Lattice calculation of the hadronic leading order contribution to the muon g – 2

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HIM Helmholtz-Institut Mainz

PRISMA+

SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

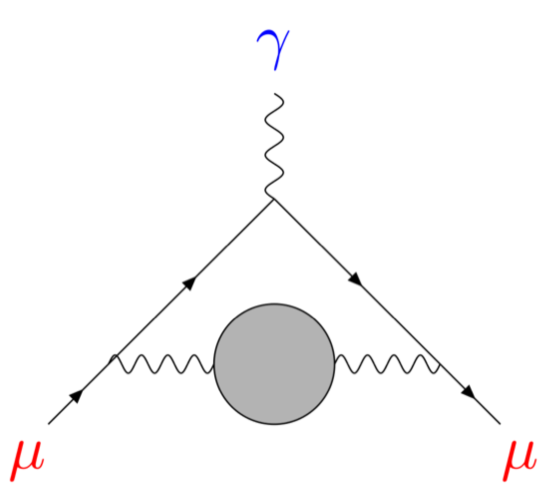
The muon anomalous magnetic moment

$$a_\mu \equiv \tfrac{1}{2}(g-2)_\mu = \begin{cases} 116\,592\,080(54)(33)\cdot 10^{-11} & \text{E821 @ BNL} \\ 116\,591\,825(34)(26)(1)\cdot 10^{-11} & \text{SM prediction} \end{cases}$$

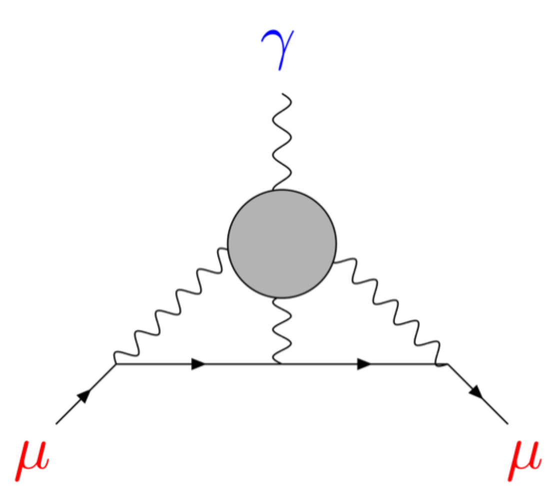
* SM estimate dominated by QED; error dominated by QCD

$$a_\mu = a_\mu^{\rm QED} + a_\mu^{\rm weak} + a_\mu^{\rm strong} + a_\mu^{\rm NP?}$$

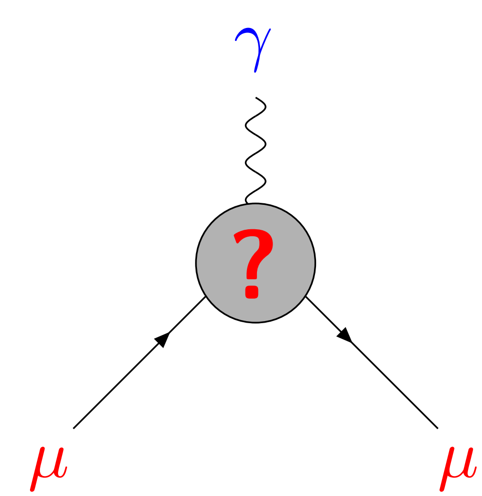
Hadronic
vacuum polarisation:



Hadronic
light-by-light scattering:



Contribution from
“New Physics”?



Theory confronts experiment

[Jegerlehner, arXiv:1705.00263]

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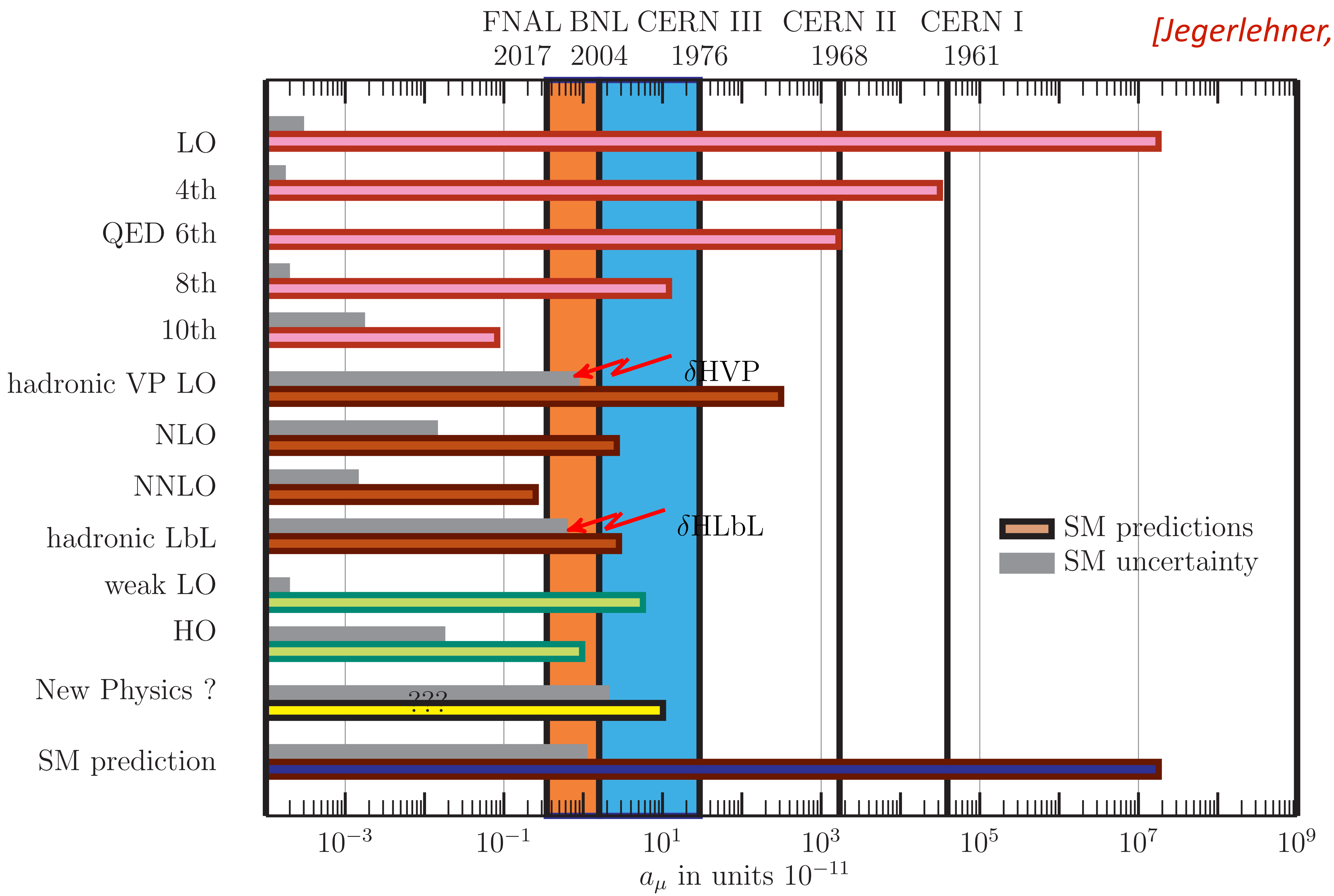

* Experimental sensitivity of E989 exceeds total theory uncertainty by far!

The muon g – 2 in lattice QCD

Motivation for first-principles approach:

* No reliance on experimental data
  — except for simple hadronic quantities to fix bare parameters

* No model dependence
  — except for chiral extrapolation and constraining the IR regime

* Can lattice QCD deliver estimates with sufficient accuracy in the coming years?

$$\delta a_\mu^{\rm hvp}/a_\mu^{\rm hvp} < 0.5\%, \qquad \delta a_\mu^{\rm hlbl}/a_\mu^{\rm hlbl} \lesssim 10\%$$

The muon g – 2 in lattice QCD

ELSEVIER

Progress in Particle and Nuclear Physics

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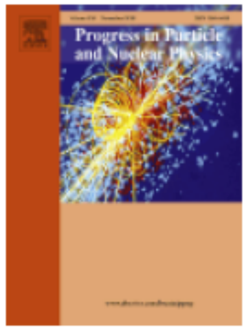


Review

Lattice QCD and the anomalous magnetic moment of the muon

Harvey B. Meyer, Hartmut Wittig

⊞ Show more

https://doi.org/10.1016/j.ppnp.2018.09.001

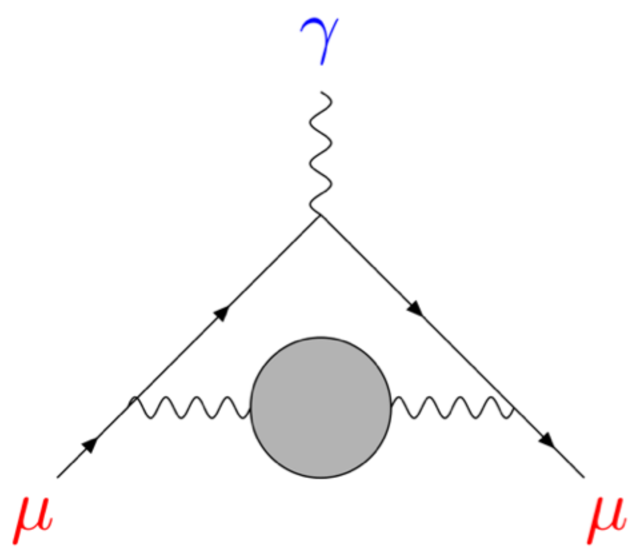
arXiv:1807.09370

The Mainz $(g-2)_\mu$ Lattice QCD project

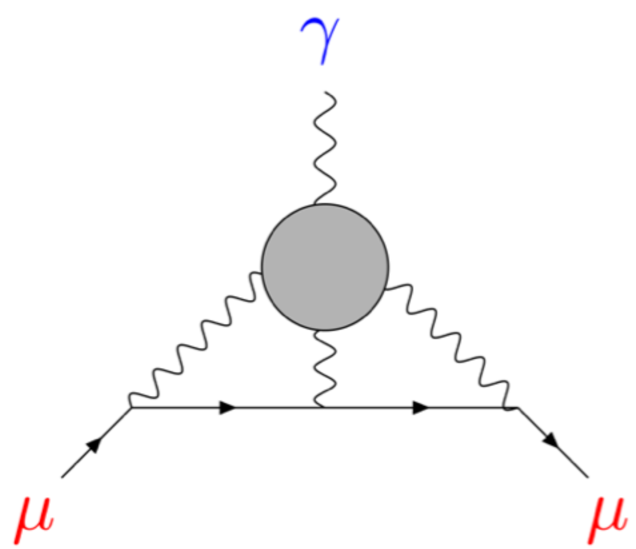
Collaborators:

M. Cè, E.-H. Chao, G. von Hippel, R.J. Hudspith, H.B. Meyer, K. Miura,
D. Mohler, A. Nyffeler, K. Ottnad, A. Risch, T. San José Perez, J. Wilhelm, HW

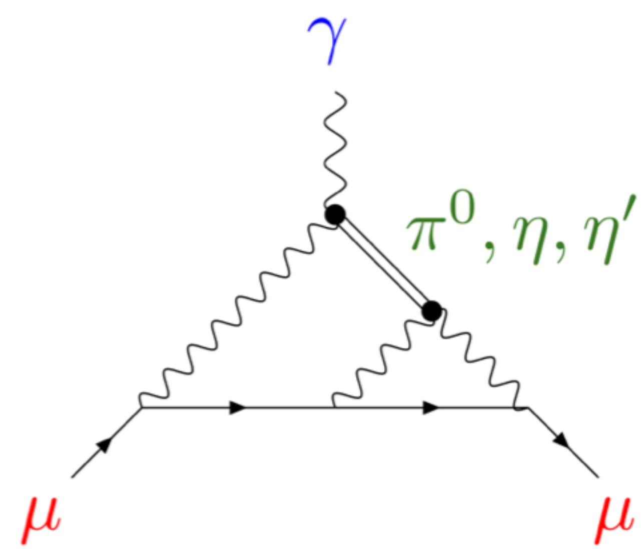
N. Asmussen, A. Gérardin, J. Green, G. Herdoíza, B. Hörz



- Direct determinations of LO a_μ^{hvp}
- Running of α and $\sin^2\theta_{\mathrm{W}}$



- Exact QED kernel
- Forward scattering amplitude



- Transition form factor for $\pi^0 \to \gamma^*\gamma^*$

Lattice QCD approach to HVP

* Vacuum polarisation tensor:

$$\Pi_{\mu\nu}(Q) = i \int d^4x\, \mathrm{e}^{iQ\cdot(x-y)} \left\langle J_\mu(x) J_\nu(y) \right\rangle \equiv (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\Pi(Q^2)$$

* Electromagnetic current:

$$J_\mu = \tfrac{2}{3}\overline{u}\gamma_\mu u - \tfrac{1}{3}\overline{d}\gamma_\mu d - \tfrac{1}{3}\overline{s}\gamma_\mu s + \ldots$$

