Dispersion relations for hadronic light-by-light scattering and the muon g-2

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Outline

- Hadronic contributions to the anomalous magnetic moment of the muon: leading-order hadronic vacuum polarization (HVP) and hadronic light-by-light (HLbL)
- Focus on a novel approach based on dispersion relations for the first data-driven determination of the hadronic light-by-light contribution
- Basic features of the formalism and first numerical results
- Summary and outlook

In collaboration with Gilberto Colangelo, Martin Hoferichter and Peter Stoffer

Introduction

- Tantalizing deviation of the high-precision measurement of $a_\mu = (g-2)_\mu/2$ by the BNL E821 experiment

$$a_\mu^{\rm exp} = 116\,592\,089(63) \times 10^{-11}$$

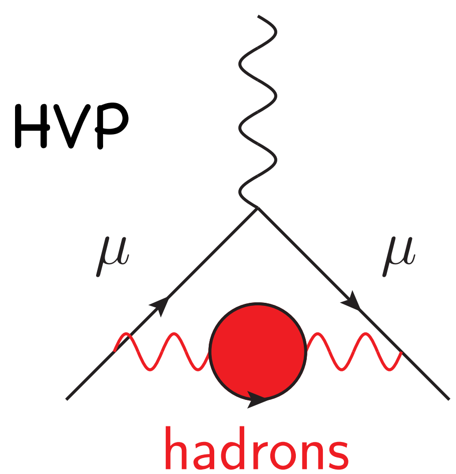
from its SM evaluation: $a_\mu^{\rm exp} - a_\mu^{\rm SM} \sim (3-4)\sigma$

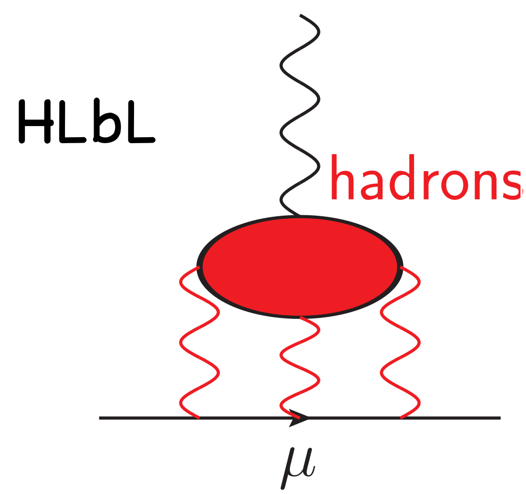
- Presently quoted theoretical and experimental uncertainties are comparable but concrete goal for Fermilab experiment E989 to reduce the error by a factor of 4

  ► calls for improved theory predictions with controlled uncertainties

Introduction

- The crucial limiting factor in the accuracy of SM predictions for a_μ is control over hadronic contributions, responsible for most of the theory uncertainty.
  The two major sources of uncertainty are the leading-order hadronic vacuum polarization contribution (HVP) and the hadronic light-by-light (HLbL)





Low-energy strong interaction effects: non-perturbative

- Two most prominent strategies for an improved determination of these contributions with controlled errors: lattice QCD and dispersion relations

HVP: dispersive approach

The most precise determination of the LO-HVP relies on a dispersive approach:

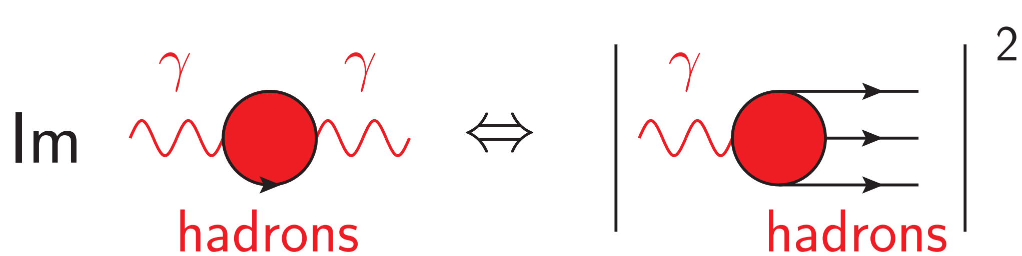
- Gauge invariance: $i\int d^4x\, e^{iq\cdot x}\langle 0|T\{j_\mu^{\rm em}(x)j_\nu^{\rm em}(0)\}|0\rangle = -(q^2 g_{\mu\nu} - q_\mu q_\nu)\,\Pi(q^2)$

  parameterized in terms of a single scalar function of one kinematic variable

- Analyticity: $\Pi^{\rm ren}(q^2) = \Pi(q^2) - \Pi(0) = \dfrac{q^2}{4\pi}\displaystyle\int_{s_{\rm thr}}^{\infty} ds\, \frac{{\rm Im}\,\Pi(s)}{s(s-q^2-i\epsilon)}$

  discontinuity along a branch cut corresponding to physical processes

- Unitarity (optical theorem):



$\propto \sigma_{\rm tot}(e^+e^- \to \text{hadrons})$

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$$\operatorname{Im}\Pi(s) = \frac{s}{4\pi\alpha(s)}\,\sigma_{\text{tot}}(e^+e^- \to \text{hadrons}) = \frac{\alpha(s)}{3}\,R^{\text{had}}(s)$$

HVP: dispersive approach

- LO-HVP is obtained by integrating the hadronic R-ratio weighted with a perturbative QED kernel:

$$a_\mu^{\mathrm{LO-HVP}} = \frac{1}{3}\left(\frac{\alpha}{\pi}\right)^2 \int_{4m_\pi^2}^{\infty} \frac{ds}{s} K(s)\, R^{\mathrm{had}}(s)$$

