz \times \frac{M_S}{M}$

Terrestrial: $f \in]1, few \times 10^3]Hz \rightarrow \frac{M}{M_S} \in [NS, 4 \times 10^3]$

LISA: $f \in [10^{-4}, 10^{-2}]Hz \rightarrow \frac{M}{M_S} \in [10^5, 10^7]$



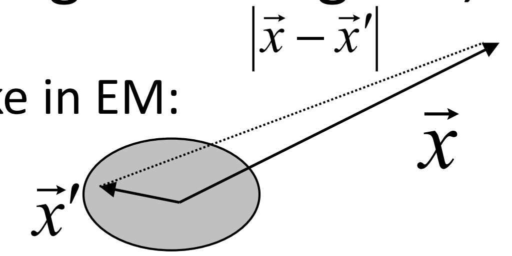
http://gwplotter.com/

Emission of GW

- Let restart from the linearized theory. The equation of field in this case is

$$\Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu} = \kappa T_{\mu\nu} \quad \text{where} \quad \kappa \equiv \frac{16\pi G}{c^4}$$

- $T_{\mu\nu}$ is the energy-momentum tensor, it respects flat space_conservation energy condition $\partial^\mu T_{\mu\nu} = 0$ and we are in the Lorentz gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$. Let suppose to be in a flat space approximation, far from the source that is generating GW, and having slow variations *v/c<<1*:
  - The equation of field can be solved using the retarded potentials like in EM:



$$\bar{h}^{\mu\nu}(t,\vec{x}) = -\frac{\kappa}{4\pi}\int \frac{T^{\mu\nu}\left(t-|\vec{x}-\vec{x}'|,\vec{x}'\right)}{|\vec{x}-\vec{x}'|} d^3x'$$

- We already encountered the multi-pole series to solve $1/\left|\vec{x}-\vec{x}'\right|$
- Introducing again the quadrupole moment

$$Q^{kl} = \int_V \left(3x'^k x'^l - r'^2 \delta^k_l\right)\rho(\vec{x}')d^3x'$$

Emission of GW

- and arresting the series to the quadrupole term we obtain:

$$\left| h_{kl}^{TT}(t,\vec{x}) \right|_{quad} = -\frac{\kappa}{8\pi r}\frac{1}{3}\ddot{Q}_{kl}^{TT}(t - r/c) = \frac{1}{r}\frac{2G}{c^4}\frac{1}{3}\ddot{Q}_{kl}^{TT}(t - r/c)$$

1/r

1.6×10^{-44} $m^{-1}kg^{-1}s^{2}$

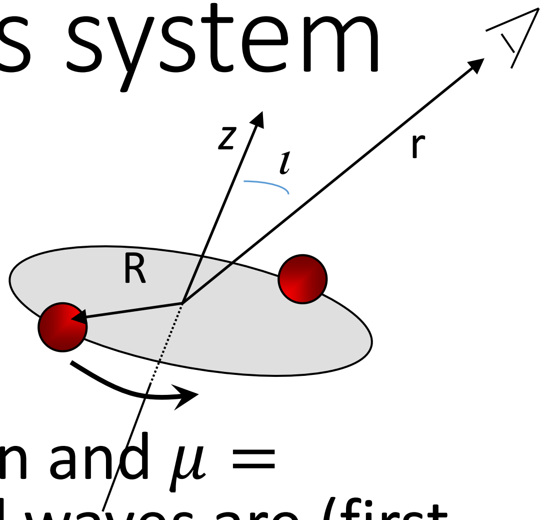
- The amplitude of the generate GW decrease linearly with the distance of the source (remember, GW detectors reveal the amplitude)
- Only sources having masses accelerated with (at least) a quadrupole moment non-null generate GW
  - The very small coefficient requires large (astronomical) masses and huge accelerations
- Why not monopole emission? → forbidden by mass conservation principle
- Why not dipole emission? → forbidden by momentum and angular momentum conservation

Luminosity

- The energy emitted (luminosity) by the accelerated masses is:

$$\mathcal{L} \equiv -\frac{dE}{dt} = \frac{G}{45c^5} \dddot{Q}^{kl} \, \dddot{Q}^{kl}$$

