FACTORISATION TOOLS FOR INFRARED SUBTRACTION

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Outline

- Introduction
- Algorithms
- Factorisation
- Counterterms
- Outlook

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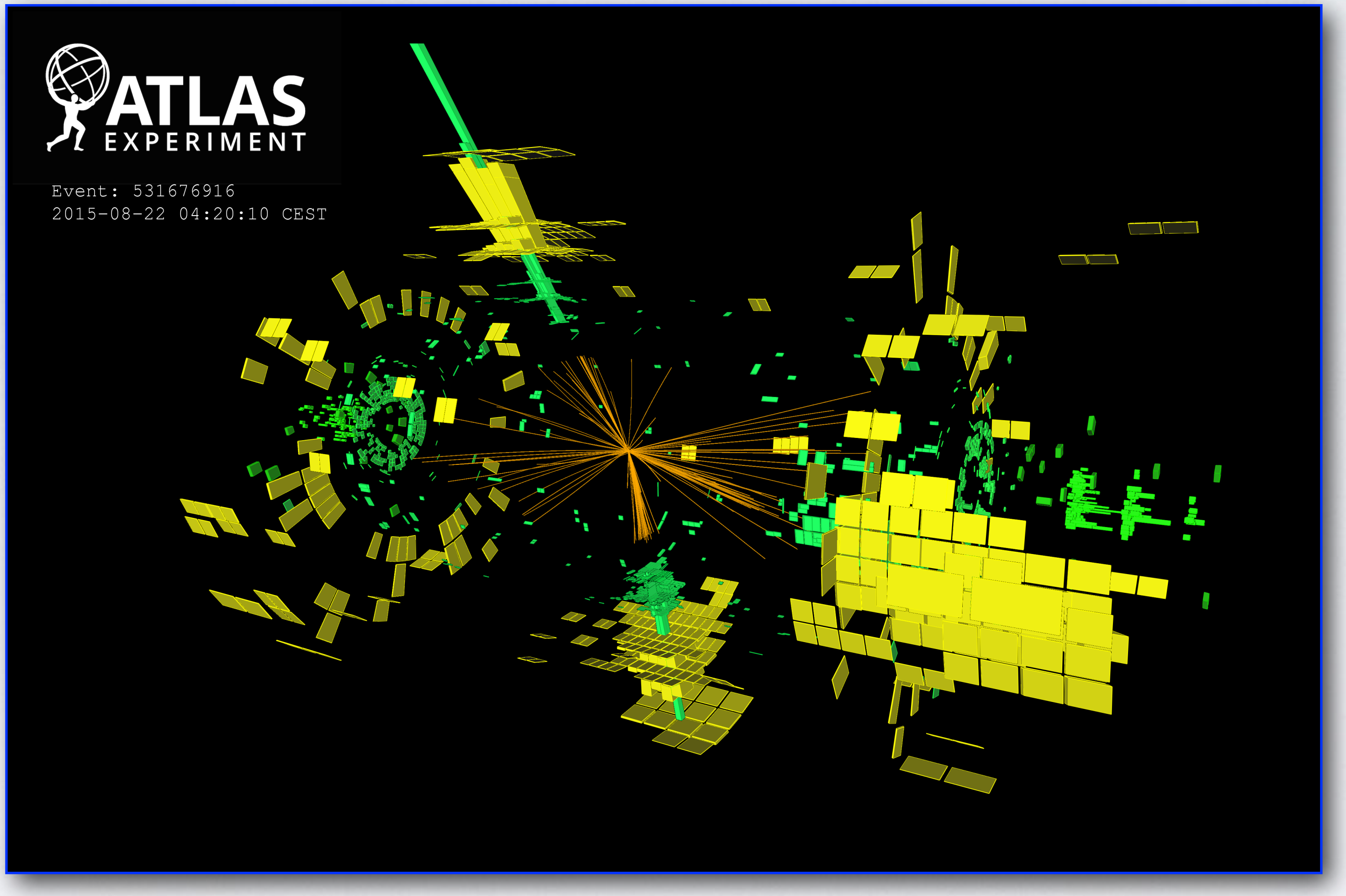
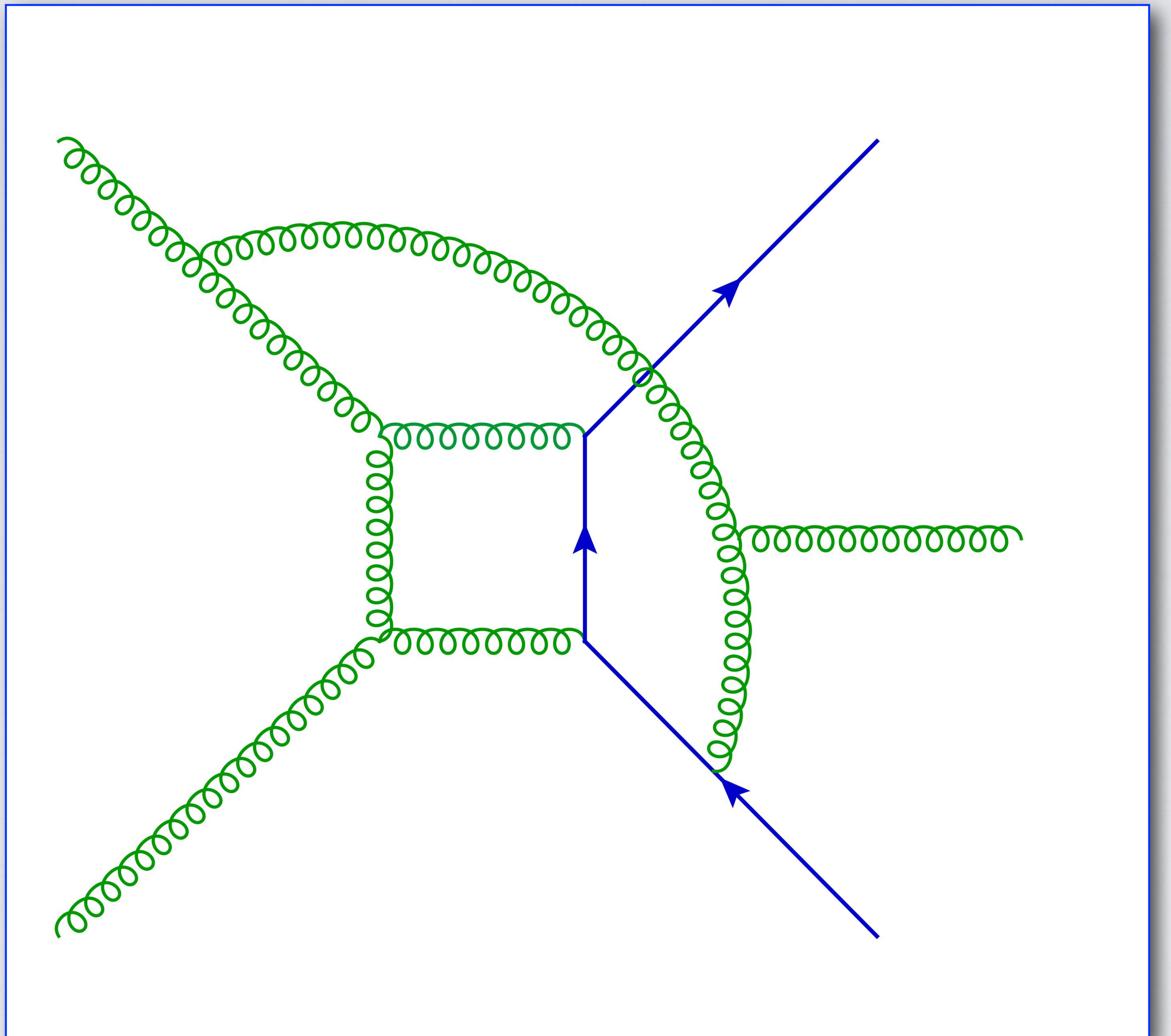
INTRODUCTION