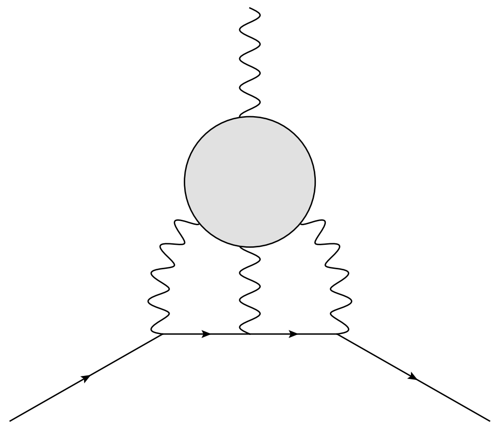
dominated by the low-energy region (in particular ππ contribution)

- Dedicated e^+e^- program (Belle II, BES-III, KLOE, BaBar, SND, CMD-3, SND, KEDR) with the goal to improve the presently quoted sub-percent accuracy.
New data are being collected and improved error analyses have been performed

▶ The HLbL contribution is emerging as a potential roadblock

Hadronic light-by-light

- Hadronic light-by-light (HLbL) is more problematic: until recently only model calculations and some high-energy and low-energy constraints



a_μ^{HLbL} in 10^{-11} units

Contribution	BPP	HKS	KN	MV	BP	PdRV	N/JN
π^0, η, η'	85±13	82.7±6.4	83±12	114±10	–	114±13	99±16
π, K loops	−19±13	−4.5±8.1	–	–	–	−19±19	−19±13
π, K loops + other subleading in N_c	–	–	–	0±10	–	–	–
axial vectors	2.5±1.0	1.7±1.7	–	22± 5	–	15±10	22± 5
scalars	−6.8±2.0	–	–	–	–	−7± 7	−7± 2
quark loops	21± 3	9.7±11.1	–	–	–	2.3	21± 3
total	83±32	89.6±15.4	80±40	136±25	110±40	105±26	116±39

Two global evaluations: Bijnens, Pallante, Prades (1995, 1996) and Hayakawa, Kinoshita, Sanda (1995, 1996)

KN = Knecht, Nyffeler; MV = Melnikov, Vainshtein; PdRV = Prades, de Rafael, Vainshtein; JN= Jegerlehner, Nyffeler

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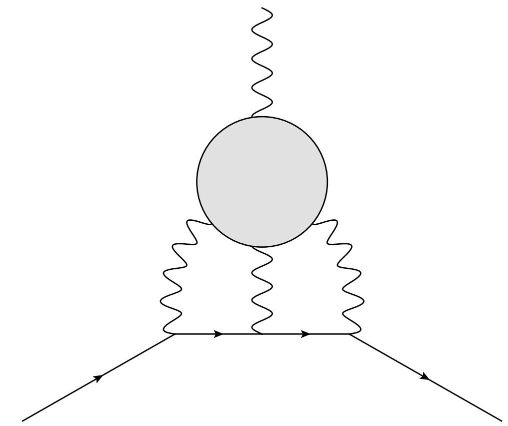

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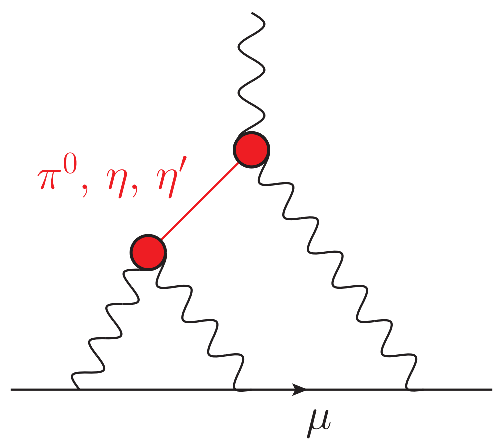

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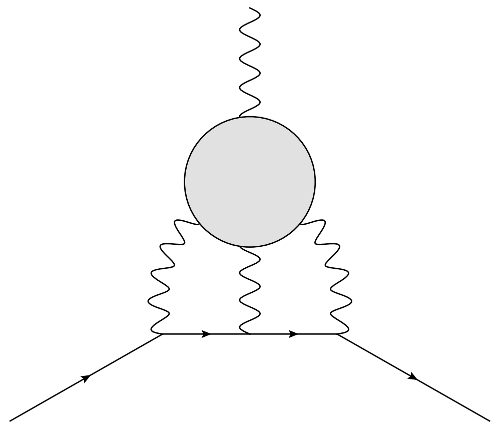

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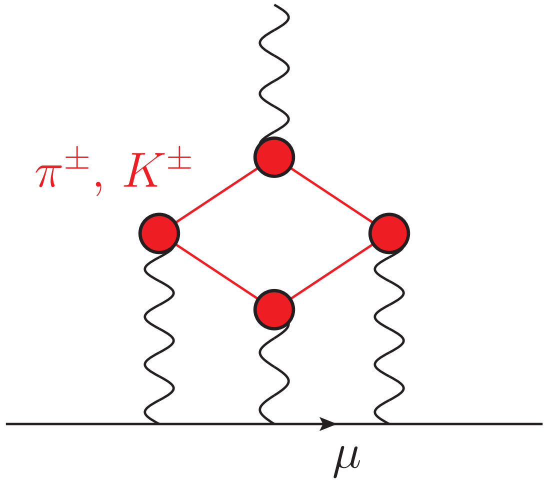

The lightest intermediate states dominate

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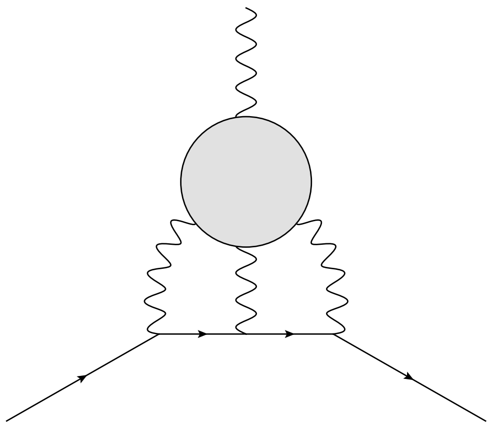

