The 4-loop slope of the Dirac form factor

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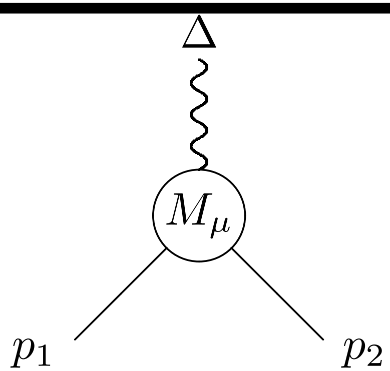


electron-photon vertex

$$M_\mu = F_1(-\Delta^2)\gamma_\mu + F_2(-\Delta^2)\frac{\sigma_{\mu\nu}}{2m}\Delta_\nu$$

Dirac Pauli

$F_1(0) \equiv 1$ charge conservation

$F_2(0) = \frac{g-2}{2}$ anomalous magnetic moment

$F_1(q^2) = 1 + F_1'(0)\frac{q^2}{m^2} + O\left(\frac{q^2}{m^2}\right)^2 + \ldots$ $F_1'(0) \neq 0$ slope *(electromagnetic charge radius)*

- $F_2(0) = (g-2)/2$ direct comparison between theory and experiment
- $F_1'(0)$ *indirect* comparison between theory and experiment
- $F_1'(0) \rightarrow$ modification of the Coulomb interaction $\rightarrow$ Energy shift in the hydrogen atom

Perturbative expansion

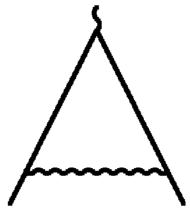
$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$

r-loop contribution to the shift of level nS:

$$\Delta E(nS, r-loop) = \frac{4(Z\alpha)^4 m}{n^3}\left(m^2 A_r\left(\frac{\alpha}{\pi}\right)^r\right) \qquad (r > 1)$$

No contribution from $F_1'(0)$ to the shift of levels P,D,F

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$



1 diagram

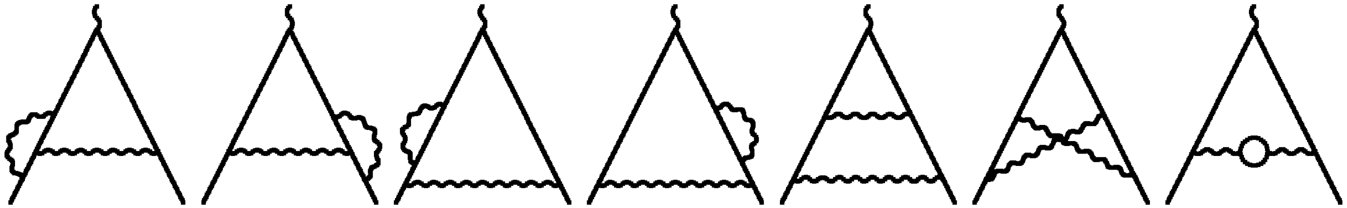
$A_1 = -\frac{1}{3}\ln\frac{\lambda}{m} - \frac{1}{8}$ I.R.divergent (λ photon mass)

Infrared divergence due to the mass-shell condition of the electron

taking into account off mass-shell effects in the expression of ΔE A_1 must be replaced with

$A_1 \to A_1' = -\frac{1}{3}\ln\frac{\Delta\epsilon}{m} - \frac{1}{8} + \frac{5}{18}$ ($\Delta\epsilon$ =Bethe energy)

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$



7 diagrams

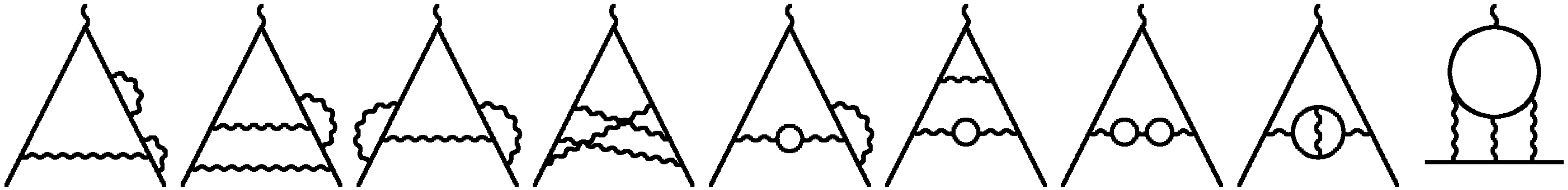
$$\begin{aligned} A_2 = & -\frac{4819}{5184} - \frac{49}{432}\pi^2 + \frac{1}{2}\pi^2 \ln 2 - \frac{3}{4}\zeta(3) \\ = & \ 0.469\,941\,487\,459\,992\ldots \end{aligned}$$

The two-loop coefficient was computed analytically by Barbieri, Mignaco and Remiddi in 1970.

