FCCP 2019

Workshop on “Flavour changing and conserving processes” 2019

NNLO QED Virtual Amplitude for the MUonE experiment

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in collaboration with:

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CSIC Consejo Superior de Investigaciones Científicas

Excelencia Severo Ochoa

Outline

Preliminaries:

- *MUonE Experiment*: Marconi's talk
- $\mu e \to \mu e$ *Scattering Amplitude* @ LO and NLO: Carloni-Calame's talk

This talk:

- NNLO anatomy + focus on Virtual Contributions
- 2-loop Amplitude Evaluation
  - Decomposition into Master Integrals: Integrands & Integrals
  - Technology exploited
  - Analytic evaluation
  - 2-loop Renormalization
- Conclusions

Motivation

Objective: precise calculation of $(g-2)_\mu \implies \alpha_\mu$.

$$\textbf{Experimental value}: a_\mu^{exp} = 116592089(63) \times 10^{-11}$$

From the **theoretical** side:

- *QED* provides more than 99.99% of the SM value of α_μ

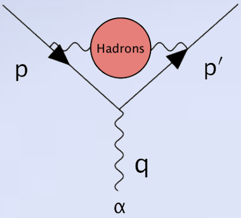
$$a_\mu^{QED} = 116584718.944(21)(77) \times 10^{-11}$$

- *Weak interactions* are the least relevant

$$a_\mu^{weak} = 156.3(1) \times 10^{-11}$$

- *Hadronic* contribution $\sim 60ppm$

$$\begin{aligned} a_\mu^{HLO} &= 6870(42) \times 10^{-11} \\ &= 6926(33) \times 10^{-11} \\ &= 6949(37)(21) \times 10^{-11} \end{aligned}$$



Upcoming validation with higher precision

- FNAL-E989 aims at $\pm 16 \times 10^{-11}$(0.14ppm)

$$\text{Relation between } a_\mu^{HLO} \to \Delta\alpha_{had}(t) \to \boxed{-\Pi_{Had}(t)}$$

Extraction of a_μ^{HLO} from μe **Elastic Scattering**

[Passera, Carloni-Calame, Trentadue, Venanzoni (2015)]

$\mu e \to \mu e$ Scattering Amplitude: overview

Target observable: **cross section** σ, related to the **scattering amplitude** $\mathcal{M}(p_i \to p_f)$

$$\sigma = \int_{LIPS} |\mathcal{M}(p_i \to p_f)|^2$$

Perturbative QFT: **Higher precision** $\implies$ calculating **higher orders** of σ

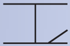
$$\sigma = \sigma_{LO} + \sigma_{NLO} + \sigma_{NNLO} + \cdots + \mathcal{O}(\alpha^n)$$

Orderwise:

$\sigma_{LO} =$ $\int \Big[\sum_i$  $_i^{*} \times$  $_i \Big] \, d\mathrm{LIPS}(p_f)$ tree-level

$\sigma_{NLO} =$

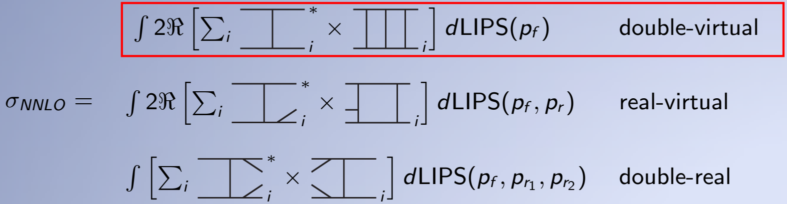
$\int 2\Re \Big[\sum_i$  $_i^{*} \times$  $_i \Big] \, d\mathrm{LIPS}(p_f)$ virtual

$\int \Big[\sum_i$  $_i^{*} \times$  $_i \Big] \, d\mathrm{LIPS}(p_f, p_r)$ real

[Alacevich, Chiesa, Montagna, Nicrosini, Piccinini, Carloni Calame (2018)]
[Fael, Passera (2019)]

$\mu e \to \mu e$ Scattering Amplitude: overview

NNLO cross section:



Double-virtual corrections are hard to compute:

- 2-loop multiscales integrals
- performing computer algebra tools needed
- unknown integrals for $m_e \neq m_\mu \neq 0$

