

# THE $Q_T$ -SUBTRACTION FORMALISM AT $N^3\text{LO}$

Leandro Cieri  
INFN Firenze

Workshop on Resummation, Evolution, Factorization (REF 2019)

27.11.2019



# OUTLINE

- Introduction and motivation
- Main idea behind the  $qT$ -subtraction method
- $qT$ -subtraction ingredients at  $N^3LO$
- Higgs Phenomenology at  $N^3LO$
- Outlook

# INTRODUCTION

- No striking manifestation of New Physics (NP) beyond the Standard Model (SM) at the LHC
- “Precision Physics” represents the key instrument to find NP
- In this framework QCD perturbative corrections play a crucial role
- Until a few years ago, the standard for such calculations was next-to-leading order (NLO) accuracy
- In the past recent years a number of growing next-to-next-to-leading order (NNLO) results were computed

# INTRODUCTION

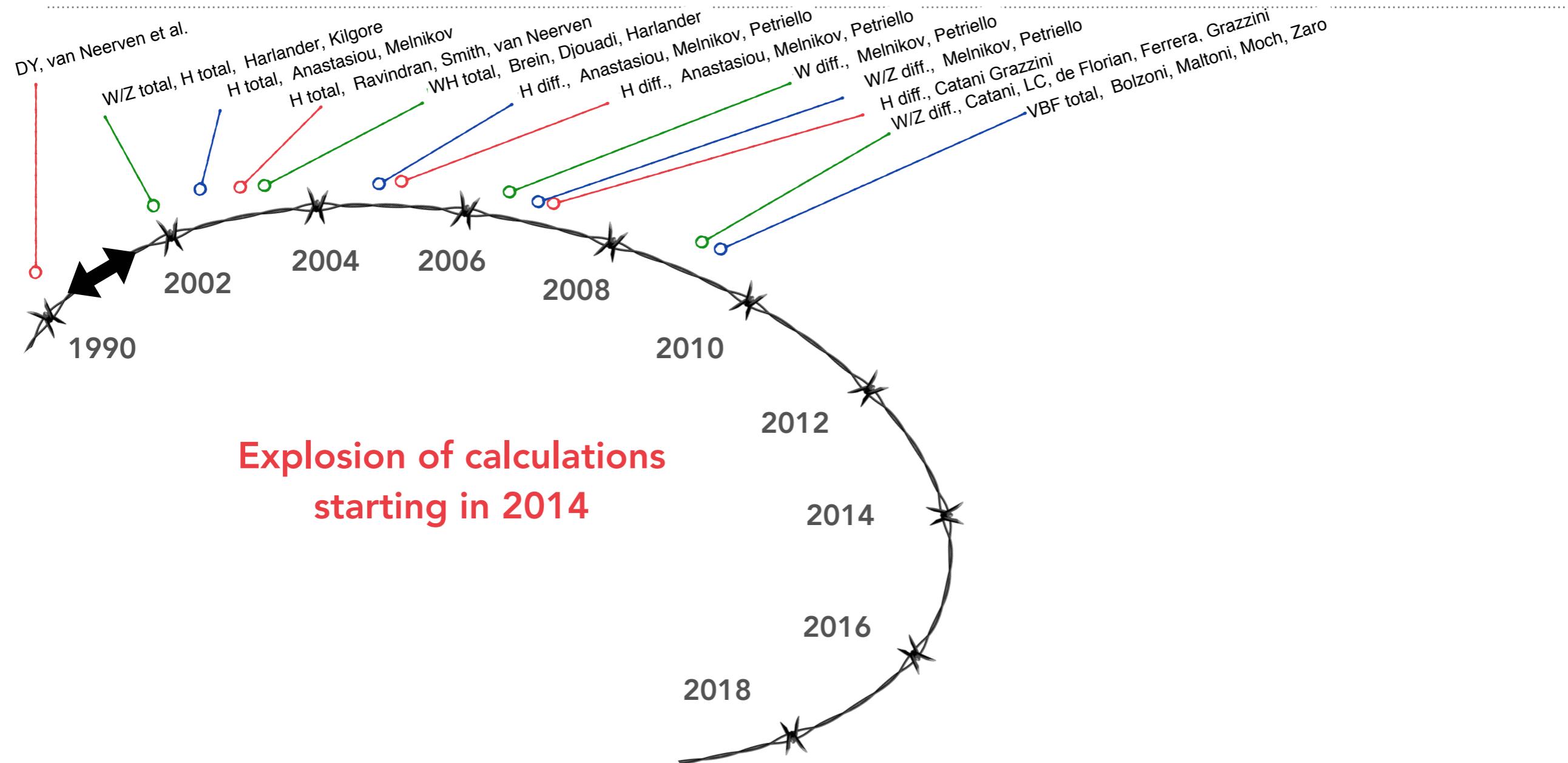
- No striking manifestation of New Physics (NP) beyond the Standard Model (SM) at the LHC
- “Precision Physics” represents the key instrument to find NP
- In this framework QCD perturbative corrections play a crucial role
- Until a few years ago, the standard for such calculations was next-to-leading order (NLO) accuracy
- In the past recent years a number of growing next-to-next-to-leading order (NNLO) results were computed



“The NNLO revolution”