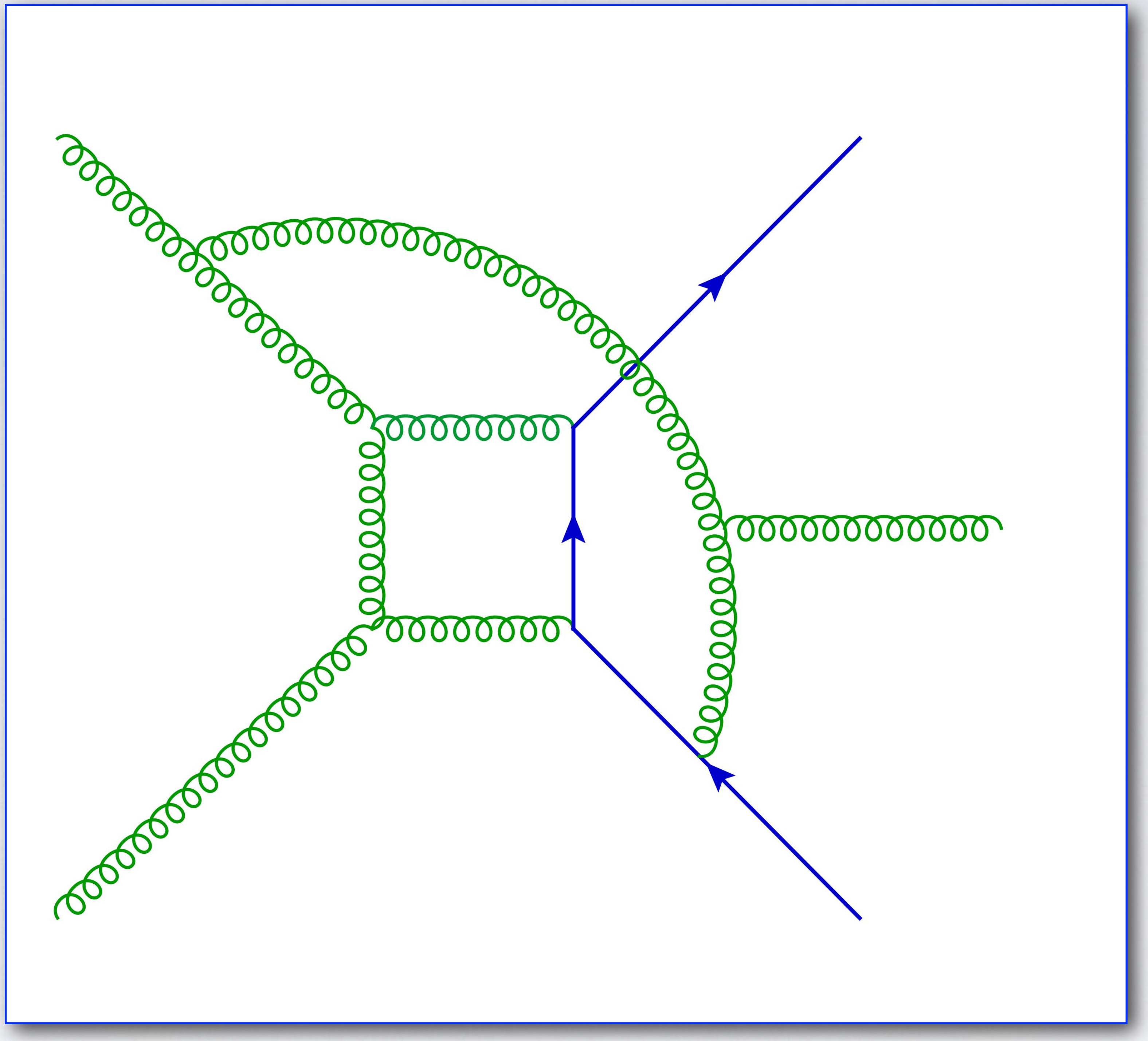


Pictorial infrared



A diagram contributing a double-virtual NNLO correction to t-tbar-jet production