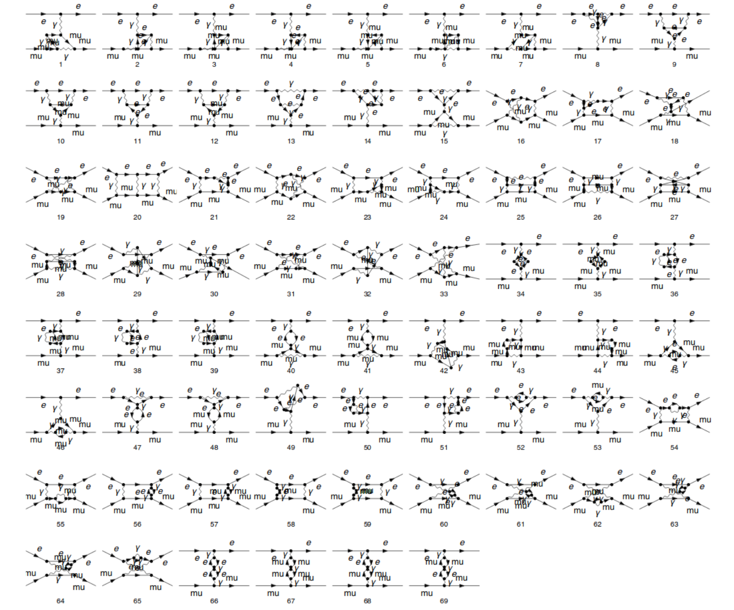
Observation: $\frac{m_e}{m_\mu} \simeq 2.3 \cdot 10^{-5} \implies m_e \to 0$.

- One less scale yields to *simplification of the algebra*
- Integrals with only m_μ dependence are *known*

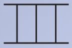
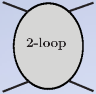
We regularize these integral using the *dimensional regularization scheme*.

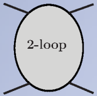
GoSam + [Chiesa, Greiner, Tramontano (2015)]

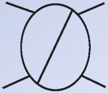
$\mu e \to \mu e$ Scattering Amplitude: 2-loop diagrams



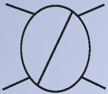
After dealing with the *tensor/Dirac algebra*, double-virtual amplitude can be expressed as

$2\Re\Big[\sum_i$  $^*_i \times$  $_i\Big] = 2\Re[\mathcal{M}^*_{(0)}\mathcal{M}_{(2)}] =$ 



$= \sum_j C'_j(s, t, m_\mu^2)$  $_j$

where

 $_j = \tilde{I}_j(s, t, m_\mu^2) = \int_{\mathbb{M}} d^d\bar{k}\, \frac{\mathcal{N}_j(\bar{p}, \bar{k})}{\mathcal{D}_j(\bar{p}, \bar{k}, m_j^2)}$

$\bar{p} = \{p_i, p_f\}$, $\bar{k} = \{k_1, k_2\}$

- Virtual contributions can be expressed as a *linear combination* of **2-loops Feynman integrals** $\tilde{I}(s, t, m_\mu^2)$.
- Coefficients $C'_j(s, t, m_\mu^2)$ contain the Feynman rules and depend also on dimension $d = 4 - 2\epsilon$

Feynman integrals $\tilde{I}(s, t, m_\mu^2)$ might have a very complex structure.
A further simplification is needed.

Idea: $d = d_{\parallel} + d_{\perp}$. Loop momenta can be parametrized into *parallel* and *transverse components* with respect to Span(p_i, p_f):

$$k_i^\mu = \sum_{j=1}^{d_\parallel} x_{\parallel ij} p_j^\mu + \lambda_i^\mu, \qquad \lambda_i^\mu = \sum_{j=d_\parallel+1}^{4} x_{\perp ij} e_j^\mu + \mu_i^\mu$$

[Collins (1984)]
[van Neerven and Vermaseren (1984)]

The integrand expressed terms of $\bar{x}_\parallel, \bar{x}_\perp, \bar{\mu}^2$ becomes

$$\int_{\mathbb{M}} d^d \bar{k} \frac{\mathcal{N}_j(\bar{p}, \bar{k})}{\mathcal{D}_j(\bar{p}, \bar{k}, m_j^2)} = \int_{\mathbb{M}} d^d \bar{k} \frac{\mathcal{N}_j(\bar{x}_\parallel, \bar{x}_\perp, \bar{\mu}^2)}{\mathcal{D}_j(\bar{x}_\parallel, \bar{\lambda}^2, m_j^2)}$$