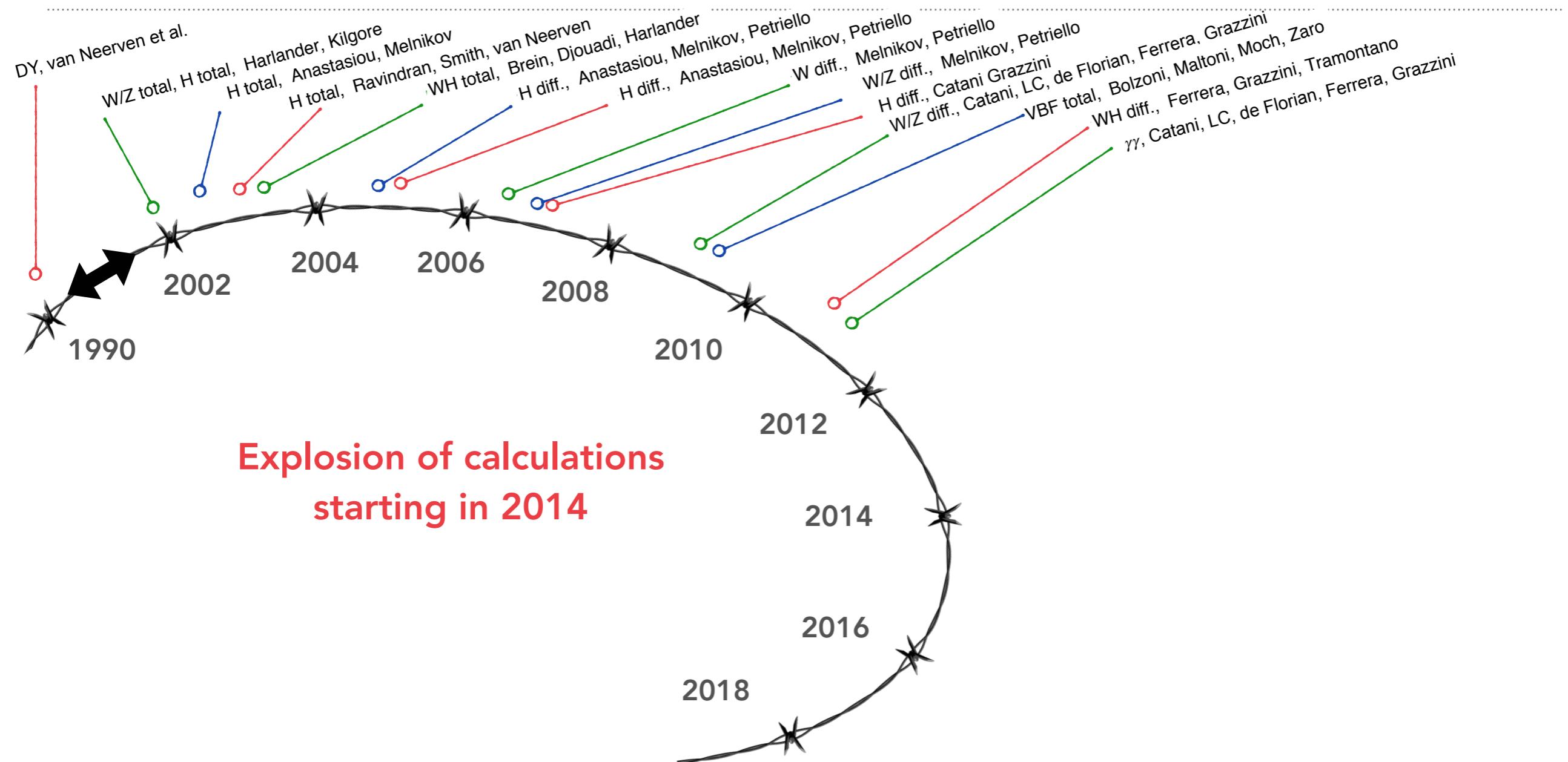
# THE NNLO STANDARD

## NNLO HADRON-COLLIDER CALCULATIONS VS. TIME



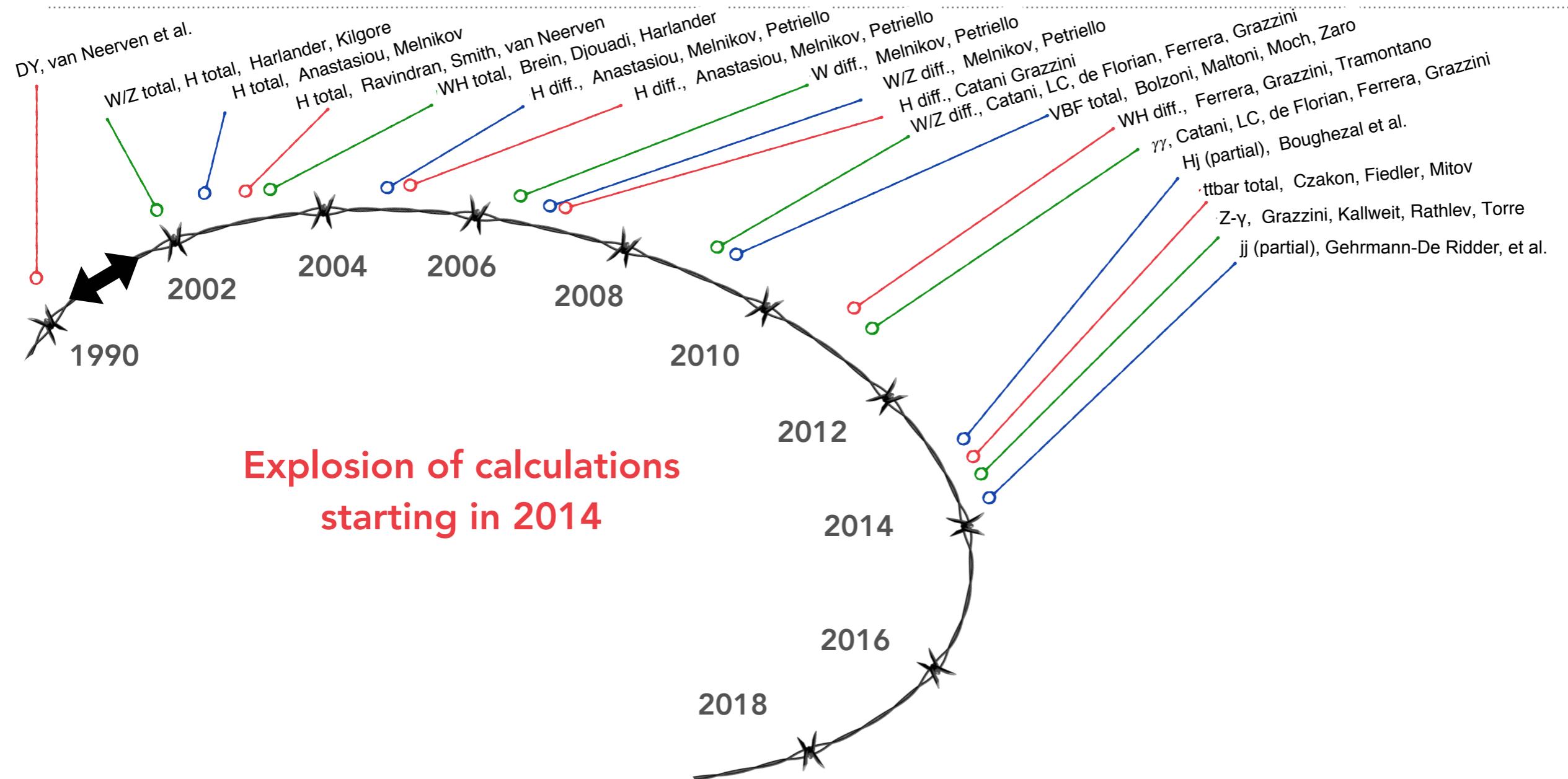
# THE NNLO STANDARD

## NNLO HADRON-COLLIDER CALCULATIONS VS. TIME



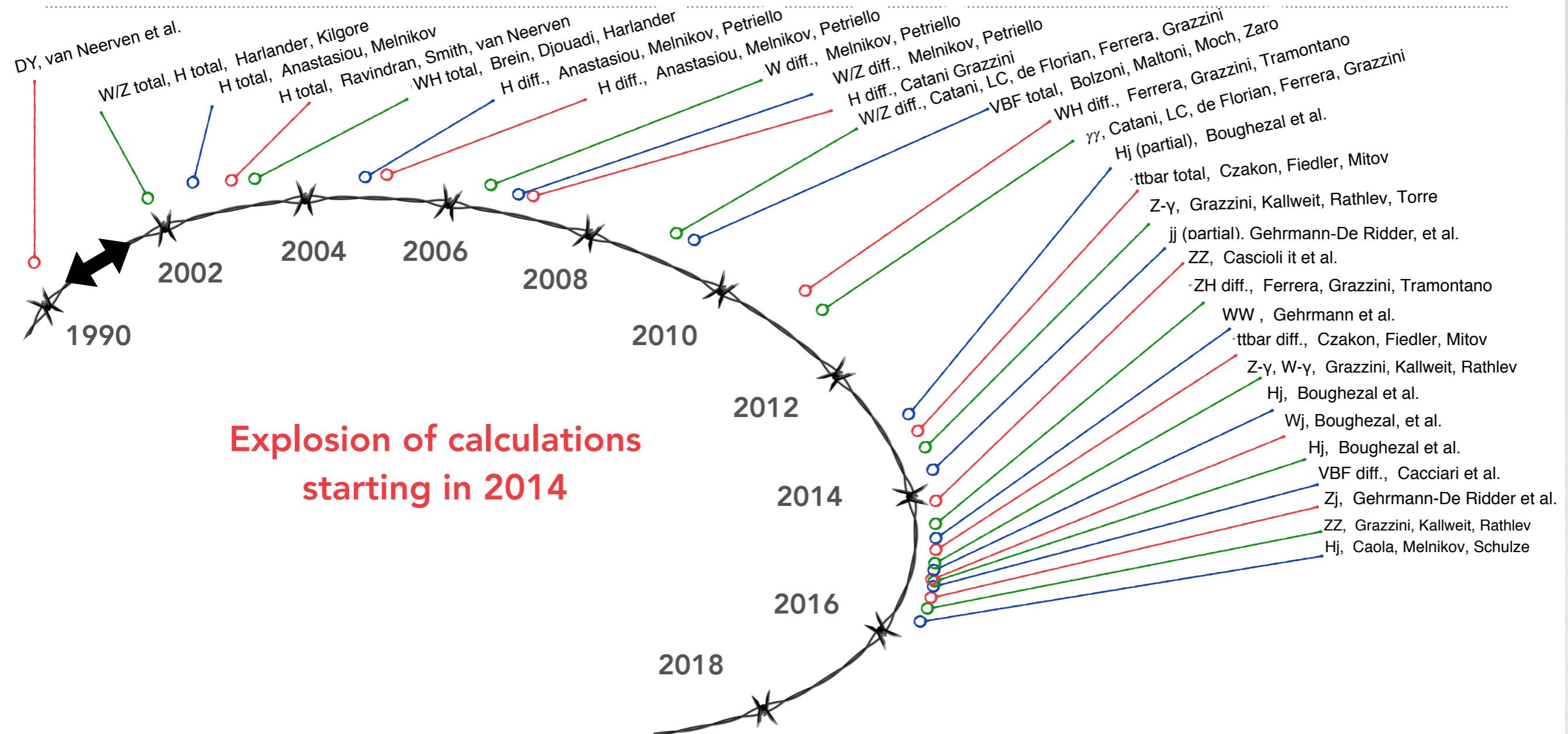
# THE NNLO STANDARD

## NNLO HADRON-COLLIDER CALCULATIONS VS. TIME



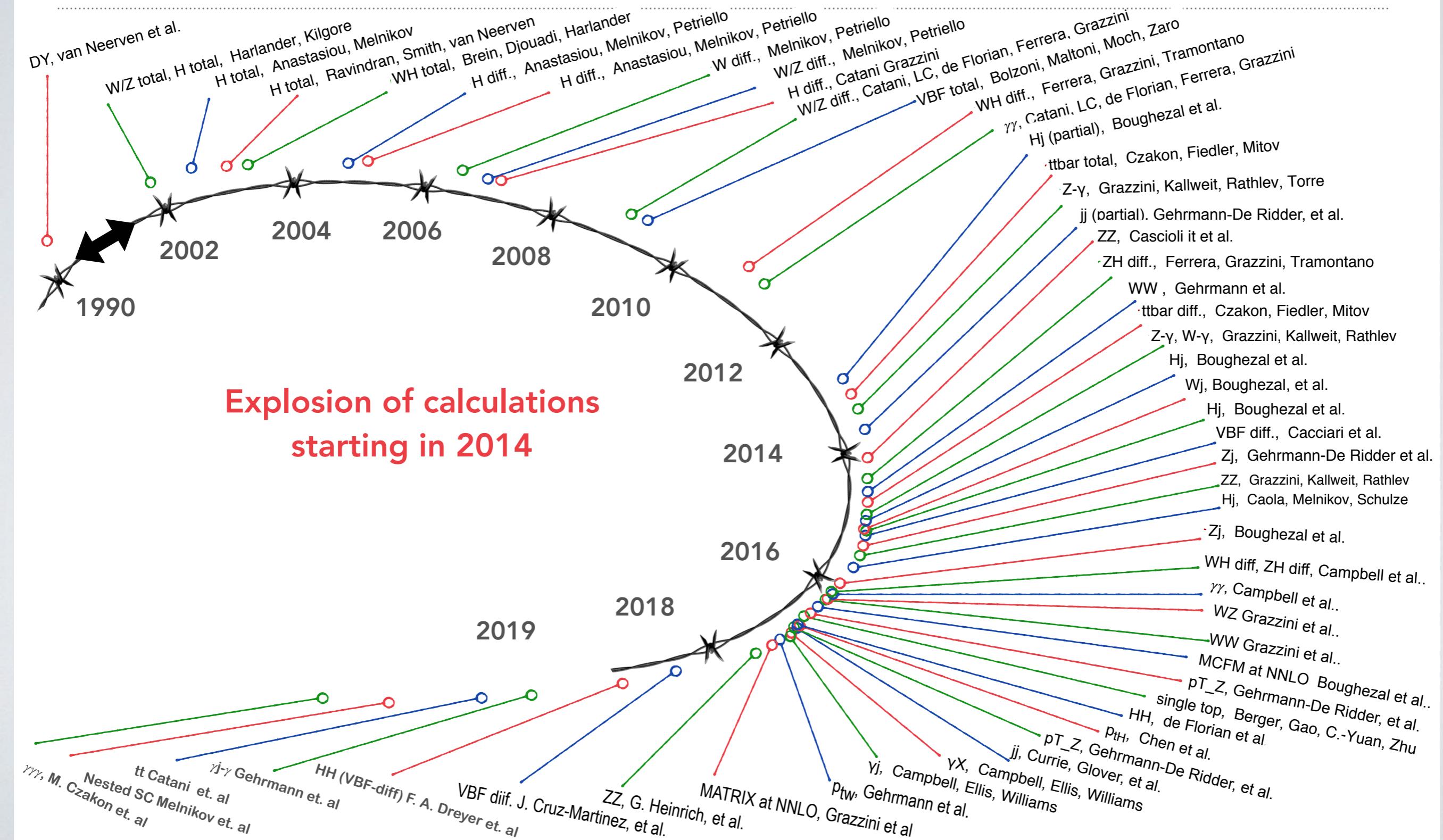
# THE NNLO STANDARD

## NNLO HADRON-COLLIDER CALCULATIONS VS. TIME



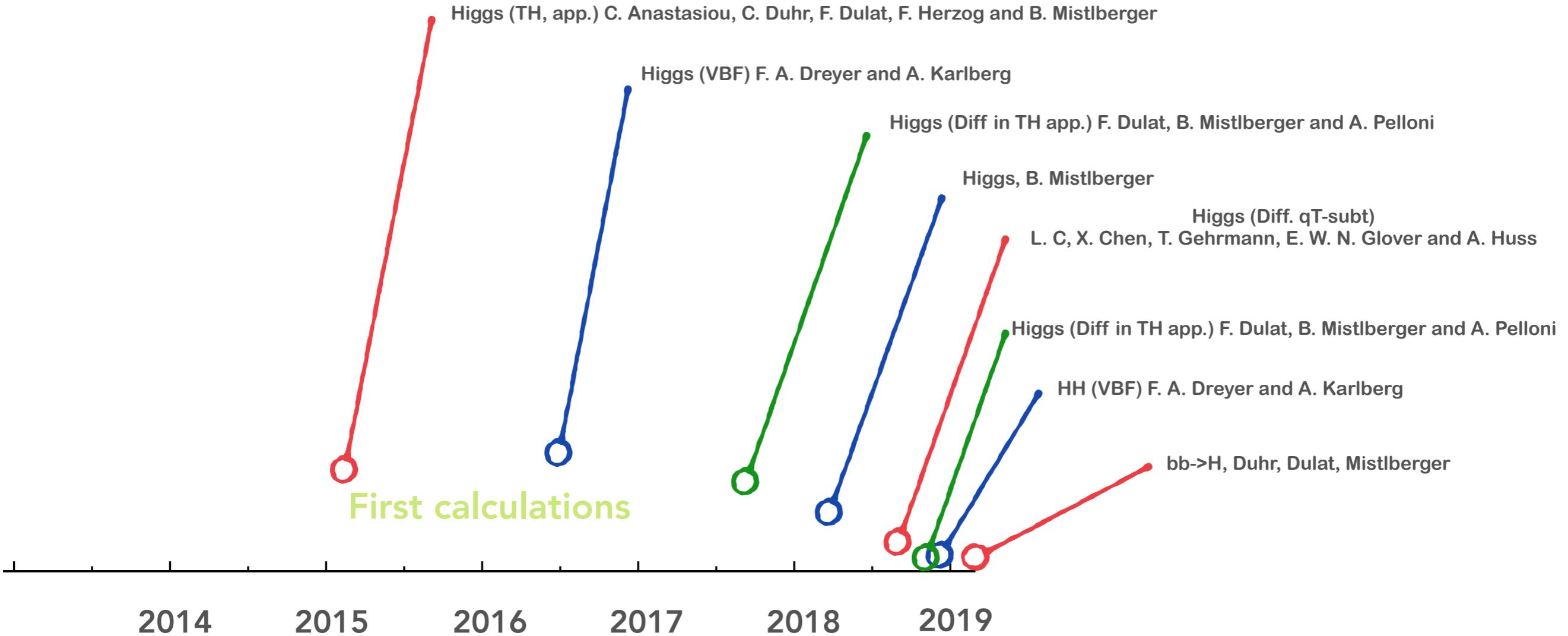
# THE NNLO STANDARD

# NNLO HADRON-COLLIDER CALCULATIONS VS. TIME



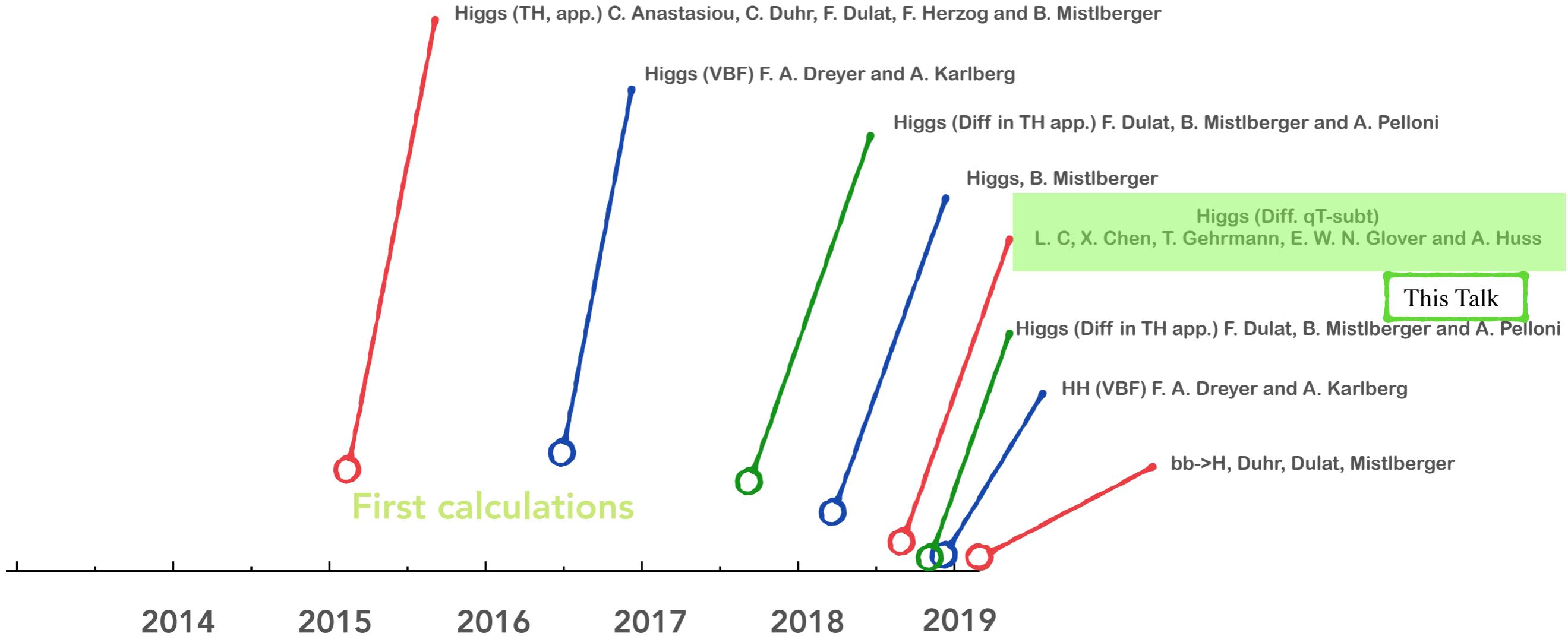
# THE N<sup>3</sup>LO ERA

## N<sup>3</sup>LO HADRON-COLLIDER CALCULATIONS VS. TIME



# THE N<sup>3</sup>LO ERA

## N<sup>3</sup>LO HADRON-COLLIDER CALCULATIONS VS. TIME



# THE QT-SUBTRACTION/RESUMMATION FORMALISM

## • NNLO+NNLL QCD

- S. Catani and M. Grazzini (2007)
- G. Bozzi, S. Catani, D. de Florian and M. Grazzini (2005)
- S. Catani and M. Grazzini (2011)
- S. Catani, L. C, D. de Florian, G. Ferrera and M. Grazzini (2013)

Colourless final states



The method

Explicit form  
qT-subt. Ingredients

Helicity-Flip  
Contributions

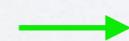
Universal relation between  
hard-virtual factors and  
two-loop scattering amplitudes

Heavy quark production



S. Catani, S. Devoto, M. Grazzini, S. Kallweit and J. Mazzitelli and H. Sargsyan (2019)

Extension at NLO+NLL QED+QCD



L. C, G. Ferrera and G. F. R. Sborlini (2018)

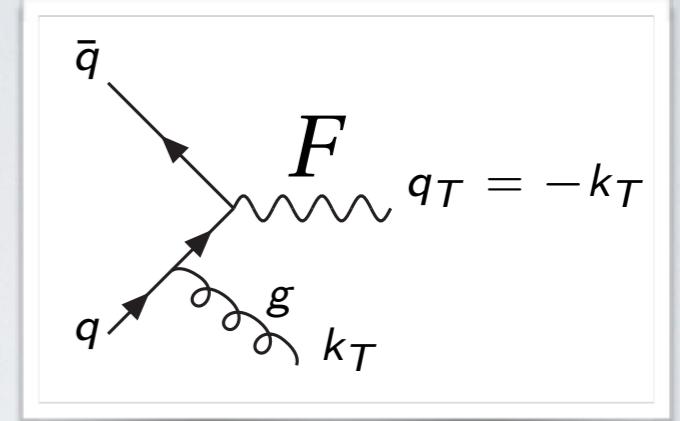


Giancarlo's Talk

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

- We consider the inclusive hard-scattering reaction

$$h_1 + h_2 \rightarrow F(Q) + X$$



- $h_i$  = hadron ( $i=1,2$ )

- Colourless final state  $F$  with transverse momentum  $q_T$  and invariant mass  $Q$

- We are interested in the QCD corrections to this process  $F$
- The basic idea behind the method

Catani, Grazzini (2007)

$$d\sigma_{N^n LO}^F(q_T \neq 0) \equiv d\sigma_{N^{n-1} LO}^{F+jets} \quad \text{with} \quad n = 1, 2, 3$$

- $d\sigma_{N^{n-1} LO}^{F+jets}$  can be computed with the well known subtraction methods at NLO and NNLO
- The remaining true  $N^n LO$  divergencies are associated to the small- $q_T$  limit

# THE $N^3LO$ QT-SUBTRACTION METHOD

- The generic form of the qT-subtraction method

Catani, Grazzini (2007)

$$d\sigma_{N^n LO}^F = \mathcal{H}_{N^n LO}^F \otimes d\sigma_{LO}^F + [d\sigma_{N^{n-1} LO}^{F+jets} - d\sigma_{N^n LO}^{F CT}] \quad \text{with } n = 1, 2, 3$$

$$d\sigma_{N^n LO}^{F CT} = \Sigma_{N^n LO}^F (q_T^2/M^2) d^2 \mathbf{q}_T \otimes d\sigma_{LO}^F$$

- How to calculate the qT-subtraction ingredients?

$$\begin{aligned} & \left( \Sigma_{c\bar{c} \leftarrow ab}^F \left( \frac{q_T^2}{M^2}; \frac{M^2}{\hat{s}}; \alpha_s \right) + \mathcal{H}_{c\bar{c} \leftarrow ab}^F \left( \frac{M^2}{\hat{s}}; \alpha_s \right) \right) \otimes d[\sigma_{c\bar{c}}^{F;(0)}]_{ab} = \frac{M^2}{s} \int_0^\infty db \frac{b}{2} J_0(bq_T) \\ & \times S_c(M, b) \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} d\hat{\sigma}_{c\bar{c}}^{F;(0)} f_{a/h_1}(x_1/z_1, b_0^2/b^2) f_{b/h_2}(x_2/z_2, b_0^2/b^2) [H^F C_1 C_2]_{c\bar{c};ab} \end{aligned}$$

- Expanding the resummation formula at fixed order

J. C. Collins, D. E. Soper and G. Sterman (1985)  
S. Catani, D. de Florian and M. Grazzini (2001)  
S. Catani and M. Grazzini (2011)

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

- Expanding at fixed-order the following expression

$$\left( \Sigma_{c\bar{c} \leftarrow ab}^F \left( \frac{q_T^2}{M^2}, \frac{M^2}{\hat{s}}; \alpha_s \right) + \mathcal{H}_{c\bar{c} \leftarrow ab}^F \left( \frac{M^2}{\hat{s}}; \alpha_s \right) \right) \otimes d[\sigma_{c\bar{c}}^{F;(0)}]_{ab} = \frac{M^2}{s} \int_0^\infty db \frac{b}{2} J_0(b q_T)$$

$$\times S_c(M, b) \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} d\hat{\sigma}_{c\bar{c}}^{F;(0)} f_{a/h_1}(x_1/z_1, b_0^2/b^2) f_{b/h_2}(x_2/z_2, b_0^2/b^2) [H^F C_1 C_2]_{c\bar{c};ab}$$

$$\Sigma^F(q_T/Q) \xrightarrow[q_T \rightarrow 0]{} \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{k=1}^{2n} \Sigma^{F(n;k)} \frac{Q^2}{q_T^2} \ln^{k-1} \frac{Q^2}{q_T^2}$$

Contribution of all large logarithmic terms  
In the small-qT limit

$$\mathcal{H}_{c\bar{c} \leftarrow ab}^F \rightarrow \sum_{n=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \mathcal{H}_{c\bar{c} \leftarrow ab}^{F(n)} \delta(q_T^2)$$

Free of large logarithmic terms  
in the small-qT limit

$$d\sigma_{N^n LO}^F = \mathcal{H}_{N^n LO}^F \otimes d\sigma_{LO}^F + [d\sigma_{N^{n-1} LO}^{F+jets} - d\sigma_{N^n LO}^{F CT}] \quad \text{with } n = 1, 2, 3$$

$$d\sigma_{N^n LO}^{F CT} = \Sigma_{N^n LO}^F (q_T^2/M^2) d^2 \mathbf{q}_T \otimes d\sigma_{LO}^F$$

# THE $N^3LO$ QT-SUBTRACTION METHOD

- Expanding at fixed-order the following expression

$$\left( \Sigma_{c\bar{c} \leftarrow ab}^F \left( \frac{q_T^2}{M^2}, \frac{M^2}{\hat{s}}; \alpha_s \right) + \mathcal{H}_{c\bar{c} \leftarrow ab}^F \left( \frac{M^2}{\hat{s}}; \alpha_s \right) \right) \otimes d[\sigma_{c\bar{c}}^{F;(0)}]_{ab} = \frac{M^2}{s} \int_0^\infty db \frac{b}{2} J_0(b q_T) \\ \times S_c(M, b) \int_{x_1}^1 \frac{dz_1}{z_1} \int_{x_2}^1 \frac{dz_2}{z_2} d\hat{\sigma}_{c\bar{c}}^{F;(0)} f_{a/h_1}(x_1/z_1, b_0^2/b^2) f_{b/h_2}(x_2/z_2, b_0^2/b^2) [H^F C_1 C_2]_{c\bar{c};ab}$$

$$x_1 = \frac{M}{\sqrt{s}} e^{+y}, \quad x_2 = \frac{M}{\sqrt{s}} e^{-y}$$

$$S_c(M, b) = \exp \left\{ - \int_{b_0^2/b^2}^{M^2} \frac{dq^2}{q^2} \left[ A_c(\alpha_s(q^2)) \ln \frac{M^2}{q^2} + B_c(\alpha_s(q^2)) \right] \right\}$$

Sudakov form factor

$$[H^{F=H} C_1 C_2]_{gg;ab} = H_g^{F=H}(\alpha_s(M^2)) [C_{ga}(z_1; \alpha_s(b_0^2/b^2)) C_{gb}(z_2; \alpha_s(b_0^2/b^2)) \\ + G_{ga}(z_1; \alpha_s(b_0^2/b^2)) G_{gb}(z_2; \alpha_s(b_0^2/b^2))]$$

Last line not present  
in qq initiated sub-  
processes

Catani, Grazzini (2011)

$$z = M^2/\hat{s}$$

$$C_{ga}(z; \alpha_s) = \delta_{ga} \delta(1-z) + \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n C_{ga}^{(n)}(z)$$

Collinear coefficient functions

$$G_{ga}(z; \alpha_s) = \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n G_{ga}^{(n)}(z)$$

Helicity-flip functions  
in gg initiated processes

$$H_g^{F=H}(\alpha_s) = 1 + \sum_{n=1}^{\infty} \left( \frac{\alpha_s}{\pi} \right)^n H_g^{F=H;(n)}$$

Hard virtual factors  
(loop-virtual contributions)

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

## Third-order ingredients

- All the ingredients which describe the cancellation of the large logarithmic terms (in the small-qT limit) are known from the previous orders (NNLO and NLO) (using convolutions written in function of Goncharov PolyLogarithms GPLS)

$$\Sigma^F(q_T/Q) \xrightarrow{q_T \rightarrow 0} \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n \sum_{k=1}^{2n} \Sigma^{F(n;k)} \frac{Q^2}{q_T^2} \ln^{k-1} \frac{Q^2}{q_T^2}$$



- The third-order contribution at qT = 0 presents two missing pieces
  - The universal relation between the three-loop virtual scattering amplitudes and the hard-virtual factor
  - The collinear coefficient functions at the third-order



# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

## Third-order ingredients at qT = 0

- The hard-virtual, collinear and helicity-flip contributions

De Florian, Grazzini (2001)

$$\begin{aligned} \mathcal{H}_{gg \leftarrow ab}^{H;(1)}(z) &= \delta_{ga} \delta_{gb} \delta(1-z) H_g^{H;(1)} + \delta_{ga} C_{gb}^{(1)}(z) + \delta_{gb} C_{ga}^{(1)}(z) , \\ \mathcal{H}_{gg \leftarrow ab}^{H;(2)}(z) &= \delta_{ga} \delta_{gb} \delta(1-z) H_g^{H;(2)} + \delta_{ga} C_{gb}^{(2)}(z) + \delta_{gb} C_{ga}^{(2)}(z) \\ &+ H_g^{H;(1)} \left( \delta_{ga} C_{gb}^{(1)}(z) + \delta_{gb} C_{ga}^{(1)}(z) \right) + \left( C_{ga}^{(1)} \otimes C_{gb}^{(1)} \right)(z) + \left( G_{ga}^{(1)} \otimes G_{gb}^{(1)} \right)(z) \end{aligned}$$

S. Catani, L. C, D. de Florian, G. Ferrera and M. Grazzini (2013)

$$\begin{aligned} \mathcal{H}_{gg \leftarrow ab}^{H;(3)}(z) &= \delta_{ga} \delta_{gb} \delta(1-z) H_g^{H;(3)} + \delta_{ga} C_{gb}^{(3)}(z) + \delta_{gb} C_{ga}^{(3)}(z) \\ &+ \left( G_{ga}^{(1)} \otimes G_{gb}^{(2)} \right)(z) + \left( G_{ga}^{(2)} \otimes G_{gb}^{(1)} \right)(z) \\ &+ H_g^{H;(1)} \left( \delta_{ga} C_{gb}^{(2)}(z) + \delta_{gb} C_{ga}^{(2)}(z) \right) + H_g^{H;(2)} \left( \delta_{ga} C_{gb}^{(1)}(z) + \delta_{gb} C_{ga}^{(1)}(z) \right) \\ &+ \left( C_{ga}^{(1)} \otimes C_{gb}^{(2)} \right)(z) + \left( C_{ga}^{(2)} \otimes C_{gb}^{(1)} \right)(z) \\ &+ H_g^{H;(1)} \left( C_{ga}^{(1)} \otimes C_{gb}^{(1)} \right)(z) + H_g^{H;(1)} \left( G_{ga}^{(1)} \otimes G_{gb}^{(1)} \right)(z) . \end{aligned}$$

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

## Third-order ingredients at qT = 0

- The hard-virtual, collinear and helicity-flip contributions

D. Gutierrez-Reyes, S. Leal-Gomez, I. Scimemi, and A. Vladimirov (2019)  
M. Luo, T. Yang, H. Xing Zhu, and Y. Jiao Zhu (2019)

These ingredients are partially known or completely missing

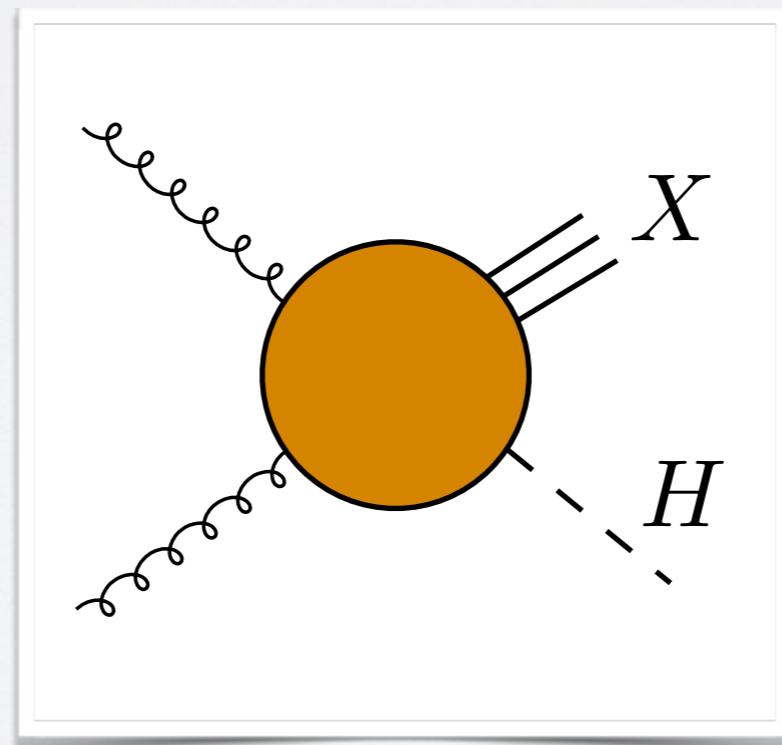
$$\begin{aligned}\mathcal{H}_{gg \leftarrow ab}^{H;(3)}(z) &= \delta_{ga} \delta_{gb} \delta(1-z) H_g^{H;(3)} + \delta_{ga} C_{gb}^{(3)}(z) + \delta_{gb} C_{ga}^{(3)}(z) \\ &\quad + \left( G_{ga}^{(1)} \otimes G_{gb}^{(2)} \right) (z) + \left( G_{ga}^{(2)} \otimes G_{gb}^{(1)} \right) (z) \\ &\quad + H_g^{H;(1)} \left( \delta_{ga} C_{gb}^{(2)}(z) + \delta_{gb} C_{ga}^{(2)}(z) \right) + H_g^{H;(2)} \left( \delta_{ga} C_{gb}^{(1)}(z) + \delta_{gb} C_{ga}^{(1)}(z) \right) \\ &\quad + \left( C_{ga}^{(1)} \otimes C_{gb}^{(2)} \right) (z) + \left( C_{ga}^{(2)} \otimes C_{gb}^{(1)} \right) (z) \\ &\quad + H_g^{H;(1)} \left( C_{ga}^{(1)} \otimes C_{gb}^{(1)} \right) (z) + H_g^{H;(1)} \left( G_{ga}^{(1)} \otimes G_{gb}^{(1)} \right) (z).\end{aligned}$$

Where we fix  $\mu_R = \mu_F = M$

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD



Up to the previous slide the discussion about the N<sup>3</sup>LO qT-subtraction formalism, was completely general. The following slides are for final processes F initiated by gluon fusion. In particular for Higgs boson production via gluon fusion in the large top-mass limit.



# THE N<sup>3</sup>LO CONTRIBUTIONS AT QT=0

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- These ingredients are partially known or completely missing

$$H_g^{H;(3)} ; \quad C_{ga}^{(3)} ; \quad G_{ga}^{(2)}$$

D. Gutierrez-Reyes, S. Leal-Gomez, I. Scimemi, and A. Vladimirov (2019)  
M. Luo, T. Yang, H. Xing Zhu, Y. Jiao Zhu (2019)

Nonetheless, within the qT-subtraction formalism,  $\mathcal{H}_{gg \leftarrow ab}^{H;(3)}$  can be reliably approximated for any hard-scattering process whose corresponding total cross section is known at N<sup>3</sup>LO.