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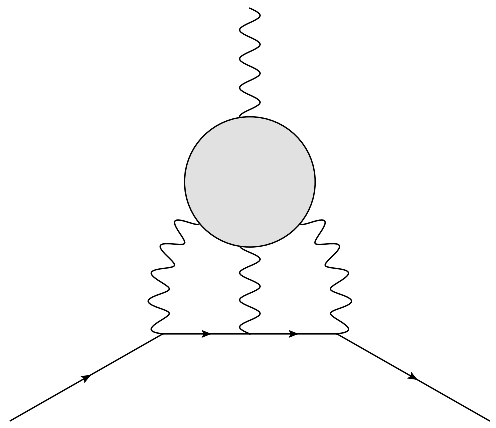
Jegerlehner (2015)

≈ 8±3

102±39

The two most often quoted estimates: Prades, de Rafael, Vainshtein (2009) and Jegerlehner, Nyffeler (2009)

Hadronic light-by-light



- Hadronic light-by-light (HLbL) is more problematic:
  until recently only model calculations and
  some high-energy and low-energy constraints

  Quoted uncertainties are guesstimates!

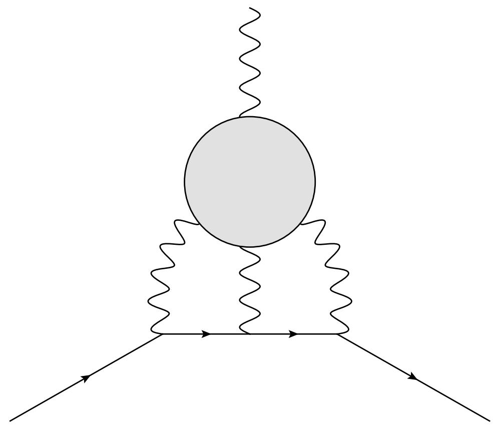
  ▶ a reliable uncertainty estimate for HLbL is still an open issue

- How to reduce model dependence and get small and reliable uncertainties?

  ▶ lattice QCD: first computations at physical pion masses with leading disconnected contributions performed (with large systematic errors due to finite volume and finite lattice spacing) RBC/UKQCD (Blum et al., 2015-2017)
  Mainz lattice group: pion-pole contribution (Gerardin, Meyer, Nyffeler, 2019)

  ▶ dispersion theory to make the evaluation as data-driven as possible

Dispersive approach to HLbL



- Exploits fundamental principles:
  - gauge invariance and crossing symmetry
  - unitarity and analyticity

  to relate HLbL to experimentally accessible quantities

- Much more challenging task than for the hadronic vacuum polarization due to the complexity of the HLbL tensor, which is the key object of our analysis

- Defines and relates single contributions to HLbL to form factors and cross sections

Colangelo, Hoferichter, Procura, Stoffer, JHEP 1505 (2015), JHEP 1704 + PRL 118 (2017)

Colangelo, Hoferichter, Procura, Stoffer, JHEP 1409 (2014)

Colangelo, Hoferichter, Kubis, Procura, Stoffer, PLB 738 (2014)

OUR FORMALISM :

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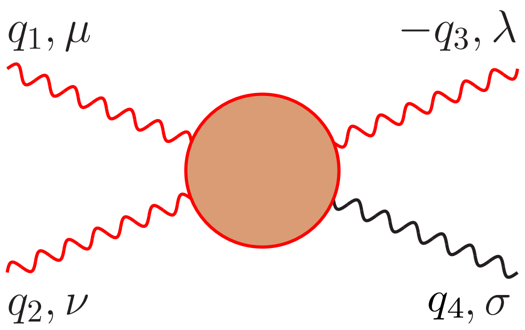
The HLbL tensor

- The fully off-shell HLbL tensor :



$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = -i \int d^4x\, d^4y\, d^4z\, e^{-i(q_1\cdot x + q_2\cdot y + q_3\cdot z)} \langle 0|T\{j^{\mu}_{\mathrm{em}}(x) j^{\nu}_{\mathrm{em}}(y) j^{\lambda}_{\mathrm{em}}(z) j^{\sigma}_{\mathrm{em}}(0)\}|0\rangle$$

- Mandelstam variables:

$$s = (q_1 + q_2)^2,\ t = (q_1 + q_3)^2,\ u = (q_2 + q_3)^2$$

- In order to extract a_μ^{HLbL}, $q_4 \to 0$ afterwards

Lorentz structure of HLbL tensor

- Based on Lorentz covariance the HLbL tensor can be decomposed in 138 structures

$$
\begin{aligned}
\Pi^{\mu\nu\lambda\sigma} &= g^{\mu\nu} g^{\lambda\sigma}\, \Pi^1 + g^{\mu\lambda} g^{\nu\sigma}\, \Pi^2 + g^{\mu\sigma} g^{\nu\lambda}\, \Pi^3 \\
&+ \sum_{\substack{k=1,2,4 \\ l=1,2,3}} g^{\mu\nu} q_k^\lambda q_l^\sigma\, \Pi^4_{kl} + \sum_{\substack{j=1,3,4 \\ l=1,2,3}} g^{\mu\lambda} q_j^\nu q_l^\sigma\, \Pi^5_{jl} + \sum_{\substack{j=1,3,4 \\ k=1,2,4}} g^{\mu\sigma} q_j^\nu q_k^\lambda\, \Pi^6_{jk} \\
&+ \sum_{\substack{i=2,3,4 \\ l=1,2,3}} g^{\nu\lambda} q_i^\mu q_l^\sigma\, \Pi^7_{il} + \sum_{\substack{i=2,3,4 \\ k=1,2,4}} g^{\nu\sigma} q_i^\mu q_k^\lambda\, \Pi^8_{ik} + \sum_{\substack{i=2,3,4 \\ j=1,3,4}} g^{\lambda\sigma} q_i^\mu q_j^\nu\, \Pi^9_{ij} \\
&+ \sum_{\substack{i=2,3,4 \\ j=1,3,4}} \sum_{\substack{k=1,2,4 \\ l=1,2,3}} q_i^\mu q_j^\nu q_k^\lambda q_l^\sigma\, \Pi^{10}_{ijkl}
\end{aligned}
$$