Quadrupolar radiation from a binary mass system in circular orbit



- Let suppose to have a system of two point-like masses, having mass m_1 and m_2, orbiting at frequency ω
- If we define ι the inclination angle of the orbit wrt the sight direction and $\mu = m_1 m_2/(m_1+m_2)$ the reduced mass the amplitudes of the emitted waves are (first term PN approximation – series development in terms of $\left(\frac{v}{c}\right)$)

$$h_+(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{gw}(t_{ret})}{c}\right)^{\frac{2}{3}}\frac{1+\cos^2 i}{2}\cos(\phi(t_{ret}))$$

$$h_\times(t) = \frac{4}{r}\left(\frac{GM_c}{c^2}\right)^{\frac{5}{3}}\left(\frac{\pi f_{gw}(t_{ret})}{c}\right)^{\frac{2}{3}}\cos i\,\sin(\phi(t_{ret}))$$

$$f_{gw}(\tau_0) = \frac{1}{\pi}\left(\frac{5}{256}\frac{1}{\tau_0}\right)^{\frac{3}{8}}\left(\frac{c^3}{GM_c}\right)^{\frac{5}{8}}$$

Chirp mass:

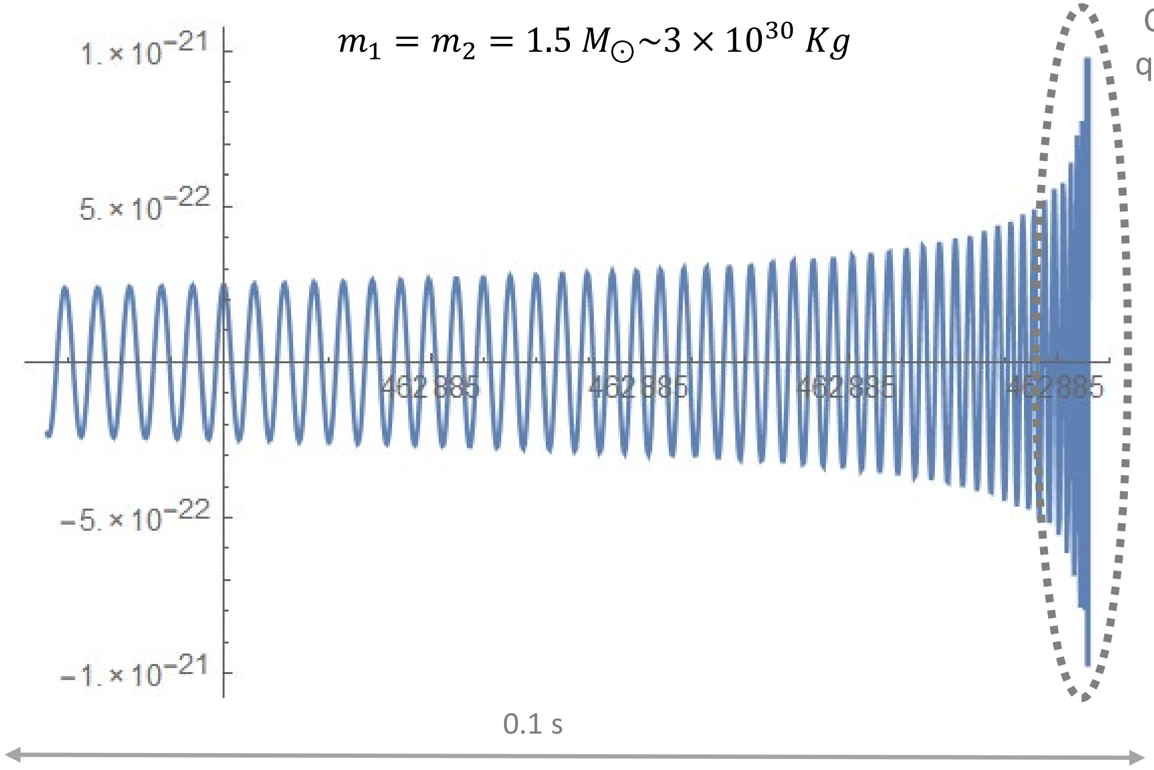
$$M_c = \mu^{\frac{3}{5}}M^{\frac{2}{5}} = \frac{(m_1m_2)^{\frac{3}{5}}}{(m_1+m_2)^{\frac{1}{5}}}$$