G. Bozzi, S. Catani, D. de Florian and M. Grazzini (2005)

- In our case we use the N<sup>3</sup>LO Higgs boson cross section recently calculated and implemented in the numerical code ihixs 2.0

B. Mistlberger (2018)

F. Dulat, A. Lazopoulos and B. Mistlberger (2018)

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- Numerical procedure to extract

$$H_g^{H;(3)} ; \quad C_{ga}^{(3)} ; \quad G_{ga}^{(2)}$$

D. Gutierrez-Reyes, S. Leal-Gomez, I. Scimemi, and A. Vladimirov (2019)

M. Luo, T. Yang, H. Xing Zhu, Y. Jiao Zhu (2019)



- Starting point is the sketchy form of the qT-subtraction method

$$d\sigma_{N^n LO}^F = \mathcal{H}_{N^n LO}^F \otimes d\sigma_{LO}^F + [d\sigma_{N^{n-1} LO}^{F+jets} - d\sigma_{N^n LO}^{F CT}] \quad \text{with } n = 1, 2, 3$$

$$\hat{\sigma}_{F ab}^{\text{tot.}}(M, \hat{s}; \alpha_s(\mu_R^2), \mu_R^2, \mu_F^2) = \int_0^\infty dq_T^2 \frac{d\hat{\sigma}_{F ab}}{dq_T^2}(q_T, M, \hat{s}; \alpha_s(\mu_R^2), \mu_R^2, \mu_F^2)$$

$$\sigma_{N^n LO}^{F(\text{tot.})} = \mathcal{H}_{N^n LO}^F \otimes \sigma_{LO}^F + \int_0^\infty dq_T^2 \frac{d\sigma_{N^n LO}^{F(\text{fin.})}}{dq_T^2}$$

$$\frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \equiv \left[ \frac{d\hat{\sigma}_{ab}^{F+jets}}{dq_T^2} - \frac{d\hat{\sigma}_{ab}^{F CT}}{dq_T^2} \right]$$

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- Subtracting two consecutive orders, we arrive to an expression that can be used to extract numerically the missing terms

$$\begin{aligned} & \left(\frac{\alpha_s}{\pi}\right)^3 \frac{M^2}{\hat{s}} \sum_c \sigma_{c\bar{c}, F}^{(0)}(\alpha_s, M) \mathcal{H}_{c\bar{c} \leftarrow ab}^{F;(3)}\left(\frac{M^2}{\hat{s}}; \frac{M^2}{\mu_R^2}, \frac{M^2}{\mu_F^2}\right) \\ &= \left\{ \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{N}^3\text{LO}} - \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{NNLO}} \right\} - \int_0^\infty dq_T^2 \left\{ \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{N}^3\text{LO}} - \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{NNLO}} \right\} \end{aligned}$$

$$\begin{aligned} \mathcal{H}_{gg \leftarrow ab}^{H;(3)}(z) &= \boxed{\delta_{g a} \delta_{g b} \delta(1-z) H_g^{H;(3)} + \delta_{g a} C_{g b}^{(3)}(z) + \delta_{g b} C_{g a}^{(3)}(z)} \\ &+ \left( G_{g a}^{(1)} \otimes G_{g b}^{(2)} \right)(z) + \left( G_{g a}^{(2)} \otimes G_{g b}^{(1)} \right)(z) \\ &+ H_g^{H;(1)} \left( \delta_{g a} C_{g b}^{(2)}(z) + \delta_{g b} C_{g a}^{(2)}(z) \right) + H_g^{H;(2)} \left( \delta_{g a} C_{g b}^{(1)}(z) + \delta_{g b} C_{g a}^{(1)}(z) \right) \\ &+ \left( C_{g a}^{(1)} \otimes C_{g b}^{(2)} \right)(z) + \left( C_{g a}^{(2)} \otimes C_{g b}^{(1)} \right)(z) \\ &+ H_g^{H;(1)} \left( C_{g a}^{(1)} \otimes C_{g b}^{(1)} \right)(z) + H_g^{H;(1)} \left( G_{g a}^{(1)} \otimes G_{g b}^{(1)} \right)(z). \end{aligned}$$

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- Subtracting two consecutive orders, we arrive to an expression that can be used to extract numerically the missing terms

$$\begin{aligned} & \left( \frac{\alpha_s}{\pi} \right)^3 \frac{M^2}{\hat{s}} \sum_c \sigma_{c\bar{c}, F}^{(0)}(\alpha_s, M) \mathcal{H}_{c\bar{c} \leftarrow ab}^{F;(3)} \left( \frac{M^2}{\hat{s}}; \frac{M^2}{\mu_R^2}, \frac{M^2}{\mu_F^2} \right) \\ &= \left\{ \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{N}^3\text{LO}} - \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{NNLO}} \right\} - \int_0^\infty dq_T^2 \left\{ \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{N}^3\text{LO}} - \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{NNLO}} \right\} \end{aligned}$$

We extract numerically these information that we put in a constant (independent of z)

$$C_{N3} \delta_{g a} \delta_{g b} \delta(1-z) \leftarrow \delta_{g a} \delta_{g b} \delta(1-z) [H_g^{H;(3)}]_{(\delta_{(2)}^{q_T})} + \delta_{g a} C_{g b}^{(3)}(z) + \delta_{g b} C_{g a}^{(3)}(z) + \left( G_{g a}^{(1)} \otimes G_{g b}^{(2)} \right)(z) + \left( G_{g a}^{(2)} \otimes G_{g b}^{(1)} \right)(z) ,$$

Exact numerical extraction

Approximation of a z function by a constant (a delta term)

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- Subtracting two consecutive orders, we arrive to an expression that can be used to extract numerically the missing terms

$$\begin{aligned} & \left( \frac{\alpha_s}{\pi} \right)^3 \frac{M^2}{\hat{s}} \sum_c \sigma_{c\bar{c}, F}^{(0)}(\alpha_s, M) \mathcal{H}_{c\bar{c} \leftarrow ab}^{F;(3)} \left( \frac{M^2}{\hat{s}}; \frac{M^2}{\mu_R^2}, \frac{M^2}{\mu_F^2} \right) \\ &= \left\{ \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{N^3\text{LO}} - \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{NNLO}} \right\} - \int_0^\infty dq_T^2 \left\{ \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{N^3\text{LO}} - \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{NNLO}} \right\} \end{aligned}$$

We extract numerically these information that we put in a constant (independent of z)

$$C_{N3} \delta_{g a} \delta_{g b} \delta(1-z) \leftarrow \delta_{g a} \delta_{g b} \delta(1-z) [H_g^{H;(3)}]_{(\delta_{(2)}^{q_T})} + \delta_{g a} C_{g b}^{(3)}(z) + \delta_{g b} C_{g a}^{(3)}(z) + \left( G_{g a}^{(1)} \otimes G_{g b}^{(2)} \right)(z) + \left( G_{g a}^{(2)} \otimes G_{g b}^{(1)} \right)(z) ,$$

Let me postpone the discussion of the universal terms inside the hard-virtual factors, since it is orthogonal to the following slides

$$H_g^{H;(3)} \equiv \tilde{H}_g^{H;(3)} + [H_g^{H;(3)}]_{(\delta_{(2)}^{q_T})}$$

Third order constant of “soft origin”, belonging to the finite part of the third order structure of the IR singularities

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- Subtracting two consecutive orders, we arrive to an expression that can be used to extract numerically the missing terms

$$\begin{aligned} & \left(\frac{\alpha_s}{\pi}\right)^3 \frac{M^2}{\hat{s}} \sum_c \sigma_{c\bar{c}, F}^{(0)}(\alpha_s, M) \mathcal{H}_{c\bar{c} \leftarrow ab}^{F;(3)}\left(\frac{M^2}{\hat{s}}; \frac{M^2}{\mu_R^2}, \frac{M^2}{\mu_F^2}\right) \\ &= \left\{ \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{N}^3\text{LO}} - \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{NNLO}} \right\} - \int_0^\infty dq_T^2 \left\{ \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{N}^3\text{LO}} - \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{NNLO}} \right\} \end{aligned}$$



Total exact Xsec. from

B. Mistlberger (2018)

Ihixs 2.0: F. Dulat, A. Lazopoulos and B. Mistlberger (2018)

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- Subtracting two consecutive orders, we arrive to an expression that can be used to extract numerically the missing terms

$$\begin{aligned} & \left(\frac{\alpha_s}{\pi}\right)^3 \frac{M^2}{\hat{s}} \sum_c \sigma_{c\bar{c}, F}^{(0)}(\alpha_s, M) \mathcal{H}_{c\bar{c} \leftarrow ab}^{F;(3)}\left(\frac{M^2}{\hat{s}}; \frac{M^2}{\mu_R^2}, \frac{M^2}{\mu_F^2}\right) \\ &= \left\{ \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{N}^3\text{LO}} - \left[ \hat{\sigma}_{F ab}^{\text{tot}} \right]_{\text{NNLO}} \right\} - \int_{qT\text{cut}^2}^{\infty} dq_T^2 \left\{ \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{N}^3\text{LO}} - \left[ \frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \right]_{\text{NNLO}} \right\} \end{aligned}$$

The lower limit in the integral has to be replaced by a technical cut, the qTcut

$$\frac{d\hat{\sigma}_{F ab}^{(\text{fin.})}}{dq_T^2} \equiv \left[ \frac{d\hat{\sigma}_{ab}^{F+\text{jets}}}{dq_T^2} - \frac{d\hat{\sigma}_{ab}^{F \text{ CT}}}{dq_T^2} \right]$$

NNLOJET

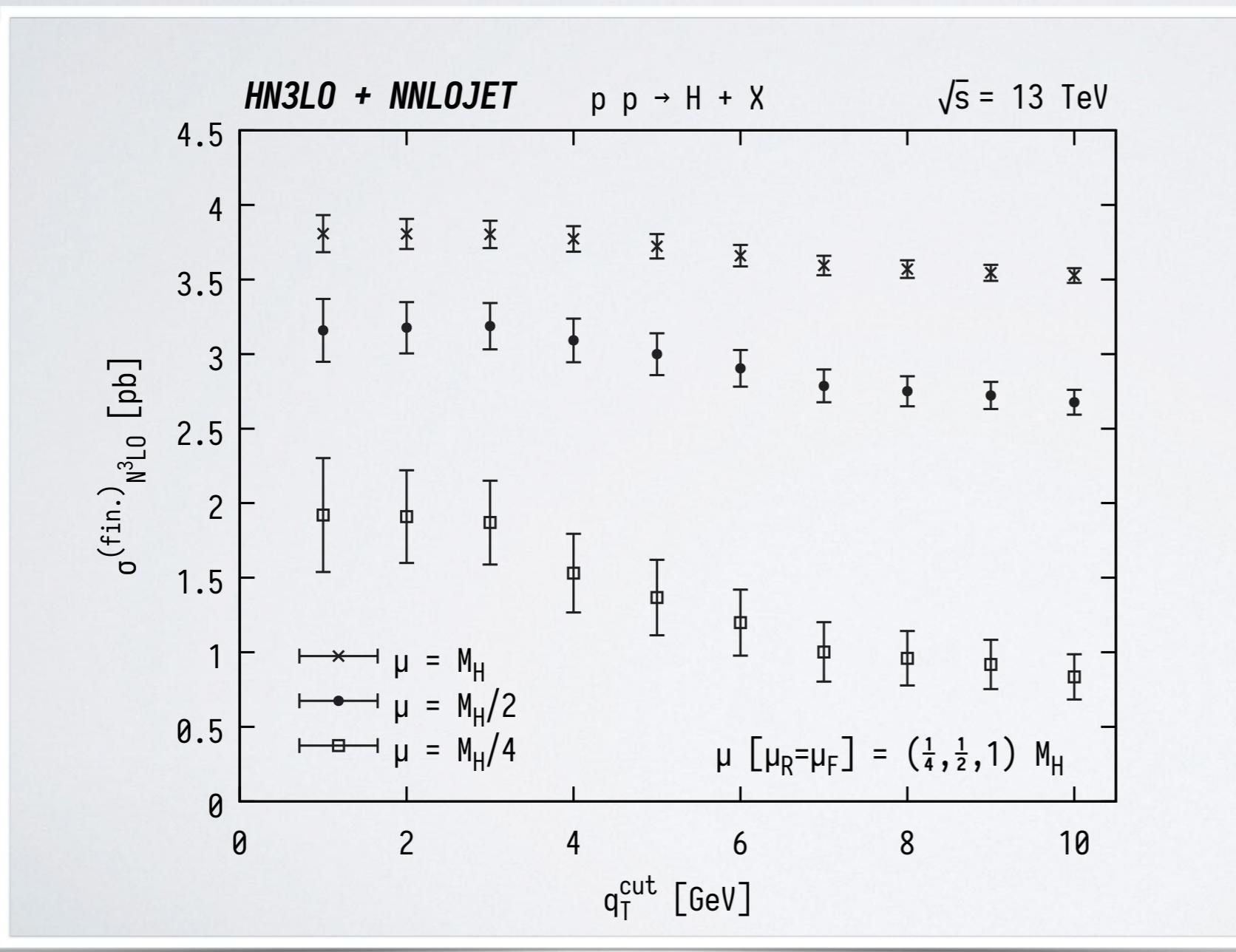
HN3LO

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- For the extraction, we consider Higgs boson production at N<sup>3</sup>LO in gluon fusion in the large top mass limit, at  $\sqrt{s} = 13$  TeV



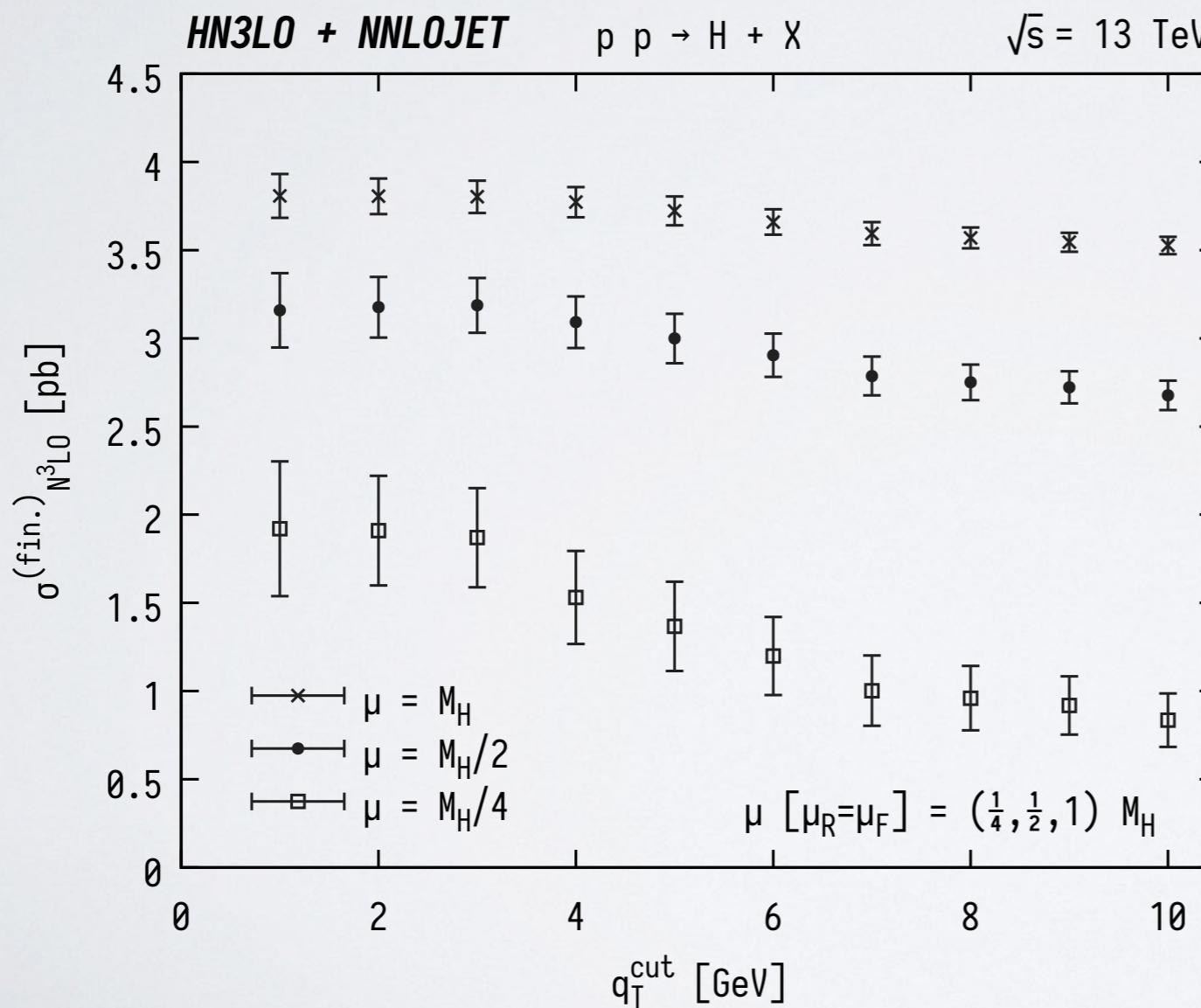
$$\int_{q_T^{\text{cut}2}}^{\infty} dq_T^2 \left[ \frac{\sigma_{N^3\text{LO}}^{F(\text{fin.})}}{q_T^2} - \frac{\sigma_{N^2\text{LO}}^{F(\text{fin.})}}{q_T^2} \right]$$

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

- For the extraction, we consider Higgs boson production at N<sup>3</sup>LO in gluon fusion in the large top mass limit, at  $\sqrt{s} = 13$  TeV



For the extraction procedure , the NNLO Xsection is taken from the analytical result, therefore, there is no systematic uncertainty from the NNLO

$$\int_{q_T^{\text{cut}2}}^{\infty} dq_T^2 \left[ \frac{\sigma_{N^3\text{LO}}^{F(\text{fin.})}}{q_T^2} - \frac{\sigma_{N^2\text{LO}}^{F(\text{fin.})}}{q_T^2} \right]$$

$$C_{N3} = -943 \pm 222$$

The size of the uncertainty bands are almost entirely due to the F+Jet calculation

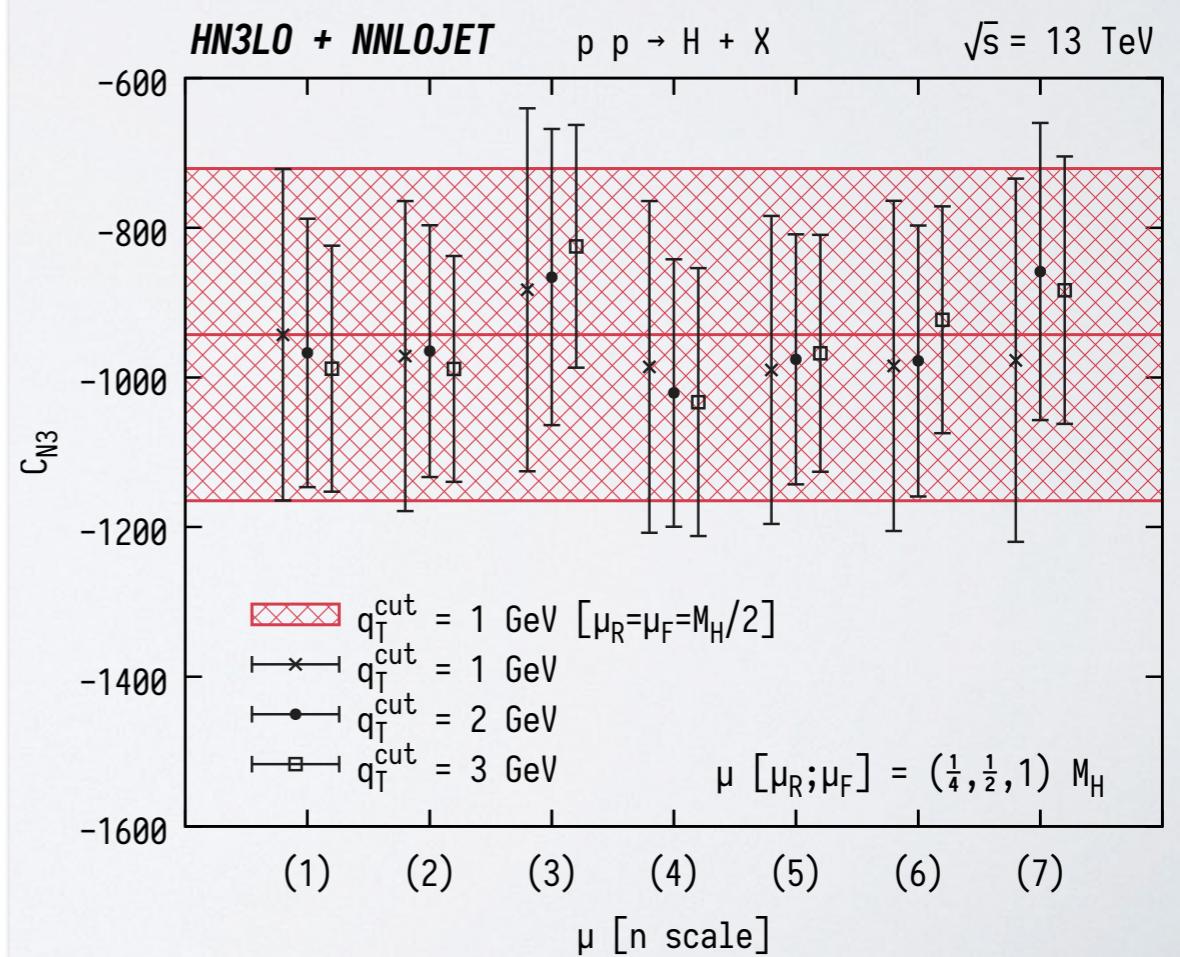
# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Third-order ingredients at qT = 0

• Repeating the same procedure for 7-point scale variation

n	$[\tilde{\mu}_R, \tilde{\mu}_F] \times M_H$	$C_{N3} (q_T^{\text{cut}} = 1 \text{ GeV})$	$C_{N3} (q_T^{\text{cut}} = 2 \text{ GeV})$	$C_{N3} (q_T^{\text{cut}} = 3 \text{ GeV})$
(1)	[1/2, 1/2]	<b>-943 ± 222</b>	-967 ± 179	-988 ± 164
(2)	[1, 1]	-971 ± 207	-965 ± 168	-989 ± 151
(3)	[1/4, 1/4]	-883 ± 243	-866 ± 198	-850 ± 162
(4)	[1/2, 1]	-986 ± 222	-1021 ± 179	-1033 ± 179
(5)	[1, 1/2]	-990 ± 206	-976 ± 167	-968 ± 158
(6)	[1/2, 1/4]	-985 ± 221	-978 ± 181	-923 ± 152
(7)	[1/4, 1/2]	-977 ± 243	-859 ± 199	-883 ± 179