2-loop g-2 coefficient:

$$\begin{aligned} C_2 = & \ \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3) \\ = & -0.328\,478\,965\,579\ldots \end{aligned}$$

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$



+63

72 diagrams

$$A_3 = -\frac{17}{24}\pi^2\zeta(3) + \frac{25}{8}\zeta(5) - \frac{217}{9}\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{103}{1080}\pi^2\ln^2 2 + \frac{3899}{25920}\pi^4 - \frac{2929}{288}\zeta(3) + \frac{41671}{360}\pi^2\ln 2 - \frac{454979}{38880}\pi^2 - \frac{77513}{186624}$$

$$= 0.171\,720\,018\,909\,775\dots$$

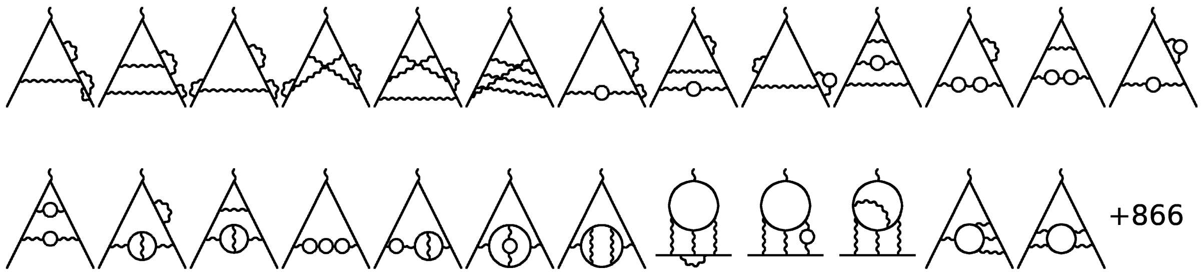
Calculated analytically by Melnikov and Ritbergen in 1999

3-loop g-2 coefficient:

$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{\ln^4 2}{24}\right) - \frac{\pi^2\ln^2 2}{24}\right] - \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

$$= 1.181\,241\,456\,587\,200\,006\,\dots$$

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$



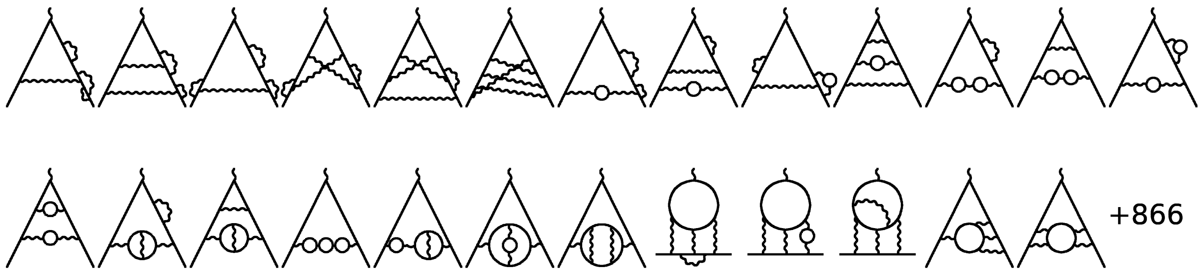
891 diagrams

$A_4 =$

0.88654567394644314583682173061031535939042403266006474536805590932084031646562892745483648632417733686
9351275874721830799687592397488846682614761175301191758483144677475267298032691740271921465153932551984
4793100495019624531372119372946716080063429980958425369584945060683836659851413873218942100123948827595
1538237865372203883496448560075689857616877564102719779603910290276615122356406105399227905150277608224
5923695043327570361335093525176476399251682267935964524928545665821844102867454764407757992111860378831
5350119800677785150747802126742479040522224733029502183107429019902991627682916022890589911642646344987
8987630727082848364358743478002455415372434008969514716831155386425591883520934780665126748875033459025
9918224556361312512411988061541553762133711228484627768486742192828968656811548030353727600787303621093
05926475295989223401783573282897174962399183352784888413242436969926422136403200684400061242352981583396
6332566753158241741448217616597381276692161976675095050740649309561361958988024564511635456757162309441
738848115650200983348479405901887854217006673782208530535419531883786100755181163... (S.L. 2019)

- New in the literature
- The high precision of the result is needed to fit analytically a very complex analytical ansatz to the numerical values by using the PSLQ algorithm.
- The successful fit is a strong reliability test of the result.

