Three-pion contribution to hadronic vacuum polarization

in collaboration with M. Hoferichter and B. Kubis

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Outline

Muon anomalous magnetic moment a_μ

- Recap hadronic light-by-light (HLbL) scattering
- Hadronic vacuum polarization (HVP)

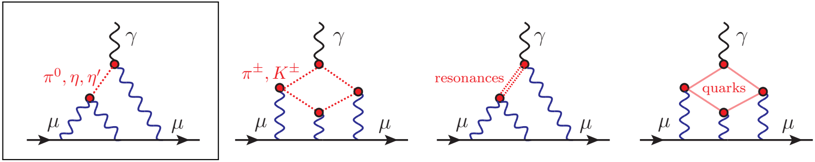
3π contribution to hadronic vacuum polarization

- Overview of the $\pi^+\pi^-\pi^0$ channel
- $\gamma^* \to 3\pi$ dispersive representation
- $e^+e^- \to 3\pi$ cross section data sets
- D'Agostini bias and unbiased fits

Fit results

Conclusions and outlook

Recap hadronic light-by-light scattering



- π^0-pole term is the largest individual contribution to HLbL
- First dispersive, complete data-driven pion-pole contribution to a_μ:

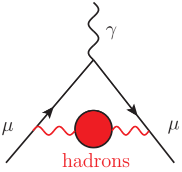
$$a_\mu^{\pi^0\text{-pole}} = 62.6(1.7)_{F_{\pi\gamma\gamma}}(1.1)_{\text{disp}}\binom{2.2}{1.4}_{\text{BL}}(0.5)_{\text{asym}} \times 10^{-11} = 62.6^{+3.0}_{-2.5} \times 10^{-11}$$

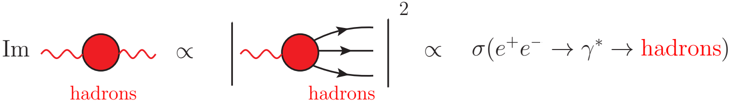
Hoferichter et al., 2018

- $\Rightarrow$ Same dispersive technique can extended to 3π contribution to HVP

Hadronic vacuum polarization

- m_μ as characteristic scale
  $\Rightarrow$ Not a perturbative QCD problem!
- Optical theorem from probability conservation
- Dispersion relations to relate to the observable





$$a_\mu^{\mathrm{HVP}} = \int_{s_{\mathrm{thr}}}^{\infty} \mathrm{d}s K(s) \sigma(e^+e^- \to \text{hadrons})$$

- Kinematic function $K(s)$: $K(s) \propto 1/s$ for large s
- $\sigma(e^+e^- \to \text{hadrons}) \propto 1/s$ for large s

Overview of the $\pi^+\pi^-\pi^0$ channel

- Second largest exclusive channel after $\pi^+\pi^-$
- Large relative discrepancy between direct data-integration works

$a_\mu^{3\pi} \times 10^{10}$ below $1.8\,\mathrm{GeV}$		
Davier et al., 2017	Keshavarzi et al., 2018	relative difference
46.20 ± 1.45	47.70 ± 0.89	3.2% (0.7% for $\pi^+\pi^-$)

Even lower value $a_\mu^{3\pi}|_{\leq 2.0\,\mathrm{GeV}} = 44.3(1.5) \times 10^{-10}$! Jegerlehner, 2017

- Other independent analyses demanding

  ⇒ Dispersive global fit function fulfilling analyticity, unitarity and QCD constraints

$\gamma^* \to 3\pi$ dispersive representation

The $\gamma^*(q) \to \pi^+(p_+)\pi^-(p_-)\pi^0(p_0)$ decay amplitude $\mathcal{F}(s,t,u;q^2)$:

$$\langle 0|j_\mu(0)|\pi^+(p_+)\pi^-(p_-)\pi^0(p_0)\rangle = -\epsilon_{\mu\nu\rho\sigma}\, p_+^\nu p_-^\rho p_0^\sigma \mathcal{F}(s,t,u;q^2)$$