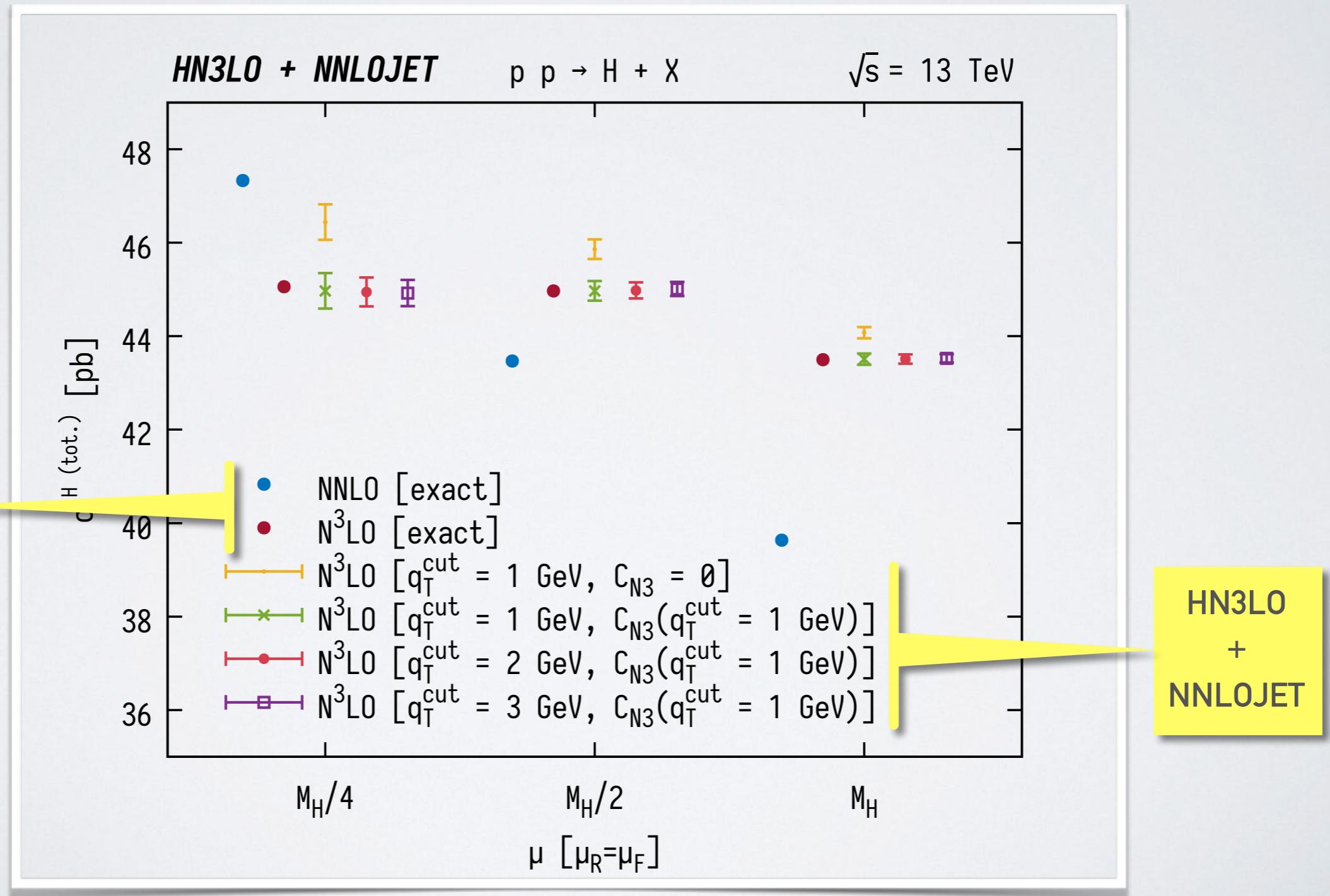
**FIRST RESULTS AT N3LO**

**N3LO TOTAL XSECTION**

# THE N<sup>3</sup>LO TOTAL XSECTION

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

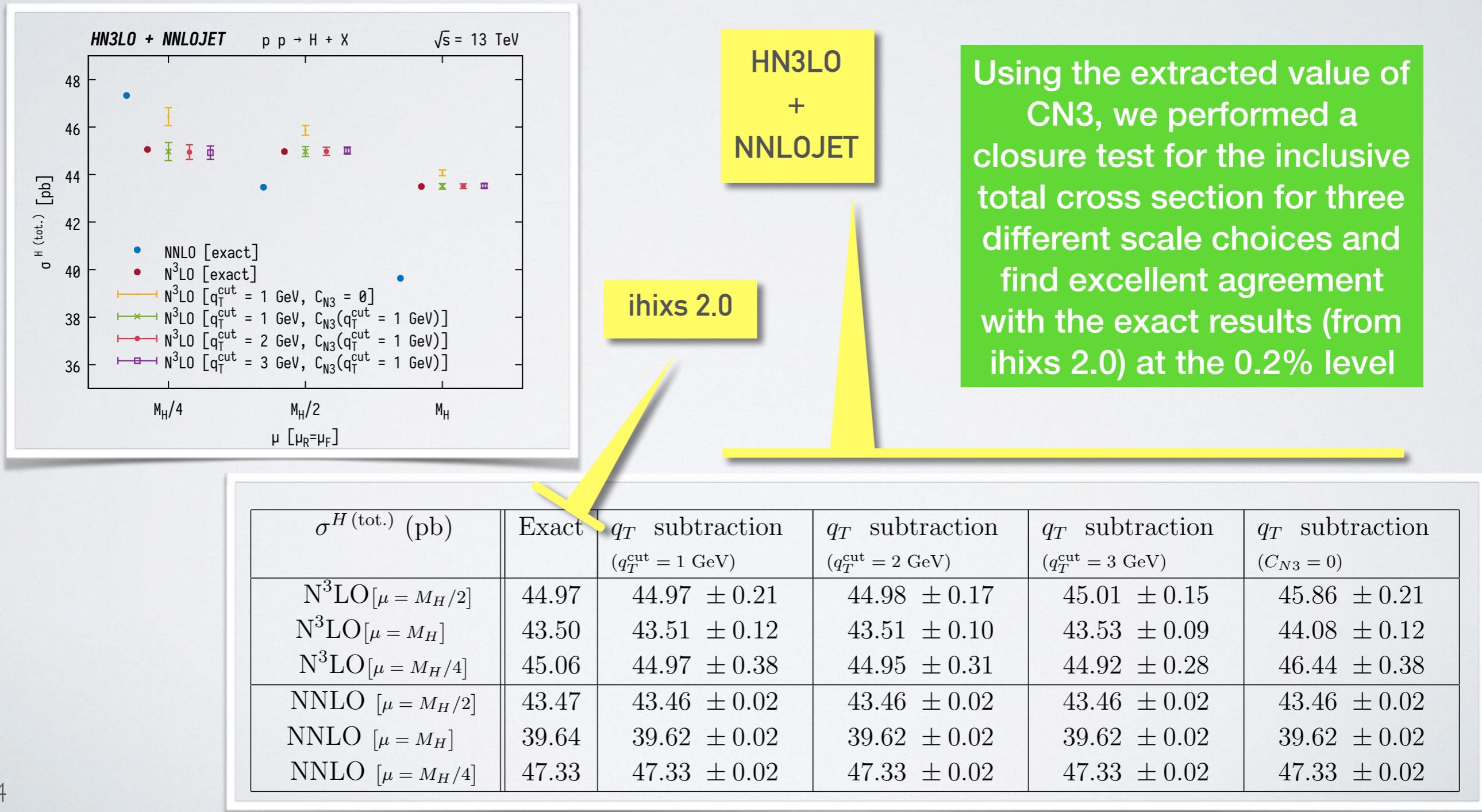
- With the extracted CN3 coefficient (only one value corresponding to the central scale) we can use the qT-subtraction method to predict total Xsections at N<sup>3</sup>LO



# THE N<sup>3</sup>LO TOTAL XSECTION

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

- With the extracted CN3 coefficient (only one value corresponding to the central scale) we can use the qT-subtraction method to predict total Xsections at N<sup>3</sup>LO



# N3LO DIFFERENTIAL DISTRIBUTIONS

# $N^3LO$ DIFFERENTIAL DISTRIBUTIONS

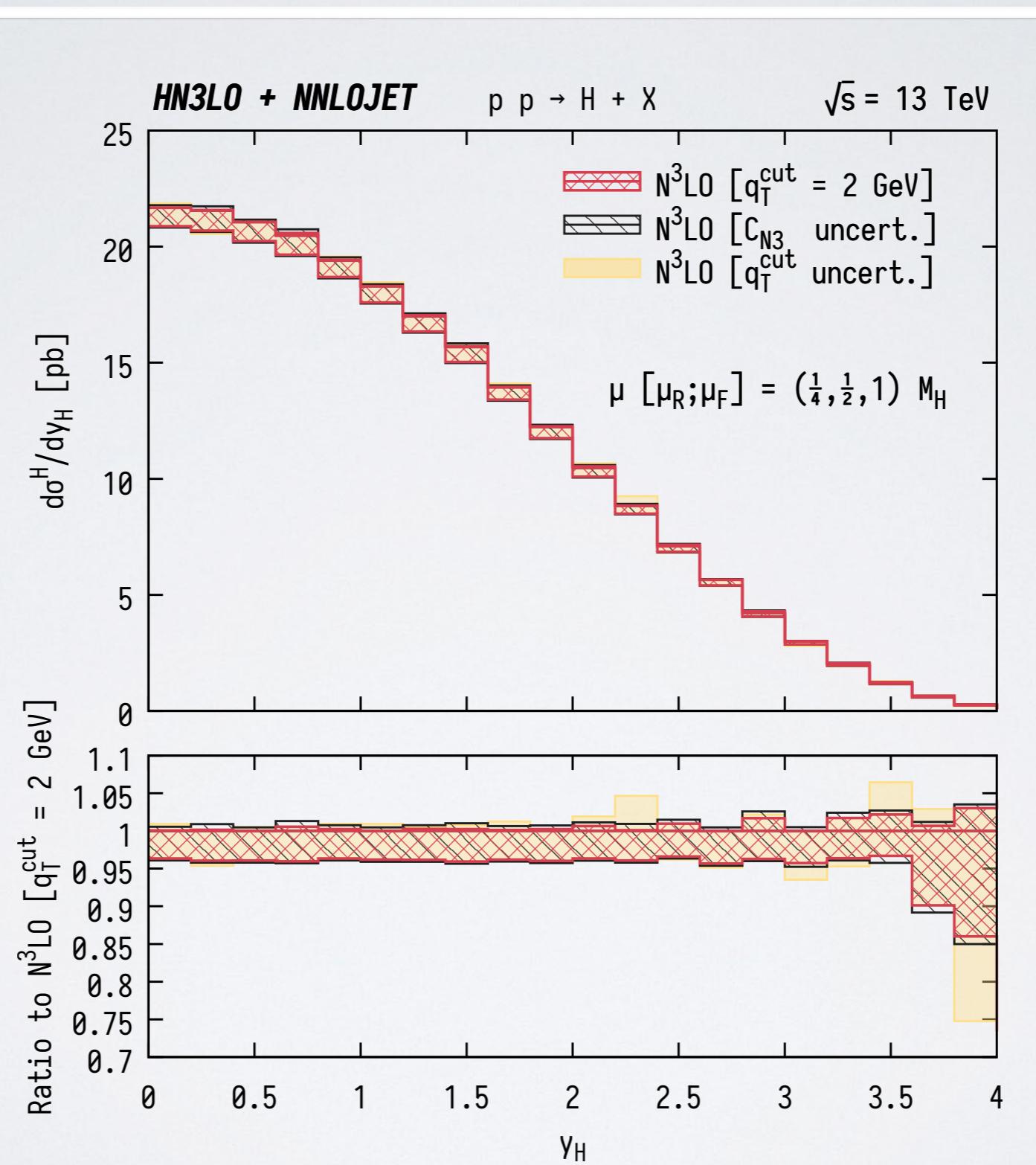
L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Uncertainties at $N^3LO$

$N^3LO$  prediction at  
 $q_T^{cut}=2\text{GeV}$   
+  $q_T^{cut}$  uncertainties +  
systematic uncertainties

$q_T^{cut}$  uncertainties  
Calculated taking three  
different predictions for  
 $q_T^{cut} = (2 \pm 1) \text{ GeV}$

The black band is  
obtained as the envelope  
of the seven-point scale  
variation at  $q_T^{cut} = 2$   
 $\text{GeV}$  considering for each  
scale the two extremal  
 $C_{N3}$  coefficients  
corresponding to its  
maximum and minimum  
statistical deviations  
 $C_{N3} = \{-1165, -721\}$



In order to produce  
smooth histograms the  
code has to run for a  
couple of months in  
2000 cores approx.

# $N^3LO$ DIFFERENTIAL DISTRIBUTIONS

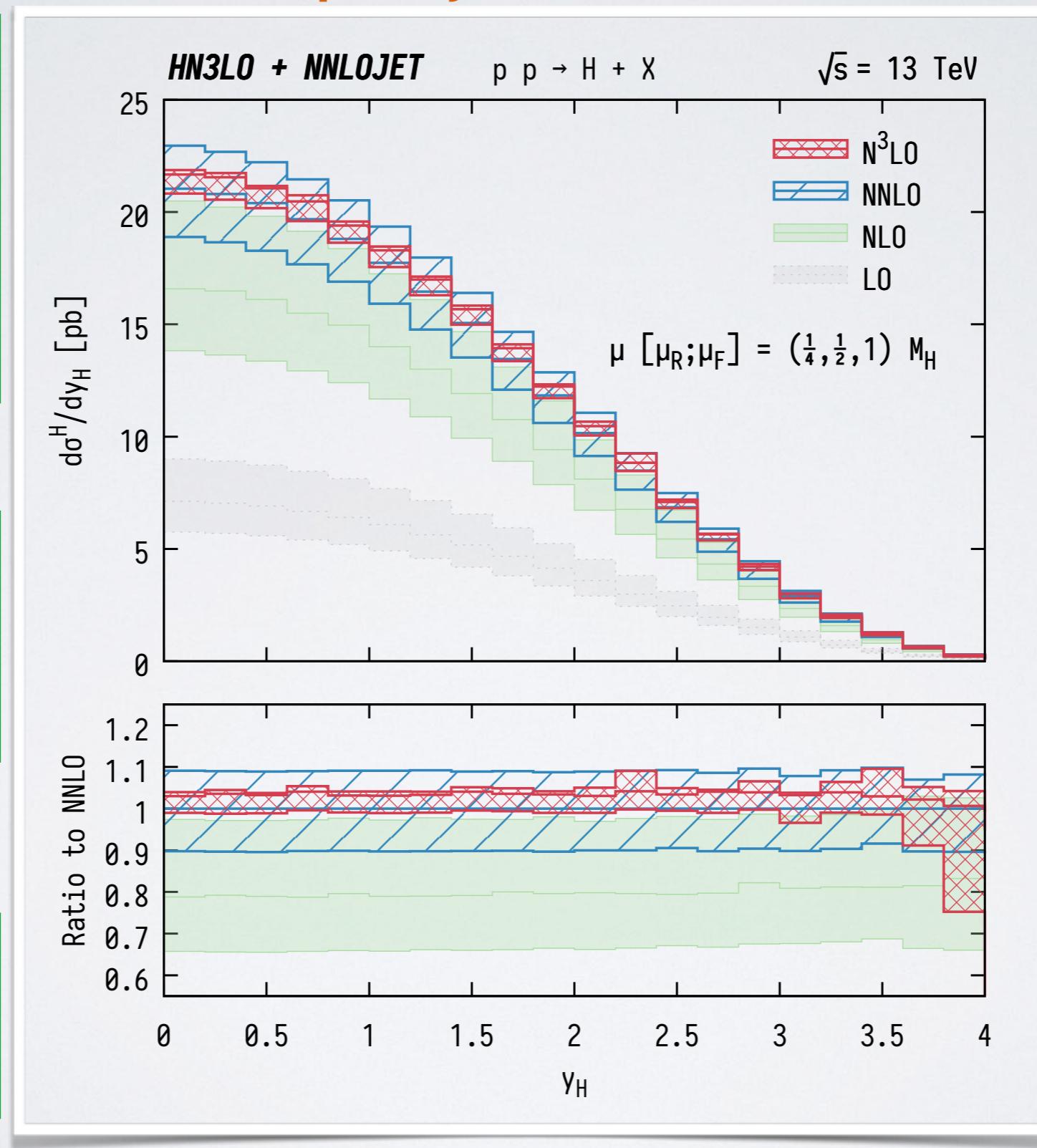
L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Rapidity distribution

We calculate the  $y_H$  distribution at  $N^3LO$  employing a seven-point scale variation and carefully assess systematic errors arising from different  $qT_{cut}$  and  $CN3$  values.

The combined theoretical uncertainty at  $N^3LO$  is at most of  $\pm 5\%$  level with respect to the central scale choice

$N^3LO$  prediction at  $qT_{cut}=2\text{GeV}$   
+  $qT_{cut}$  uncertainties +  
systematic uncertainties

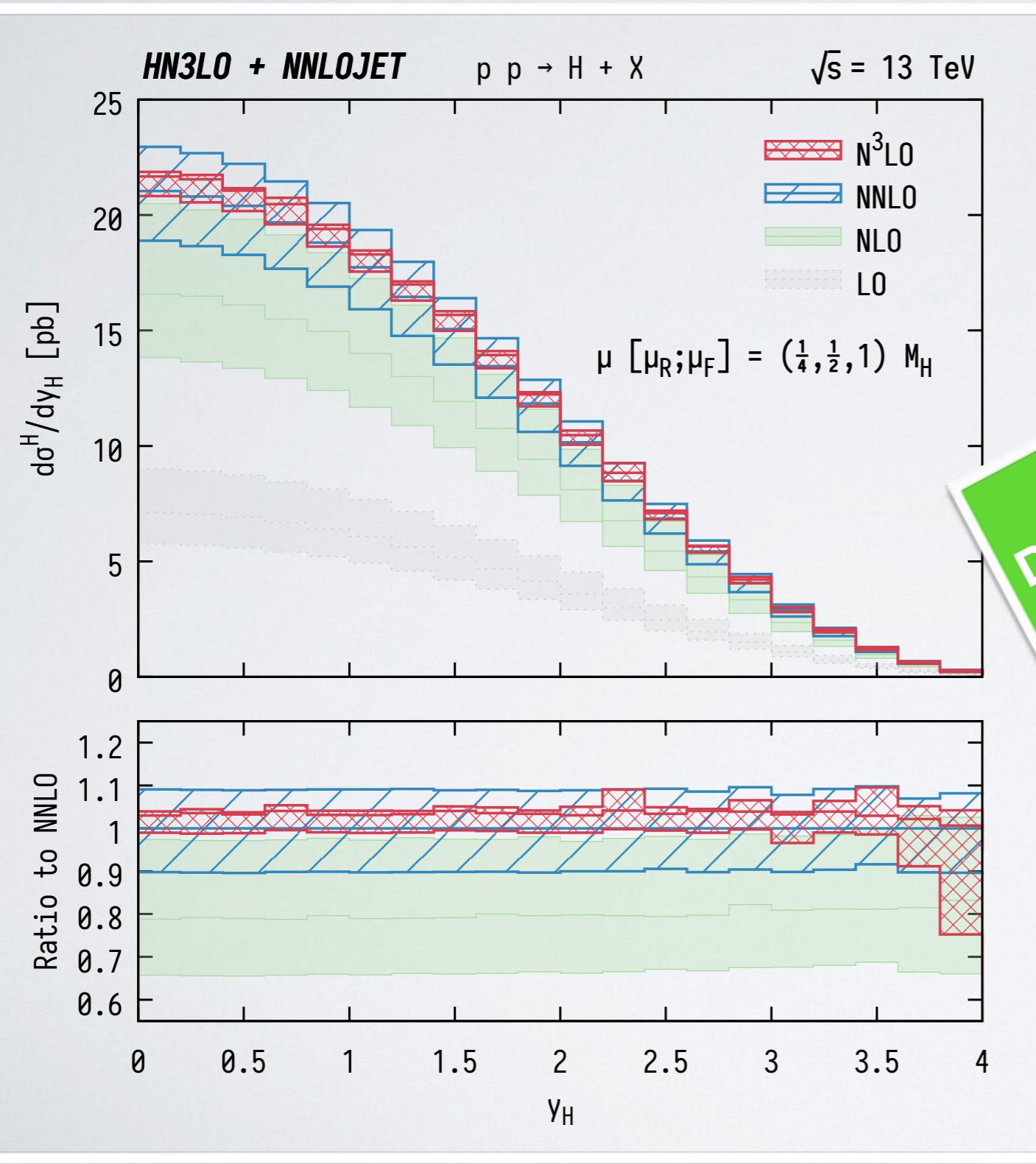


Compared to the NNLO  $y_H$  distributions, we observe a large reduction of theory uncertainties by more than 50% at  $N^3LO$ . The scale variation band at  $N^3LO$  stays within the NNLO band with a flat K-factor of about 1.034 in the central rapidity region ( $|y_H| \leq 3.6$ ).