$$F_2(0) = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + \ldots$$



891 diagrams

$C_4 =$

```
-1.91224576492644557415264716743983005406087339065872534517132984800603844398065170614276089270000363158
37558415331473270056378514912854539190280432705027382230434557895704556272930994129669976027778221157847
20339064151908166527097970867438115012155147972274322164273431927975958607405005783738496070187432831402
48380251922494607422985589304635061404922526634310944240002356356881280620645494013224977594300429288836
76174889923691518087808698970526357853375377696411702453619601349757449436126848617516260683238718674730
38315059627418780153055148794005369777983694642786843269184311758895811597435669504330483490736134265864
99531163878117434753854234883640855844418822372174567068710418233074305174430557394596117155085896114899
52612660612469940731184039274723400234649695317354825848179982240973737107736574046451352112309124252811
11372153021544537210148111211598489708842232798797204842014451228284515165852365617865945926009917330317
21302865467212345340500349104700728924487200616044261325449069000431915198230047488181494311038495378299
4062967586787538524978194698979313216219797575067670114290489796208505...
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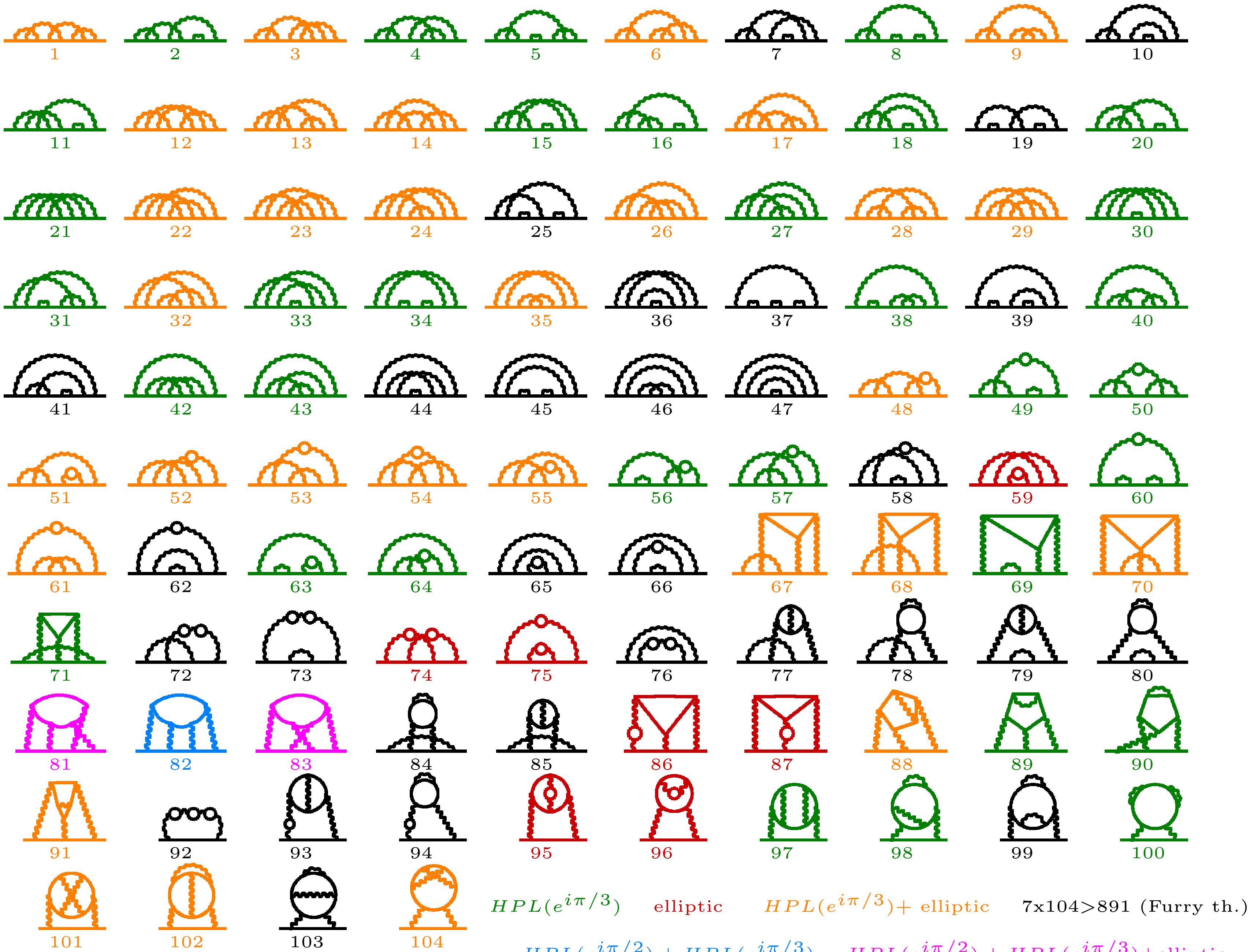
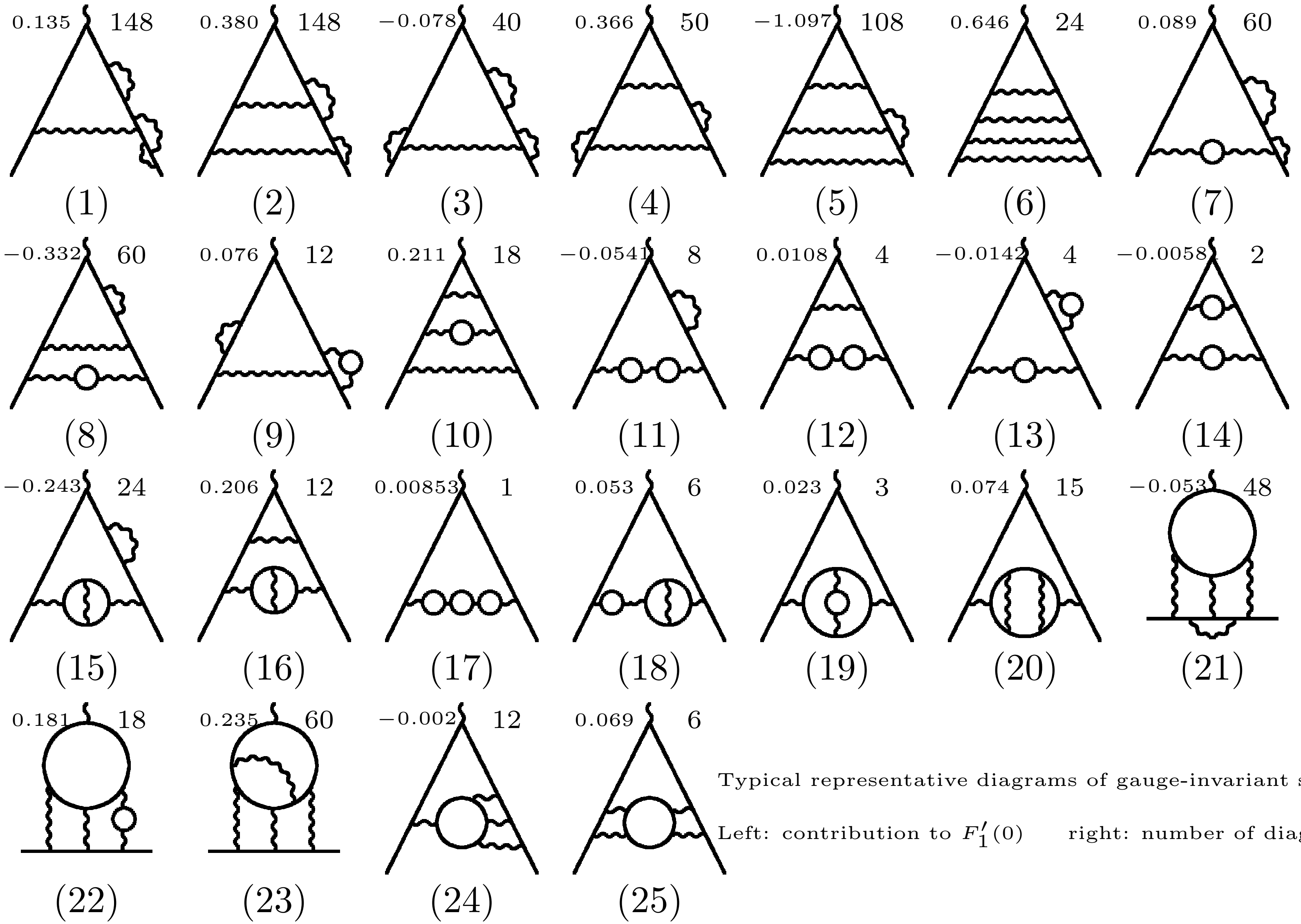
(S.L. 2017)

A coloured view of the 891 diagrams arranged in insertions into 104 self-mass diagrams



891 diagrams arranged into 25 gauge-invariant sets



Typical representative diagrams of gauge-invariant sets

Left: contribution to $F_1'(0)$ right: number of diagrams

loop	$F_1'(0)$	$F_2(0)$
1	∞	0.5
2	0.469941487459	-0.328478965579
3	0.171720018909	1.181241456587
4	0.886545673946	-1.912245764926
5		6.737(159)

positive — alternating signs

2019: analytical fit of the 4-loop QED contribution to $F_1'(0)$

$$
\begin{aligned}
A_4 =& -\frac{92473962293}{19752284160} - \frac{6619898477}{21772800}\zeta(2) - \frac{12334741}{132300}\zeta(3) + \frac{97832509}{90720}\zeta(2)\ln 2 - \frac{241619904061}{391910400}\zeta(4) + \frac{4572662443}{12247200}\ln^2 2\,\zeta(2) - \frac{1449791143}{3061800}\left(a_4 + \frac{1}{24}\ln^4 2\right) + \frac{90355973}{134400}\zeta(5) \\
& + \frac{1173056009}{9072000}\zeta(3)\zeta(2) - \frac{8548241}{30240}\zeta(4)\ln 2 - \frac{68168}{135}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) - \frac{244603373713}{52254720}\zeta(6) - \frac{8082848863}{24192000}\zeta^2(3) + \frac{26062}{27}a_6 - \frac{18215}{27}b_6 + \frac{18215}{27}a_5\ln 2 \\
& - \frac{18215}{27}\zeta(5)\ln 2 + \frac{402152509}{189000}a_4\zeta(2) + \frac{159693503}{72000}\zeta(3)\zeta(2)\ln 2 - \frac{328317209}{302400}\zeta(4)\ln^2 2 - \frac{18215}{162}\zeta(3)\ln^3 2 + \frac{188648503}{1512000}\zeta(2)\ln^4 2 - \frac{21671}{6480}\ln^6 2 - \frac{7224951103}{1741824}\zeta(7) \\
& - \frac{1267114025}{387072}\zeta(4)\zeta(3) - \frac{427145}{504}a_4\zeta(3) - \frac{2749470791}{387072}\zeta(5)\zeta(2) + \frac{1420289}{180}a_5\zeta(2) + \frac{116987}{21}a_7 - \frac{116987}{63}b_7 + \frac{256321}{756}d_7 + \frac{971827}{128}\zeta(6)\ln 2 + \frac{607282}{189}a_6\ln 2 \\