- Scalar functions encode the hadronic dynamics and depend on 6 kinematic variables

- In 4 space-time dimensions there are 2 linear relations among these 138 structures

Eichmann, Fischer, Heupel, Williams (2014)

- This set of functions is hugely redundant: Ward identities imply 95 linear relations among these scalar functions (kinematic zeros)

Lorentz structure of HLbL tensor

- Following Bardeen and Tung (1968) - "BT"- we contracted the HLBL tensor with

$$I_{12}^{\mu\nu} = g^{\mu\nu} - \frac{q_2^{\mu} q_1^{\nu}}{q_1 \cdot q_2}, \quad I_{34}^{\lambda\sigma} = g^{\lambda\sigma} - \frac{q_4^{\lambda} q_3^{\sigma}}{q_3 \cdot q_4}$$

  - 95 structures project to zero

- $1/q_1 \cdot q_2$ and $1/q_3 \cdot q_4$ poles eliminated by taking linear combinations of structures

- This procedure introduces kinematic singularities in the scalar functions: degeneracies in these BT Lorentz structures, e.g. as $q_1 \cdot q_2 \to 0$, $q_3 \cdot q_4 \to 0$

$$\sum_k c_k^i T_k^{\mu\nu\lambda\sigma} = q_1 \cdot q_2 X_i^{\mu\nu\lambda\sigma} + q_3 \cdot q_4 Y_i^{\mu\nu\lambda\sigma}$$

Lorentz structure of HLbL tensor

- Following Tarrach (1975) we extended BT set to incorporate $X_i^{\mu\nu\lambda\sigma}$, $Y_i^{\mu\nu\lambda\sigma}$ to obtain a ("BTT") generating set of structures even for $q_1 \cdot q_2 \to 0,\, q_3 \cdot q_4 \to 0$

$$\Pi^{\mu\nu\lambda\sigma}(q_1, q_2, q_3) = \sum_{i=1}^{54} T_i^{\mu\nu\lambda\sigma} \Pi_i(s, t, u; q_j^2)$$

- Lorentz structures are manifestly gauge invariant
- crossing symmetry is manifest (only 7 genuinely different structures, the remaining ones being obtained by crossing)
- the BTT scalar functions are free of kinematic singularities and zeros: their analytic structure is dictated by dynamics only.
  This makes them suitable for a dispersive treatment

OUR FORMALISM :

- The HLbL tensor: gauge invariance and crossing symmetry
- Master formula for the HLbL contribution to $(g-2)_\mu$
- Dispersive representation of scalar functions at fixed photon virtualities

Master formula for a_μ^{HLbL}

- From $\Pi_{\mu\nu\lambda\sigma}$ to a_μ^{HLbL} :

By expanding the photon-muon vertex function around $q_4 = 0$,

$$a_\mu^{\mathrm{HLbL}} = -\frac{1}{48 m_\mu} \mathrm{Tr}\left((\not{p} + m_\mu)[\gamma^\rho, \gamma^\sigma](\not{p} + m_\mu)\Gamma_{\rho\sigma}^{\mathrm{HLbL}}(p) \right)$$

Aldin, Brodsky, Dufner, Kinoshita (1970)

where $p^2 = m_\mu^2$ and

$$\Gamma_{\rho\sigma}^{\mathrm{HLbL}}(p) = e^6 \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \gamma^\mu \frac{(\not{p} + \not{q}_1 + m_\mu)}{(p+q_1)^2 - m_\mu^2} \gamma^\lambda \frac{(\not{p} - \not{q}_2 + m_\mu)}{(p-q_2)^2 - m_\mu^2} \gamma^\nu \times \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2} \frac{\partial}{\partial q_4^\rho} \Pi_{\mu\nu\lambda\sigma}(q_1, q_2, q_4 - q_1 - q_2) \Big|_{q_4 = 0}$$

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- Since there are no kinematic singularities in the BTT scalar functions, the limit $q_4 \to 0$ can be taken explicitly

$$a_\mu^{\mathrm{HLbL}} = -\frac{e^6}{48 m_\mu} \int \frac{d^4 q_1}{(2\pi)^4} \frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1+q_2)^2} \frac{1}{(p+q_1)^2 - m_\mu^2} \frac{1}{(p-q_2)^2 - m_\mu^2}$$
$$\times \mathrm{Tr}\left((\not{p} + m_\mu)[\gamma^\rho, \gamma^\sigma](\not{p} + m_\mu)\gamma^\mu(\not{p} + \not{q}_1 + m_\mu)\gamma^\lambda(\not{p} - \not{q}_2 + m_\mu)\gamma^\nu \right)$$
$$\times \sum_{i=1}^{54} \left(\frac{\partial}{\partial q_4^\rho} T^i_{\mu\nu\lambda\sigma}(q_1, q_2, q_4 - q_1 - q_2) \right) \Bigg|_{q_4=0} \Pi_i(q_1, q_2, -q_1 - q_2)$$

Master formula for a_μ^{HLbL}

- We obtained a general master formula

$$a_\mu^{\mathrm{HLbL}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\,Q_1^3 Q_2^3 \sum_{i=1}^{12} T_i(Q_1, Q_2, \tau)\bar{\Pi}_i(Q_1, Q_2, \tau)$$