# $N^3\text{LO}$ DIFFERENTIAL DISTRIBUTIONS

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Rapidity distribution

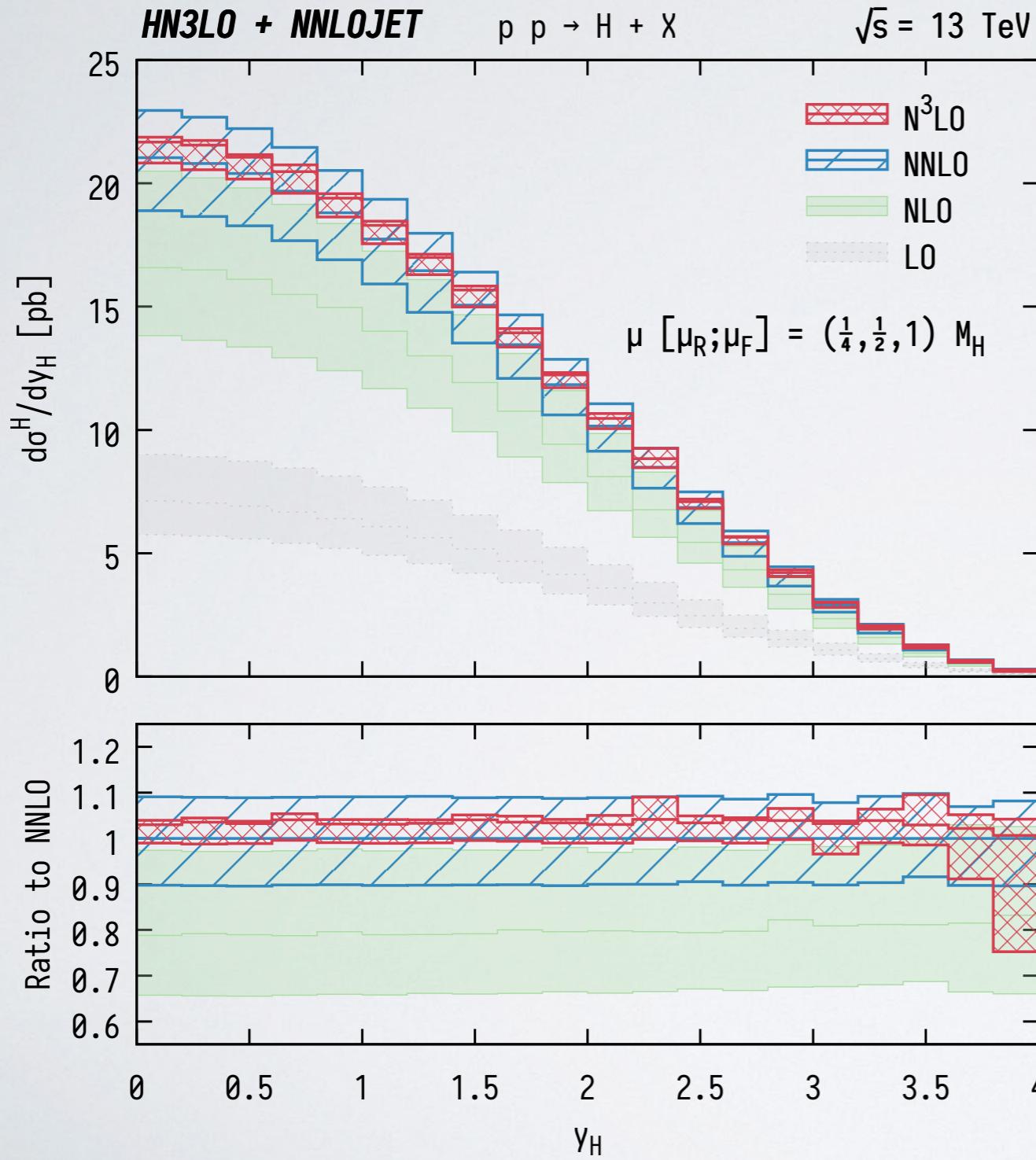


In agreement with  
Dulat, Mistlberger and Pelloni (2018)  
[Higgs production in TH expansion  
approximation]  
Which is far from a trivial test!!

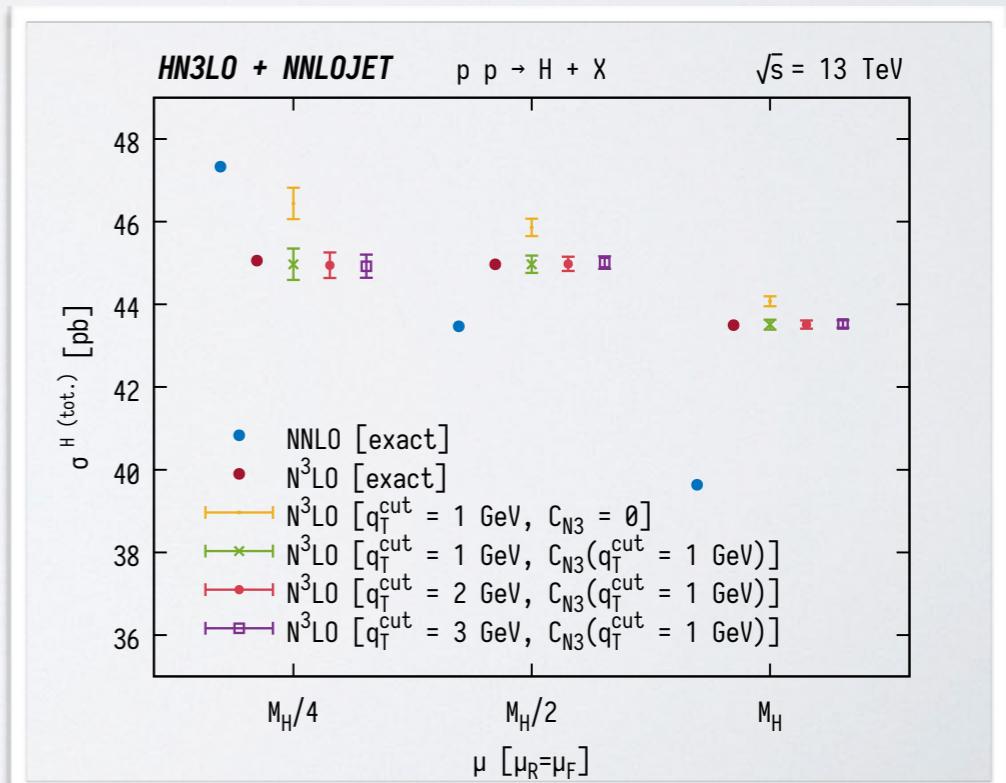
# N<sup>3</sup>LO DIFFERENTIAL DISTRIBUTIONS

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Rapidity distribution



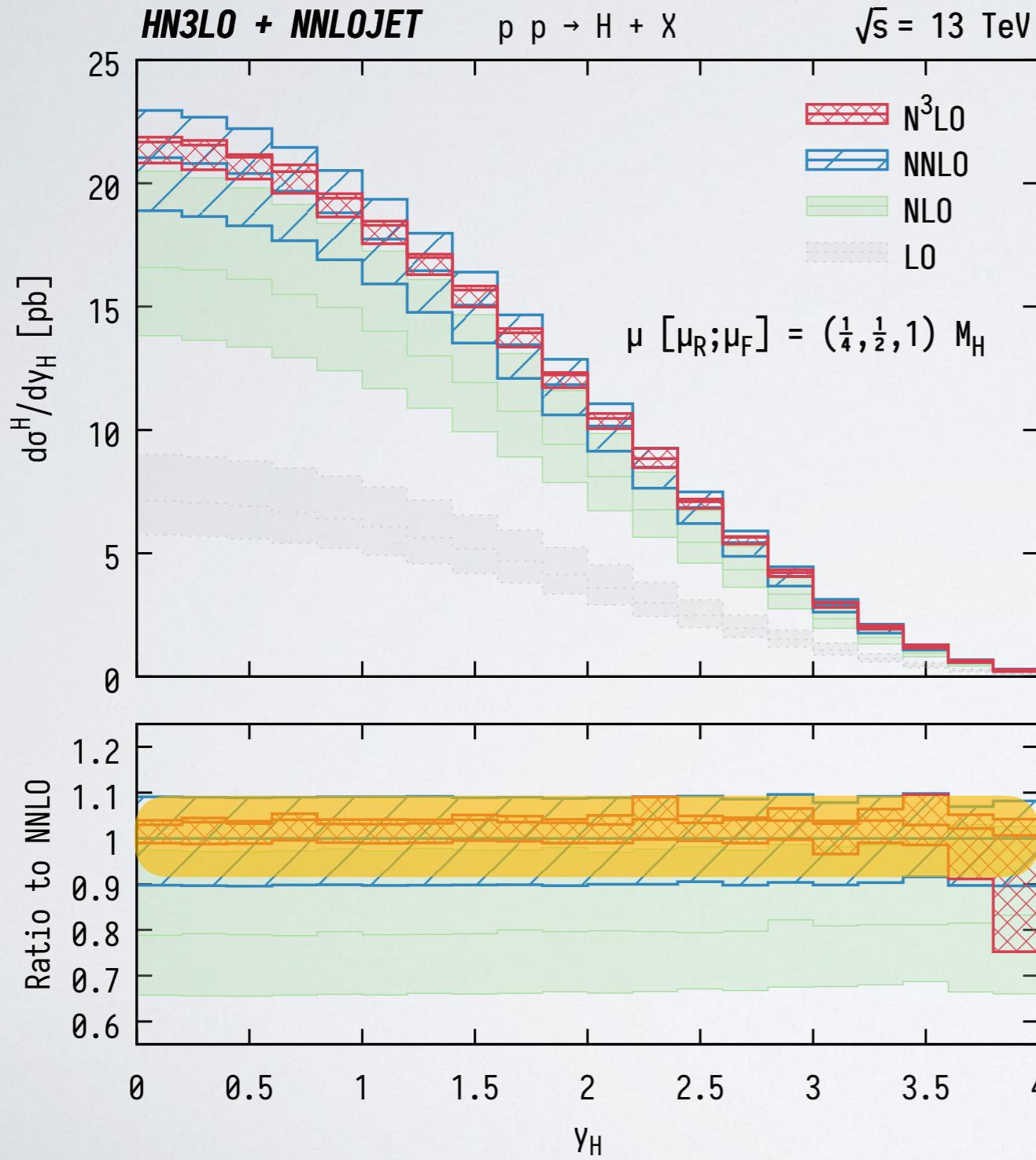
The central prediction at N<sup>3</sup>LO, almost coincides with the upper edge of the band, as was already observed for the total cross section



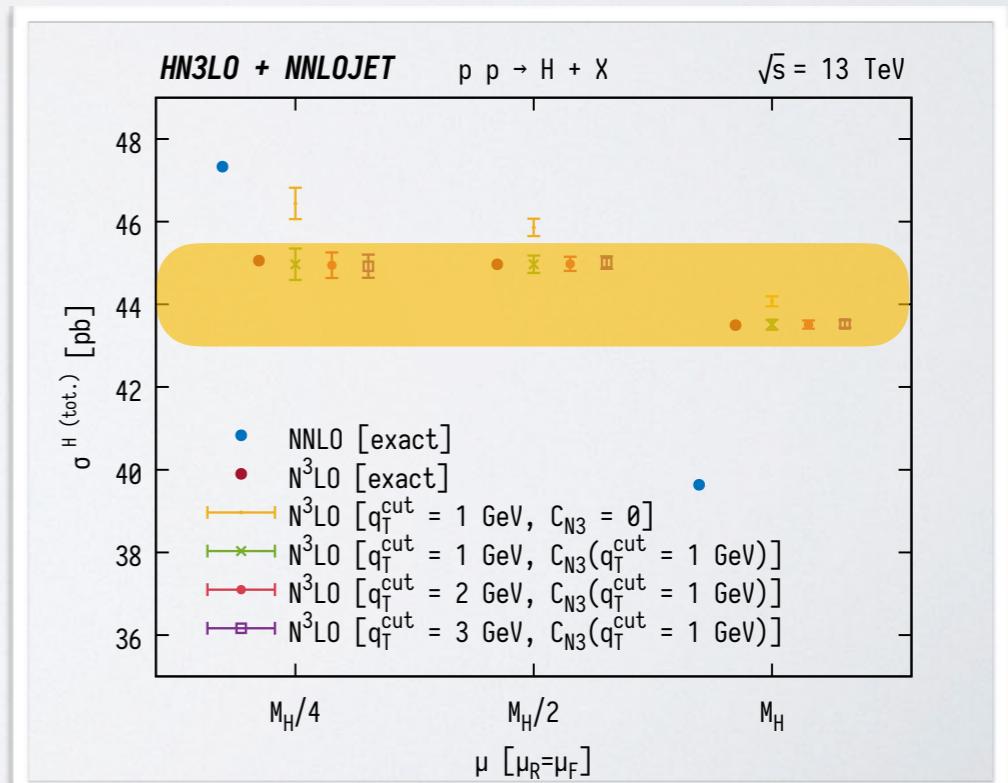
# N<sup>3</sup>LO DIFFERENTIAL DISTRIBUTIONS

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## Rapidity distribution



The central prediction at N<sup>3</sup>LO, almost coincides with the upper edge of the band, as was already observed for the total cross section



# CONCLUSIONS AND OUTLOOK

- We presented results at  $N^3LO$  for completely differential distributions for observables measured at the LHC  
**(in particular for Higgs boson production in gluon fusion)**
- The qT-subtraction method at  $N^3LO$ , presented here, can be applied to other processes initiated by gluon fusion (if the three-loop scattering amplitudes are known),  
**i.e the CN3 in gluon fusion is universal**

We can describe IR safe differential observables,  
not only the rapidity, and apply any set of cuts on the final state

# CONCLUSIONS AND OUTLOOK

- We presented results at N<sup>3</sup>LO for completely differential distributions for observables measured at the LHC  
**(in particular for Higgs boson production in gluon fusion)**
- The qT-subtraction method at N<sup>3</sup>LO, presented here, can be applied to other processes initiated by gluon fusion (if the three-loop scattering amplitudes are known),  
**i.e the CN3 in gluon fusion is universal**
- More work is necessary in order to calculate the CN3 coefficient for processes initiated by qqb annihilation, since the total cross section at N<sup>3</sup>LO for Drell-Yan is not known
- In the future, two elements are necessary in order to complete analytically the qT-subtraction method at N<sup>3</sup>LO
  - The universal relation between the three-loop virtual scattering amplitudes and the hard virtual factor
  - The third order collinear functions

# CONCLUSIONS AND OUTLOOK

- The approximation related to the CN3 coefficient in our approach **can be easily replaced by the full analytical results for the coefficients once available**
- Our results can be applied to perform  $qT$ -resummation at the corresponding level of logarithmic accuracy. **i.e without reweighing by the total cross section**

# THANK YOU!

# BACKUP SLIDES

# N<sup>3</sup>LO DIFFERENTIAL DISTRIBUTIONS

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

- The precedent results (CN3 coefficient) allow us to compute more exclusive observables (distributions) at N<sup>3</sup>LO
- Before starting with the N<sup>3</sup>LO computation, a couple of questions are in order (regarding CN3): **uncertainty introduced by the proposed numerical strategy?**
- In order to answer this question we can repeat the numerical strategy, but at one order less (NNLO) where all the coefficients are known analytically

$$\begin{aligned} C_N \ \delta_{g a} \delta_{g b} \delta(1 - z) &\leftarrow \delta_{g a} \delta_{g b} \delta(1 - z) [H_g^{H;(2)}]_{(\delta_{(1)}^{q_T})} \\ &+ \delta_{g a} C_{g b}^{(2)}(z) + \delta_{g b} C_{g a}^{(2)}(z) + \left( G_{g a}^{(1)} \otimes G_{g b}^{(1)} \right)(z) \end{aligned}$$

NNLO TEST

# NNLO DIFFERENTIAL DISTRIBUTIONS

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

- In order to answer this question we can repeat the numerical strategy, at one order less (NNLO) where all the coefficient are known analytically

$$\begin{aligned}\mathcal{H}_{gg \leftarrow ab}^{H;(2)}(z) = & \delta_{ga} \delta_{gb} \delta(1-z) H_g^{H;(2)} + \delta_{ga} C_{gb}^{(2)}(z) + \delta_{gb} C_{ga}^{(2)}(z) + H_g^{H;(1)} \left( \delta_{ga} C_{gb}^{(1)}(z) + \delta_{gb} C_{ga}^{(1)}(z) \right) \\ & + \left( C_{ga}^{(1)} \otimes C_{gb}^{(1)} \right)(z) + \left( G_{ga}^{(1)} \otimes G_{gb}^{(1)} \right)(z) .\end{aligned}$$

Only scale independent part

$$\begin{aligned}C_N \delta_{ga} \delta_{gb} \delta(1-z) \leftarrow & \delta_{ga} \delta_{gb} \delta(1-z) \left[ H_g^{H;(2)} \right]_{(\delta_{(1)}^{q_T})} \\ & + \delta_{ga} C_{gb}^{(2)}(z) + \delta_{gb} C_{ga}^{(2)}(z) + \left( G_{ga}^{(1)} \otimes G_{gb}^{(1)} \right)(z)\end{aligned}$$

Second order constant of “soft origin”, belonging to the finite part of  
the second-order structure of the IR singularities

$$H_g^{H;(2)} \equiv \tilde{H}_g^{H;(2)} + \left[ H_g^{H;(2)} \right]_{(\delta_{(1)}^{q_T})}$$

$$C_{N2} = 28 \pm 1$$

# NNLO DIFFERENTIAL DISTRIBUTIONS

## NNLO exercise

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

In order to assess the intrinsic uncertainty of the method we can repeat the numerical strategy, at one order less (NNLO) where all the coefficients are known analytically

The uncertainties at the differential distributions are at the 0.2% level!

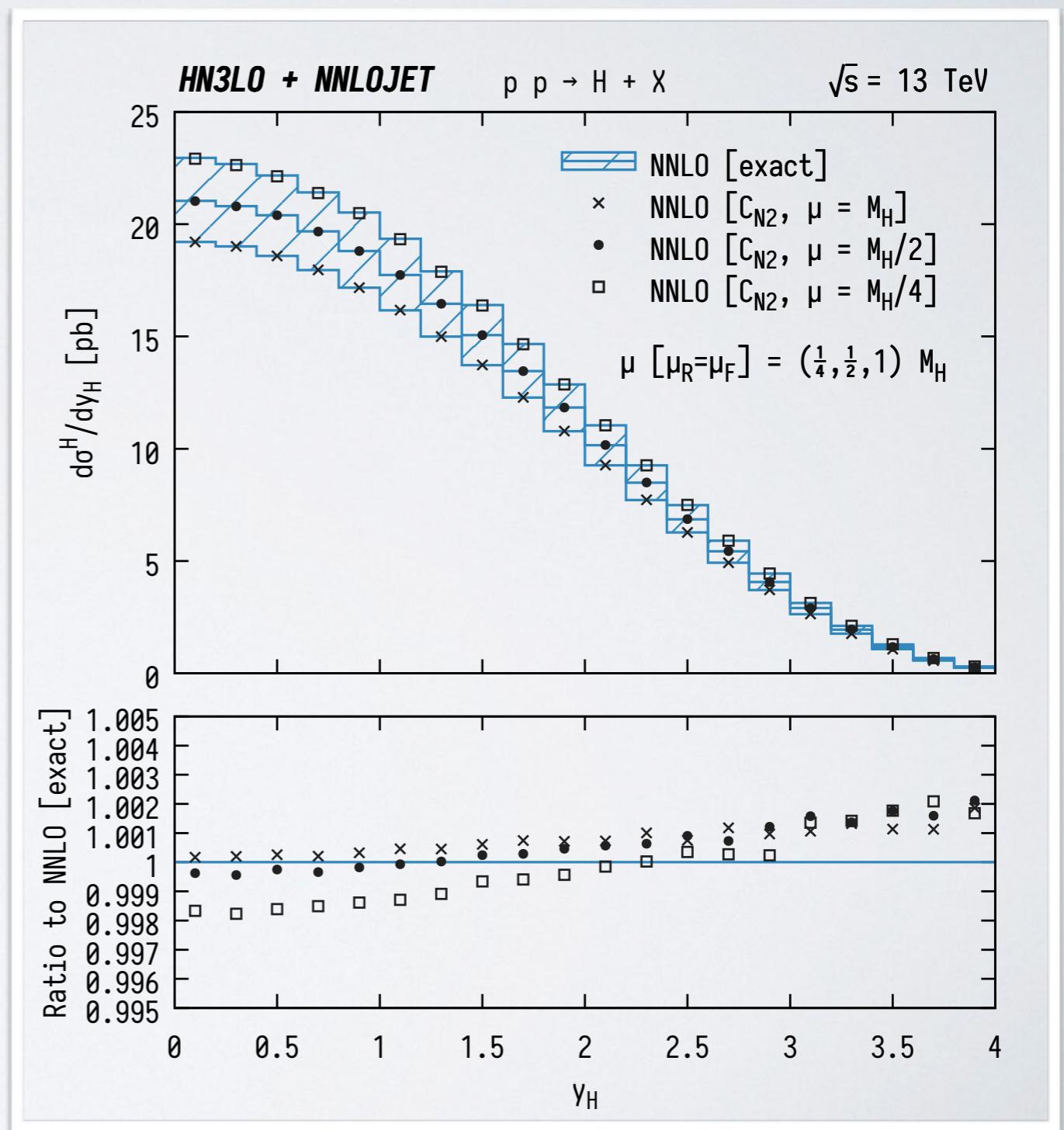
This is the expected level of precision in our approximation

With the exception of the scale  $M_H/4$ , the rest of the scales show their best behaviour in the low rapidity region and they start to separate from the exact result at large rapidities, which is what we expect.

$$\mathcal{H}_{c\bar{c} \leftarrow ab}^{F;(2)} \left( H_c^{F;(2)}; C_{ab}^{(2)}; G_{ga}^{(1)} \right) \leftarrow \mathcal{H}_{c\bar{c} \leftarrow ab}^{F;(2)} (C_{N2})$$