$q = p_+ + p_- + p_0$; s, t & u are Mandelstam variables

Decompose into single-variable functions:

$$\mathcal{F}(s,t,u;q^2) = \mathcal{F}(s,q^2) + \mathcal{F}(t,q^2) + \mathcal{F}(u,q^2)$$

Normalization from the Wess–Zumino–Witten (WZW) anomaly:

$$\mathcal{F}(0,0,0;0) = \frac{1}{4\pi^2 F_\pi^3} \equiv F_{3\pi}$$

$F_\pi = 92.28(9)\,\mathrm{MeV}$: pion decay constant Tanabashi et al., 2018

$\gamma^* \to 3\pi$ dispersive representation

Discontinuity equation:

$$\text{disc}\,\mathcal{F}(s,q^2) = 2i(\mathcal{F}(s,q^2) + \hat{\mathcal{F}}(s,q^2))\theta(s-4M_\pi^2)\sin\delta_1^1(s)\,e^{-i\delta_1^1(s)}$$

- $\mathcal{F}(s,q^2)$: right-hand cut
- $\hat{\mathcal{F}}(s,q^2)$: left-hand cut; angular averages of $\mathcal{F}(t,q^2)$ & $\mathcal{F}(u,q^2)$
- $\delta_1^1(s)$: $\pi\pi$ P-wave phase shift from Roy equations

$\gamma^* \to 3\pi$ dispersive representation

A once-subtracted dispersive solution to the discontinuity equation:

Hoferichter et al., 2014

$$\mathcal{F}(s,q^2) = a(q^2)\Omega(s)\Big\{1 + \frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\hat{\mathcal{F}}(s',q^2)\sin\delta_1^1(s')}{s'(s'-s)|\Omega(s')|}\Big\}$$

$$\Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} \mathrm{d}s' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}$$

is the Omnès function

Omnès, 1958

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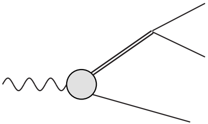
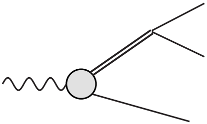
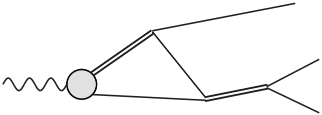
$\hat{\mathcal{F}}(s,q^2)$ absent:

$\mathcal{F}(s,q^2) =$ 

$\hat{\mathcal{F}}(s,q^2)$ present:

$\mathcal{F}(s,q^2) =$  $+$  $+ \cdots$

- Incorporated crossed-channel interactions

$\gamma^* \to 3\pi$ dispersive representation

$a(q^2)$ fit to different $e^+e^- \to 3\pi$ cross-section data with parameterization:

$$a(q^2) = \frac{F_{3\pi}}{3} + \frac{q^2}{\pi}\int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \frac{\operatorname{Im}\mathcal{A}(s')}{s'(s'-q^2)} + C_n(q^2)$$

$$\mathcal{A}(q^2) = \sum_V \frac{c_V}{M_V^2 - q^2 - i\sqrt{q^2}\,\Gamma_V(q^2)}, \qquad V = \omega, \phi, \omega', \omega''$$

$$C_n(q^2) = \sum_{i=1}^{n} c_i \big(z(q^2)^i - z(0)^i\big), \qquad z(q^2) = \frac{\sqrt{s_{\text{inel}} - s_1} - \sqrt{s_{\text{inel}} - q^2}}{\sqrt{s_{\text{inel}} - s_1} + \sqrt{s_{\text{inel}} - q^2}}$$

$\gamma^* \to 3\pi$ dispersive representation

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- S-wave cusp eliminated
- Exact implementation of $\gamma^* \to 3\pi$ anomaly:

$$\frac{F_{3\pi}}{3} = \frac{1}{\pi}\int_{s_{\text{thr}}}^{\infty} \mathrm{d}s' \frac{\operatorname{Im} a(s')}{s'}$$