$$\left.\begin{aligned} m_1 &= m_2 = 1.4M_\Theta = 1.4\cdot\left(2\cdot 10^{30}\,kg\right) \\ r &= 10\,\mathrm{MPc} = 10\cdot 10^6\cdot\left(3\cdot 10^{16}\,m\right) \\ R &= 50\,km \end{aligned}\right\} \Rightarrow h \approx 1.5\times 10^{-21}\left(\frac{f}{100Hz}\right)^2$$

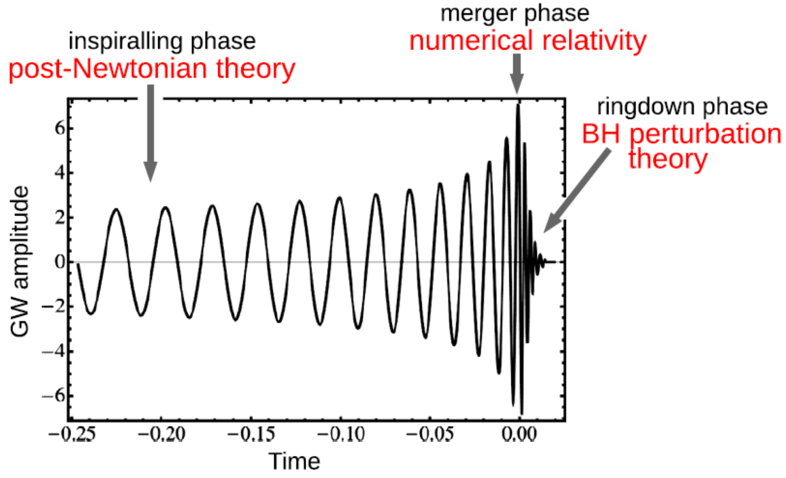
$$\Delta L = h\cdot L \Rightarrow \Delta L \approx 5\times 10^{-18}\,@100Hz$$

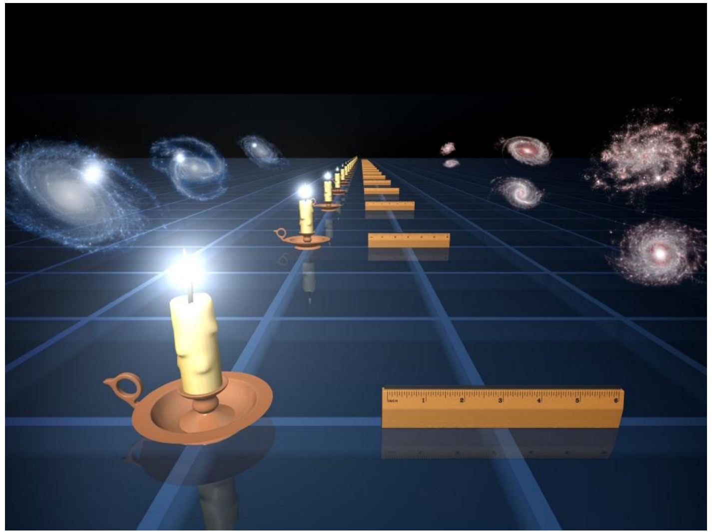
Chirping waveform: Standard sirens!