Approximated numerical extraction

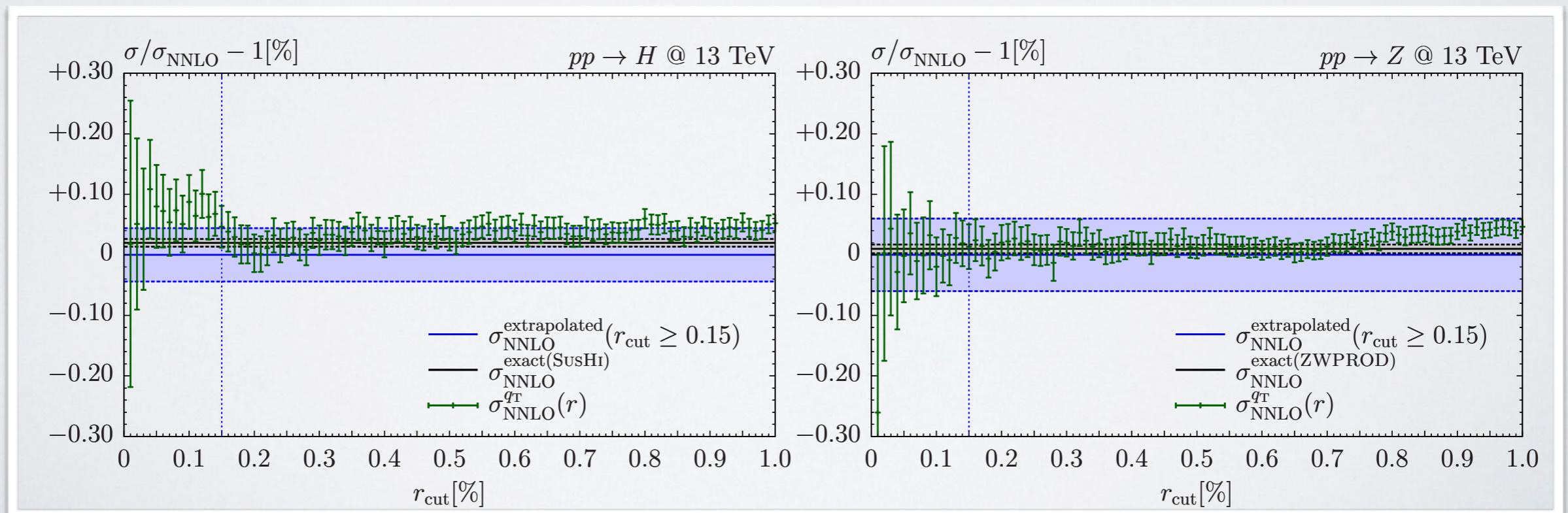
Exact numerical extraction



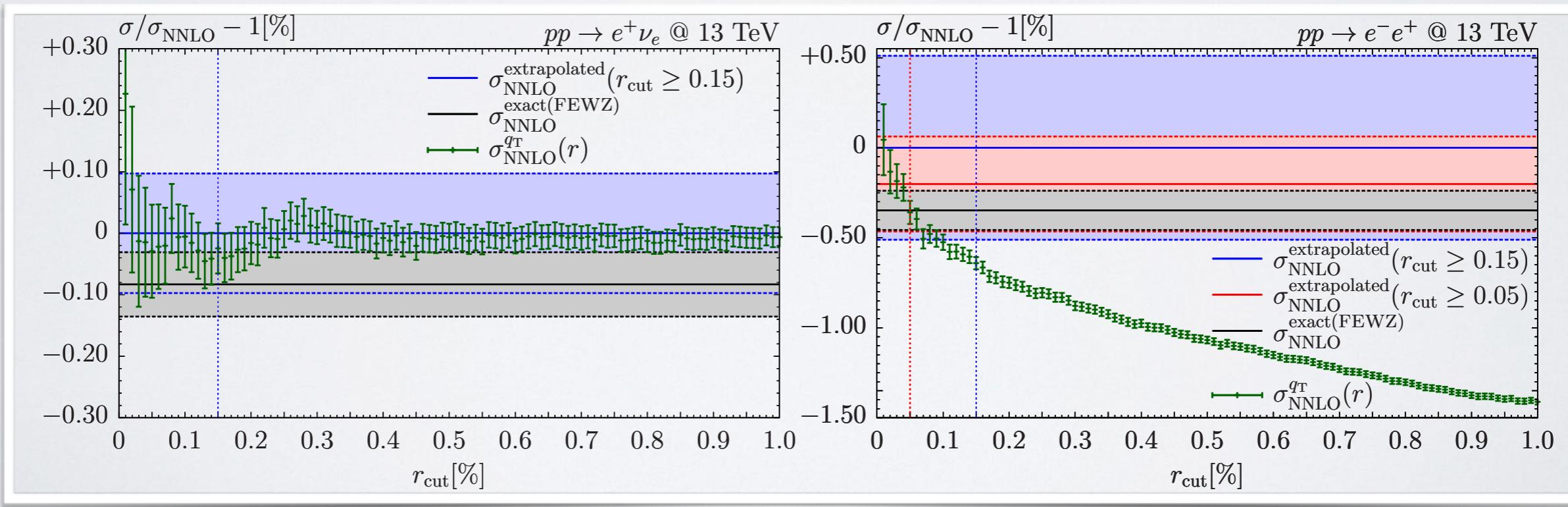
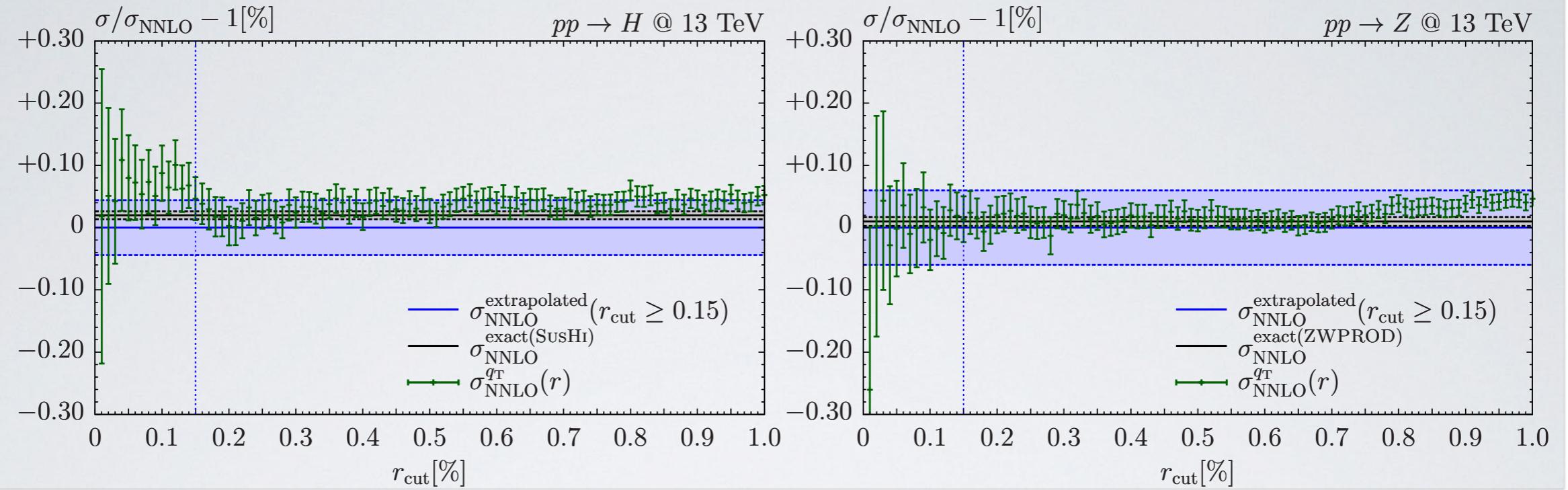
# N3LO DIFFERENTIAL DISTRIBUTIONS

# MOTIVATION

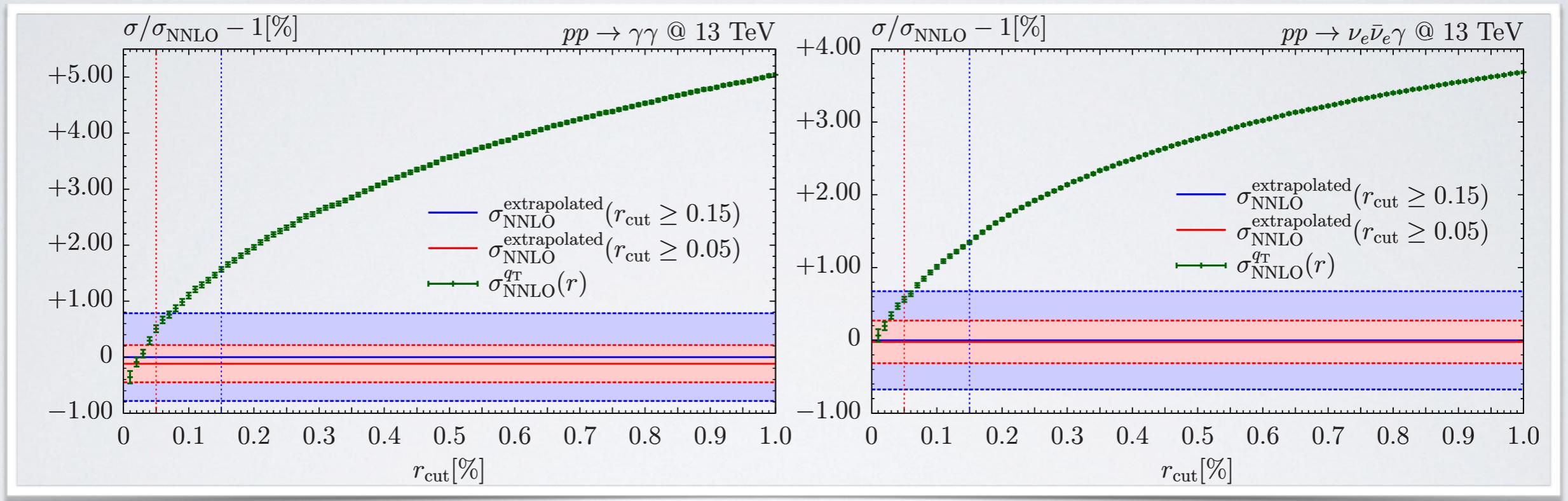
- Local subtractions are independent of any regularising parameter
- “Slicing” methods require the use of a cutoff to separate the different IR regions
- Such separation of the phase space introduces instabilities in the numerical evaluation of cross sections and differential distributions, and some care has to be taken in order to obtain stable and reliable results



# MOTIVATION

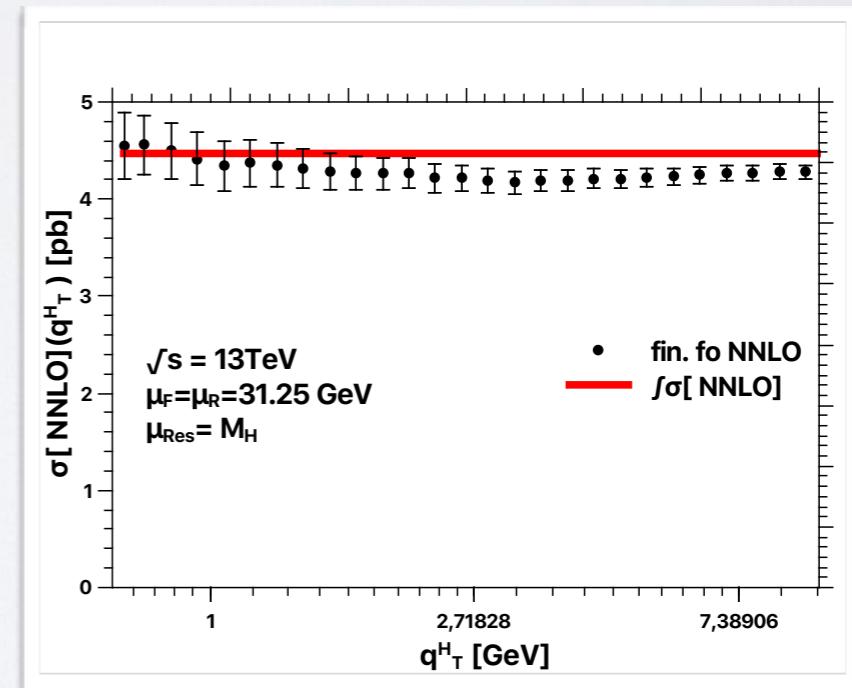
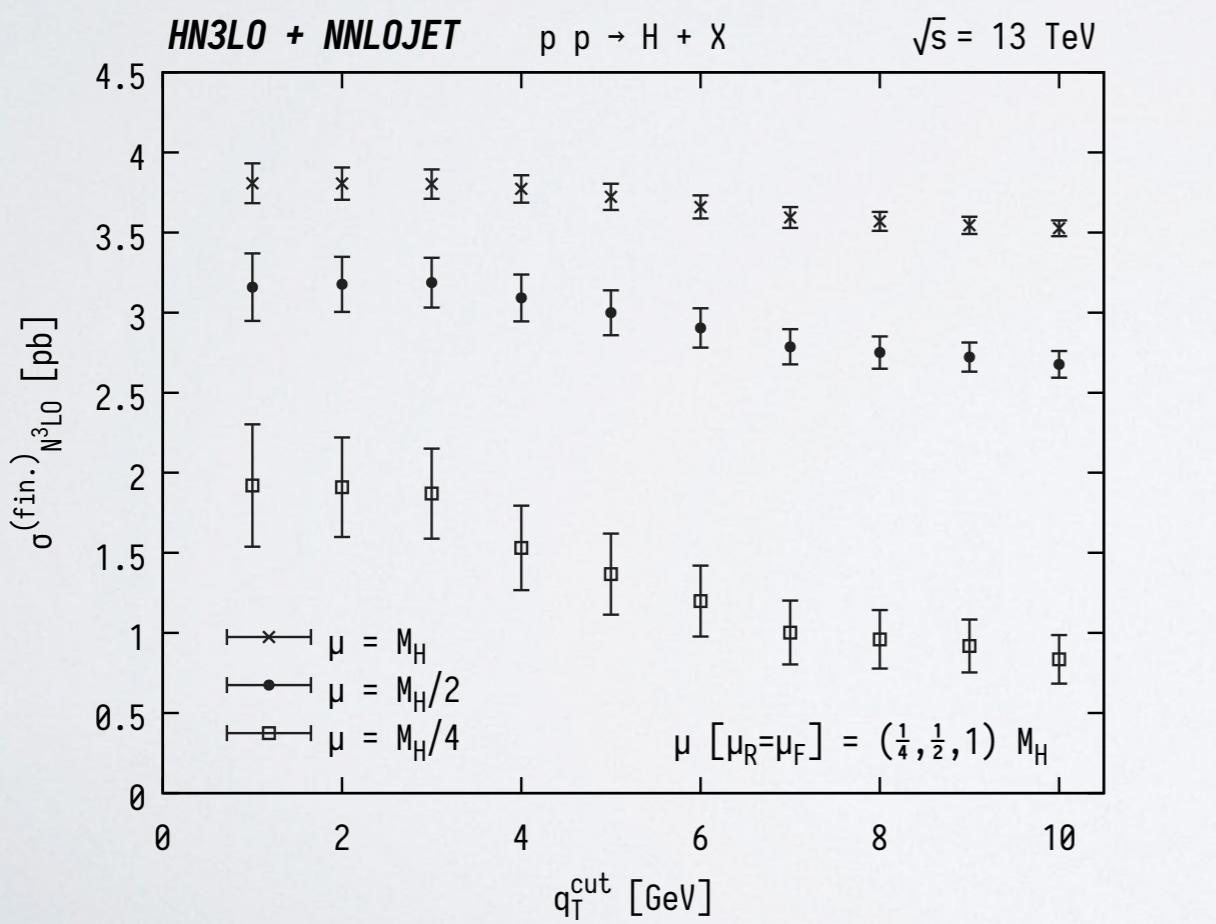
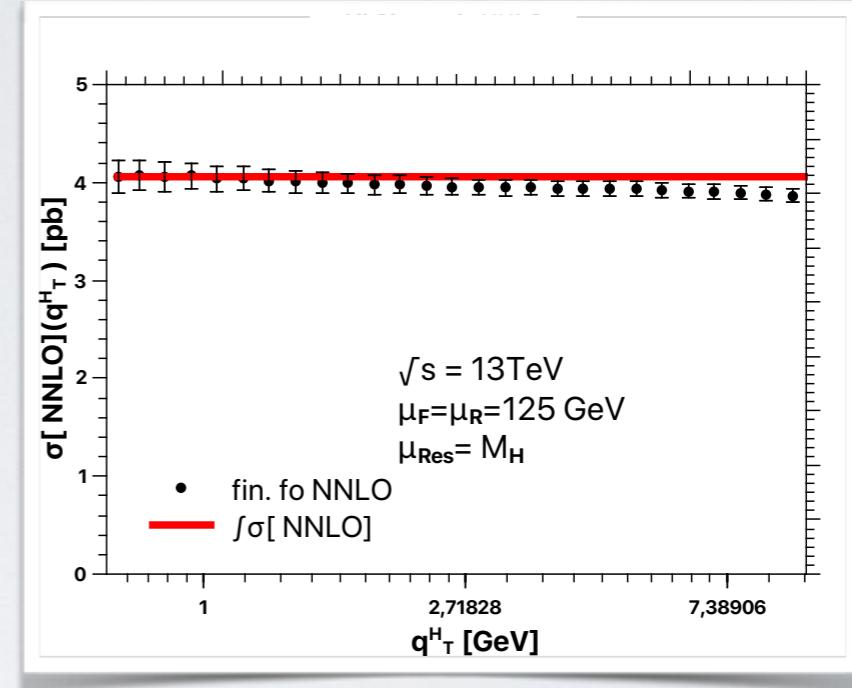
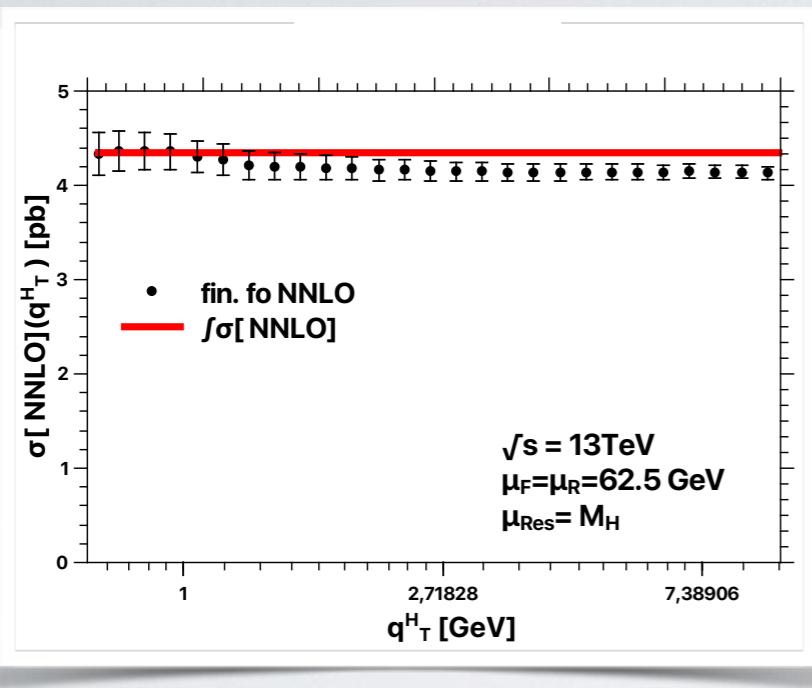


# MOTIVATION



M. Grazzini, S. Kallweit and M. Wiesemann (2017)

# MOTIVATION



# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

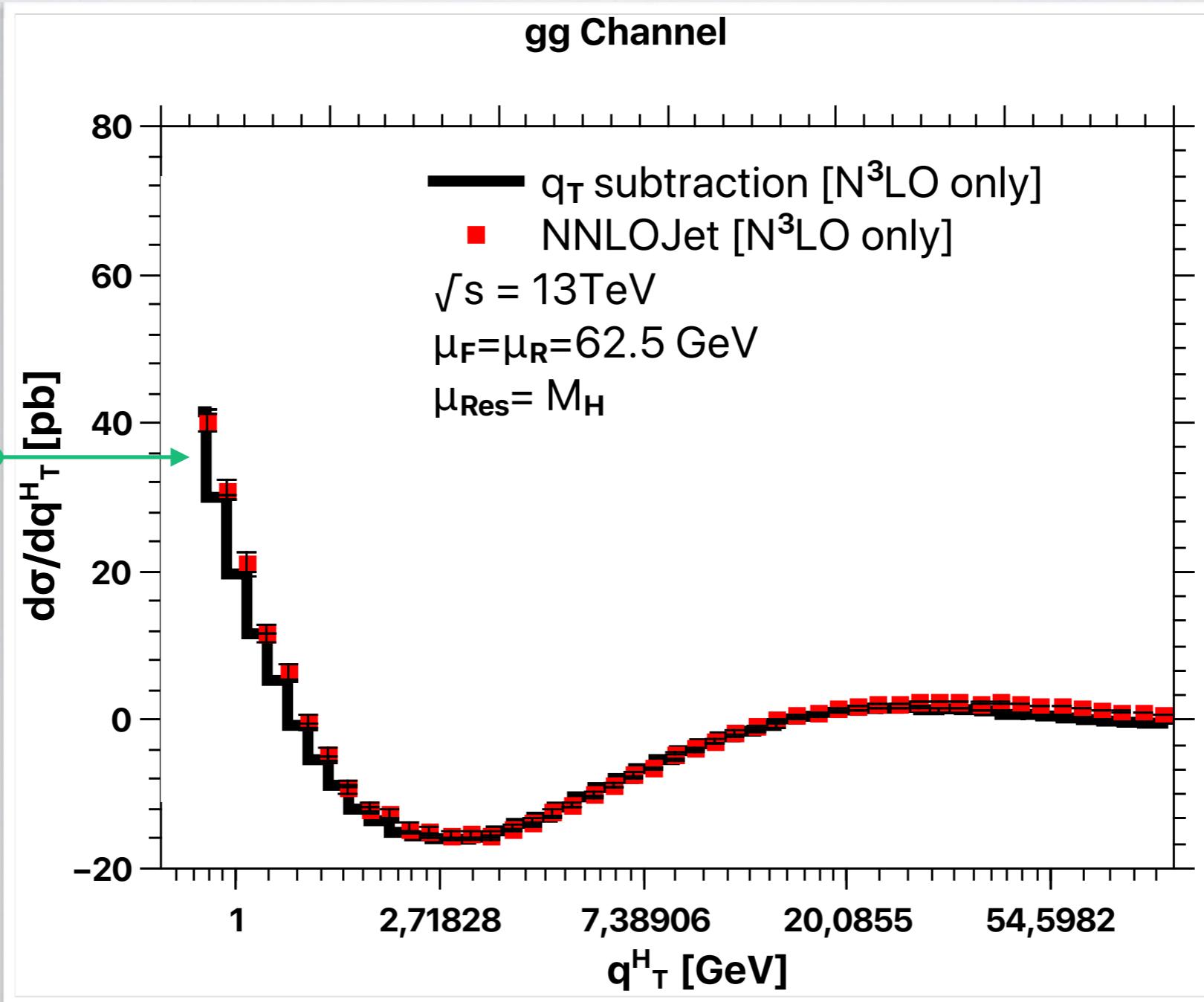
L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## First results divergent behaviour at the third-order

The presence of large logarithms in the small- $q_T$  limit requires  $q_T$ -resummation for this observable

The sign in the exponent of the Sudakov predicts the sign of the divergent behaviour

NLO > 0  
NNLO < 0  
N3LO > 0



Higgs boson production in gluon fusion at  $\text{Sqrt}[s] = 13\text{TeV}$  at the LHC, in the large  $m_{\text{top}}$  limit