& - \frac{256321}{378}b_6\ln 2 - \frac{1794247}{3456}\zeta^2(3)\ln 2 + \frac{104041}{20}a_4\zeta(2)\ln 2 - \frac{1888991}{24192}\zeta(5)\ln^2 2 + \frac{75222353}{60480}\zeta(3)\zeta(2)\ln^2 2 + \frac{256321}{378}a_5\ln^2 2 - \frac{9699379}{6048}\zeta(4)\ln^3 2 - \frac{2574883}{36288}\zeta(3)\ln^4 2 \\
& + \frac{37144753}{226800}\zeta(2)\ln^5 2 - \frac{218465}{127008}\ln^7 2 + \sqrt{3}\Bigg[-\frac{14186171}{194400}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{103023803}{583200}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& + \frac{916598}{76545}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{458299}{36855}\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{10540877}{442260}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{178619489}{3980340}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{1833196}{45927}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{12563350487}{2579260320}\zeta(5)\pi \\
& + \frac{533401067}{459270}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{844343}{18900}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{844343}{28350}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{458299}{21870}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{458299}{14580}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{263673944}{295245}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) - \frac{39924629}{6889050}\zeta(3)\zeta(2)\pi + \frac{844343}{1224720}\zeta(4)\pi\ln 2 - \frac{844343}{11340}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 - \frac{844343}{7560}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 + \frac{458299}{275562}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2 \\
& + \frac{19130869}{367416}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^2 2\Bigg] + \frac{212671}{2400}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) - \frac{1031987}{14400}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{507}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& - 507\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{13689}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{68445}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{13689}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{507}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) \\
& - \frac{1521}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{24505}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi - \frac{295}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{295}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) - \frac{2655}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
& - \frac{2655}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) - \frac{295}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{885}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{1117}{36}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right) + \frac{38424}{125}\zeta(2)\mathrm{Cl}_2^2\left(\frac{\pi}{2}\right) \\