$e^+e^- \to 3\pi$ cross section data sets

Experiment	Region of $\sqrt{s}\,\mathrm{GeV}$	# data points	Normalization uncertainty
SND 2002	$[0.98, 1.38]$	67	5.0% or 5.4%
SND 2003	$[0.66, 0.97]$	49	3.4% or 4.5%
SND 2015	$[1.05, 1.80]$	31	3.7%
CMD-2 1995	$[0.99, 1.03]$	16	4.6%
CMD-2 1998	$[0.99, 1.03]$	13	2.3%
CMD-2 2004	$[0.76, 0.81]$	13	1.3%
CMD-2 2006	$[0.98, 1.06]$	54	2.5%
DM1 1980	$[0.75, 1.10]$	26	3.2%
ND 1991	$[0.81, 1.39]$	28	10% or 20%
DM2 1992	$[1.34, 1.80]$	10	8.7%
BaBar 2004	$[1.06, 1.80]$	30	all systematics

- Normalization-type systematic uncertainties are assumed to be 100% correlated
- Normalization uncertainties produce a biased fit for an empirical full covariance-matrix minimization

D'Agostini bias and unbiased fits

Simple example of overall normalization uncertainty inducing a bias:

D'Agostini, 1994

$y_1 = 8.0 \pm 2\%$ & $y_2 = 8.5 \pm 2\%$, normalization error of $\epsilon = 10\%$

- Covariance matrix

$$V = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} y_1^2 & y_1 y_2 \\ y_1 y_2 & y_2^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 + \varepsilon^2 y_1^2 & \varepsilon^2 y_1 y_2 \\ \varepsilon^2 y_1 y_2 & \sigma_2^2 + \varepsilon^2 y_2^2 \end{pmatrix}$$

- $\chi^2 = \Delta^{\mathrm{T}} V^{-1} \Delta$, $\Delta = \begin{pmatrix} y_1 - \hat{y} \\ y_2 - \hat{y} \end{pmatrix}$

- $\Rightarrow \hat{y} = 7.87 \pm 0.81 < y_1 \,\&\, y_2$ **?**

D'Agostini bias and unbiased fits

- Happens when the data values are rescaled independently of their errors
- Smaller data points are assigned a smaller uncertainty than larger ones

A better covariance matrix:

$$\mathrm{V} = \begin{pmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{pmatrix} + \varepsilon^2 \begin{pmatrix} \hat{y}^2 & \hat{y}^2 \\ \hat{y}^2 & \hat{y}^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 + \varepsilon^2 \hat{y}^2 & \varepsilon^2 \hat{y}^2 \\ \varepsilon^2 \hat{y}^2 & \sigma_2^2 + \varepsilon^2 \hat{y}^2 \end{pmatrix}$$

General iterative solution: NNPDF collaboration, 2010

$$\mathrm{V}_{n+1}(i,j) = \mathrm{V}^{\mathsf{stat}}(i,j) + \frac{\mathrm{V}^{\mathsf{syst}}(i,j)}{y_i y_j} f_n(x_i) f_n(x_j)$$

Fit results

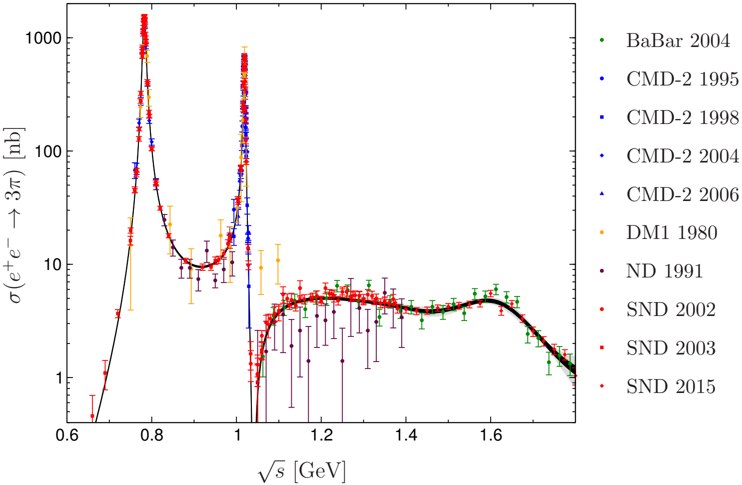
- Resonance parameters $M_\omega, \Gamma_\omega, M_\phi, \Gamma_\phi, c_\omega$, c_ϕ, $c_{\omega'}$ & $c_{\omega''}$
- Conformal parameters c_1, c_2 & c_3
- Energy rescaling $\sqrt{s} \to \sqrt{s} + \xi(\sqrt{s} - 3M_\pi)$