* Convolution integral over Euclidean momenta: *[Lautrup & de Rafael; Blum]*

$$a_\mu^{\mathrm{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2\, f(Q^2)\hat{\Pi}(Q^2), \quad \hat{\Pi}(Q^2) = 4\pi^2 \left(\Pi(Q^2) - \Pi(0)\right)$$

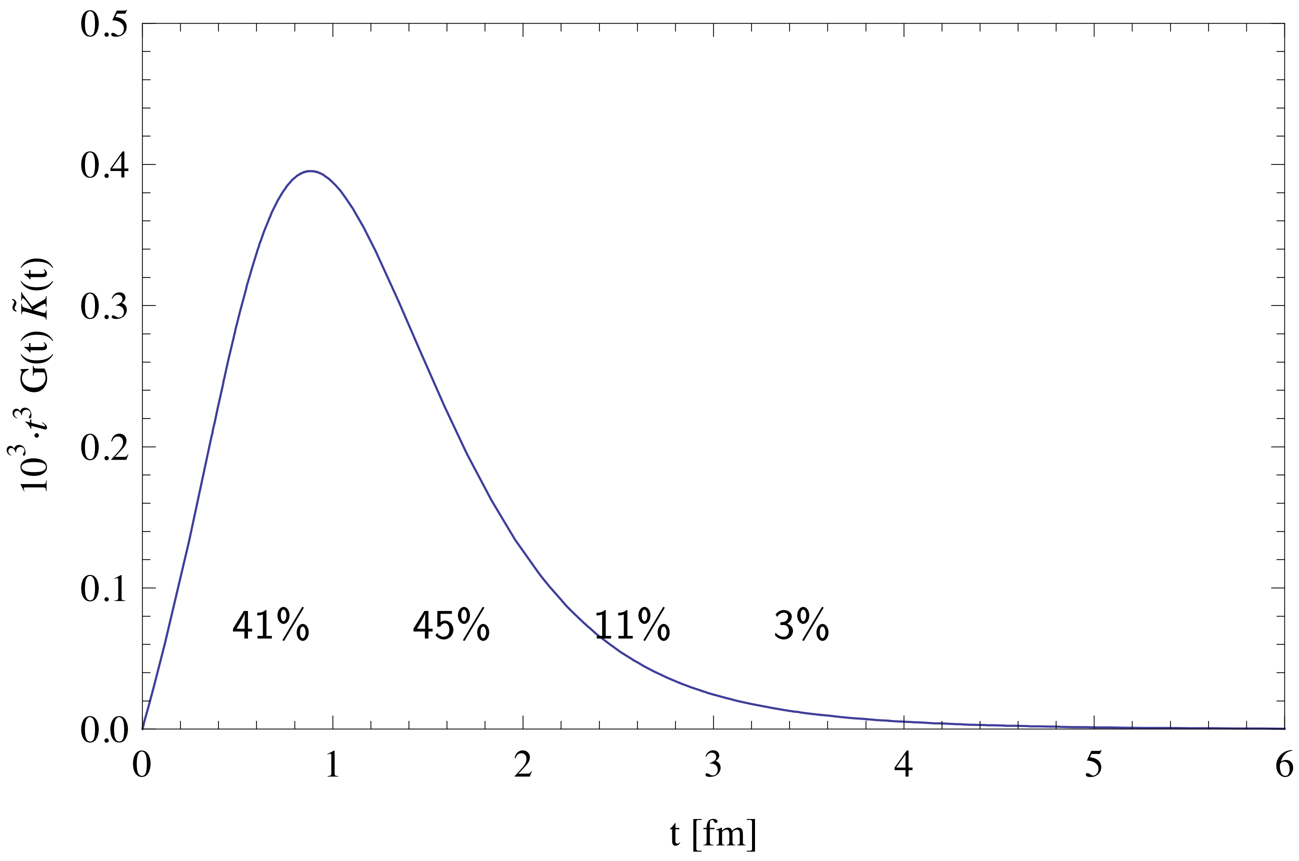
* Weight function $f(Q^2)$ strongly peaked near muon mass

Lattice QCD approach to HVP

* **Time-momentum representation (TMR):** *[Bernecker & Meyer, EPJC (2011)]*

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{K}(x_0)\, G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$$

$$\tilde{K}(x_0) = 4\pi^2 \int_0^\infty dQ^2\, f(Q^2) \left[x_0^2 - \frac{4}{Q^2} \sin^2\left(\tfrac{1}{2} Q x_0\right) \right]$$



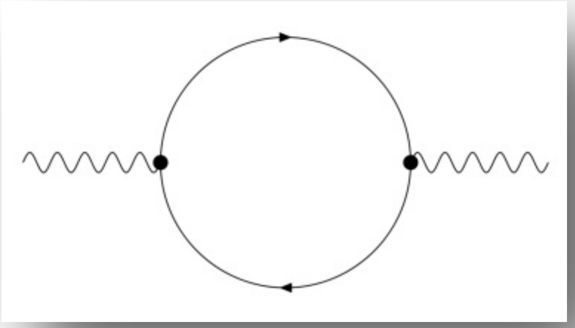
- Significant contribution from tail of $G(x_0)$
- Exponentially increasing noise-to-signal ratio:

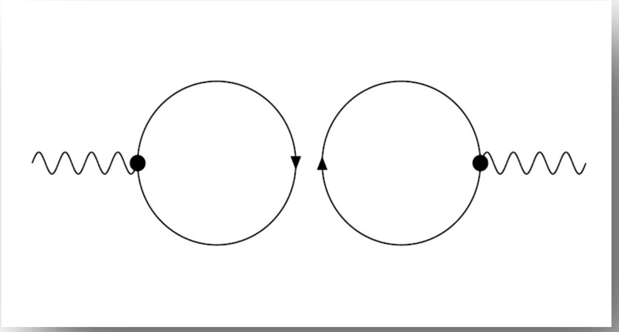
$$R_{\rm NS} \propto \exp\{(m_{\rm V} - m_\pi) x_0\}$$

Lattice QCD approach to HVP

Challenges:

* Statistical accuracy at the sub-percent level required
* Control infrared regime of vector correlator: $G(x_0)$ at large x_0
* Perform comprehensive study of finite-volume effects
* Include quark-disconnected diagrams





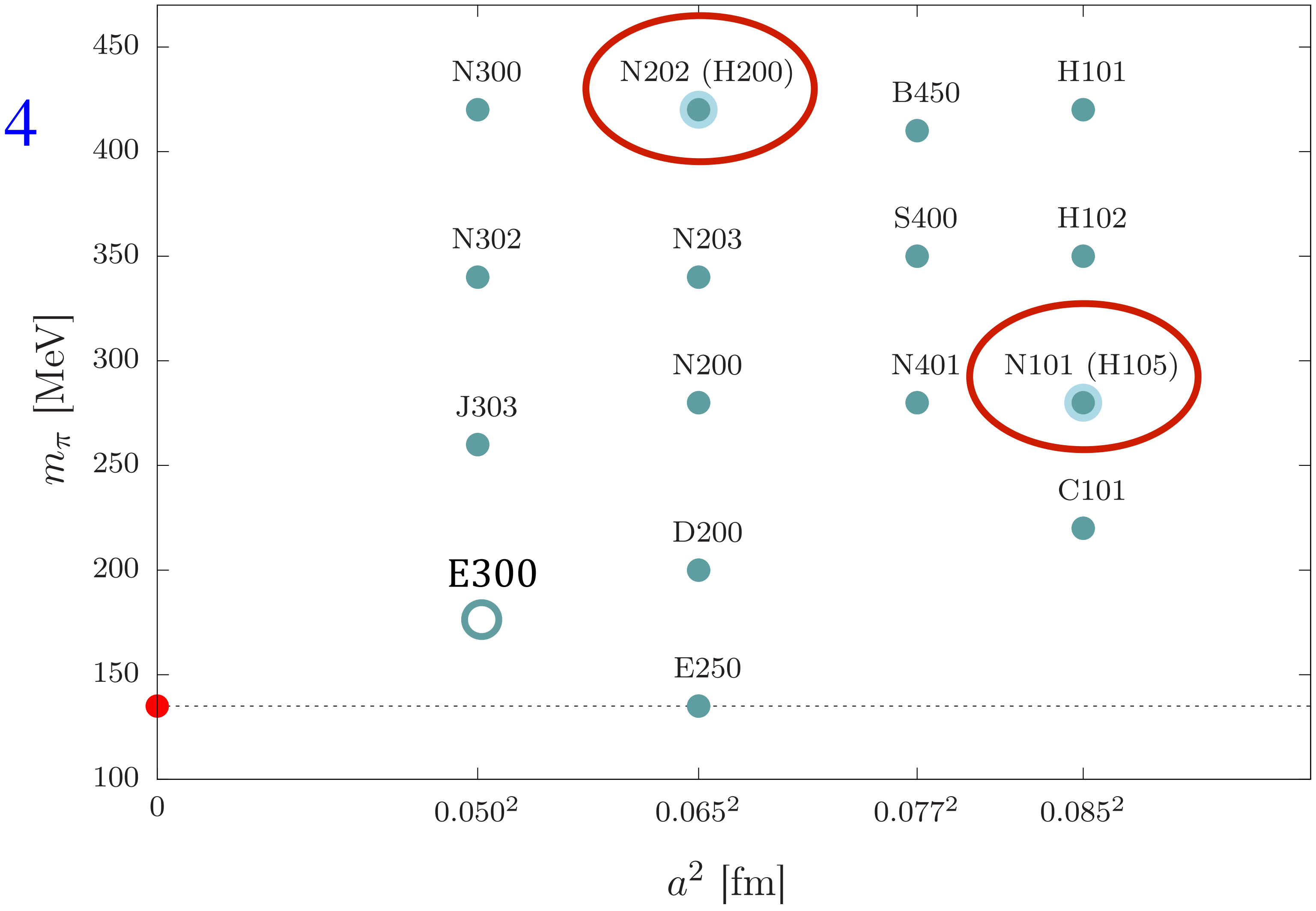
* Include isospin breaking: $m_u \neq m_d$, QED corrections

Features of our calculation

Gérardin et al., Phys. Rev. D100 (2019) 034513, arXiv:1904.03120

- $N_f = 2+1$ flavours of $O(a)$ improved Wilson quarks
- Four values of the lattice spacing: $a = 0.085, 0.077, 0.065, 0.050\,\mathrm{fm}$
- Pion masses and volumes:

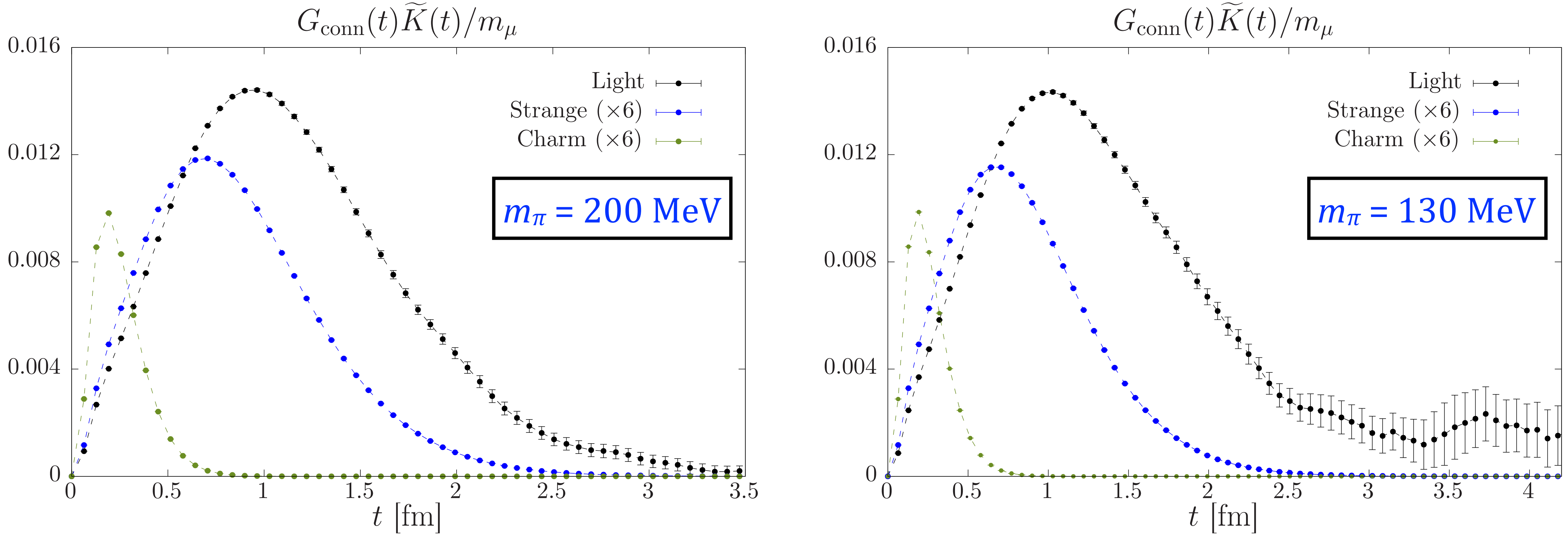
  $m_\pi^{\min} \approx 135\,\mathrm{MeV}, \quad m_\pi L > 4$
- Check finite-volume effects
- Additional ensembles: lighter pion masses at smaller lattice spacings