The N3LO CT is computed without approximation

The CT is shown here at  $q_T > 0$

$$S_c(M, b) = \exp \left\{ - \int_{b_0^2/b^2}^{M^2} \frac{dq^2}{q^2} \left[ A_c(\alpha_s(q^2)) \ln \frac{M^2}{q^2} + B_c(\alpha_s(q^2)) \right] \right\}$$

$$\Sigma^F(q_T/Q) \xrightarrow{q_T \rightarrow 0} \sum_{n=1}^{\infty} \left( \frac{\alpha_S}{\pi} \right)^n \sum_{k=1}^{2n} \Sigma^{F(n;k)} \frac{Q^2}{q_T^2} \ln^{k-1} \frac{Q^2}{q_T^2}$$

# THE $N^3LO$ QT-SUBTRACTION METHOD

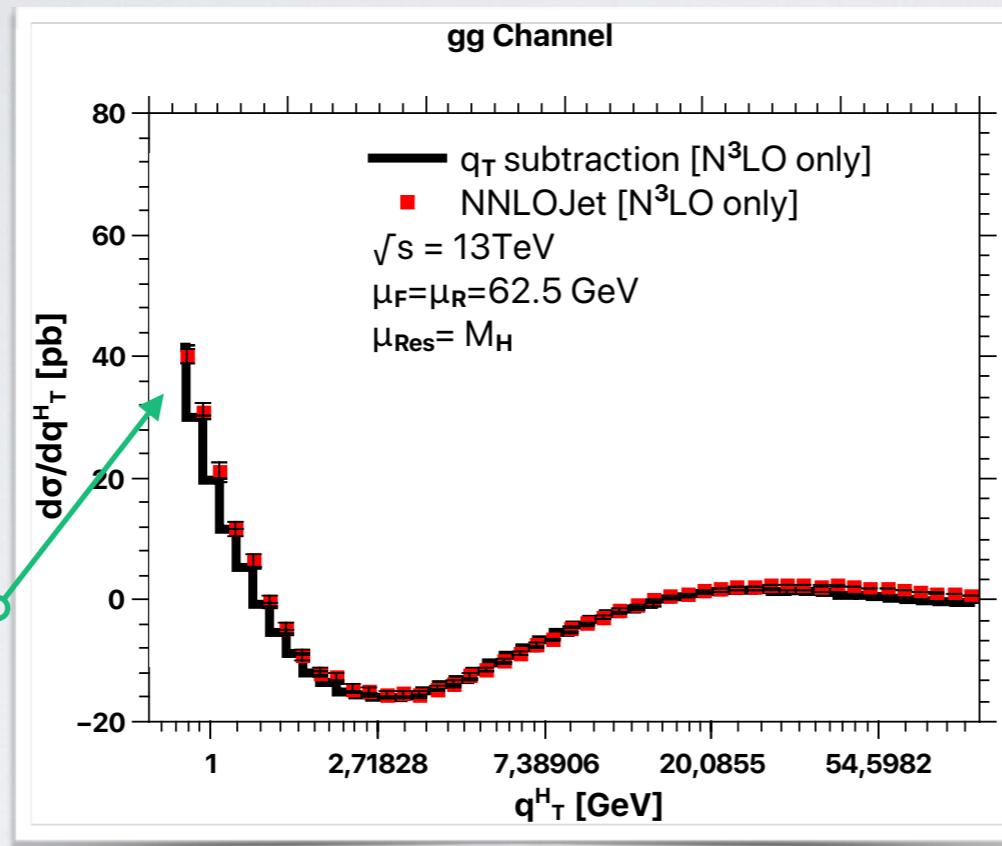
L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## First results divergent behaviour at the third-order

The presence of large logarithms in the small- $q_T$  limit requires  $q_T$ -resummation for this observable

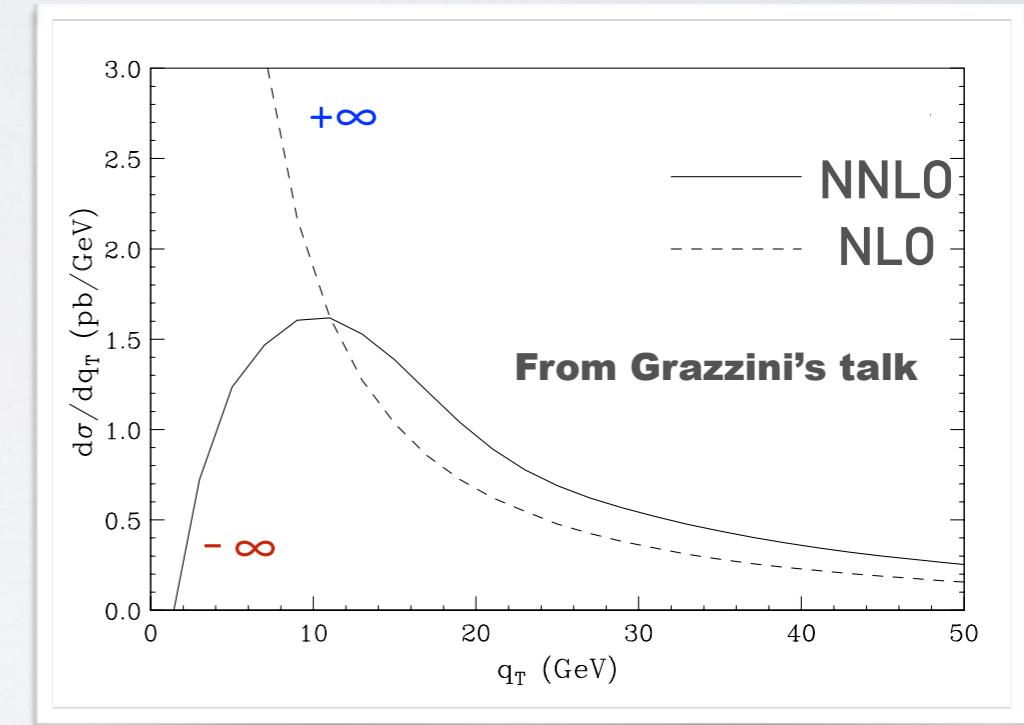
The sign in the exponent of the Sudakov predicts the sign of the divergent behaviour

NLO  $> 0$   
NNLO  $< 0$   
N3LO  $> 0$



$$\begin{aligned} \text{NLO: } & \frac{d\sigma}{dq_T} \rightarrow +\infty \quad \text{as} \quad q_T \rightarrow 0 \\ \text{NNLO: } & \frac{d\sigma}{dq_T} \rightarrow -\infty \quad \text{as} \quad q_T \rightarrow 0 \\ \text{N}^3\text{LO: } & \frac{d\sigma}{dq_T} \rightarrow +\infty \quad \text{as} \quad q_T \rightarrow 0 \end{aligned}$$

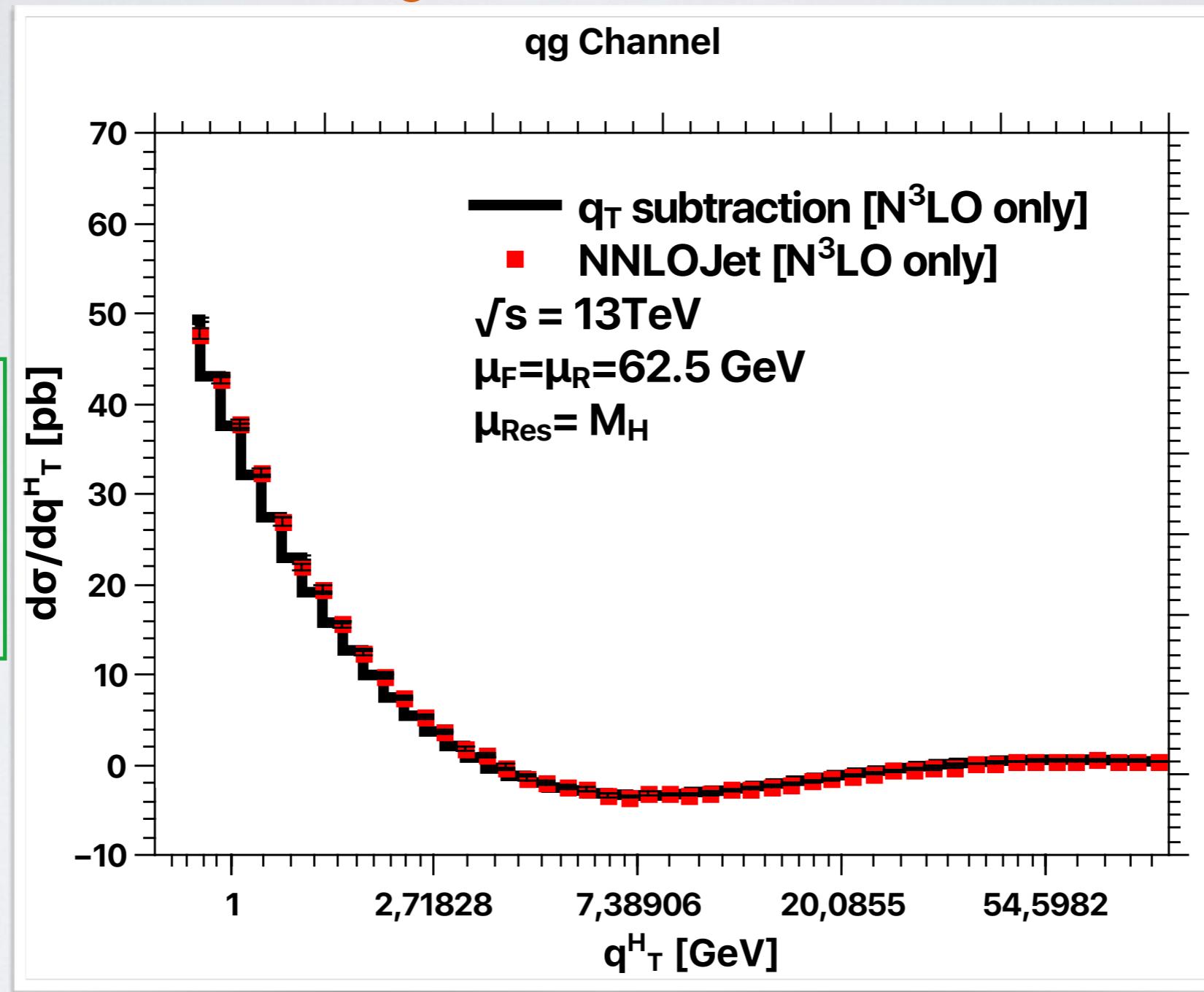
$$S_c(M, b) = \exp \left\{ - \int_{b_0^2/b^2}^{M^2} \frac{dq^2}{q^2} \left[ A_c(\alpha_s(q^2)) \ln \frac{M^2}{q^2} + B_c(\alpha_s(q^2)) \right] \right\}$$



# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## First results divergent behaviour at the third-order



In the off-diagonal channels, the first power (most divergent) in the logarithmic expansion is not present.

Higgs boson production in gluon fusion at  $\text{Sqrt}[s] = 13 \text{ TeV}$  at the LHC, in the large  $m_{\text{top}}$  limit

The N3LO CT is computed without approximation

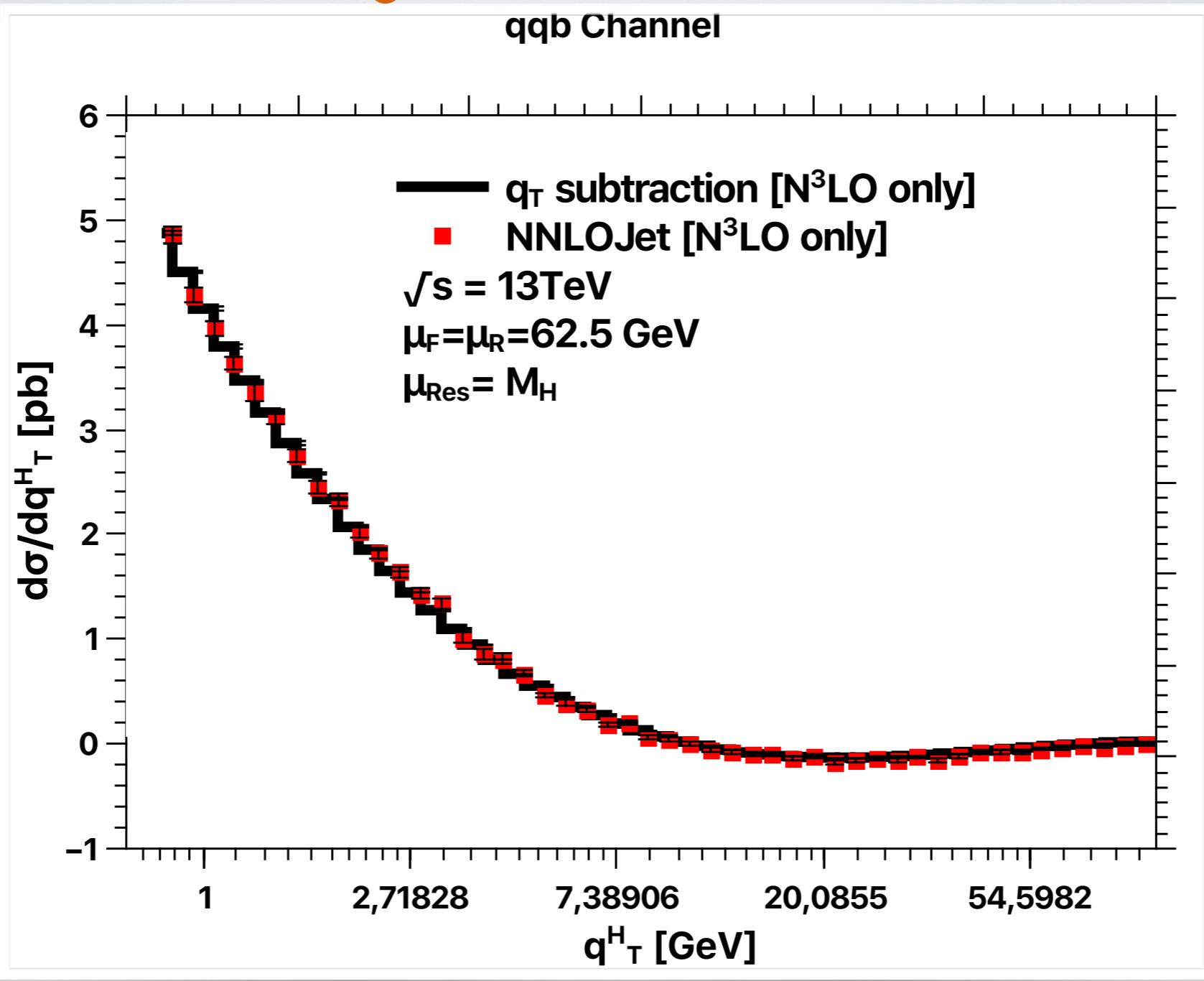
The CT is shown here at  $q_T > 0$  !

$$\Sigma^F(q_T/Q) \xrightarrow[q_T \rightarrow 0]{} \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n \sum_{k=1}^{2n} \Sigma^{F(n;k)} \frac{Q^2}{q_T^2} \ln^{k-1} \frac{Q^2}{q_T^2}$$

# THE N<sup>3</sup>LO QT-SUBTRACTION METHOD

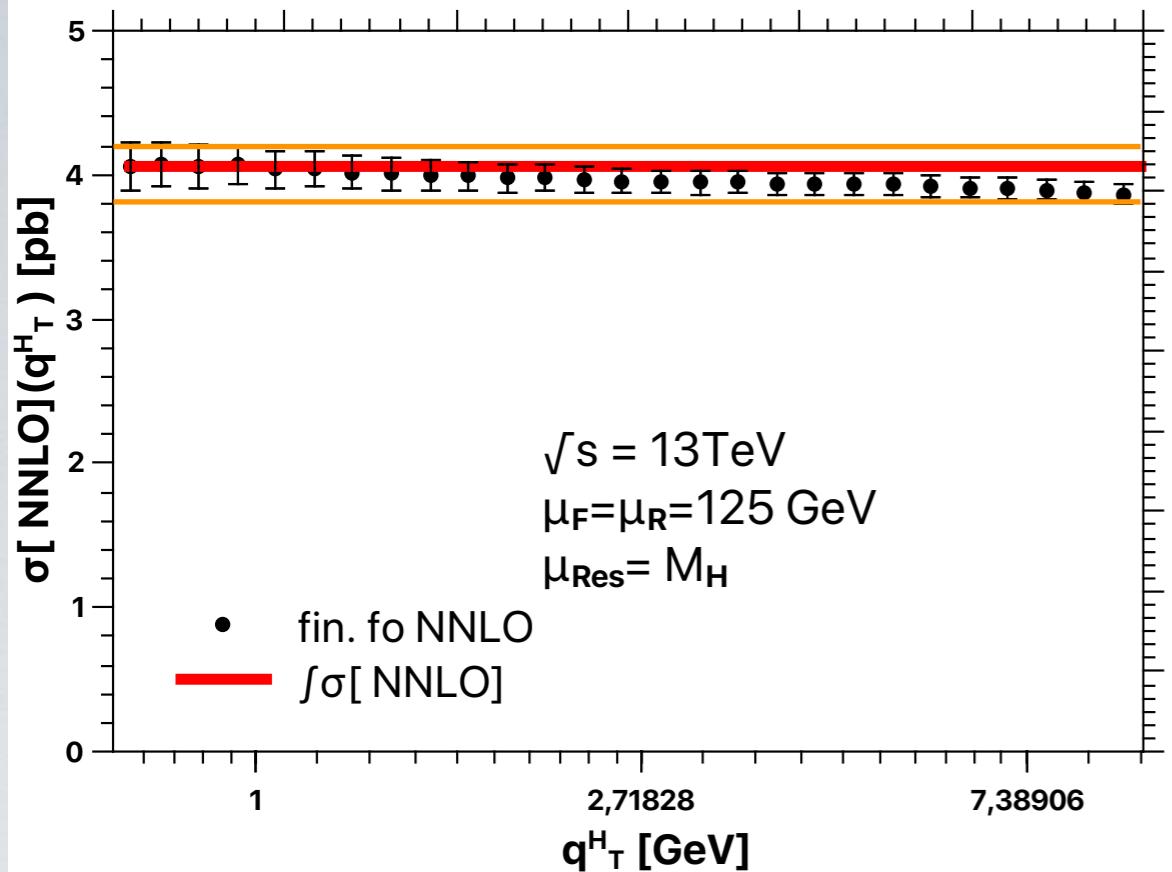
L. C, X. Chen, T. Gehrmann, E. W. N. Glover, and A. Huss (2018)

## First results divergent behaviour at the third-order

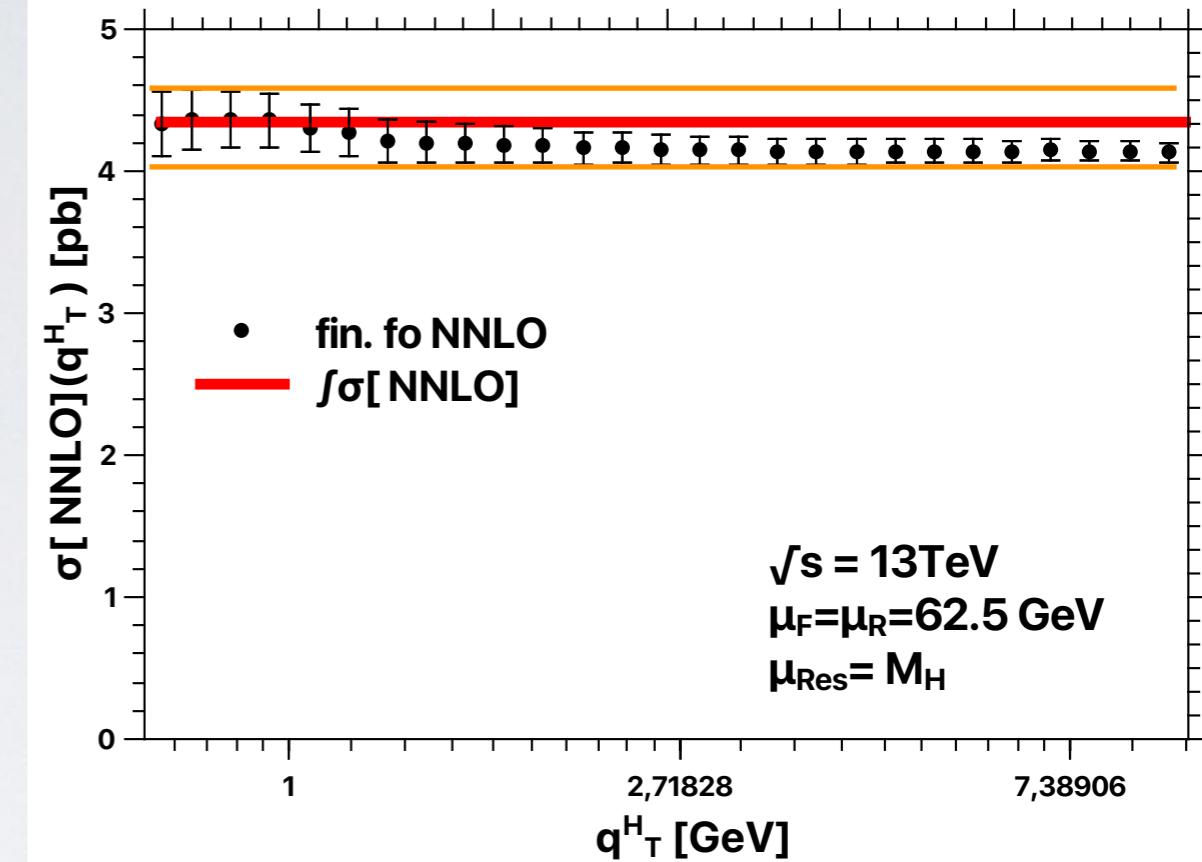


$$\Sigma^F(q_T/Q) \xrightarrow[q_T \rightarrow 0]{} \sum_{n=1}^{\infty} \left(\frac{\alpha_S}{\pi}\right)^n \sum_{k=1}^{2n} \Sigma^{F(n;k)} \frac{Q^2}{q_T^2} \ln^{k-1} \frac{Q^2}{q_T^2}$$