	This work
χ^2/dof	430.8/305=1.41

- Fit errors are inflated by the scale factor $S = \sqrt{\chi^2/\text{dof}}$

Fit results

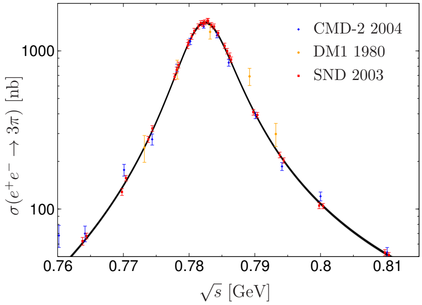
Fit to $e^+e^- \to 3\pi$ data up to $1.8\,\mathrm{GeV}$:

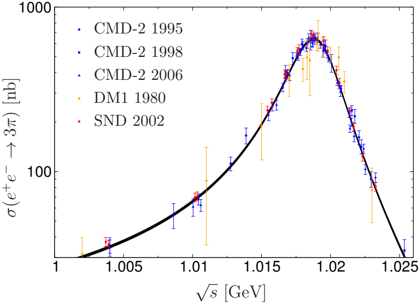


- VP removed from the cross section
- Black and Gray bands represent fit and total uncertainties

Fit results

Enlarged ω and ϕ regions:





Fit results

Central result for the 3π contribution to HVP:

$$\boxed{a_\mu^{3\pi}|_{\leq 1.8\,\mathrm{GeV}} = 46.2(6)(6) \times 10^{-10} = 46.2(8) \times 10^{-10}}$$

- Interpolation errors $\Rightarrow$ main discrepancy between different groups

Threshold region $a_\mu^{3\pi}|_{\leq 0.66\,\mathrm{MeV}} = 0.02 \times 10^{-10}$

- Twice the estimate from WZW action+vector meson dominance model

Kuraev and Silagadze, 1995, Ahmedov et al., 2002

Fit results

In combination with the 2π channel: Colangelo et al., 2018

$$a_\mu^{2\pi}|_{\leq 1.0\,\mathrm{GeV}} + a_\mu^{3\pi}|_{\leq 1.8\,\mathrm{GeV}} = 541.2(2.7) \times 10^{-10}$$

- 80% of HVP imposing analyticity and unitarity constraints

Adding the rest of HVP: Davier et al., 2017, Keshavarzi et al., 2018

$$a_\mu^{\mathsf{HVP}} = 692.3(3.3) \times 10^{-10}$$

Together with other contributions:

- Reaffirms the $(g-2)_\mu$ anomaly at the level of 3.4σ!

Conclusions and outlook

- Dispersive global fit function for the 3π channel
  - Provided another independent analysis
  - Resolved main tension in 3π, reaffirmed tension in a_μ
  - Removed D'Agostini bias in the fits
  - Incorporated analyticiy, unitarity and low-energy constraints
- Final state radiation, $\gamma^*\gamma^* \to \pi\pi$, $\cdots$

Much obliged for your attention!

"$g - 2$ is not an experiment: it is a way of life."