Controlling the infrared regime

- TMR integrand and its long-distance behaviour:

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{K}(x_0)\, G(x_0), \qquad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$$



- Large-x_0 regime still statistics-limited for $x_0 \gtrsim 2.5$ fm

[Gérardin et al., arXiv:1904.03120]

Controlling the infrared regime

* Long-distance regime of $G(x_0)$ dominated by the iso-vector channel:

$$G(x_0) = G^{\rho\rho}(x_0) + G^{I=0}(x_0)$$

* Three methods to constrain long-distance regime:

1. Dedicated calculation of the iso-vector correlator:

$$G^{\rho\rho}(x_0, L) \stackrel{x_0\to\infty}{=} \sum_n |A_n|^2\, \mathrm{e}^{-\omega_n x_0}, \qquad \omega_n = 2\sqrt{m_\pi^2 + k^2}$$

2. "Bounding method": $0 \leq G(x_0) \leq G^{\rho\rho}(x_0^{\mathrm{cut}})\, \mathrm{e}^{-\omega_1(x_0 - x_0^{\mathrm{cut}})}$

   [Lehner 2016, Borsanyi et al., PRL 121 (2018) 022002]

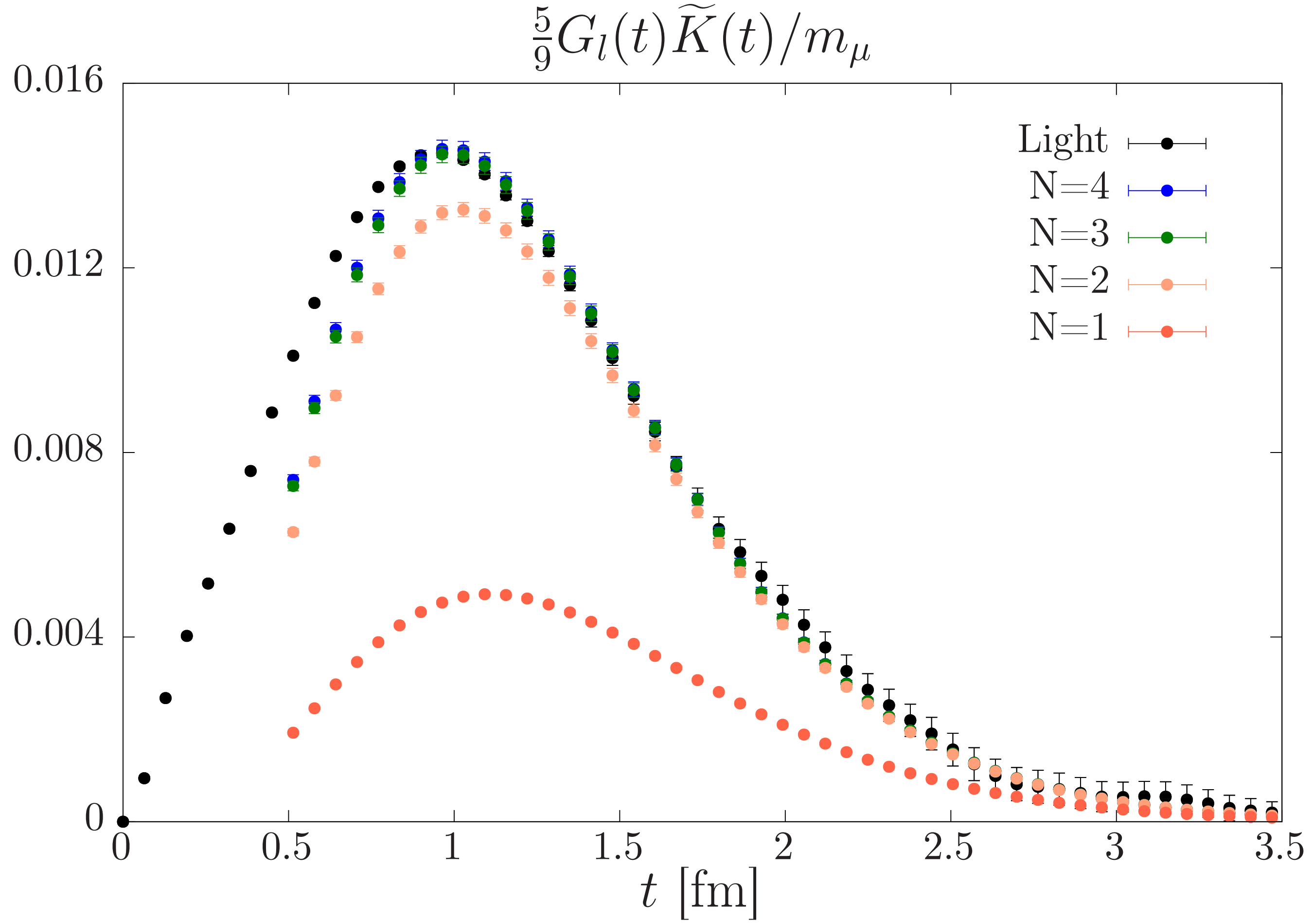
3. Determination of the timelike pion form factor *[Meyer, PRL 107 (2011) 072002]*

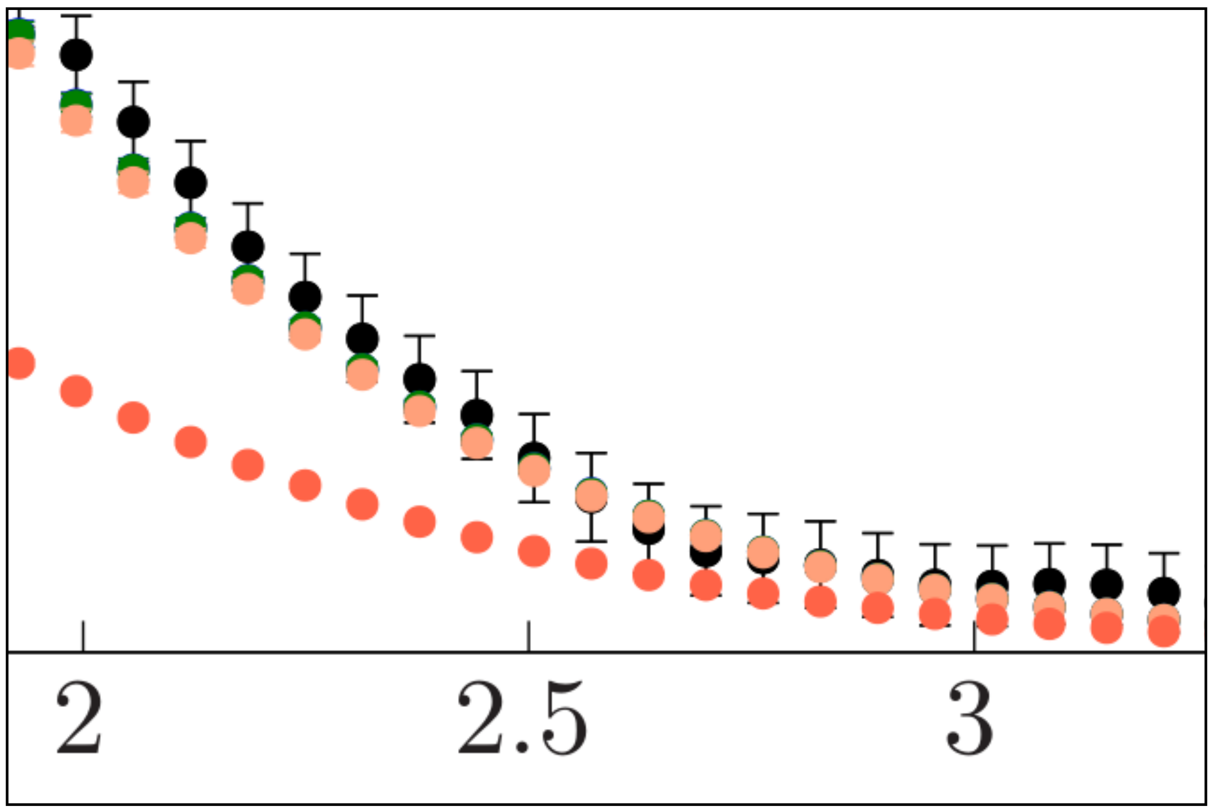
$$\delta_1(k) + \phi\Big(\frac{kL}{2\pi}\Big) = 0 \bmod \pi, \qquad |A_n|^2 = \frac{2k^5}{3\pi\omega^2}\frac{|F_\pi(\omega)|^2}{\left\{k\phi'(k) + k\delta_1'(k)\right\}}$$

Method 1: Iso-vector vector correlator

* Solve GEVP to determine iso-vector correlator at large x_0

$$G^{\rho\rho}(x_0, L) \overset{x_0\to\infty}{=} \sum_n |A_n|^2\, \mathrm{e}^{-\omega_n x_0}, \qquad \omega_n = 2\sqrt{m_\pi^2 + k^2}$$