Pictorial infrared



$$\frac{1}{\epsilon^4}$$

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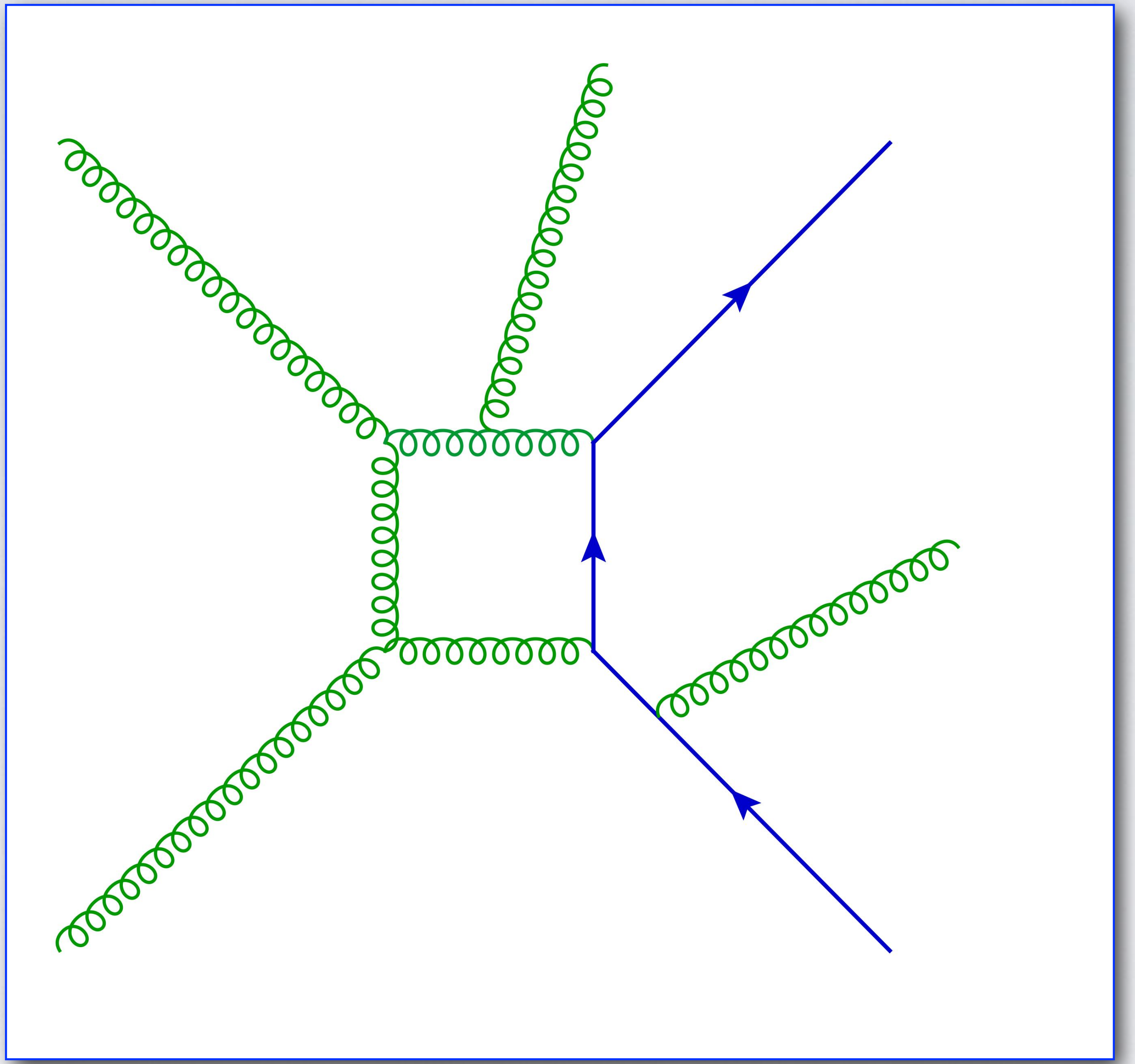