- $Q_i^2 = -q_i^2$ are Euclidean momenta and $Q_1 \cdot Q_2 = Q_1\, Q_2\, \tau$: space-like kinematics

- We calculated the integration kernels T_i .
  The scalar functions $\bar{\Pi}_i$ are linear combinations of the BTT Π_i

- Our goal: dispersive representation of HLbL scalar functions at fixed photon virtualities to be evaluated at the reduced kinematics in the master formula,

$$s = -Q_3^2 = -Q_1^2 - 2Q_1Q_2\tau - Q_2^2, \quad t = -Q_2^2, \quad u = -Q_1^2,$$
$$q_1^2 = -Q_1^2, \quad q_2^2 = -Q_2^2, \quad q_3^2 = -Q_3^2 = -Q_1^2 - 2Q_1Q_2\tau - Q_2^2, \quad q_4^2 = 0$$

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Mandelstam representation

- Analytic properties of scalar functions at fixed photon virtualities: contributions from unitarity in s-, t-, u-channel, in the form of poles and branch cuts.
  For example, a fixed-t dispersion relation :

$$\Pi_i^t(s,t,u) = c_i^t + \frac{\rho_{i;s}^t}{s-M_\pi^2} + \frac{\rho_{i;u}^t}{u-M_\pi^2} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\mathrm{Im}_s \Pi_i^t(s',t,u')}{s'-s} + \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} du' \frac{\mathrm{Im}_u \Pi_i^t(s',t,u')}{u'-u}$$

- By symmetrizing over all channels, the Mandelstam representation is obtained

$$\Pi_i = \frac{1}{\pi^2}\int ds' dt' \frac{\rho_i^{st}(s',t')}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

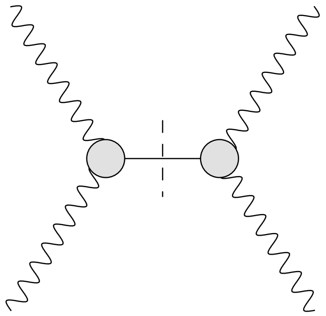
- Very complex analytic structure: approximations are required. The lightest states are expected to be the most important (in agreement with model calculations)

One- and two-pion intermediate states

- When exploiting unitarity, we considered the 2 lowest-lying contributions: one- and two-pion intermediate states in all channels

$$\Pi_{\mu\nu\lambda\sigma} = \Pi^{\pi^0\text{-pole}}_{\mu\nu\lambda\sigma} + \Pi^{\text{box}}_{\mu\nu\lambda\sigma} + \bar{\Pi}_{\mu\nu\lambda\sigma} + \ldots$$

one-pion intermediate state :



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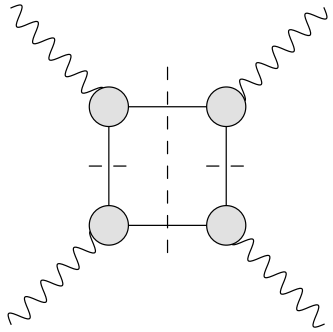
two-pion intermediate state in both channels :



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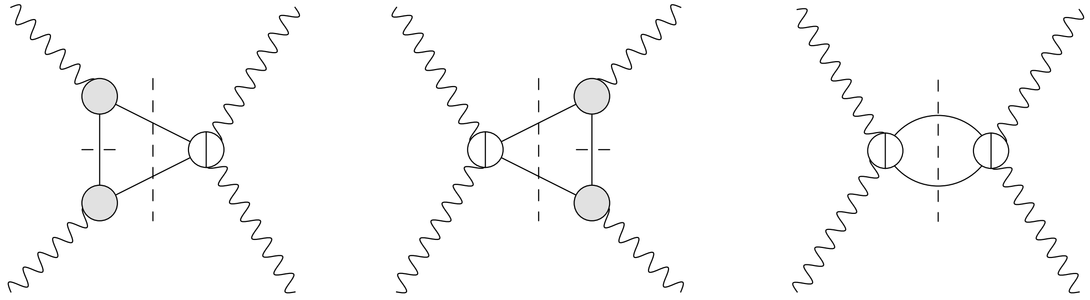
two-pion state only in the direct channel:



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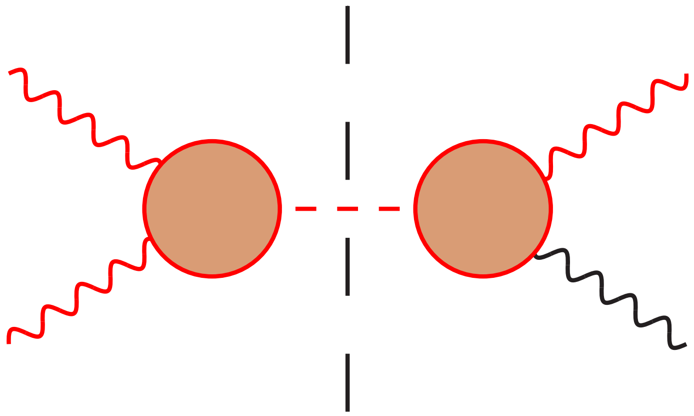
higher intermediate states: ongoing work