$m_\pi = 200\,\mathrm{MeV}$

Method 1: Iso-vector vector correlator

* Solve GEVP to determine iso-vector correlator at large x_0

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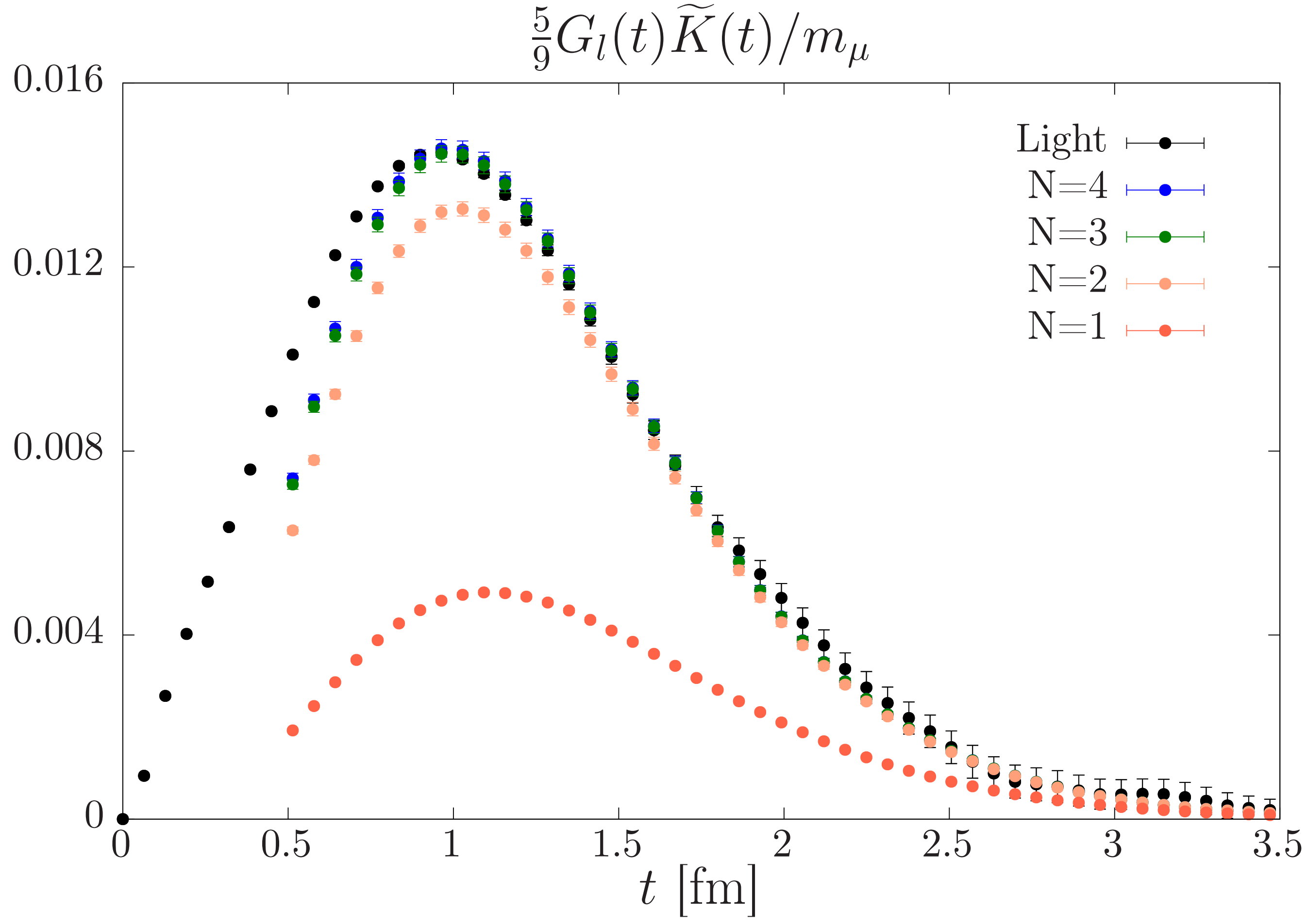
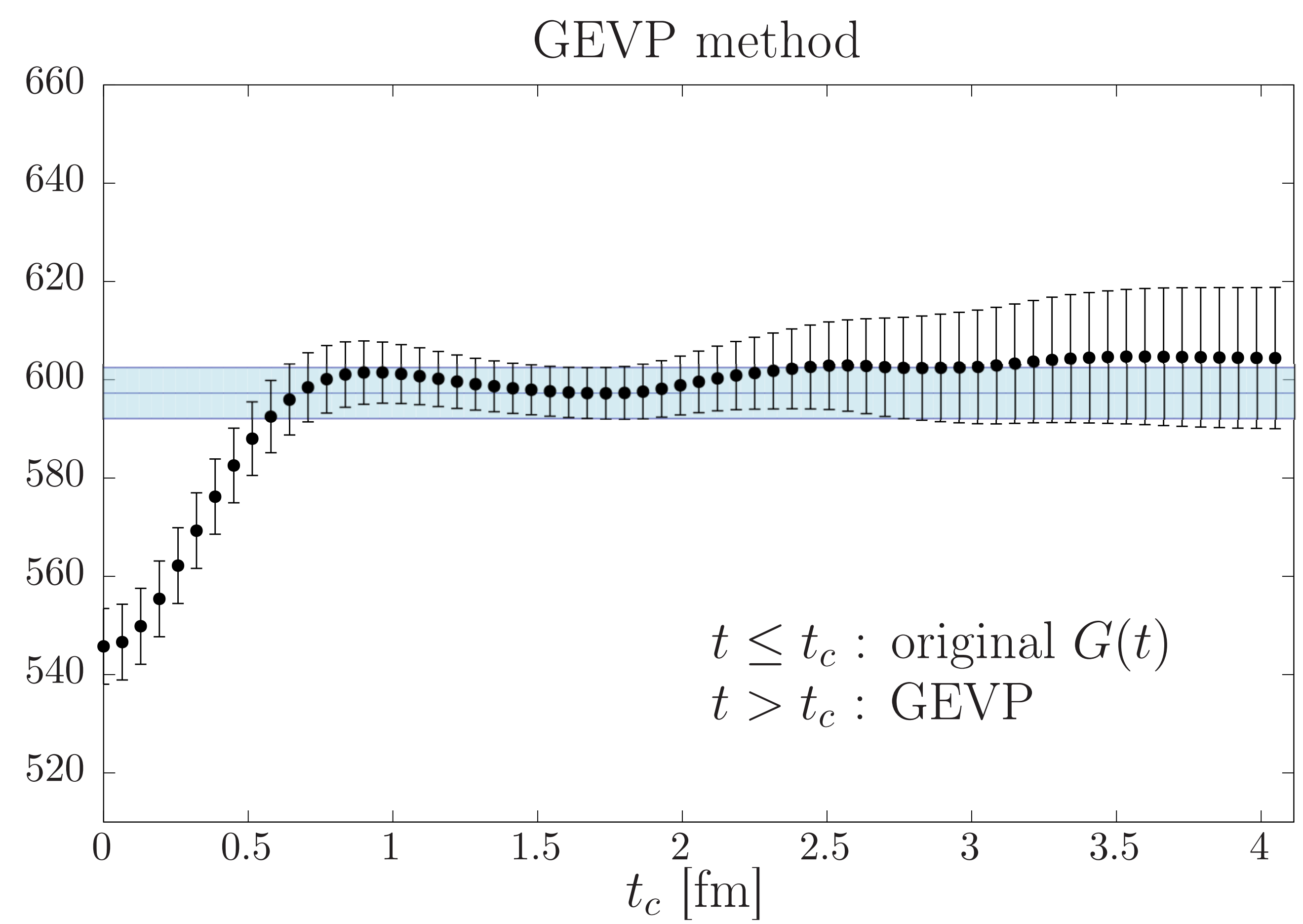




* Optimise choice of $x_0^{\rm cut}$

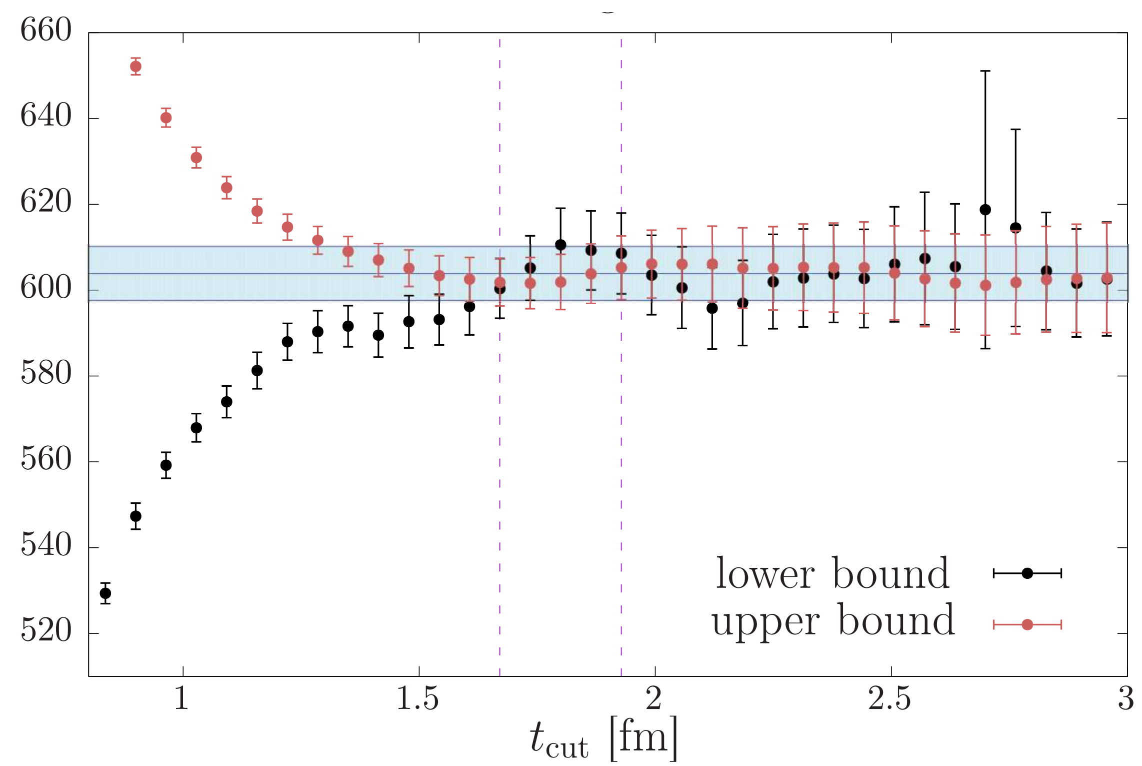
$m_\pi = 200\,\mathrm{MeV}$

Method 2: Bounding method

* Positivity of spectral sum for $G^{\rho\rho}(x_0)$ implies:

$\Rightarrow$ $$0 \leq G(x_0) \leq G^{\rho\rho}(x_0^{\rm cut})\,{\rm e}^{-\omega_1(x_0-x_0^{\rm cut})}$$

ω_1: ground state energy

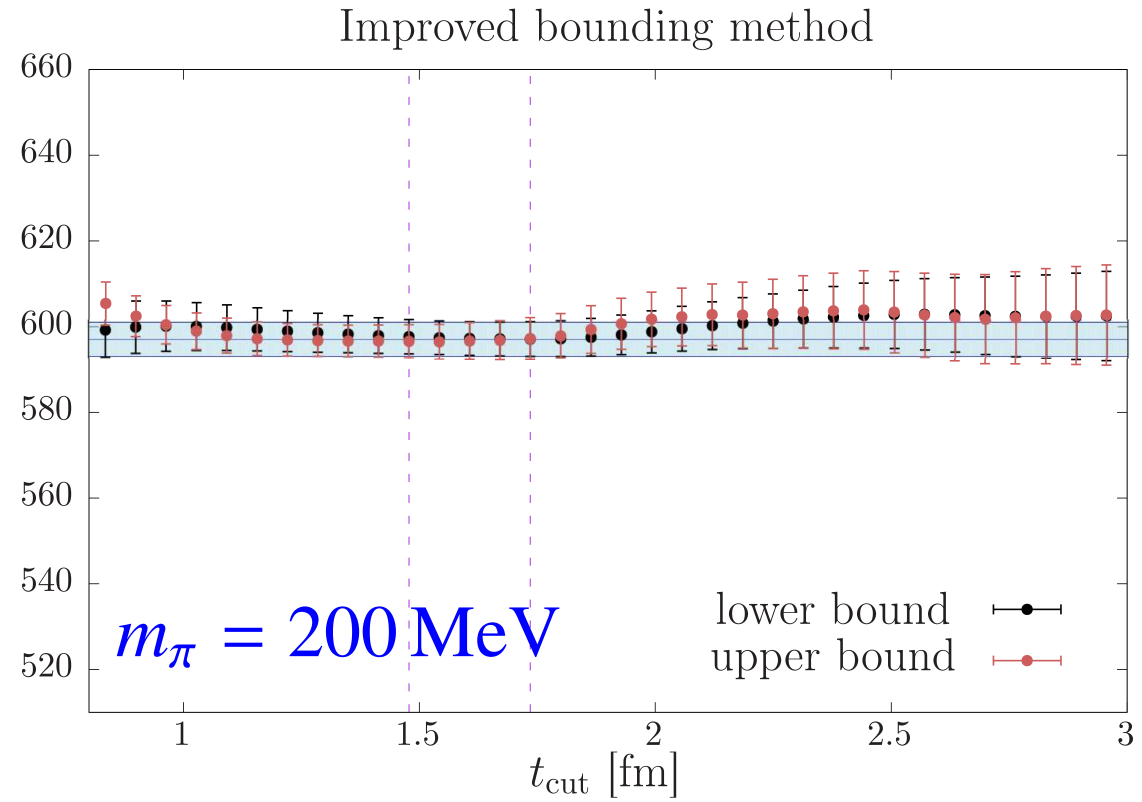


* Improved bounding method

$$\tilde{G}(x_0) \equiv G(x_0) - \sum_{n=1}^{N-1} |A_n|^2\,{\rm e}^{-\omega_n x_0}$$

$\Rightarrow$ $$0 \leq \tilde{G}(x_0) \leq \tilde{G}(x_0^{\rm cut})\,{\rm e}^{-\omega_N(x_0-x_0^{\rm cut})}$$

ω_N: energy of $N^{\rm th}$ state



Finite-volume effects

* Finite-volume correction

$$a_\mu^{\rm hvp}(\infty) - a_\mu^{\rm hvp}(L) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, \tilde{K}(x_0)\, [G(x_0,\infty) - G(x_0,L)]$$

* Iso-vector correlator in infinite volume

$$G^{\rho\rho}(x_0,\infty) = \int_0^\infty d\omega\, \omega^2 \rho(\omega^2) \mathrm{e}^{-\omega|x_0|}, \quad \rho(\omega^2) = \frac{1}{48\pi^2}\left(1 - \frac{4m_\pi^2}{\omega^2}\right)^{3/2} |F_\pi(\omega)|^2$$

* Finite volume:

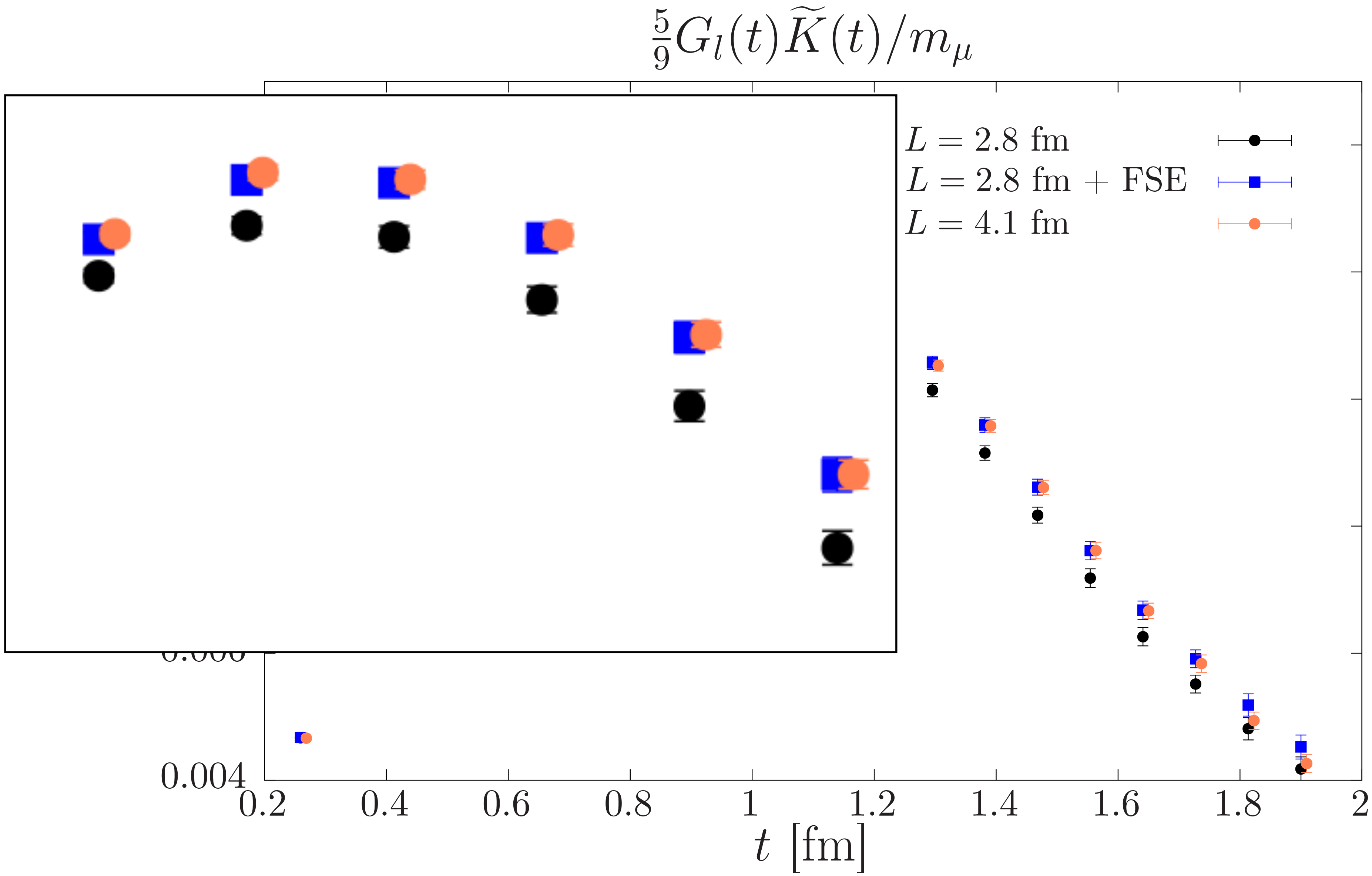
$$G^{\rho\rho}(x_0,L) \overset{x_0\to\infty}{=} \sum_n |A_n|^2\, \mathrm{e}^{-\omega_n x_0}, \qquad |A_n|^2 = \frac{2k^5}{3\pi\omega^2} \frac{|F_\pi(\omega)|^2}{\left\{k\phi'(k) + k\delta_1'(k)\right\}}$$

* Direct calculation of $F_\pi(\omega)$

* Use Gounaris-Sakurai parameterisation of $F_\pi(\omega)$ in terms of (m_ρ, Γ_ρ)

Finite-volume effects

* Cross check at $m_\pi = 280\,\mathrm{MeV}$ (H105, N101: $m_\pi L = 3.8, 5.8$)



* Finite-size effects well described by GS parameterisation of $F_\pi(\omega)$

Scale setting

* Lattice scale enters determination of $a_\mu^{\rm hvp}$

$$a_\mu^{\rm hvp} = a_\mu^{\rm hvp}(M_\mu; M_u, M_d, M_s, \ldots), \quad M_\mu = m_\mu/\Lambda, \quad M_u = m_u/\Lambda, \ldots$$

Λ : sets the lattice scale

$$M_\mu \frac{\partial a_\mu^{\rm hvp}}{\partial M_\mu} = -a_\mu^{\rm hvp} + \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0\, G(x_0)\, J(x_0), \quad J(x_0) = x_0 \tilde{K}'(x_0) - \tilde{K}(x_0)$$

$$\frac{\Delta a_\mu^{\rm hvp}}{a_\mu^{\rm hvp}} = \left| \underbrace{\frac{M_\mu}{a_\mu^{\rm hvp}} \frac{\partial a_\mu^{\rm hvp}}{\partial M_\mu}}_{1.8} + \underbrace{\frac{M_\pi}{a_\mu^{\rm hvp}} \frac{\partial a_\mu^{\rm hvp}}{\partial M_\pi}}_{-0.18(6)} + \ldots \right| \frac{\Delta\Lambda}{\Lambda}$$

* Our calculation: $\Lambda^{-1} = \sqrt{8t_0} = 0.415(4)(2)\,\text{fm}$ *[Bruno et al., PRD 95 (2017) 074504]*

Connected light quark contribution

* Chiral and continuum extrapolation:

$$a_\mu^{\rm hvp}(a, m_\pi) = a_\mu^{\rm hvp}\Big|_{\rm phys} + Ba^2 + Cm_\pi^2 + D\ln m_\pi^2 \qquad (m_\pi \to 0 \text{ at fixed } m_\mu)$$

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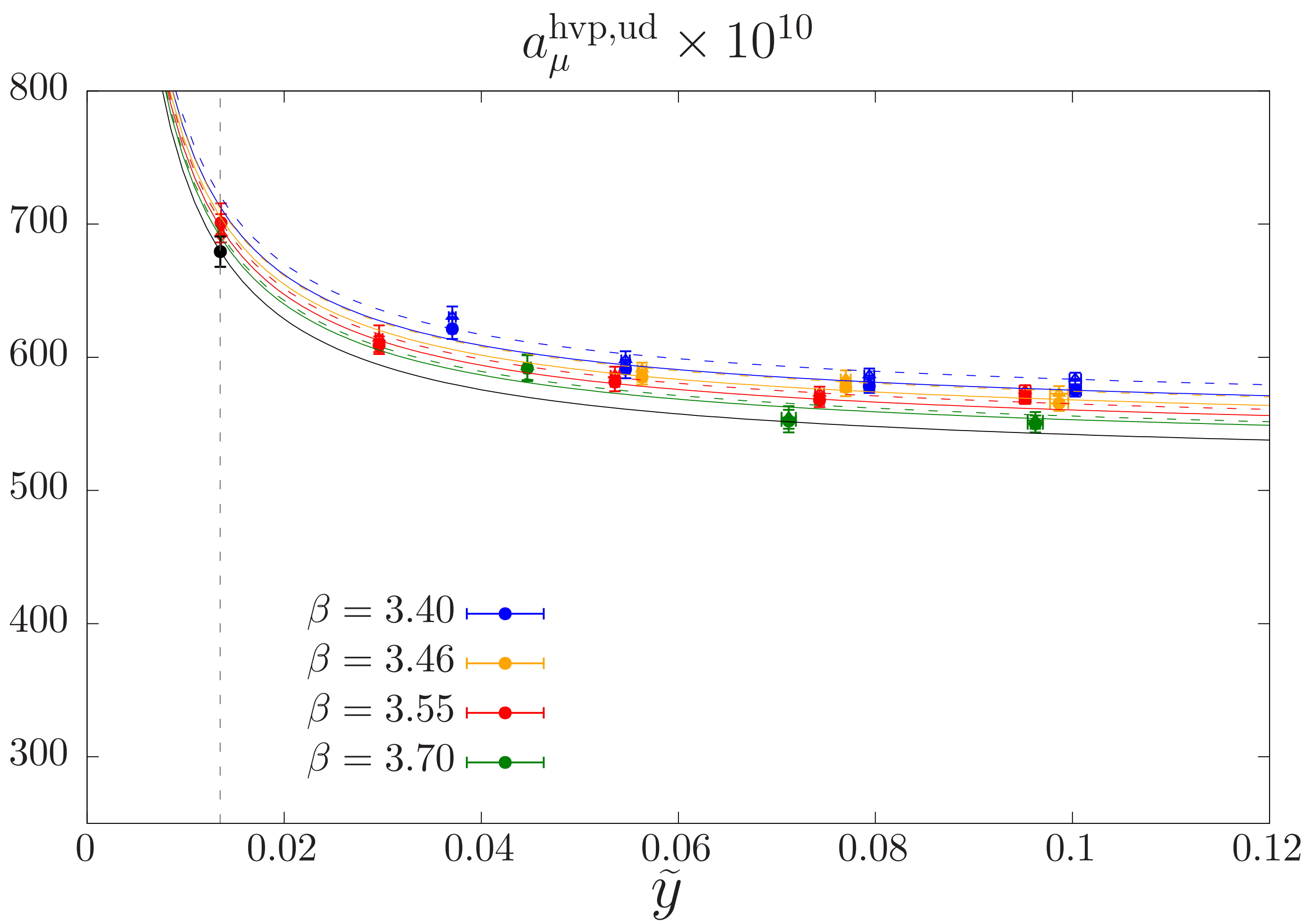

- Dashed / solid lines: different discretisations of the vector current

$$(a_\mu^{\rm hvp})^{ud}_{\rm conn} = (674 \pm 12 \pm 5) \cdot 10^{-10}$$

- Systematic error: different *ansätze* in chiral extrapolation
- Statistical error dominated by scale setting uncertainty

Connected light quark contribution

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$$a_\mu^{\rm hvp}(a,m_\pi) = a_\mu^{\rm hvp}\Big|_{\rm phys} + Ba^2 + Cm_\pi^2 + D\ln m_\pi^2 \qquad (m_\pi \to 0 \text{ at fixed } m_\mu)$$

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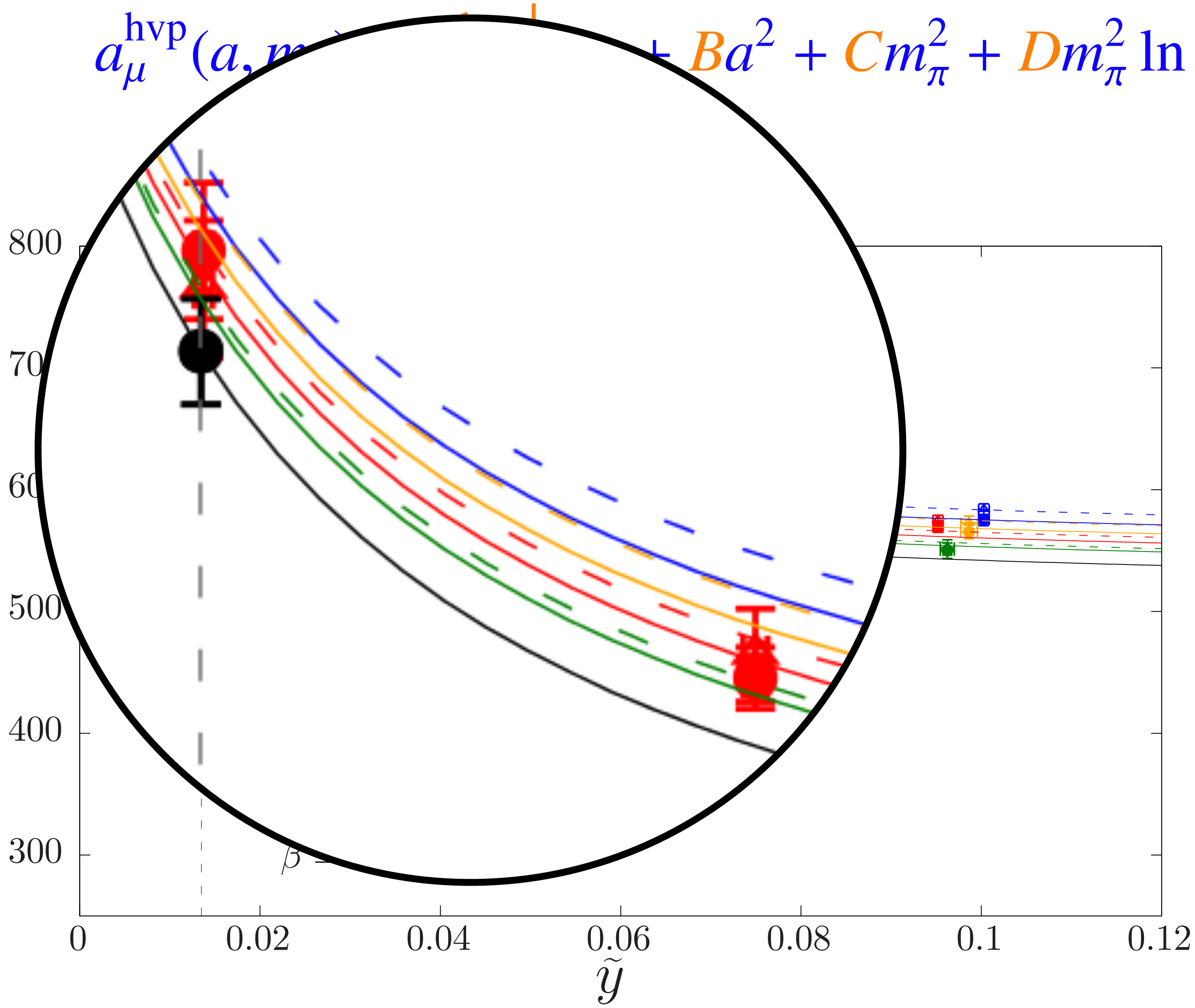