& - 118\left(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) + \sqrt{3}\Bigg[\pi\left(+\frac{5581729229}{362880000}B_3 + \frac{1233637481}{1399680000}C_3\right) \\
& - \frac{11495611}{3265920}\pi f_2(0,0,1) + \pi\left(\frac{751}{972}\ln 2\, f_2(0,0,1) - \frac{365478661}{24494400}f_2(0,2,0) + \frac{119022487}{5443200}f_2(0,1,1) - \frac{119022487}{14515200}f_2(0,0,2)\right) - \frac{751}{729}\zeta(2)f_1(0,0,1) \\
& + \pi\Bigg(-\frac{1735283}{497664}\zeta(2)f_2(0,0,1) + \frac{1105}{108}\ln 2\, f_2(0,0,2) - \frac{2210}{81}\ln 2\, f_2(0,1,1) + \frac{4420}{243}\ln 2\, f_2(0,2,0) - \frac{1104271}{497664}f_2(0,0,3) + \frac{272833}{41472}f_2(0,1,2) - \frac{4011005}{497664}f_2(0,2,1) \\
& + \frac{8417635}{2239488}f_2(0,3,0) + \frac{157753}{248832}f_2(1,0,2) + \frac{354323}{248832}f_2(1,1,1) - \frac{298711}{124416}f_2(1,2,0) - \frac{157753}{497664}f_2(2,0,1) - \frac{98285}{248832}f_2(2,1,0)\Bigg)\Bigg] + \zeta(2)\Bigg(-\frac{4629335}{165888}f_1(0,0,2) \\
& + \frac{112357}{1536}f_1(0,1,1) - \frac{99731}{1944}f_1(0,2,0) + \frac{157753}{41472}f_1(1,0,1)\Bigg) + \frac{174623}{288000}C_{81a} + \frac{29479}{7200}C_{81b} - \frac{43}{6}C_{81c} + \frac{10871}{14400}C_{83a} - \frac{157}{1620}C_{83b} - \frac{95}{24}C_{83c}
\end{aligned}
$$

$$
\begin{aligned}
C_4 =& \frac{1243127611}{130636800} + \frac{30180451}{25920}\zeta(2) - \frac{255842141}{2721600}\zeta(3) - \frac{8873}{3}\zeta(2)\ln 2 + \frac{6768227}{2160}\zeta(4) + \frac{19063}{360}\zeta(2)\ln^2 2 + \frac{12097}{90}\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{2862857}{6480}\zeta(5) - \frac{12720907}{64800}\zeta(3)\zeta(2) \\
& - \frac{221581}{2160}\zeta(4)\ln 2 + \frac{9656}{27}\left(a_5 + \frac{1}{12}\zeta(2)\ln^3 2 - \frac{1}{120}\ln^5 2\right) + \frac{191490607}{46656}\zeta(6) + \frac{10358551}{43200}\zeta^2(3) - \frac{40136}{27}a_6 + \frac{26404}{27}b_6 - \frac{700706}{675}a_4\zeta(2) - \frac{26404}{27}a_5\ln 2 \\
& + \frac{26404}{27}\zeta(5)\ln 2 - \frac{63749}{50}\zeta(3)\zeta(2)\ln 2 - \frac{40723}{135}\zeta(4)\ln^2 2 + \frac{13202}{81}\zeta(3)\ln^3 2 - \frac{253201}{2700}\zeta(2)\ln^4 2 + \frac{7657}{1620}\ln^6 2 + \frac{2895304273}{435456}\zeta(7) + \frac{670276309}{193536}\zeta(4)\zeta(3) + \frac{85933}{63}a_4\zeta(3) \\
& + \frac{7121162687}{967680}\zeta(5)\zeta(2) - \frac{142793}{18}a_5\zeta(2) - \frac{195848}{21}a_7 + \frac{195848}{63}b_7 - \frac{116506}{189}d_7 - \frac{4136495}{384}\zeta(6)\ln 2 - \frac{1053568}{189}a_6\ln 2 + \frac{233012}{189}b_6\ln 2 + \frac{407771}{432}\zeta^2(3)\ln 2 \\
& - \frac{8937}{2}a_4\zeta(2)\ln 2 + \frac{833683}{3024}\zeta(5)\ln^2 2 - \frac{3995099}{6048}\zeta(3)\zeta(2)\ln^2 2 - \frac{233012}{189}a_5\ln^2 2 + \frac{1705273}{1512}\zeta(4)\ln^3 2 + \frac{602303}{4536}\zeta(3)\ln^4 2 - \frac{1650461}{11340}\zeta(2)\ln^5 2 + \frac{52177}{15876}\ln^7 2 \\