The pion-pole contribution

- From the unitarity relation with only π^0 intermediate state, the pole residues in each channel are given by products of doubly-virtual and singly-virtual pion transition form factors ($\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$, input for our analysis)



$$a_\mu^{\pi^0\text{-pole}} = \frac{2\alpha^3}{3\pi^2}\int_0^\infty dQ_1 \int_0^\infty dQ_2 \int_{-1}^1 d\tau \sqrt{1-\tau^2}\, Q_1^3 Q_2^3 \Big(T_1(Q_1,Q_2,\tau)\bar{\Pi}_1^{\pi^0\text{-pole}}(Q_1,Q_2,\tau) + T_2(Q_1,Q_2,\tau)\bar{\Pi}_2^{\pi^0\text{-pole}}(Q_1,Q_2,\tau)\Big)$$

with

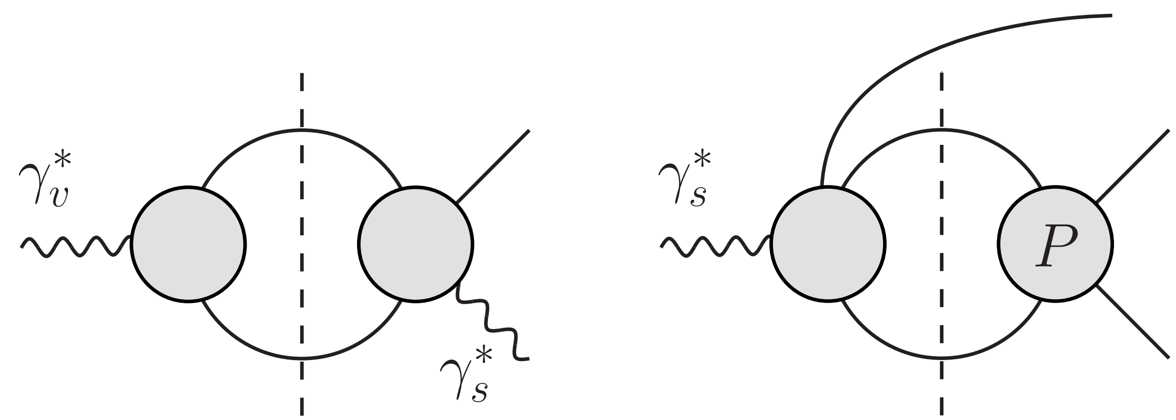
$$\bar{\Pi}_1^{\pi^0\text{-pole}} = -\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_2^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_3^2,0)}{Q_3^2+M_\pi^2} \qquad \bar{\Pi}_2^{\pi^0\text{-pole}} = -\frac{\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_1^2,-Q_3^2)\mathcal{F}_{\pi^0\gamma^*\gamma^*}(-Q_2^2,0)}{Q_2^2+M_\pi^2}$$

The pion-pole contribution

- From the unitarity relation with only π^0 intermediate state, the pole residues in each channel are given by products of doubly-virtual and singly-virtual pion transition form factors ($\mathcal{F}_{\gamma^*\gamma^*\pi^0}$ and $\mathcal{F}_{\gamma^*\gamma\pi^0}$, input for our analysis)

- These form factors can be reconstructed dispersively using

  - pion vector form factor
  - $\gamma^* \to 3\pi$ amplitude
  - elastic $\pi\pi$ scattering amplitude



Hoferichter, Kubis, Leupold, Niecknig, Schneider (2014)

$$\longrightarrow \quad a_\mu^{\pi^0-\text{pole}} = 62.6^{+3.0}_{-2.5} \times 10^{-11}$$

Hoferichter, Hoid, Kubis, Leupold, Schneider (2018)

- Pseudoscalar poles with higher masses can be treated analogously

Pion-box contribution

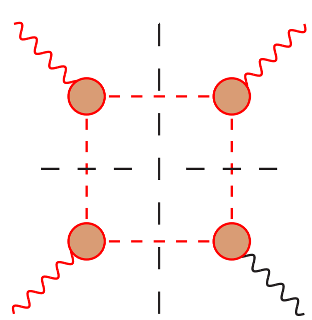
- Defined by simultaneous two-pion cuts in two channels

- Contribution to scalar functions as dispersive integral of double spectral functions

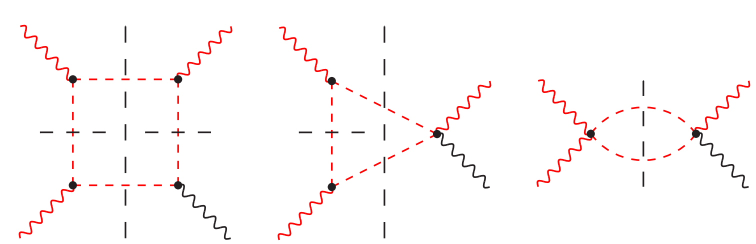
$$\Pi_i = \frac{1}{\pi^2} \int ds' dt' \frac{\rho_i^{st}(s',t')}{(s'-s)(t'-t)} + (t \leftrightarrow u) + (s \leftrightarrow u)$$

- Dependence on q_i^2 carried by the pion vector FFs for each off-shell photon

- One-loop sQED projected onto the BTT structures fulfills the same Mandelstam representation of the pion box, the only difference being the pion vector FFs :