- The denominator is $\mathcal{D}_j(\bar{x}_\parallel, \bar{\lambda}^2, m_j^2) = D_{j1}^{a_{j1}} \cdots D_{jk}^{a_{jk}}$, product of **inverse propagators**.
- Expressing $x_{\parallel i}$ in terms of a combination of D_j:

$$x_{\parallel i} = \sum_j \alpha_{ij} D_j + f_i(s, t, m_\mu^2), \qquad \lambda_{ij} = \sum_k \beta_{ijk} D_k + f_{ij}(s, t, m_\mu^2)$$

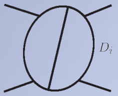
$\lambda_i^\mu \lambda_{j\mu} = \lambda_{ij}^2,\ \bar{\lambda}^2 = \{\lambda_{ij}^2, \forall i, j\}$

[Mastrolia, Peraro, Primo (2016)]
[Mastrolia, Peraro, Primo, Torres Bobadilla (2016)]

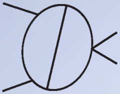
The numerator becomes

$$\mathcal{N}_j(\bar{x}_\parallel, \bar{x}_\perp, \bar{\mu}^2) = \sum_{i=1}^{n_{den}} \mathcal{N}_{ji}(\bar{x}_\parallel, \bar{x}_\perp, \bar{\mu}^2) D_i + \Delta_j(\bar{x}_\parallel, \bar{x}_\perp, \bar{\mu}^2)$$

$\Longrightarrow$

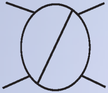


$= \sum_{i=1}^{n_{den}}$



$+ \dfrac{\Delta_j(\bar{x}_\parallel, \bar{x}_\perp, \bar{\mu}^2)}{D_1 \cdots D_{n_{den}}}$

- *Tranverse* degrees of freedom can be **integrate**
- This decomposition can be *iterated* for any new $\mathcal{N}_{ji}(\bar{x}_\parallel, \bar{x}_\perp, \bar{\mu}^2)$



$$= \sum_{i=1}^{N} \sum_{|\bar{b}_j|=i} \frac{\Delta_{\bar{b}_j}(x_{\parallel, ISP})}{D_1^{b_{j1}} \cdots D_{n_{den}}^{b_{jn_{den}}}}$$

Furthermore: *Integrals* have **more relations** than *integrands*.

[Mastrolia, Peraro, Primo (2016)]
[Mastrolia, Peraro, Primo, Torres Bobadilla (2016)]

Decomposing scattering amplitude: Integration-by-parts Identities (IBPs)

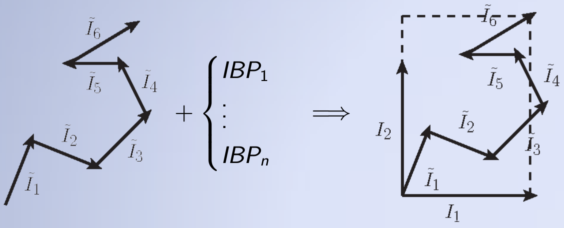
Exploiting d-**dimensionality** and **invariance** of the integrals under **shifts/rotations of the loop momenta**, a new set of relations can be build

$$\int f(k) d^d k = \int e^{v^\mu \frac{\partial}{\partial k^\mu}} f(k) d^d k \quad \Longrightarrow \quad \int d^d \bar{k} \frac{\partial}{\partial k^\mu} \left[\frac{v^\mu}{D_1^{b_{j1}} \dots D_{n_{den}}^{b_{jn_{den}}}} \right] = 0$$

$v^\mu = \{\bar{p}, \bar{k}\}$

These are known as **Integration-by-parts identities** (IBPs).