- The frequency sweep *univocally* identifies the (chirp) mass
- The amplitude defines the luminosity distance:
  - Standard ~~candles~~ sirens!
- In effect is more complex:
  - Chirp mass identification is ambiguous because of the signal red-shift
  - Amplitude measurement is entangled with orbit plane orientation
  - Merging and ringdown phases need numerical relativity & BH perturbation theory (or EOS of NS)



Close to merger, the quasi-circular motion assumption is no longer valid

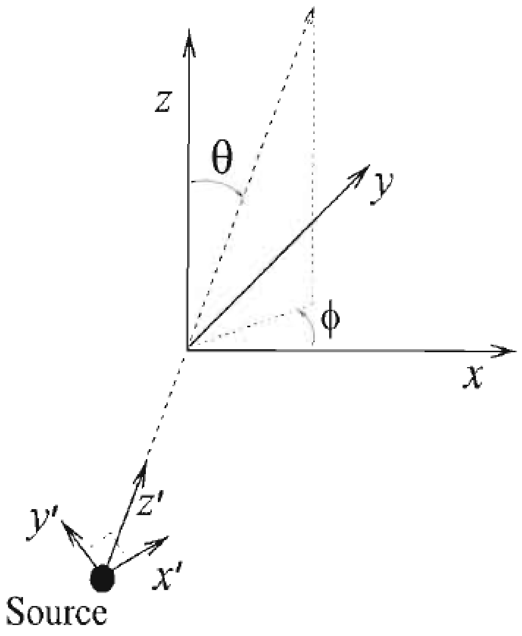




3G+LISA-1

Detector acceptance: antenna pattern

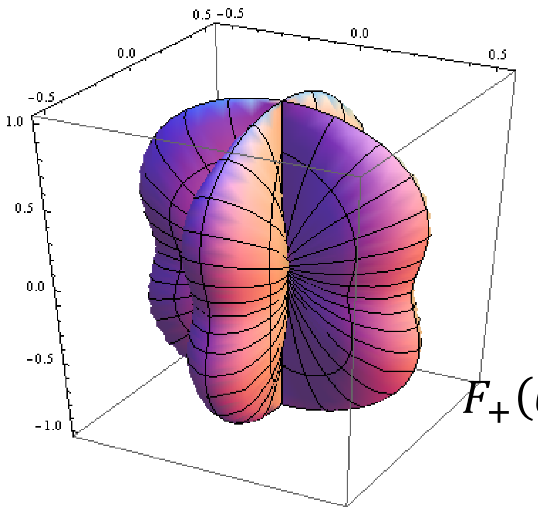
- Interferometric GW aren't isotropic in their acceptance
- For example if a wave arrives along the detector plane with 45 degrees wrt the arms, the detector is blind
- In general the detector (xy) measures a combination of the two polarisations according to the formula



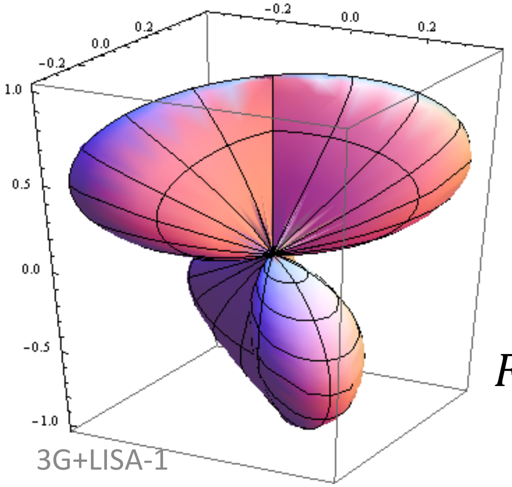
$$h = F_+ h_+ + F_\times h_\times$$

$$F_+(\theta, \varphi) = \frac{1}{2}\left(1 + \cos^2\theta\right)\cos 2\varphi$$

$$F_\times(\theta, \varphi) = \cos\theta \sin 2\varphi$$



$F_+(\theta, \varphi)$



$F_\times(\theta, \varphi)$

Sources of GW

- We seen that to generate a GW we need a system of large masses accelerated with a quadrupolar component
- Binary systems:
  - Composed by black holes (BH-BH)
  - Composed by Neutron Stars (NS-NS)
  - Composed by a BH and a NS (BH-NS)
- Isolated NS
- Supernova Explosion (Sne)

- Let see (some of) these sources through the LIGO-Virgo detections and the perspectives on the 3rd generation of GW observatories