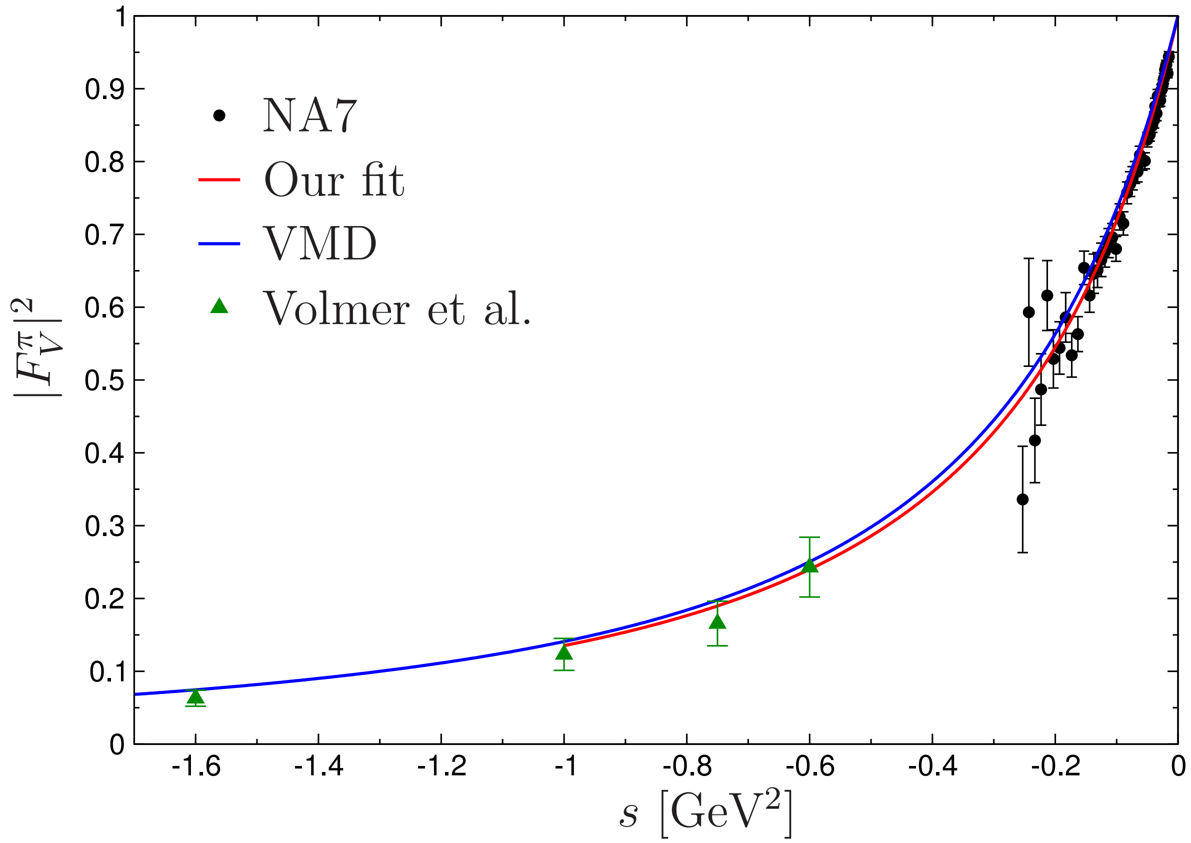


$$\equiv F_\pi^V(q_1^2) F_\pi^V(q_2^2) F_\pi^V(q_3^2)$$

$\times$ []

Numerics for the pion-box contribution

- The only input: pion vector form factor in the space-like region

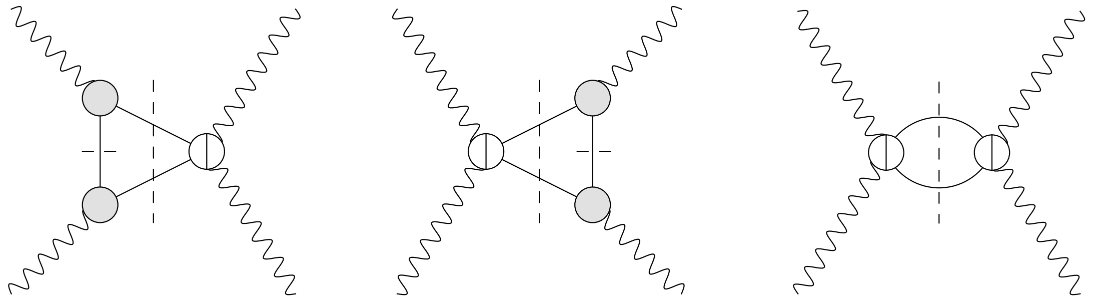


- Numerical results: $a_\mu^{\pi-\text{box}} = -15.9(2) \times 10^{-11}$ vs $a_\mu^{K-\text{box},\text{VMD}} \simeq -0.5 \times 10^{-11}$

- Rapid convergence: $Q_{\text{max}} = \{1, 1.5\}$ GeV $\Rightarrow$ $a_\mu^{\pi\text{-box}} = \{95, 99\}\%$ of full result

The remaining ππ contribution

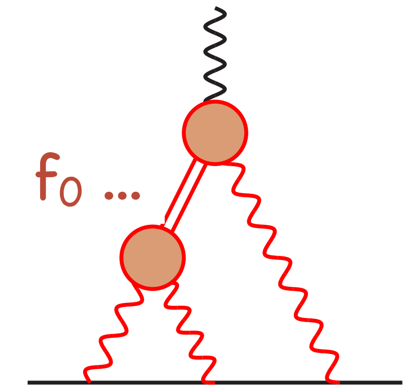
- Two-pion cut only in the direct channel: LH cut due to multi-particle intermediate states in the crossed channel neglected



- Unitarity relates this contribution to the subprocess $\gamma^*\gamma^{(*)} \to \pi\pi$

- By generalizing previous analyses of $\gamma\gamma \to \pi\pi$ and $\gamma\gamma^* \to \pi\pi$ Moussallam et al. (2010, 2013) our goal is a dispersive reconstruction (based on analyticity, unitarity and crossing) of helicity partial waves for $\gamma^*\gamma^* \to \pi\pi$ Colangelo, Hoferichter, MP, Stoffer (2014)

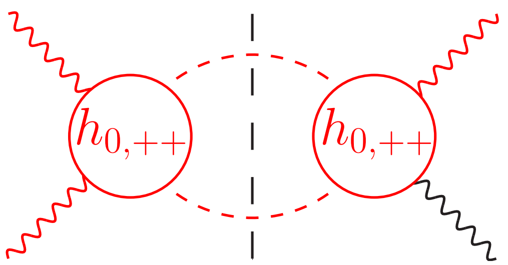
- The solution of the resulting coupled set of dispersion relations involves elastic ππ phase shifts, which allows for the summation of ππ rescattering effects in the direct channel (effects of resonances coupling to ππ)



The remaining $\pi\pi$ contribution

- Contribution to $a_\mu^{\rm HLbL}$ from $\gamma^*\gamma^* \to \pi\pi$ helicity partial waves:

$$\text{Im}\, h^J_{++,++}(s; q_1^2, q_2^2; q_3^2, 0) = \frac{\sigma(s)}{16\pi} h^*_{J,++}(s; q_1^2, q_2^2)\, h_{J,++}(s; q_3^2, 0)$$



projecting onto BTT basis determines Im Π_i , from which Π_i for master formula.

