Euclidean Partons?

G.C. Rossi, M. Testa

Phys. Rev. D 96, 014507 (2017)
Phys. Rev. D 98, 054028 (2018)

Deep Inelastic Scattering

$$W(q^2, q \cdot p) = \int d^4x \, e^{iq \cdot x} \langle p | j(x) j(0) | p \rangle$$
$$j(x) \equiv \text{``}\phi(x)^2\text{''}$$

$\langle p | j(x) j(0) | p \rangle$ is a distribution, i.e. a singular function with integrable singularities, so that in the Bjorken limit

$$q^2 \to -\infty$$
$$\omega = -\frac{q^2}{2p \cdot q}$$

$W(q^2, q \cdot p)$ will not vanish exponentially (Riemann-Lebesgue lemma) and will be determined by the singularities of the Wightman function

The Euclidean Connection

We start defining the Minkowski amplitude

$$T(q,p) \equiv \int d^4x\, e^{iqx} < p|T(j(x)j(0))|p >$$

We can perform the *change of variables*

$$\begin{aligned} x^0 &= -ix_E^0 \\ \mathbf{r} &= \mathbf{r}_E \end{aligned}$$

so that

$$T(q^2,\omega) = -i \int d^4x_E\, e^{q^0 x_E^0} < p|T_E(j(x_E)j(0))|p > e^{-i\mathbf{q}\cdot\mathbf{r}},$$

where in the Bj limit

$$q^0 = \frac{(-q^2)}{2m}\omega$$

This is the expression of the Minkowski amplitude in terms of euclidean quantities and is valid, provided it is well defined.

Due to the presence of the growing exponential $e^{q^0 x_E^0}$ we must worry about the behaviour of the euclidean T-product as $x_E^0 \to +\infty$. If we define

$$F(x_E^0) \equiv \int d\mathbf{r} < p|T_E(j(x_E)j(0))|p > e^{-i\mathbf{q}\cdot\mathbf{r}} \, ,$$

we have

$$F(x_E^0) \to_{x_E^0 \to +\infty} (2\pi)^3 \sum_n | < n|j(0)|p > |^2 e^{-(E_n - m)x_E^0} \delta(\mathbf{p}_n - \mathbf{q}) \, .$$

so that the condition for the validity of the change of variables is

$$q^0 < E_n - m = \sqrt{M_n^2 + \mathbf{q}^2}$$

from which it follows

$$q^2 + 2mq^0 + m^2 = (q+p)^2 < M_n^2$$

which shows that the deep inelastic region cannot be reached from euclidean data

The Canonical Light-Cone approach

In analogy with free field theory, the light-cone singularity was assumed of the form

$$(2\pi)^4 W(q^2, q\cdot p) = \int d^4x e^{-iq\cdot x} < p| : \phi(0)\phi(x) : |p > \Delta(x)$$

where

$$\Delta(x) \equiv \int \frac{d\mathbf{k}}{2|\mathbf{k}|} e^{ik\cdot x} = \int d^4k\, \delta(k^2)\theta(k^0) e^{ikx} ,$$

In the canonical parton model the quantity $< p| : \phi(0)\phi(x) : |p >$ is regular and we have

$$< p| : \phi(0)\phi(x) : |p >= \sum_{n=0}^{+\infty} \frac{x^{\mu_1} \dots x^{\mu_n}}{n!} \langle p|\phi(0)\partial_{\mu_1} \dots \partial_{\mu_n}\phi(0)|p\rangle$$

$$\langle p|\phi(0)\partial_{\mu_1} \dots \partial_{\mu_n}\phi(0)|p\rangle = A_n p^{\mu_1} \dots p^{\mu_n} + B_n \dots g^{\mu_l \mu_k} \dots$$

so that

$$< p| : \phi(0)\phi(x) : |p >= \sum_{n=0}^{+\infty} \frac{A_n (x \cdot p)^n}{n!} + B_n \dots x^2 \dots$$

and, when evaluated at $x^2 \approx 0$, we have

$$\tilde{f}(p \cdot x) \equiv < p| : \phi(0)\phi(x) : |p > |_{x^2=0} = \int_{-\infty}^{+\infty} d\lambda f(\lambda) e^{-i\lambda p \cdot x}$$

$$f(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{f}(p \cdot x) e^{i\lambda p \cdot x} d(p \cdot x)$$

which leads to

$$W(q^2, q \cdot p) \approx \frac{\omega f(\omega)}{-q^2} \,,$$

giving the structure functions in terms of the Fourier transform of the bilocal matrix element.

The structure function is completely determined by the A_n form factors.