& + \sqrt{3}\left[-\frac{14101}{480}\mathrm{Cl}_4\left(\frac{\pi}{3}\right) - \frac{169703}{1440}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,-1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{494}{27}\mathrm{Im}H_{0,0,0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\right. \\
& + 19\mathrm{Im}H_{0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{437}{12}\mathrm{Im}H_{0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{29812}{297}\mathrm{Cl}_6\left(\frac{\pi}{3}\right) + \frac{4940}{81}a_4\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{520847}{69984}\zeta(5)\pi - \frac{129251}{81}\zeta(4)\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{892}{15}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) \\
& - \frac{1784}{45}\mathrm{Im}H_{0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1729}{54}\zeta(3)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1729}{36}\zeta(3)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{837190}{729}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\zeta(2) + \frac{25937}{4860}\zeta(3)\zeta(2)\pi - \frac{223}{243}\zeta(4)\pi\ln 2 \\
& \left. + \frac{892}{9}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2)\ln 2 + \frac{446}{3}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2)\ln 2 - \frac{7925}{81}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2)\ln^2 2 + \frac{1235}{486}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\ln^4 2\right] + \frac{13487}{60}\left(\mathrm{Re}H_{0,0,0,1,0,1}\left(e^{i\frac{\pi}{3}}\right) + \mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right) \\
& + \frac{136781}{360}\mathrm{Cl}_2^2\left(\frac{\pi}{3}\right)\zeta(2) + \frac{651}{4}\mathrm{Re}H_{0,0,0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + 651\mathrm{Re}H_{0,0,0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right) - \frac{17577}{32}\mathrm{Re}H_{0,0,1,0,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) - \frac{87885}{64}\mathrm{Re}H_{0,0,0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right) \\
& - \frac{17577}{8}\mathrm{Re}H_{0,0,0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{651}{4}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right) + \frac{1953}{8}\mathrm{Cl}_4\left(\frac{\pi}{3}\right)\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right) + \frac{31465}{176}\mathrm{Cl}_6\left(\frac{\pi}{3}\right)\pi + \frac{211}{4}\mathrm{Re}H_{0,1,0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) \\
& + \frac{211}{2}\mathrm{Re}H_{0,0,1,1,-1}\left(e^{i\frac{\pi}{3}}\right)\zeta(2) + \frac{1899}{16}\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{1899}{8}\mathrm{Re}H_{0,0,1,1,1}\left(e^{i\frac{2\pi}{3}}\right)\zeta(2) + \frac{211}{4}\mathrm{Im}H_{0,1,-1}\left(e^{i\frac{\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) \\
& + \frac{633}{8}\mathrm{Im}H_{0,1,1}\left(e^{i\frac{2\pi}{3}}\right)\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\zeta(2) - \frac{28276}{25}\zeta(2)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)^2 + 104\left(4\mathrm{Re}H_{0,1,0,1,1}\left(e^{i\frac{\pi}{2}}\right)\zeta(2) + 4\mathrm{Im}H_{0,1,1}\left(e^{i\frac{\pi}{2}}\right)\mathrm{Cl}_2\left(\frac{\pi}{2}\right)\zeta(2) - 2\mathrm{Cl}_4\left(\frac{\pi}{2}\right)\zeta(2)\pi + \mathrm{Cl}_2^2\left(\frac{\pi}{2}\right)\zeta(2)\ln 2\right) \\