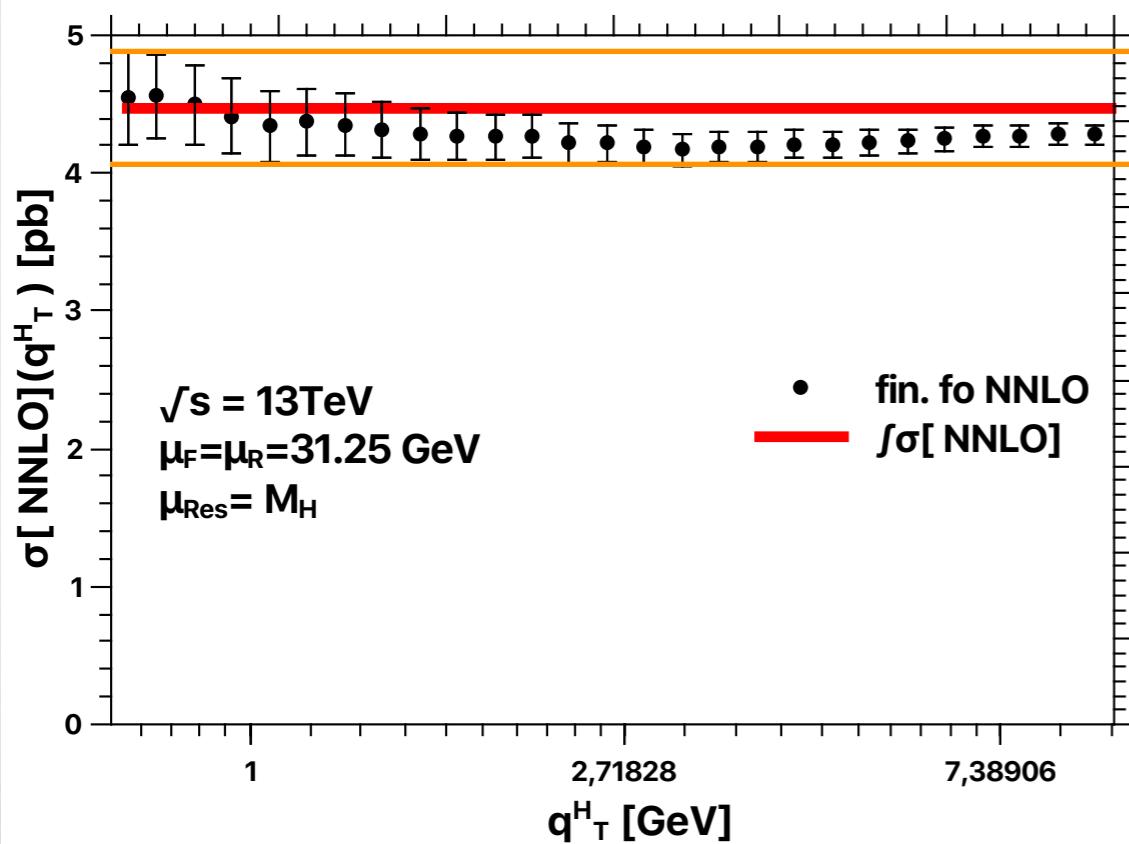
### All Channels NNLO



### All Channels



### All Channels



	$\sigma_{\text{NNLO}} [\text{pb}]$ Best	$\sigma_{\text{NNLO}} [\text{pb}]$ Worst
$\mu = M_H/2$	$43.47(\pm 0.3\%)$	$43.47(\pm 0.7\%)$
$\mu = M_H$	$39.64(\pm 0.2\%)$	$39.64(\pm 0.5\%)$
$\mu = M_H/4$	$47.33(\pm 0.4\%)$	$39.64(\pm 0.84\%)$

The variation is equivalent to  $\text{rcut} = (0.4; 8)\%$

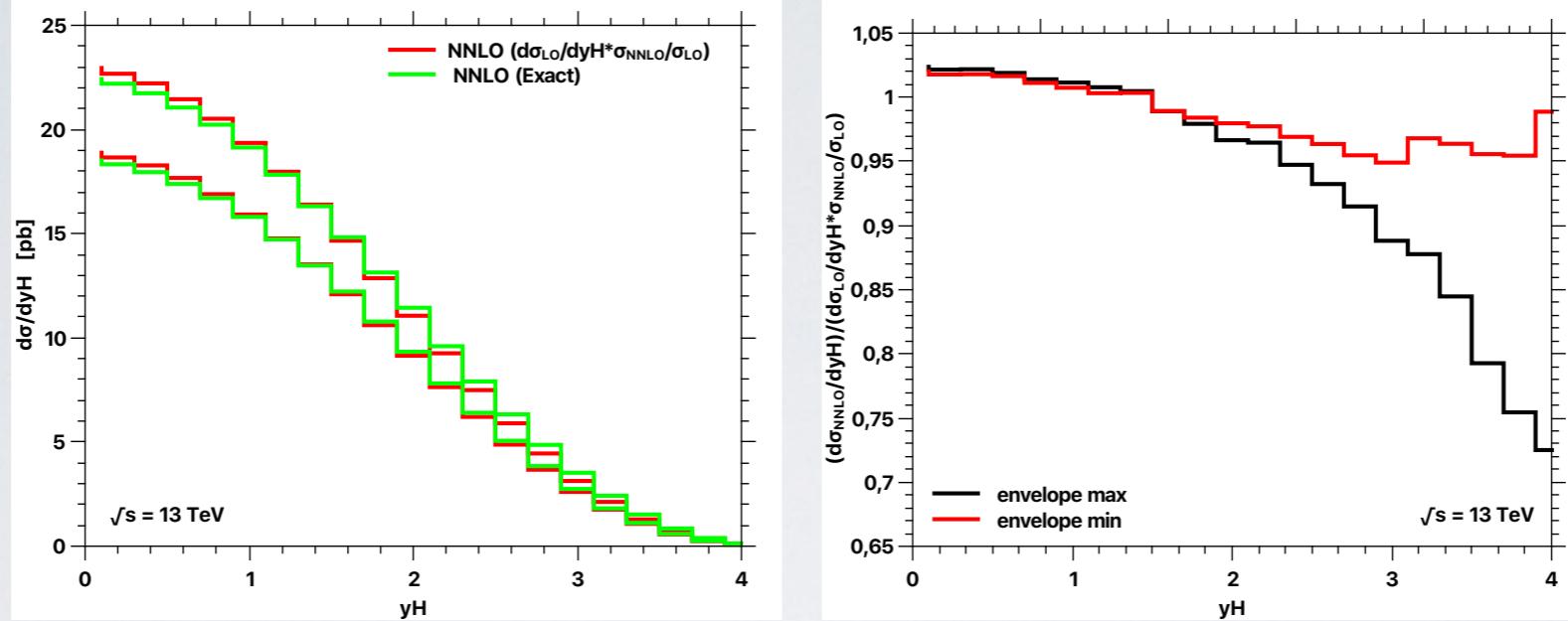


Figure A: The rapidity distribution of the Higgs boson at NNLO. We compare the exact result with the proposed approximation in the referee report detailed in Eq. (3). We use the 7-point scale variation as explained in the paper.

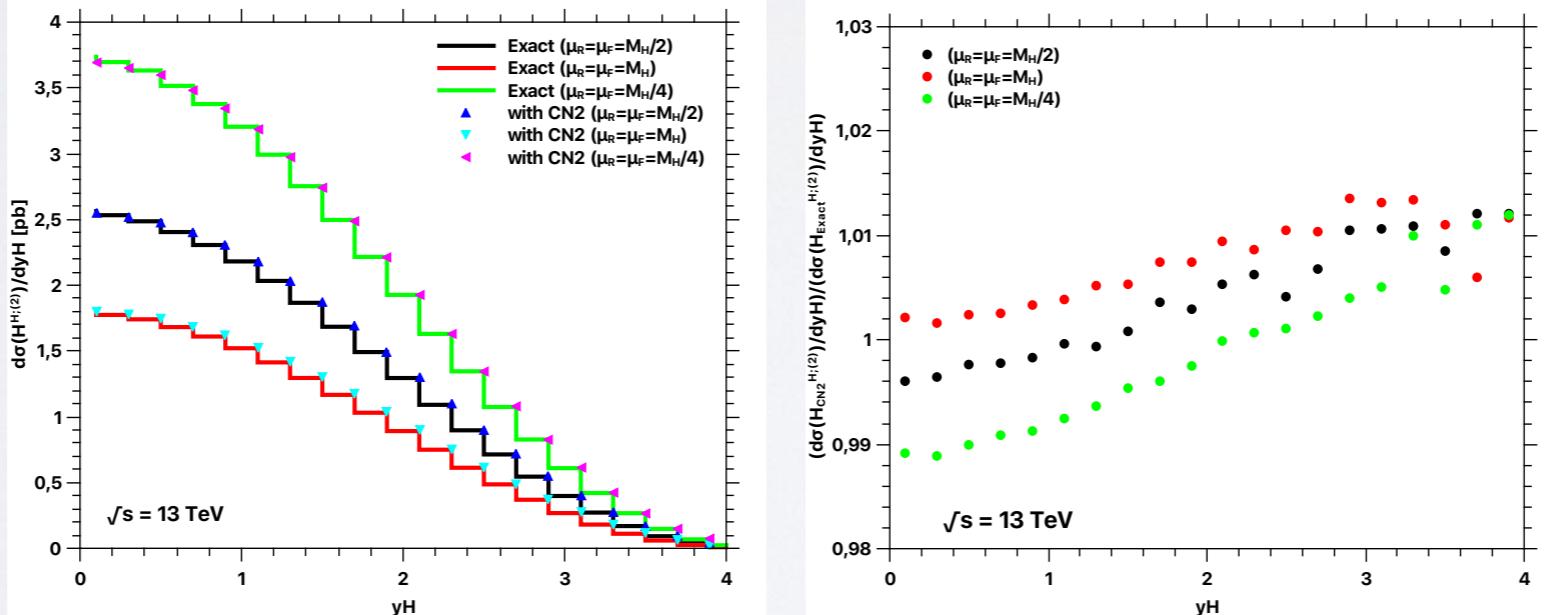


Figure D: The rapidity distribution of the coefficient functions  $\mathcal{H}_{\text{CN2}}^{H;(2)}$  and  $\mathcal{H}_{\text{exact}}^{H;(2)}$  and the corresponding ratio plot. For this particular example at NNLO, we employ the three-point scale variation:  $\mu = \mu_R = \mu_F = \{M_H/4, M_H/2, M_H\}$ .

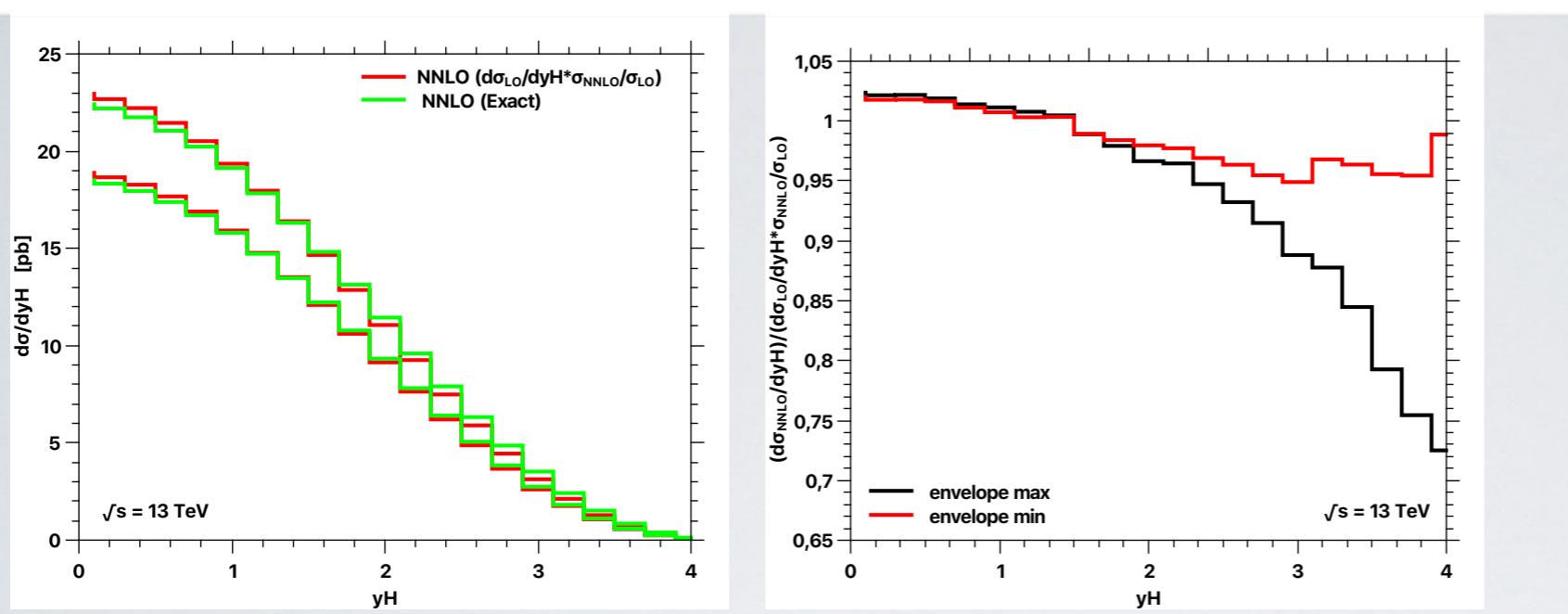
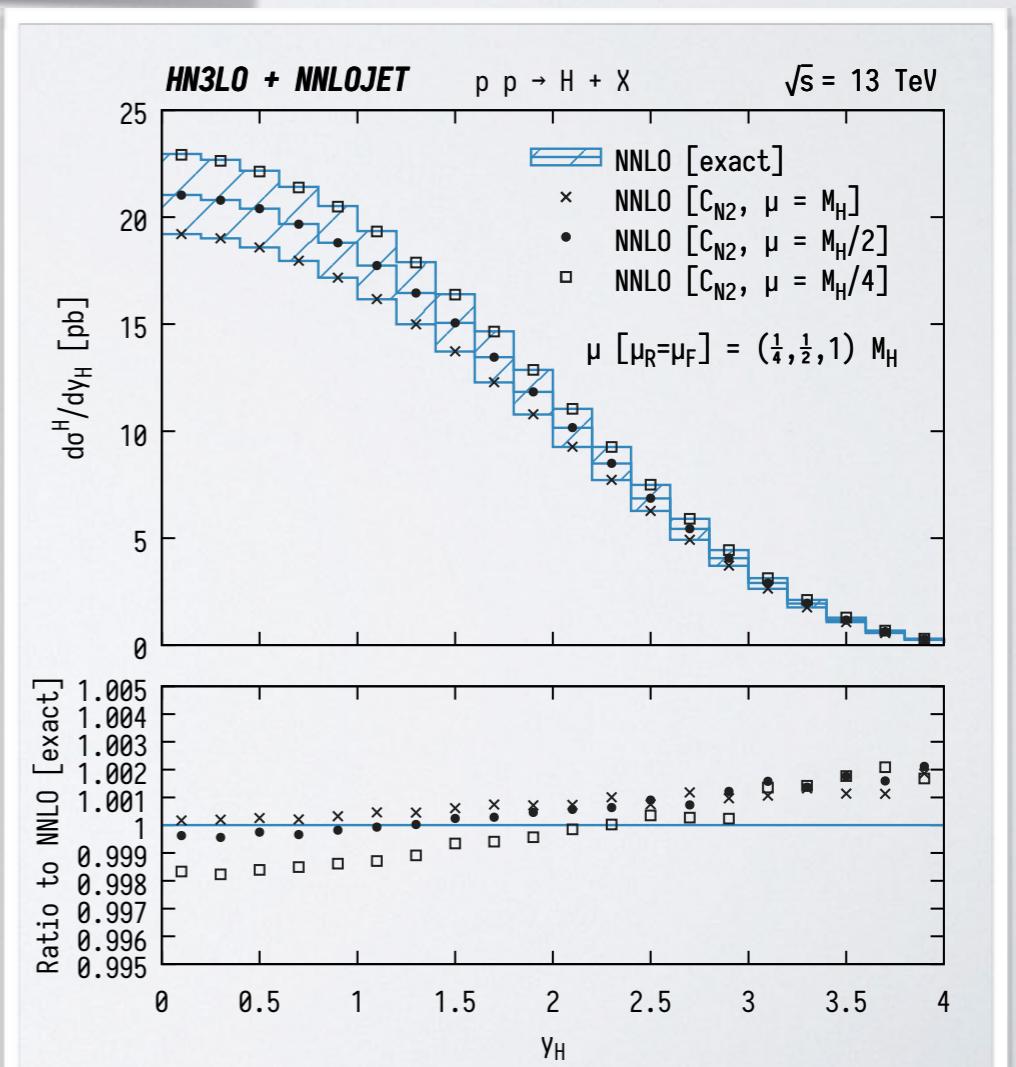


Figure A: The rapidity distribution of the Higgs boson at NNLO. We compare the exact result with the proposed approximation in the referee report detailed in Eq. (3). We use the 7-point scale variation as explained in the paper.

The uncertainties at the differential distributions are at the 0.2% level!  
 Therefore concerning a global rescaling, we are comparing a 0.2% level of precision vs. a 30% of deviation from the exact result



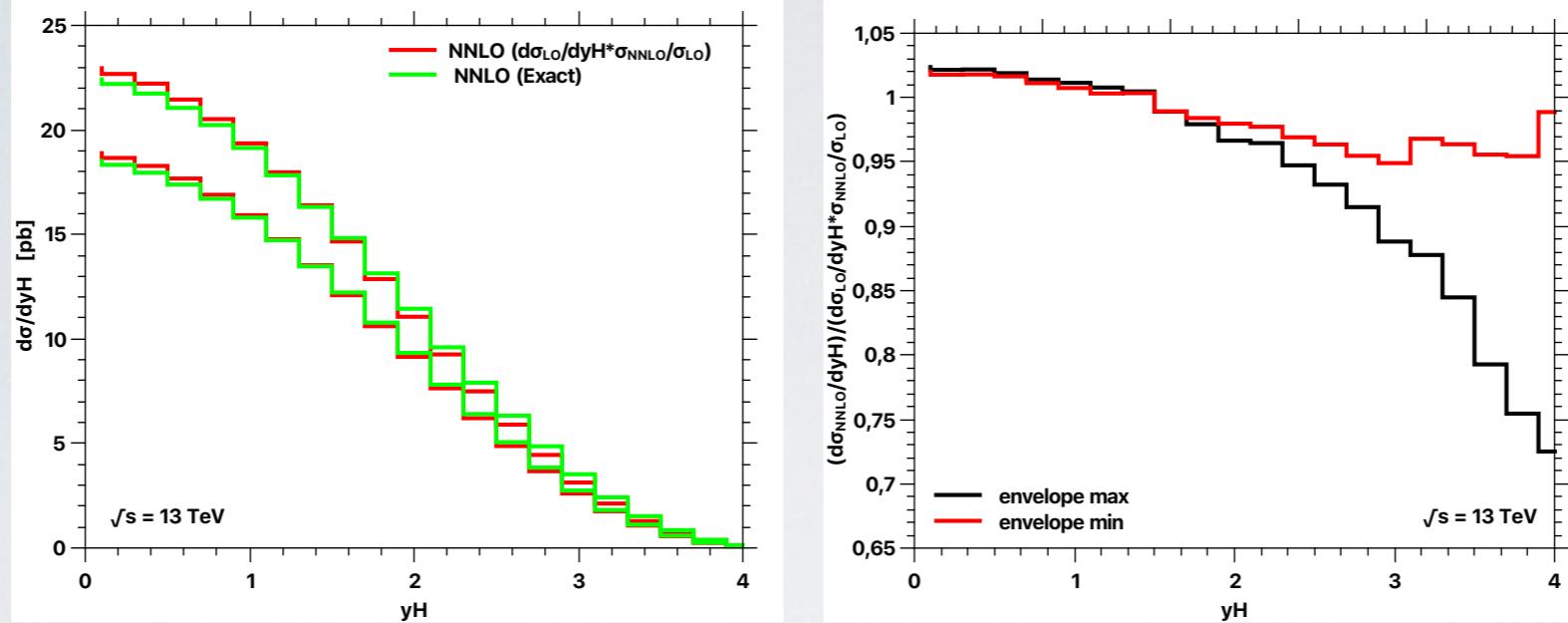


Figure A: The rapidity distribution of the Higgs boson at NNLO. We compare the exact result with the proposed approximation in the referee report detailed in Eq. (3). We use the 7-point scale variation as explained in the paper.

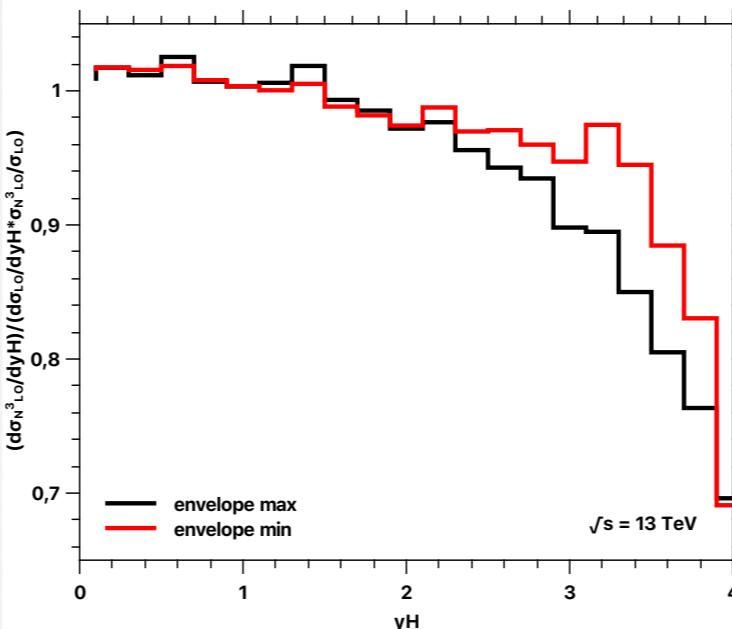


Figure B: Ratio between the  $N^3\text{LO}$  rapidity distribution as calculated in the paper (red band of Fig. 5 in the paper) and the suggested approximation in the referee report detailed in Eq. (3). We use the 7-point scale variation as described in the paper.