The Real Life Light-Cone approach

In QCD (as in every renormalizable Field Theory) the singularity structure is more complicated

$$\langle p|j(x)j(0)|p\rangle \approx_{x^2\approx 0}$$
$$\Delta(x^2)\sum_{n=0}^{+\infty}\alpha_n(\mu^2x^2)x^{\mu_1}\ldots x^{\mu_n}\langle p|\tilde{O}^{(n)}_{\mu_1\ldots\mu_n}(0)|p\rangle$$

where $\tilde{O}^{(n)}_{\mu_1\ldots\mu_n}(0)$ denote a renormalized version of

$$O^{(n)}_{\mu_1\ldots\mu_n}(0) = \phi(0)\partial_{\mu_1}\ldots\partial_{\mu_n}\phi(0)$$

The $O^{(n)}_{\mu_1\ldots\mu_n}(0)$'s are in general **not multiplicatively renormalizable** (but they are in dimensional regularization)

After renormalization we have

$$\langle p|\tilde{O}^{(n)}_{\mu_1\ldots\mu_n}(0)|p\rangle =$$
$$= A^{(n)}(\mu)\, p_{\mu_1}\ldots p_{\mu_n} + B^{(n)}(\mu) p_{\mu_1}\ldots g_{\mu_i\mu_j}\ldots p_{\mu_n}$$

$$W(q^2, q\cdot p) \to_{Bj} \frac{\omega f(\omega, q^2)}{-q^2}$$

We will consider $-q^2 = \mu^2$ (Evolution)

We have

$$A^{(n)}(\mu) = \int_{-1}^{+1} d\omega f(\omega,\mu^2)\omega^n = \int_{-\infty}^{+\infty} d\omega f(\omega,\mu^2)\omega^n$$

The $A^{(n)}(\mu)$ are measured quantities and can be computed on the lattice as matrix elements of appropriately renormalized local operators
Before renormalization the A_n are logarithmically divergent
It is a difficult mathematical problem to reconstruct the structure function by the knowledge of the moments (Stiltjes problem)

Partons and Light-Cone expansion in Lattice regularized QCD

As previously discussed, the deep inelastic scattering process cannot be simulated in the euclidean region starting with the currents, but a very interesting proposal by Xiangdong Ji reintroduces the idea bilocal operators in the space-like region

The basic formula of the approach is based on the $P_z \to \infty$ limit

$$f(\omega) = \lim_{P_z \to \infty} \tilde{F}(\omega, P_z)$$

where

$$\tilde{F}(\omega, P_z) = \frac{P_z}{2\pi} \int_{-\infty}^{+\infty} dz e^{iP_z z \omega} \langle P_z | \phi(0)\phi(z) | P_z \rangle =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tilde{z} e^{i\tilde{z}\omega} \langle P_z | \phi(0)\phi(\tilde{z}/P_z) | P_z \rangle \,,$$

and $\tilde{z} \equiv P_z z$

How can we be sure that $f(\omega)$ is the correct structure function?
It must satisfy **necessary** (and **sufficient**) conditions

1. $f(\omega)$ must be u.v. finite;
2. the support of $f(\omega)$ must be contained in $(-1, +1)$
3. The moments of $f(\omega)$ must be related to the matrix elements of the renormalized local operators generated by the bilocal

$$A^{(n)}(\mu) = \int_{-1}^{+1} d\omega f(\omega, \mu^2)\omega^n = \int_{-\infty}^{+\infty} d\omega f(\omega, \mu^2)\omega^n$$

As for the condition 1, $\langle P_z|\phi(0)\phi(z)|P_z\rangle$ can be easily made u.v. finite distribution through an harmless logarithmic wave function renormalization. After that it becomes a well defined distribution and is only logarithmically divergent as $z \to 0$ (integrable)
Therefore the Fourier transform of the renormalized bilocal, $\tilde{F}(\omega, P_z)$, is u.v. finite

Condition 2 on the support is difficult to check
We will assume it is satisfied although it appears not to be in numerical simulations

Condition 3 is more tricky

We start from the definition

$$\tilde{F}(\omega, P_z) = \frac{P_z}{2\pi} \int_{-\infty}^{+\infty} dz e^{izP_z\omega} \langle P_z|\phi(0)\phi(z)|P_z\rangle =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\tilde{z} e^{i\tilde{z}\omega} \langle P_z|\phi(0)\phi(\tilde{z}/P_z)|P_z\rangle$$

and invert it

$$\langle P_z|\phi(0)\phi(\tilde{z}/P_z)|P_z\rangle = \int_{-\infty}^{+\infty} d\omega e^{-i\tilde{z}\omega}\tilde{F}(\omega, P_z)$$

We can take the n-th derivative with respect to $\tilde{z}$ at $\tilde{z} = 0$

$$(-i)^n \int_{-\infty}^{+\infty} d\omega\ \omega^n \tilde{F}(\omega, P_z) = \frac{1}{(P_z)^n}\langle P_z|\phi(0)\frac{\partial^n \phi}{\partial z^n}(0)|P_z\rangle\,,$$

which clearly shows the origin of the u.v. divergencies coming from power divergent trace terms

This argument shows that, even if $\langle P_z|\phi(0)\phi(z)|P_z\rangle$ is only logarithmically divergent as $z \to 0$, the moments of $\tilde{F}(\omega, P_z)$ will, in general, exhibit power divergencies: the moments are not quantities of a distribution-theoretical nature.