John Adams (CERN Director General 1971 - 1980)

Backup

The cross section in terms of the $\gamma^* \to 3\pi$ amplitude $\mathcal{F}(s,t,u;q^2)$:

$$\sigma_{e^+e^-\to 3\pi}(q^2) = \alpha^2 \int_{s_{\text{min}}}^{s_{\text{max}}} \mathrm{d}s \int_{t_{\text{min}}}^{t_{\text{max}}} \mathrm{d}t \, \frac{(s-4M_\pi^2)\,\lambda(q^2,M_\pi^2,s)\sin^2\theta_s}{768\,\pi\,q^6} |\mathcal{F}(s,t,u;q^2)|^2$$

Källén function $\lambda(x,y,z) = x^2 + y^2 + z^2 - 2(xy + yz + xz)$

Integration boundaries

$$s_{\text{min}} = 4M_\pi^2, \qquad s_{\text{max}} = \left(\sqrt{q^2} - M_\pi\right)^2,$$

$$t_{\text{min/max}} = (E_-^* + E_0^*)^2 - \left(\sqrt{E_-^{*2} - M_\pi^2} \pm \sqrt{E_0^{*2} - M_\pi^2}\right)^2$$

and

$$E_-^* = \frac{\sqrt{s}}{2}, \qquad E_0^* = \frac{q^2 - s - M_\pi^2}{2\sqrt{s}}.$$

Backup

Fits to the data combination:

	diagonal			full		
p_{conf}	2	3	4	2	3	4
χ^2/dof	361.3/306	354.6/305	354.0/304	443.7/306	430.8/305	430.7/304
	= 1.18	= 1.16	= 1.16	= 1.45	= 1.41	= 1.42
p-value	0.02	0.03	0.03	4×10^{-7}	3×10^{-6}	2×10^{-6}
$\mathcal{M}_\omega$ [MeV]	782.60(4)	782.60(4)	782.60(4)	782.63(2)	782.63(2)	782.63(2)
Γ_ω [MeV]	8.75(6)	8.79(6)	8.77(6)	8.69(3)	8.71(3)	8.71(3)
$\mathcal{M}_\phi$ [MeV]	1019.23(2)	1019.22(2)	1019.22(2)	1019.20(1)	1019.20(1)	1019.20(1)
Γ_ϕ [MeV]	4.34(4)	4.32(4)	4.32(4)	4.24(3)	4.23(3)	4.23(3)
c_ω [GeV^{-1}]	2.87(1)	2.89(1)	2.88(1)	2.85(2)	2.86(2)	2.86(2)
c_ϕ [GeV^{-1}]	−0.395(3)	−0.394(3)	−0.394(3)	−0.388(3)	−0.386(3)	−0.386(3)
$c_{\omega'}$ [GeV^{-1}]	−0.18(3)	−0.09(5)	−0.08(5)	−0.17(3)	−0.07(4)	−0.06(4)
$c_{\omega''}$ [GeV^{-1}]	−1.65(8)	−1.52(10)	−1.55(10)	−1.65(8)	−1.52(8)	−1.53(10)
c_1 [GeV^{-3}]	−0.35(10)	−0.22(11)	−0.24(11)	−0.31(10)	−0.12(11)	−0.14(12)
c_2 [GeV^{-3}]	−1.28(4)	−1.39(6)	−1.33(9)	−1.24(4)	−1.36(5)	−1.34(9)
c_3 [GeV^{-3}]	—	−0.48(8)	−0.51(9)	—	−0.47(7)	−0.48(8)
c_4 [GeV^{-3}]	—	—	1.39(9)	—	—	1.41(9)
$10^4 \times \xi$	1.9(7)	1.8(7)	1.8(7)	1.3(5)	1.3(5)	1.3(5)
$10^{10} \times a_\mu^{3\pi}\|_{\leq 1.8\,\text{MeV}}$	46.65(21)	46.70(21)	46.67(22)	45.87(47)	46.16(47)	46.10(50)

Backup

Final result for the ω and ϕ parameters:

$$M_\omega = 782.63(3)\,\text{MeV}, \qquad \Gamma_\omega = 8.71(6)\,\text{MeV}$$
$$M_\phi = 1019.20(2)\,\text{MeV}, \qquad \Gamma_\phi = 4.23(4)\,\text{MeV}$$

In comparison to PDG:

$$M_\omega = 782.65(12)\,\text{MeV}, \qquad \Gamma_\omega = 8.49(8)\,\text{MeV}$$
$$M_\phi = 1019.461(16)\,\text{MeV}, \qquad \Gamma_\phi = 4.249(13)\,\text{MeV}$$

- VP-subtracted parameters!