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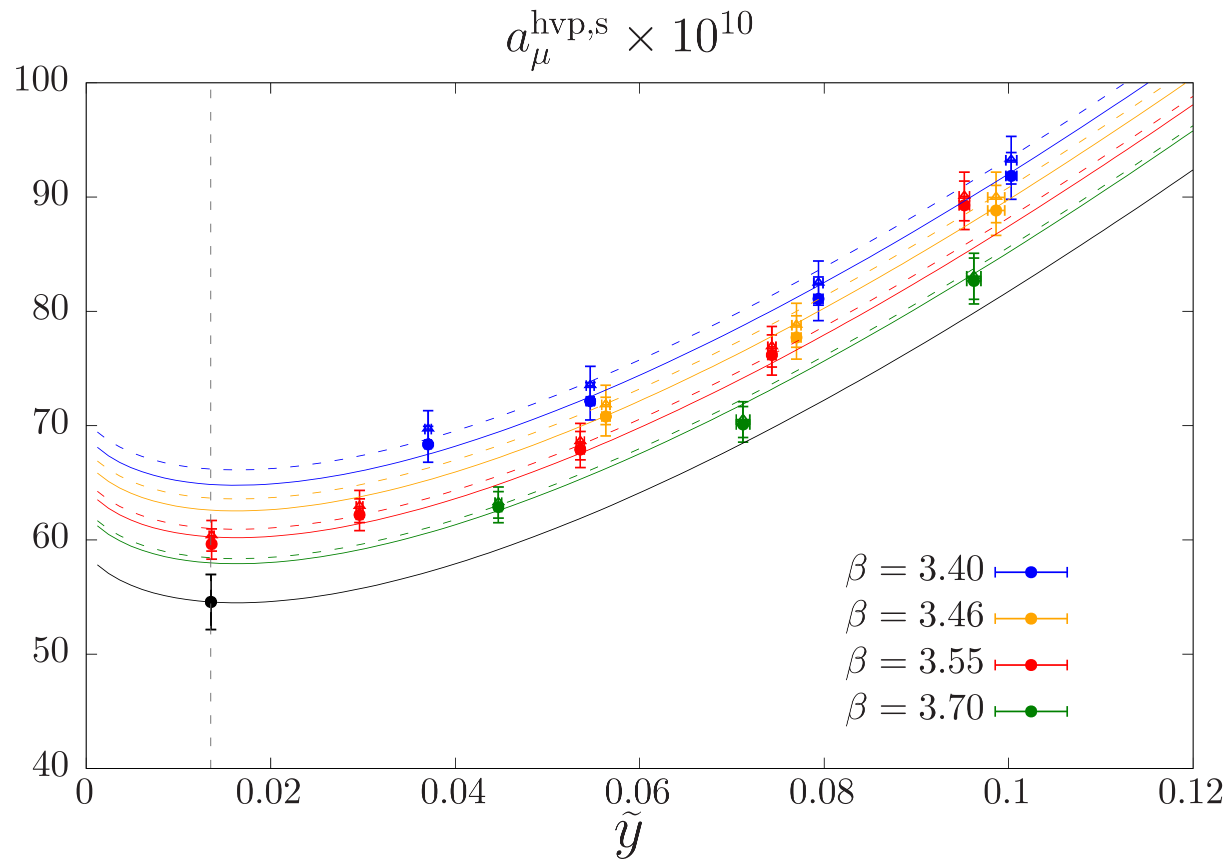
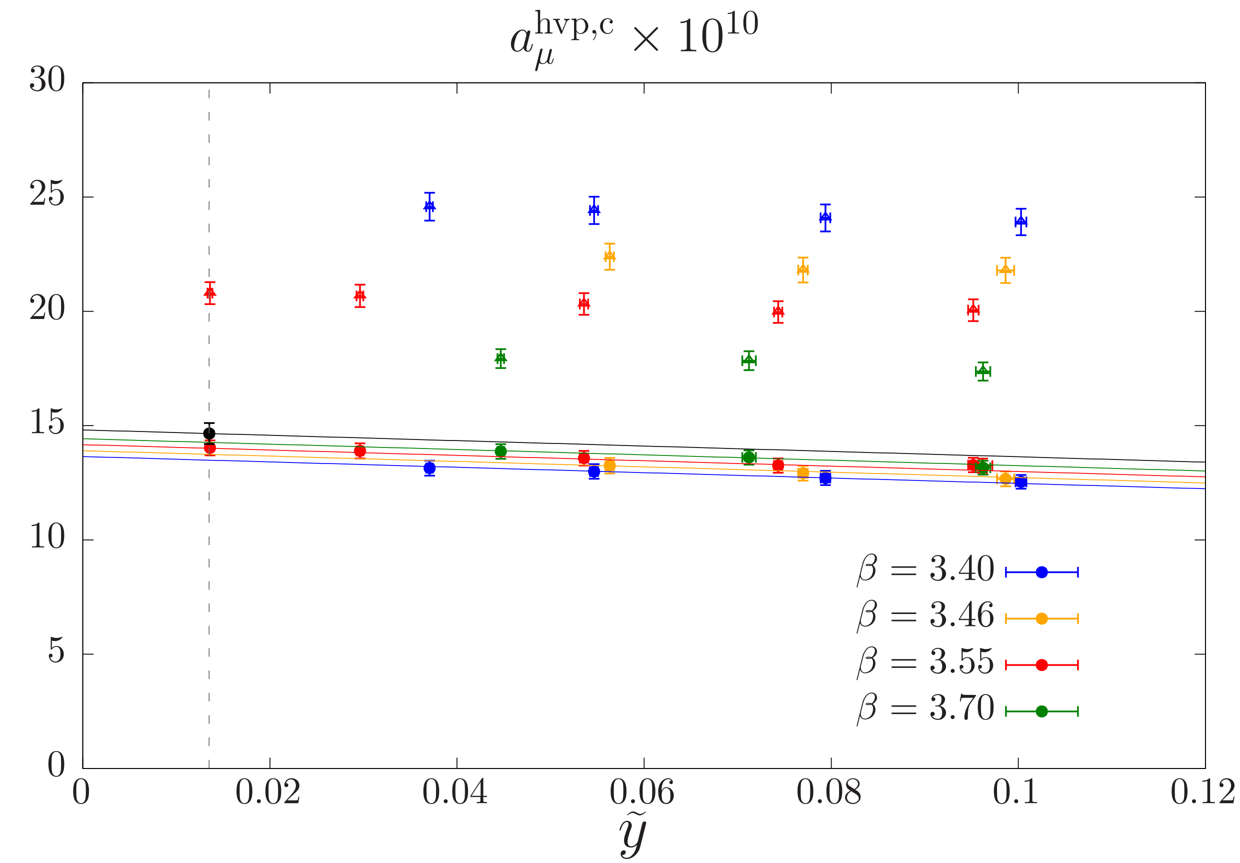
- Systematic error: different *ansätze* in chiral extrapolation
- Statistical error dominated by scale setting uncertainty

Connected strange and charm contributions

* Chiral and continuum extrapolation:





$$(a_\mu^{\rm hvp})^s = (54.5 \pm 2.4 \pm 0.6)\cdot 10^{-10}$$

$$(a_\mu^{\rm hvp})^c = (14.66 \pm 0.45 \pm 0.06)\cdot 10^{-10}$$

* Error dominated by scale setting uncertainty

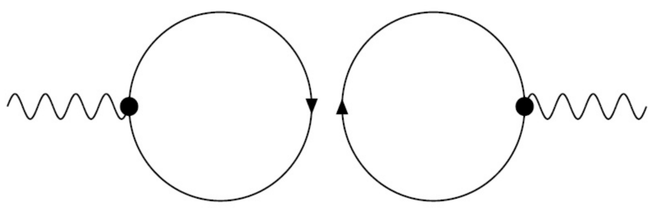
* Larger discretisation effects observed for charm quark

Disconnected diagrams

* Iso-scalar contribution…

  …admits a positive spectral representation

  …contains disconnected contributions



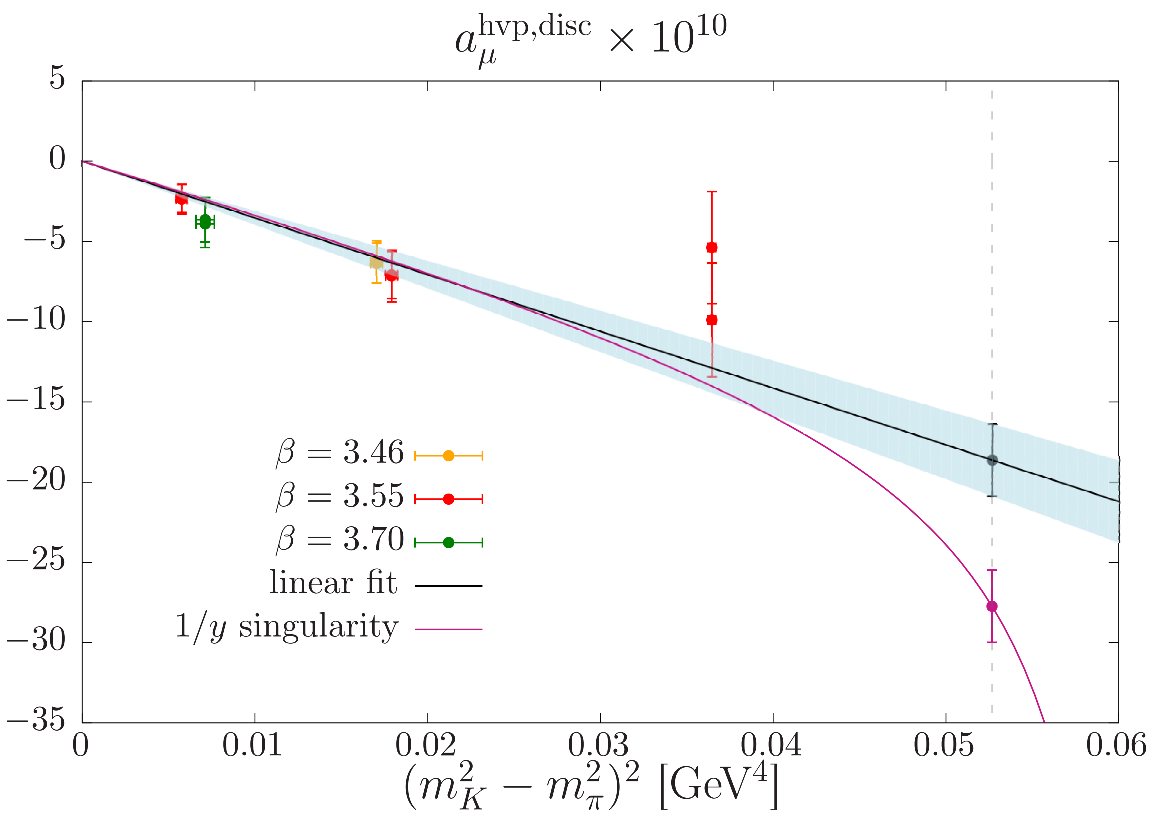
⇒ Apply bounding method to iso-scalar correlator (uds) quarks:

$$0 \leq G^{I=0}(x_0) \leq G^{I=0}(x_0^{\rm cut})\, {\rm e}^{-m_\rho(x_0-x_0^{\rm cut})}$$

$$(a_\mu^{\rm hvp})_{\rm disc} = (a_\mu^{\rm hvp})^{I=0} - \frac{1}{10}(a_\mu^{\rm hvp})^{ud}_{\rm con} - (a_\mu^{\rm hvp})^{s}_{\rm con}$$