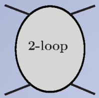
- IBPs generate a *linear sistem of relations* between integrals
- N_{Eq} independent equations $< N_I$ integrals $\Longrightarrow$ $N_I - N_{Eq}$ *building block integrals*



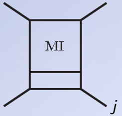
[Chetyrkin, Tkachev (1981)]
[Laporta (2000)]

Decomposing scattering amplitude: Master integrals

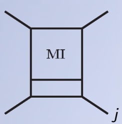
AID + IBPs yield to a **complete decomposition** of the **Scattering Amplitude** into a *minimal* set of integrals, the so-called **Master Integrals** (MIs) $I_j(s, t, m_\mu^2)$:



$$= \sum_j C_j(s, t, m_\mu^2)$$



- MIs are *integrals of rational functions*

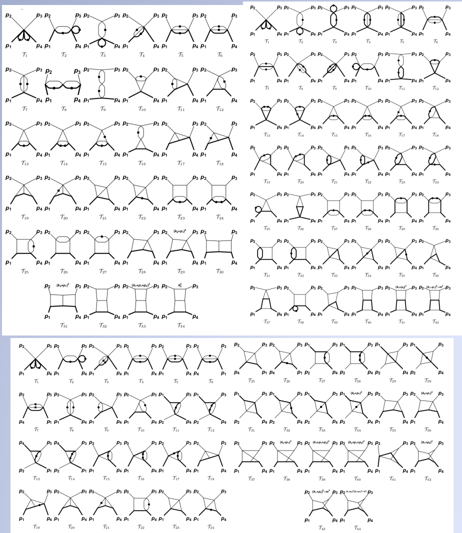


$$= \int_{\mathbb{M}} d^d \bar{k} \frac{\prod_{k=1}^{r} (ISP_k)^{s_k}}{D_1^{a_1} \cdots D_7^{a_7}}$$

  where ISPs stands for **I**rreducible **S**calar **P**roducts

- MIs may contain UV/IR *divergencies* that need to be extracted
- **Evaluation** of Master integrals is needed to obtain the values of the amplitude

Decomposing scattering amplitude: Master integrals



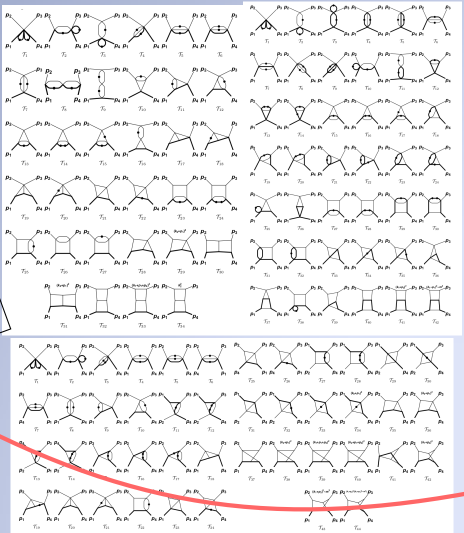
PDEs Method
and
Magnus Exponential

+

Boundary conditions
and
Analytical continuation

[Mastrolia, Passera, Primo, Schubert (2017)]
[Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)]

Decomposing scattering amplitude: Master integrals



Integral Analytically Known

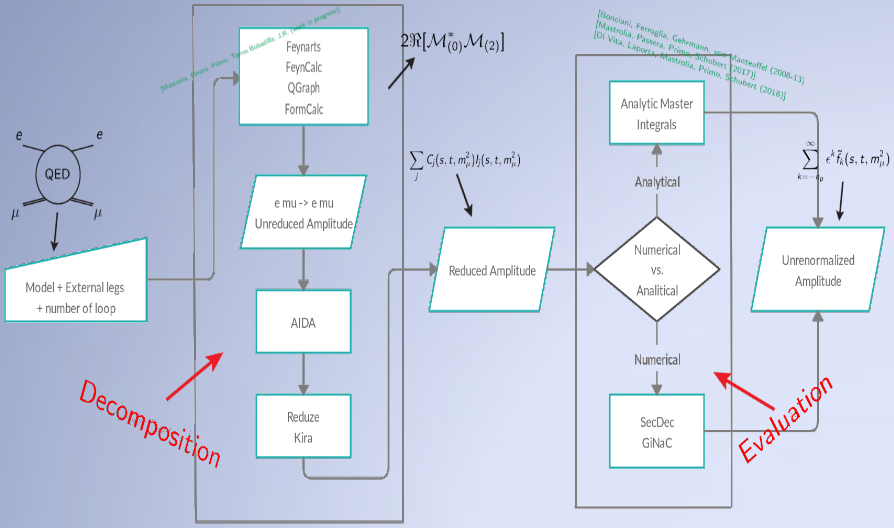
PDEs Method and **Magnus Exponential**

+

Boundary conditions and **Analytical continuation**

[Mastrolia, Passera, Primo, Schubert (2017)]
[Di Vita, Laporta, Mastrolia, Primo, Schubert (2018)]