& + \sqrt{3}\left[\pi\left(-\frac{28458503}{691200}B_3 + \frac{250077961}{18662400}C_3\right) + \frac{483913}{77760}\pi f_2(0,0,1) + \pi\left(\frac{4715}{1944}\ln 2\, f_2(0,0,1) + \frac{270433}{10935}f_2(0,2,0) - \frac{188147}{4860}f_2(0,1,1) + \frac{188147}{12960}f_2(0,0,2)\right)\right. \\
& + \pi\left(\frac{826595}{248832}\zeta(2)f_2(0,0,1) - \frac{5525}{432}\ln 2\, f_2(0,0,2) + \frac{5525}{162}\ln 2\, f_2(0,1,1) - \frac{5525}{243}\ln 2\, f_2(0,2,0) + \frac{526015}{248832}f_2(0,0,3) - \frac{4675}{768}f_2(0,1,2) + \frac{1805965}{248832}f_2(0,2,1)\right. \\
& \left.\left. - \frac{3710675}{1119744}f_2(0,3,0) - \frac{75145}{124416}f_2(1,0,2) - \frac{213635}{124416}f_2(1,1,1) + \frac{168455}{62208}f_2(1,2,0) + \frac{75145}{248832}f_2(2,0,1) + \frac{69245}{124416}f_2(2,1,0)\right)\right] - \frac{4715}{1458}\zeta(2)f_1(0,0,1) \\
& + \zeta(2)\left(\frac{2541575}{82944}f_1(0,0,2) - \frac{556445}{6912}f_1(0,1,1) + \frac{54515}{972}f_1(0,2,0) - \frac{75145}{20736}f_1(1,0,1)\right) - \frac{541}{300}C_{81a} - \frac{629}{60}C_{81b} + \frac{49}{3}C_{81c} - \frac{327}{160}C_{83a} + \frac{49}{36}C_{83b} + \frac{37}{6}C_{83c}
\end{aligned}
$$

Numerical cancellations

There are strong numerical cancellations in the analytical expressions of A_4 and C_4

$$A_4 \text{ contains } -\frac{2749470791}{387072}\zeta(2)\zeta(5) = -12115.862\ldots$$

$$C_4 \text{ contains } \quad \frac{7121162687}{967680}\zeta(2)\zeta(5) = \quad 12552.092\ldots$$

$$F_1'(0) = A_1\left(\frac{\alpha}{\pi}\right) + A_2\left(\frac{\alpha}{\pi}\right)^2 + A_3\left(\frac{\alpha}{\pi}\right)^3 + A_4\left(\frac{\alpha}{\pi}\right)^4 + \dots$$

Experimental result

$$\frac{\Delta E(2S_{1/2} - 2P_{1/2})(exper.)}{h} = 1057847 \pm 9 \text{ KHz}$$

Most recent theoretical value

$$\frac{\Delta E(2S_{1/2} - 2P_{1/2})(theory)}{h} = 1057834.12 \pm 0.23 \pm 0.13 \text{ KHz}$$

Effect of $F_1'(0)$

$$\Delta\nu = \frac{(Z\alpha)^4 mc^2}{2h}\left[(mc)^2\left(\frac{\alpha}{\pi}\right)^n A_n\right]$$

$$\Delta\nu(A_2) = \quad 442.018 \text{ KHz}$$

$$\Delta\nu(A_3) = \quad 0.37702 \text{ KHz}$$

$$\Delta\nu(A_4) = \quad 0.004521 \text{ KHz} \qquad < 0.26 < 9$$

The method used is the same used for 4-loop g-2:
extraction of contribution $\rightarrow$ algebraic reduction to master integrals $\rightarrow$ numerical calculation of master integrals

- The 891 vertex diagrams are the same of 4-loop g-2
- Therefore the master integrals are the same; I used the numerical values already calculated with 1100 digits of precision.
- The big difference is the algebraic reduction to M.I.: the derivative generates Feynman integrals with one power more in the denominator and two power more in the numerator $\rightarrow$ huge sizes!
- For g-2 each I.B.P. system contained tipically 5×10^6 identities with a size up to 100GB. Total size 3TB.