A simple toy model of what is happening is provided by

$$\langle P_z|\phi(0)\phi(\tilde{z}/P_z)|P_z\rangle \approx \log|z|$$

which shows how the bare local operators are more and more divergent with increasing n

The corresponding "structure function" is obtained through the formula

$$\int_{-\infty}^{+\infty} e^{iz\omega} \log|z| dz = -\frac{1}{2|\omega|}$$

and the moments are infinite because of the failure to reproduce the support property

Matching

The approach proposed by Ji does not identify directly the Fourier transform of the bilocal with the physical structure function. In fact one starts with the Fourier transform in the presence of the regulator $\Lambda \approx 1/a$

$$\tilde{F}(\omega, P_z, \Lambda) = \frac{P_z}{2\pi} \int_{-\infty}^{+\infty} dz e^{iP_z \omega z} \langle P_z | \phi(0) \phi(z) | P_z \rangle |_\Lambda$$

a quantity denoted as a Quasi-PDF. As already discussed the moments of the Quasi-PDF are u.v. divergent

A “matching procedure” is then applied to $\tilde{F}(\omega, P_z, \Lambda)$ through a condition of the form

$$F(\omega, \mu) = \int_{\omega}^{+\infty} \frac{dx}{x} Z(\frac{\omega}{x}, \Lambda, P_z) \tilde{F}(x, P_z, \Lambda)$$

where $Z(\frac{\omega}{x}, \Lambda, P_z)$ is computed in perturbation theory through the requirement that $F(x, \mu)$ be u.v. finite

However the convolution property of the Mellin transform implies

$$\int_0^{+\infty} d\omega F(\omega,\mu)\omega^n = \int_0^{+\infty} dx x^n Z(x,\Lambda) \int_0^{+\infty} dx \tilde{F}(x,P_z,\Lambda)x^n =$$
$$\equiv Z_n\left(\frac{\Lambda}{\mu}\right) \int_0^{+\infty} dx \tilde{F}(x,P_z,\Lambda)x^n$$

which reads

$$\int_0^{+\infty} d\omega F(\omega,\mu)\omega^n = \frac{Z_n(\Lambda/\mu)}{(P_z)^n} \langle P_z|\phi(0)\frac{\partial^n\phi}{\partial z^n}(0)|P_z\rangle|_\Lambda$$

This clearly shows the multiplicative nature of the matching condition. The problem is that the Z_n should be the renormalization constants which make the operators

$$Z_n(\Lambda/\mu)\,\phi(0)\frac{\partial^n\phi}{\partial z^n}(0)$$

finite. However these operators are not multiplicatively renormalizable due to the presence of divergent trace terms, which require actual subtractions and not only multiplications

Truncation as a cure for u.v. divergencies

Suppose that, in order to solve the problem of u.v. divergencies, we truncate the structure functions saying that we only consider the restriction of the structure function to the interval $\omega \in (-1, +1)$.
In other words we compute the moments as $\int_{-1}^{+1} d\omega e^{-i\tilde{z}\omega} \tilde{F}(\omega, P_z)$.
We have

$$\int_{-1}^{+1} d\omega e^{-i\tilde{z}\omega} \tilde{F}(\omega, P_z) = \frac{1}{2\pi} \int_{-1}^{+1} d\omega \int_{-\infty}^{+\infty} d\tilde{z}' e^{-i(\tilde{z}-\tilde{z}')\omega} \langle P_z | \phi(0)\phi(\tilde{z}'/P_z) | P_z \rangle =$$

$$= \frac{1}{\pi} \int_{-\infty}^{+\infty} d\tilde{z}' \, \frac{\sin(\tilde{z}-\tilde{z}')}{\tilde{z}-\tilde{z}'} \langle P_z | \phi(0)\phi(\tilde{z}'/P_z) | P_z \rangle$$

so that the computation of a moment restricted to $(-1, +1)$ corresponds to the matrix element

$$(-i)^n \int_{-1}^{+1} d\omega \, \omega^n \tilde{F}(\omega, P_z) =$$

$$= \frac{d^n}{d\tilde{z}^n} \Big[\frac{1}{\pi} \int_{-\infty}^{+\infty} d\tilde{z}' \, \frac{\sin(\tilde{z}-\tilde{z}')}{\tilde{z}-\tilde{z}'} \langle P_z | \phi(0)\phi(\tilde{z}'/P_z) | P_z \rangle \Big]_{\tilde{z}=0}$$

which are not the matrix elements of local operators any more.

IN CONCLUSION

THE STRATEGY TO COMPUTE STRUCTURE FUNCTIONS FROM LATTICE QCD, ALTHOUGH VERY INTERESTING, IS STILL MISSING SOME ESSENTIAL INGREDIENT

THANK YOU FOR YOUR ATTENTION