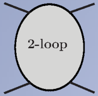
Decomposing scattering amplitude: Automation



[Maierhoefer, Usovitsch, Uwer (2017)]
[von Manteuffel, Studerus (2016)]
[Carter, Heinrich (2010)]
AIDA [Mastrolia, Primo, Torres Bobadilla, Peraro, Schubert, Mattiazzi, Ronca (in progress)]

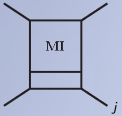
Evaluating Scattering amplitude: Renormalization

Amplitude still needs to be **renormalized** $\Longrightarrow$ Choosing $\overline{\text{MS}}$ *renormalization scheme*



AID $\Downarrow$ ISPs

$$\sum_j C_j(s,t,m_\mu^2)$$



Analytical $\Downarrow$ Evaluation

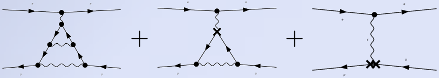
$$\sum_{k=-n_p}^{\infty} \epsilon^k \tilde{f}_k(s,t,m_\mu^2)$$

$\overline{MS}$ $\Downarrow$ Ward Id's

$$f_0(s,t,m_\mu^2) + O(\epsilon)$$

Same technology (decomposition + analytical evaluation) can be applied to evaluate the counterterms:

- renormalization @2-loop $\Longrightarrow$ 1-loop and tree-level counterterms



Ward Identities for QED $\Longrightarrow$ only *Field Renormalization* needed + *LSZ Factor residues*.

$$Z_e = Z_A^{-\frac{1}{2}}$$

[Di Vita, Schubert (in progress)]

Conclusions

$e^- \mu^+ \to e^- \mu^+$ @2-loop Scattering Amplitude:

- Decomposition:
  - Generation of Diagrams + interference with LO amplitude
  - Integrand Decomposition @2-loop implemented in AIDA
  - Integration-by-parts id's $\Longrightarrow$ linear combination of *Master Integrals*
  - All the previous steps are completely automated
- Evaluation:
  - Analytical Evaluation $\Longrightarrow$ Done
  - Numerical Evaluation $\Longrightarrow$ Check
- Renormalization:
  - UV conterterms in $\overline{\text{MS}}$ scheme $\Longrightarrow$ UV finite 2-loop amplitude

To do:

- Complete NNLO Amplitude
  - Real-Virtual + Double-Real Amplitude
  - Check UV/IR finiteness
- NNLO Amplitude with $m_e \neq 0$

Outlook:

- Applications on $gg \to t\bar{t}$ @2-loop

Thank you

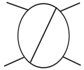
for your attention

Backup slides

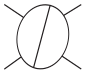
Evaluating Scattering amplitude: differential equations

Master integrals fulfil a system of first order differential equations through IBPs

$$\frac{\partial}{\partial \bar{x}_i}$$



$$= \frac{\partial}{\partial x_i} I_j(\bar{x}, \epsilon) = \sum_j B'_{ij}(\bar{x}, \epsilon) \tilde{I}_j(\bar{x}, \epsilon) \overset{\text{IBP}}{=} \sum_j B_{ij}(\bar{x}, \epsilon)$$



$$_j$$

$\bar{x} = \{s, t, m_\mu^2\}$

Given n master integrals, the basis $\bar{I}(\bar{x}, \epsilon) = (I_1, \ldots, I_n)$:

$$\frac{\partial}{\partial \bar{x}} \bar{I}(\bar{x}, \epsilon) = \mathbb{B}(\bar{x}, \epsilon) \bar{I}(\bar{x}, \epsilon)$$

$\mathbb{B}(\bar{x}, \epsilon)$ has some properties:

- It has upper triangular form.
- Its entries are rational functions of its arguments.