- For $F_1'(0)$ each I.B.P. system contains tipically 500×10^6 identities with a size up to 1TB. Total size > 30TB.
- The number of identities of each system is similar to what I would expect in a calculation of 5-loop g-2.

- For the calculation I used my program `SYS`.
- Calculation was done mainly on clusters of the University of Zurich, on big-memory machines of the ITP of Zurich, with memory sizes from 384GB to 768GB (thank to Thomas Gehrmann for the use of these computers). Some smaller parts were done on the CloudVeneto infrastructure.
- A disk crash (in Zurich) destroyed completely 10TB of data (6 months of work). The disk system was working in RAID6 mode, able to survive to the simultaneous loss of two disks. *Three* broke at the same time.
- Parallelization of code with threads and MPI was heavily used.
- I did not detect calculation errors due to the hardware
- I *did* detect errors due to some (nasty!) software bug in the library OpenMPI used for parallelization with full-thread support. For example: assigning two different polynomial multiplication to two cores, e.g. $a * b$ and $x * y$, to be executed in parallel, every 2-3 weeks the results were switched, e.g. $a * b = xy$ and $x * y = ab$.
- This kind of errors was found by discovering that identities supposed to be trivially zero (70% of the total) were actually not.

- The 4-loop slope was calculated for the first time.
- The value is positive, and of the order of 1
- It allows to determine the size of the first $\alpha^4(Z\alpha)^4$ contribution to the Lamb-Shift.
- This shift is not important at the current level of experimental precision, but it might become in the future.
- The analytical structure is almost identical to the 4-loop g-2.
- The experience gained is useful for the preparation of a calculation of a 5-loop g-2

The End

Backup: A simple example of analytical fit by using the PSLQ algorithm

$$
\begin{aligned}
G_7 = & -2342.2075141060230754235225405907927098853287320565594708 07 \\
& 359481483571384691680645591697318599261483194890419734356986 \\
& 640536482839180927737599376306979737829110608311707671767935 \\
& 983139125960766918329923883871930584868496516072868729243183 \\
& 317800519694759939914751761141283435810030791136838793708071 \\
& 157346099787020302357526852412095436287332846448926242430503 \\
& 236449547474407307581291123637921078586418676517549877972867 \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
= & \frac{1671597}{512} - \frac{4381}{96}\pi^2 - \frac{22193}{24}\zeta(3) - 144\,\pi^2\ln 2 - \frac{3617}{240}\pi^4 - \frac{71}{2}\zeta(5) \\
& - \frac{393}{2}\pi^2\zeta(3) - \frac{869}{162}\pi^6 - 24\pi^4\ln^2 2 + 576\pi^2 a_4 + 24\pi^2\ln^4 2 - \frac{803}{2}\zeta(3)^2 \\
& + 504\pi^2\zeta(3)\ln 2 - \frac{1735}{4}\zeta(7) + \frac{799}{6}\pi^2\zeta(5) - \frac{661}{180}\pi^4\zeta(3)
\end{aligned}
$$

black: ansatz (the input)

brown: coefficients found by PSLQ (the output)

This particular fit can be found using input data with a minimum precision of 415 digits.