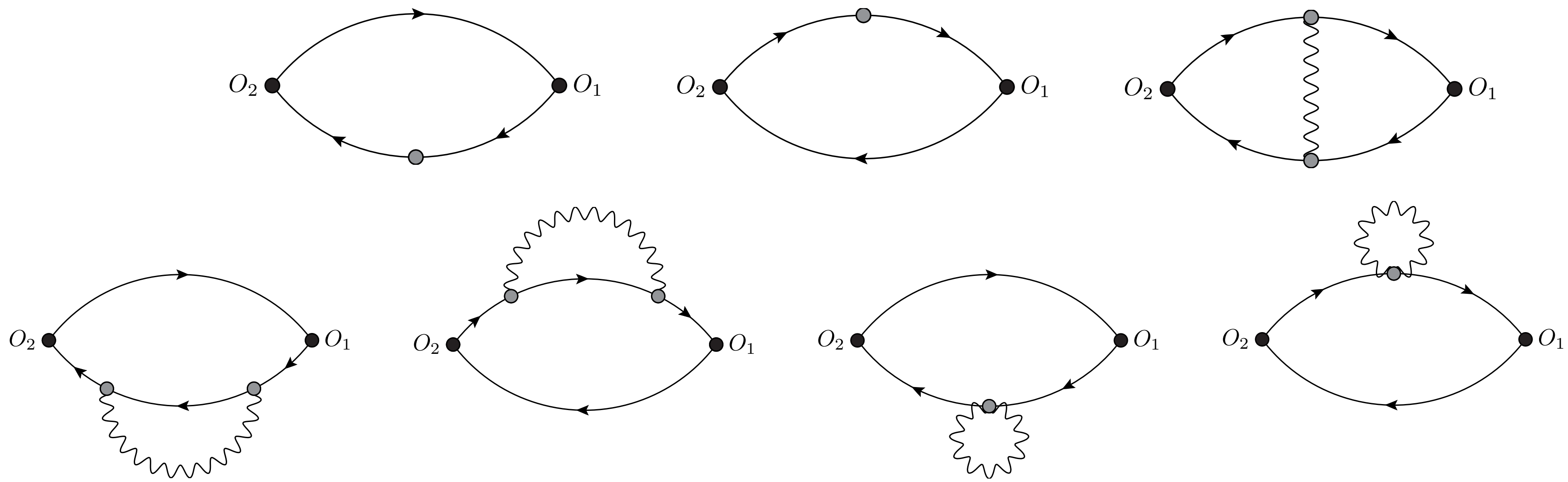
* Physical point:

$$(a_\mu^{\rm hvp})_{\rm disc} = (-23.2 \pm 2.2 \pm 4.5) \cdot 10^{-10}$$



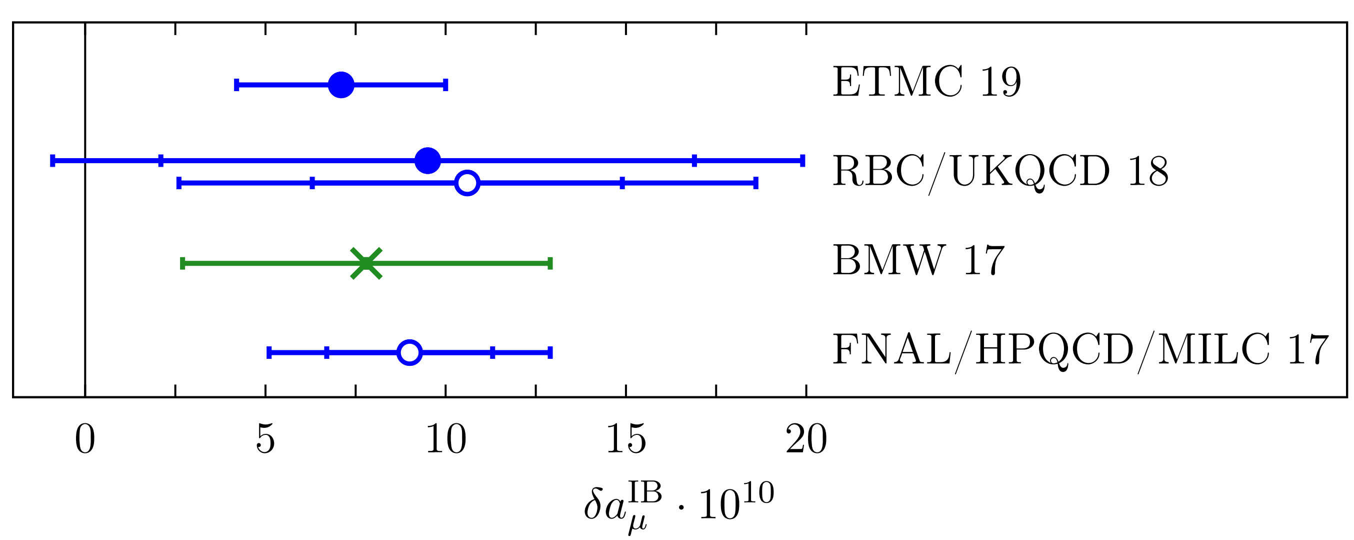
Isospin breaking effects

- Stochastic vs. perturbative method
- Expand correlator in powers of $(m_d - m_u)$ and $\alpha = e^2/4\pi$



[Divitiis et al., arXiv:1110.6294, arXiv:1303.4896; Giusti et al., arXiv:1704.06561; Risch, HW, arXiv:1811.00895]

Isospin breaking corrections:



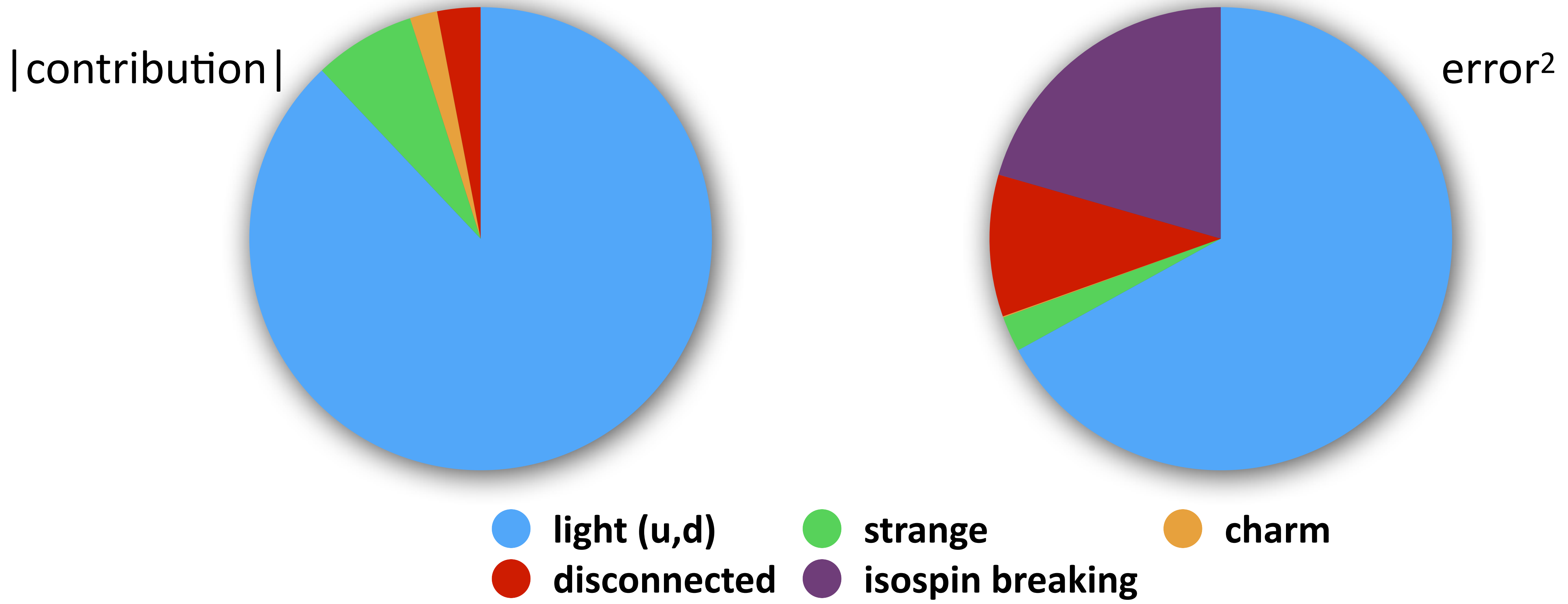
Results and comparison

* Final result at the physical point [Gérardin et al., arXiv:1904.03120]

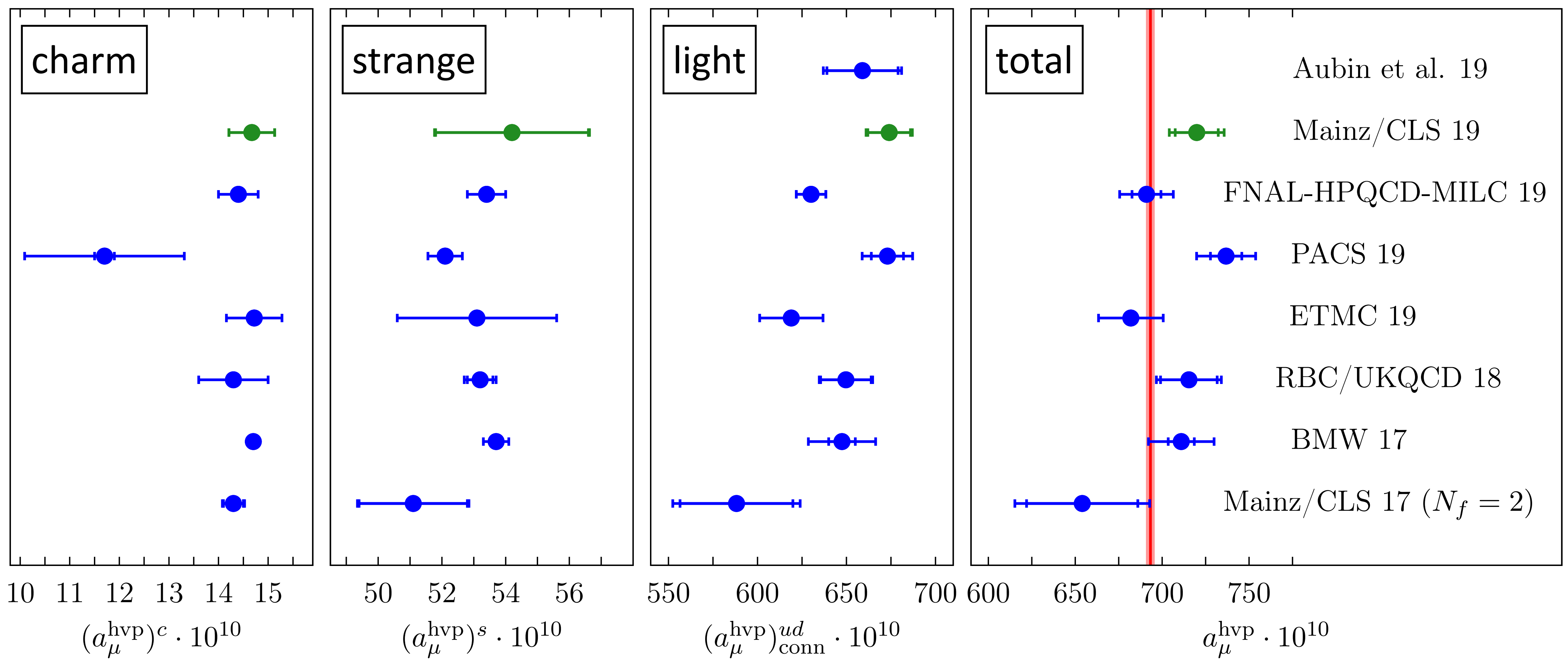
$$a_\mu^{\text{hvp}} = (720 \pm 12.4_{\text{stat}} \pm 9.9_{\text{syst}}) \times 10^{-10}$$

(total uncertainty: 2.2%)

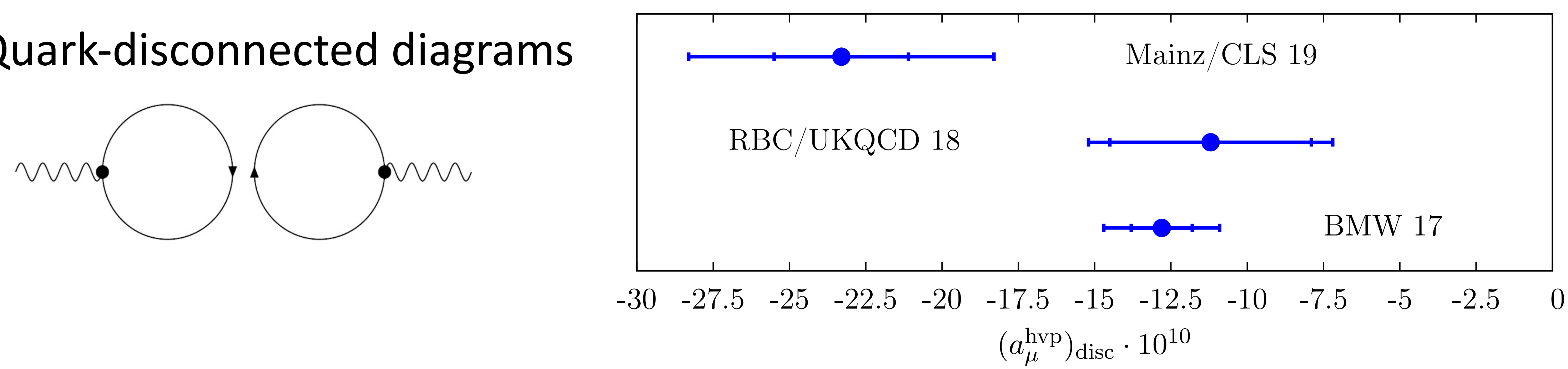
* Individual contributions:



Results and comparison

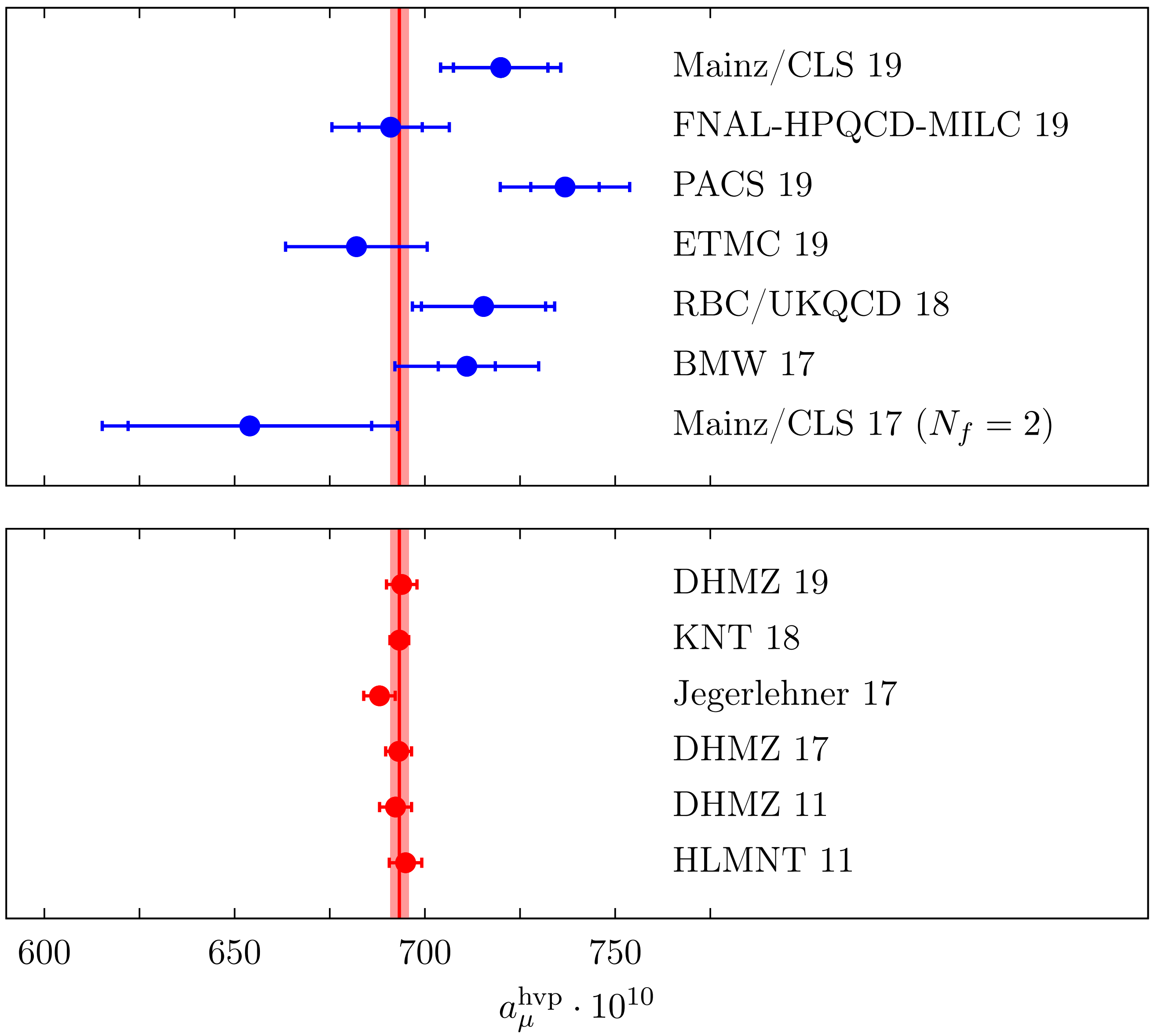


Quark-disconnected diagrams



Results and comparison

* **Lattice QCD** vs. **dispersion theory**:



Our result:

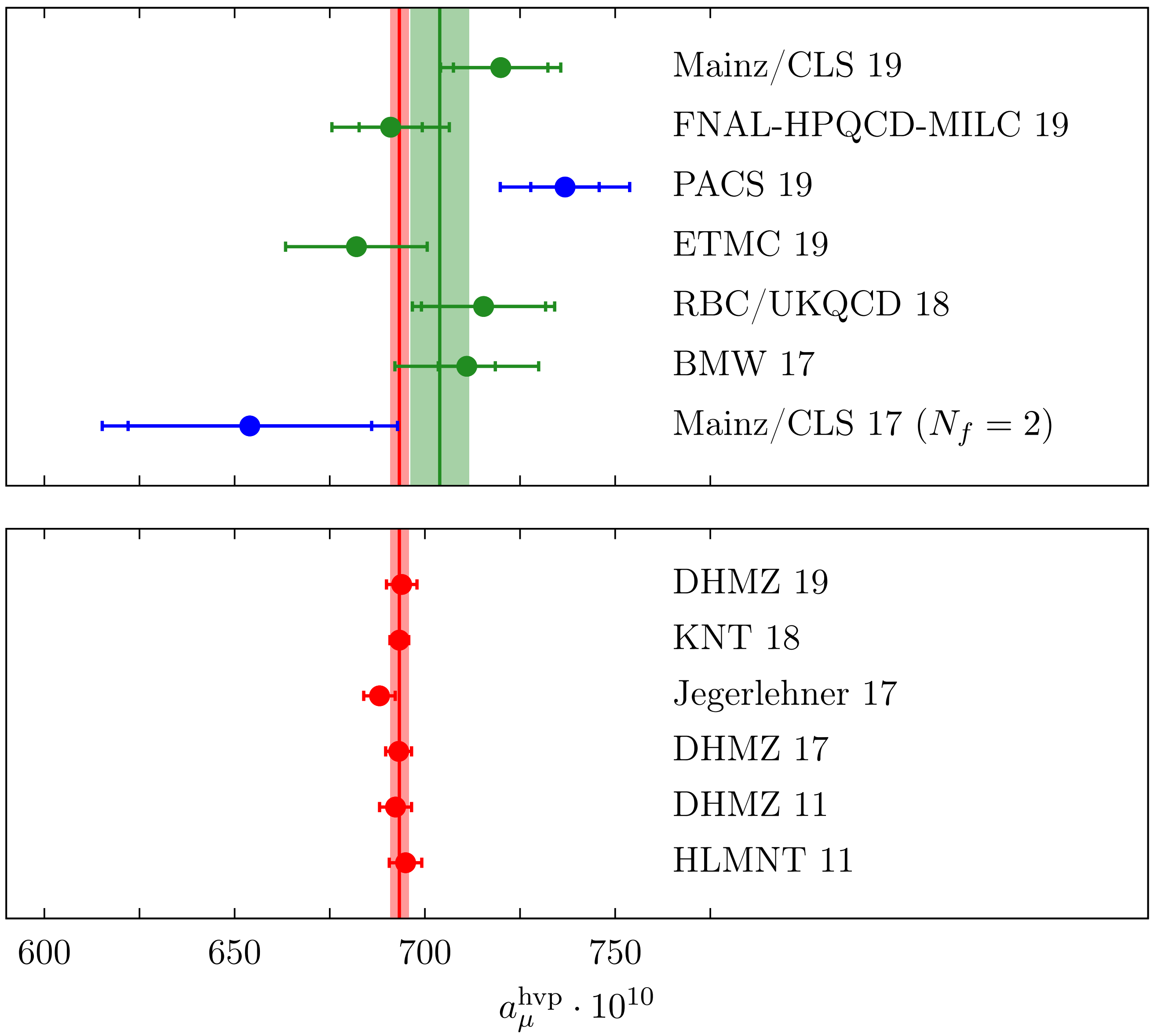
$$a_\mu^{\rm hvp} = (720.0 \pm 15.8) \cdot 10^{-10}$$

Dispersion theory (KNT 18):

$$(a_\mu^{\rm hvp})_{\rm disp} = (693.3 \pm 2.5) \cdot 10^{-10}$$

Results and comparison

* **Lattice QCD average** vs. **dispersion theory**:



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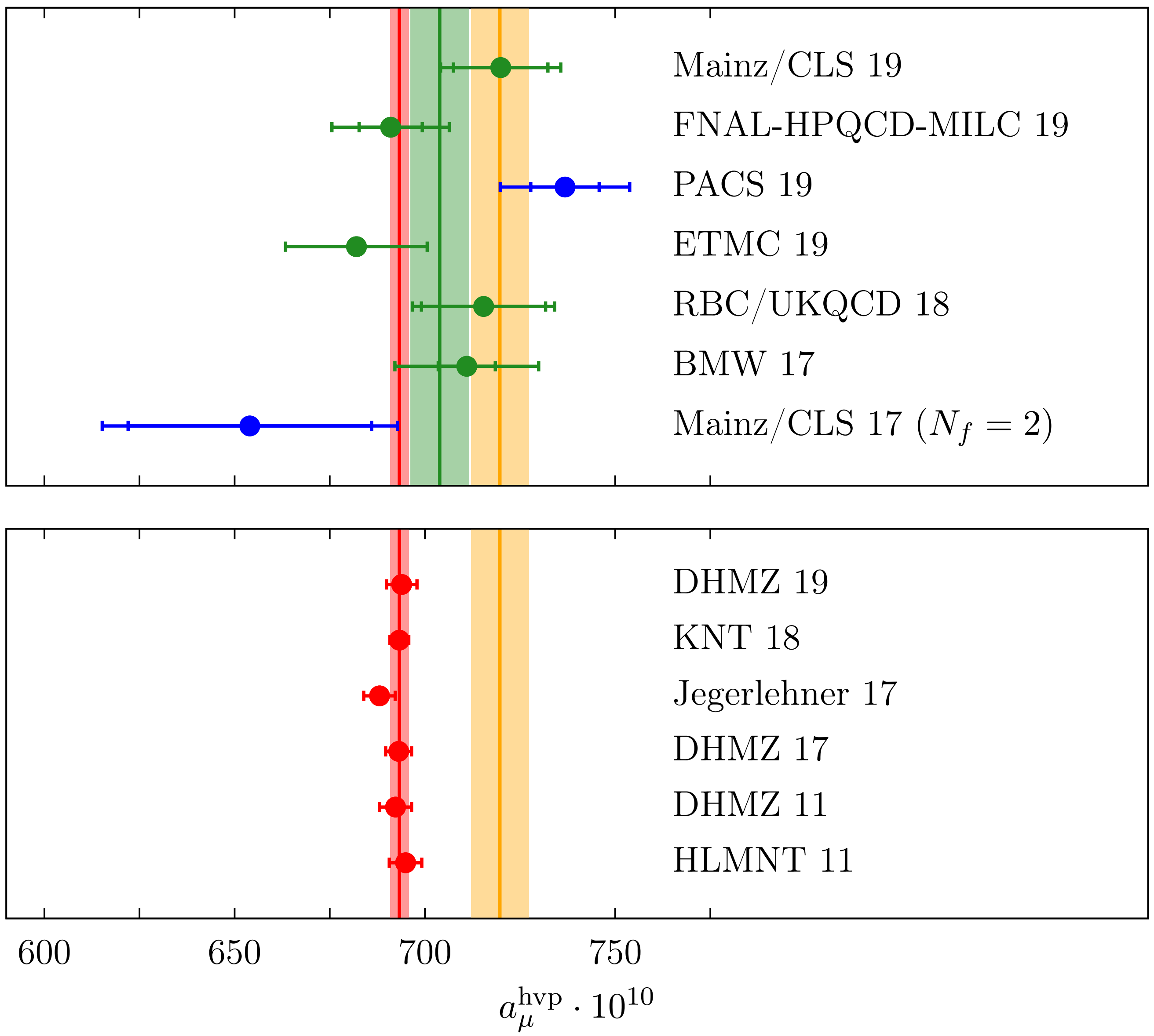
Global lattice average:

$$(a_\mu^{\rm hvp})_{\rm lat} = (703.9 \pm 7.7) \cdot 10^{-10}$$

Results and comparison

* **Lattice QCD average** vs. **dispersion theory**:



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$$a_\mu^{\rm hvp} = (720.0 \pm 15.8) \cdot 10^{-10}$$

Dispersion theory (KNT 18):

$$(a_\mu^{\rm hvp})_{\rm disp} = (693.3 \pm 2.5) \cdot 10^{-10}$$

Global lattice average:

$$(a_\mu^{\rm hvp})_{\rm lat} = (703.9 \pm 7.7) \cdot 10^{-10}$$

"No new physics":

$$(a_\mu^{\rm hvp})_{\rm NNP} = (a_\mu^{\rm hvp})_{\rm disp} + (a_\mu^{\rm exp} - a_\mu^{\rm SM})$$

Summary & Outlook

- Individual calculations achieve overall precision of $\approx 2\,\%$ in $a_\mu^{\rm hvp}$
- Light quark contribution dominant — magnitude and uncertainty
  - → increase overall statistics
  - → improve determination of long-distance contribution
  - → improve precision of scale setting
- Quark-disconnected and isospin-breaking contributions crucial for credible and competitive overall precision

Mainz/CLS effort

- Add ensembles at smaller lattice spacing and (near-)physical m_π
- Improve precision of quark-disconnected contribution
- Include isospin breaking; improve scale setting uncertainty