MIs = Solution of PDE + Boundary Condition

Evaluating Scattering amplitude: differential equations

Rotation of the MIs basis $\bar{I}(\bar{x},\epsilon) = \mathbb{R}(\bar{x},\epsilon)\bar{J}(\bar{x},\epsilon)$:

$$\frac{\partial}{\partial x}\bar{J}(\bar{x},\epsilon) = \mathbb{R}^{-1}(\bar{x},\epsilon)\left[\mathbb{B}(\bar{x},\epsilon)\mathbb{R}(\bar{x},\epsilon) - \frac{\partial}{\partial \bar{x}}\mathbb{R}(\bar{x},\epsilon)\right]\bar{J}(\bar{x},\epsilon)$$

If $\mathbb{R}(\bar{x},\epsilon)$ casts the PDE into

$$\frac{\partial}{\partial x}\bar{J}(\bar{x},\epsilon) = \epsilon\mathbb{A}(\bar{x})\bar{J}(\bar{x},\epsilon) \implies d\bar{J}(\bar{x},\epsilon) = \epsilon d\mathbb{A}(\bar{x})\bar{J}(\bar{x},\epsilon)$$

- System casted into *canonical form*
- System decoupled order-by-order

$$\bar{J}(\bar{x},\epsilon) = \sum \epsilon^k \bar{J}^{(k)}(\bar{x}), \quad d\bar{J}^{(k)}(\bar{x}) = d\mathbb{A}(\bar{x})\bar{J}^{(k-1)}(\bar{x})$$

- Its solution can be found iteratively (Chen's iterated integral):

$$\bar{J}^{(k)}(\bar{x}) = \int_\gamma d\mathbb{A}(\bar{x})\bar{J}(\bar{x})^{(k-1)} = \int_\gamma \underbrace{d\mathbb{A}(\bar{x})\cdots d\mathbb{A}(\bar{x})}_{k}\bar{J}(\bar{x}_0)$$

where $\bar{J}(\bar{x}_0)$ a boundary condition and γ a path that connects $\bar{x}_0$ to $\bar{x}$

Evaluating Scattering amplitude: Magnus Exponential

General solution:

$$\bar{J}(\bar{x},\epsilon) = Pe^{\epsilon\int_\gamma dA}\bar{J}(\bar{x}_0) = \left(1 + \sum_{k=1}^{\infty}\epsilon^k \int_\gamma \underbrace{d\mathbb{A}(\bar{x})\cdots d\mathbb{A}(\bar{x})}_{k}\right)\bar{J}(\bar{x}_0)$$

How to find the rotation matrix $\mathbb{R}(\bar{x},\epsilon) \implies$ *Magnus Exponential*
Ansatz: $\mathbb{B}(\bar{x},\epsilon) = \mathbb{B}_0(\bar{x}) + \epsilon\mathbb{B}_1(\bar{x})$. A matrix $\mathbb{R}(\bar{x},\epsilon)$ that satisfies

$$\frac{\partial}{\partial\bar{x}}\mathbb{R}(\bar{x}) = \mathbb{B}_0(\bar{x})\mathbb{R}(\bar{x})$$

$$\implies \mathbb{R}(\bar{x})\frac{\partial}{\partial x}\bar{J}(\bar{x},\epsilon) + \cancel{\frac{\partial}{\partial\bar{x}}\mathbb{R}(\bar{x})\bar{J}(\bar{x},\epsilon)} = \left[\cancel{\mathbb{B}_0(\bar{x})} + \epsilon\mathbb{B}_1(\bar{x})\right]\mathbb{R}(\bar{x})\bar{J}(\bar{x},\epsilon)$$

$$\implies \frac{\partial}{\partial x}\bar{J}(\bar{x},\epsilon) = \epsilon\mathbb{R}^{-1}(\bar{x})\mathbb{B}_1(\bar{x})\mathbb{R}(\bar{x})\bar{J}(\bar{x},\epsilon) = \epsilon\mathbb{A}(\bar{x})\bar{J}(\bar{x},\epsilon)$$

PDE for $\mathbb{B}_0(\bar{x})$ admits a solution in terms of the Magnus Exponential

$$\mathbb{R}(\bar{x}) = e^{\sum_k \Omega_k[\mathbb{B}_0](\bar{x})}, \qquad \begin{aligned} \Omega_1[\mathbb{B}_0](\bar{x}) &= \int_\gamma dx_1\mathbb{B}_0(\bar{x}_1) \\ \Omega_2[\mathbb{B}_0](\bar{x}) &= \int_\gamma dx_1 dx_2[\mathbb{B}_0(\bar{x}_1), \mathbb{B}_0(\bar{x}_2)] \\ &\vdots \end{aligned}$$