- We solved dispersion relations for $\gamma^*\gamma^* \to \pi\pi$ S-waves taking:

  - pion pole as only LH singularity and phenomenological $\pi\pi$ phase shifts

$a_\mu^{\rm HLbL}$ in 10^{-11} units

	Λ	1 GeV	1.5 GeV	2 GeV	∞
$f_0(500)$ →	$I = 0$	-9.2	-9.5	-9.3	-8.8
	$I = 2$	2.0	1.3	1.1	0.9

$$a_{\mu,J=0}^{\pi\pi,\pi\text{-pole LHC}} = -8(1) \times 10^{-11}$$

Outlook for dispersive a_μ^{HLbL}

A more precise, data-driven SM evaluation of HLbL using dispersion relations is feasible!

Ongoing and future work:

- Rescattering contributions for higher partial waves to account for prominent features in the cross sections for photon-photon to two mesons.
  Extension of the solution of partial-wave dispersion relations for $\gamma^*\gamma^* \to \pi\pi$ to D-waves to capture effects of $f_2(1270)$ beyond narrow width approximation
  Hoferichter and Stoffer (2019)

- Contributions from higher intermediate states

- Systematic study of all short-distance constraints on HLbL

Summary and outlook about HLbL

- Dispersive approach to HLbL scattering based on general principles: gauge invariance and crossing symmetry, unitarity and analyticity

- Derivation of a set of BTT structures free of kinematic singularities and zeros

- Derivation of a general master formula for a_μ^{HLbL} in terms of BTT functions

- Single- and double-pion intermediate states are taken into account.
  Results can be extended to other pseudoscalar poles and two-meson states

- Numerical results for pion box and S-wave ππ rescattering: small uncertainties

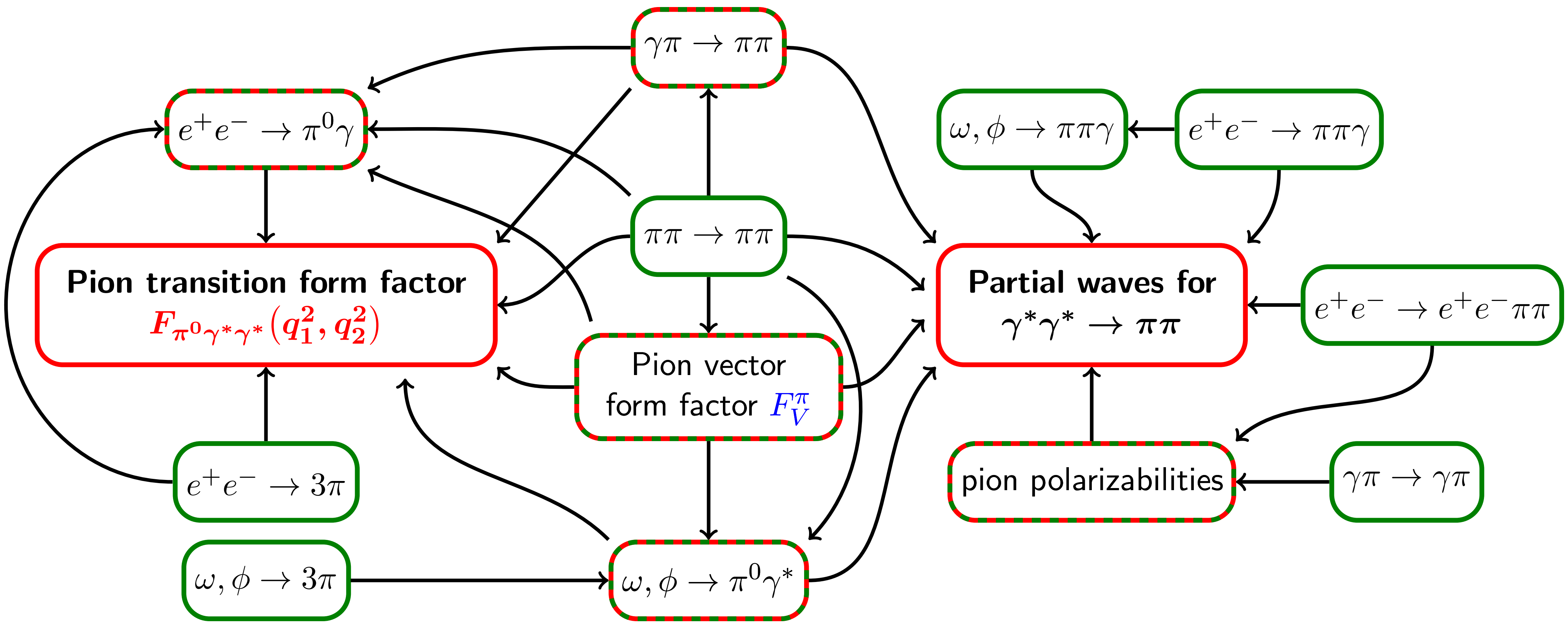
- Next: Refined analysis of two-meson states (include kaons, coupled channel $\pi\pi/K\bar{K}$ system, include D-waves, generalize to heavier LH cuts).
  Study higher intermediate states in the direct channel (3 pions, axials).
  Investigate and incorporate QCD short-distance constraints.

- First step towards a reduction of model dependence of HLbL: relations with experimentally accessible (or dispersively reconstructed) quantities

Additional slides

A roadmap for HLbL

Colangelo, Hoferichter, Kubis, MP, Stoffer (2014)



Artwork by M. Hoferichter