A diagram contributing a real-virtual NNLO correction to t-tbar-jet production

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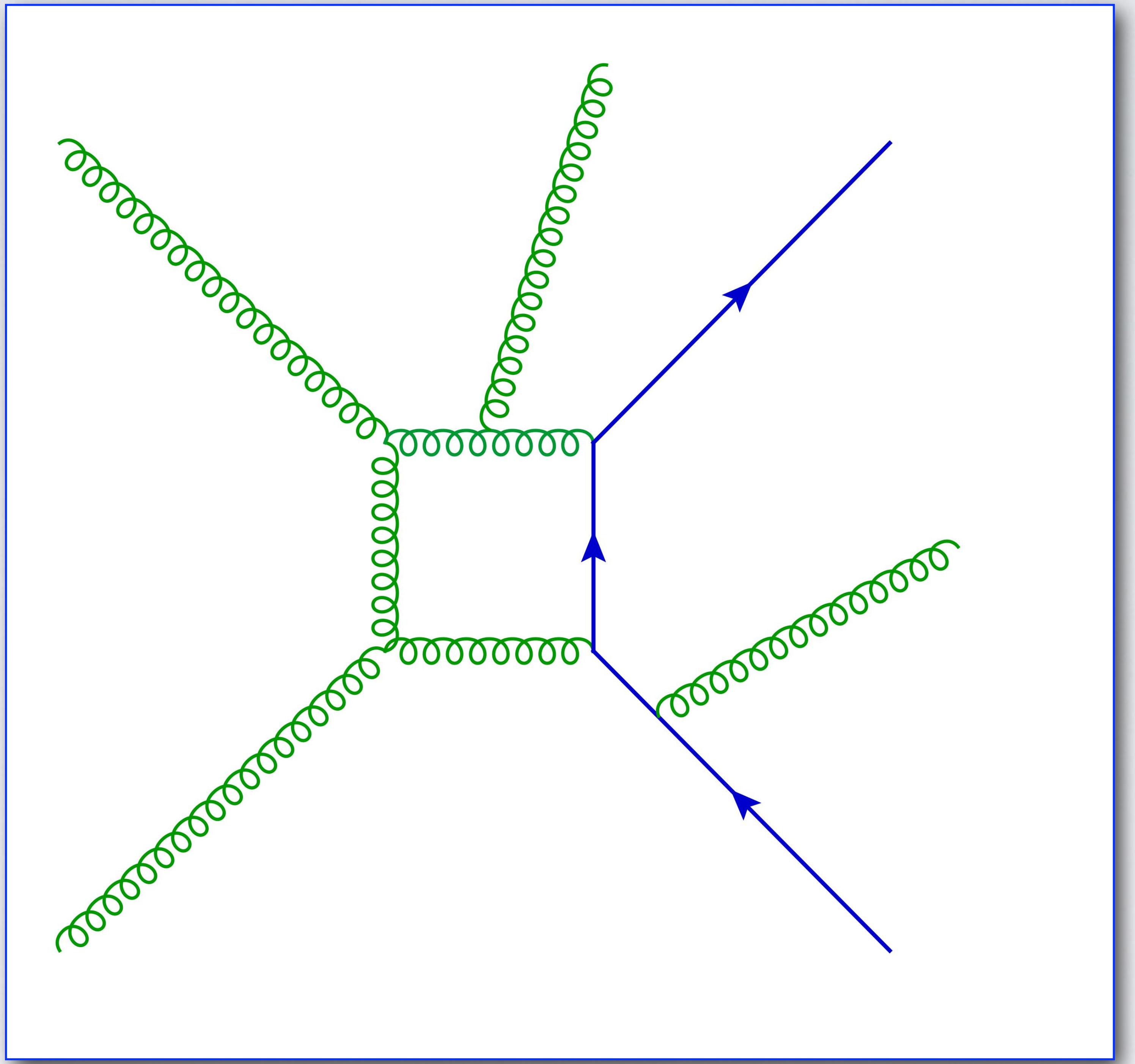

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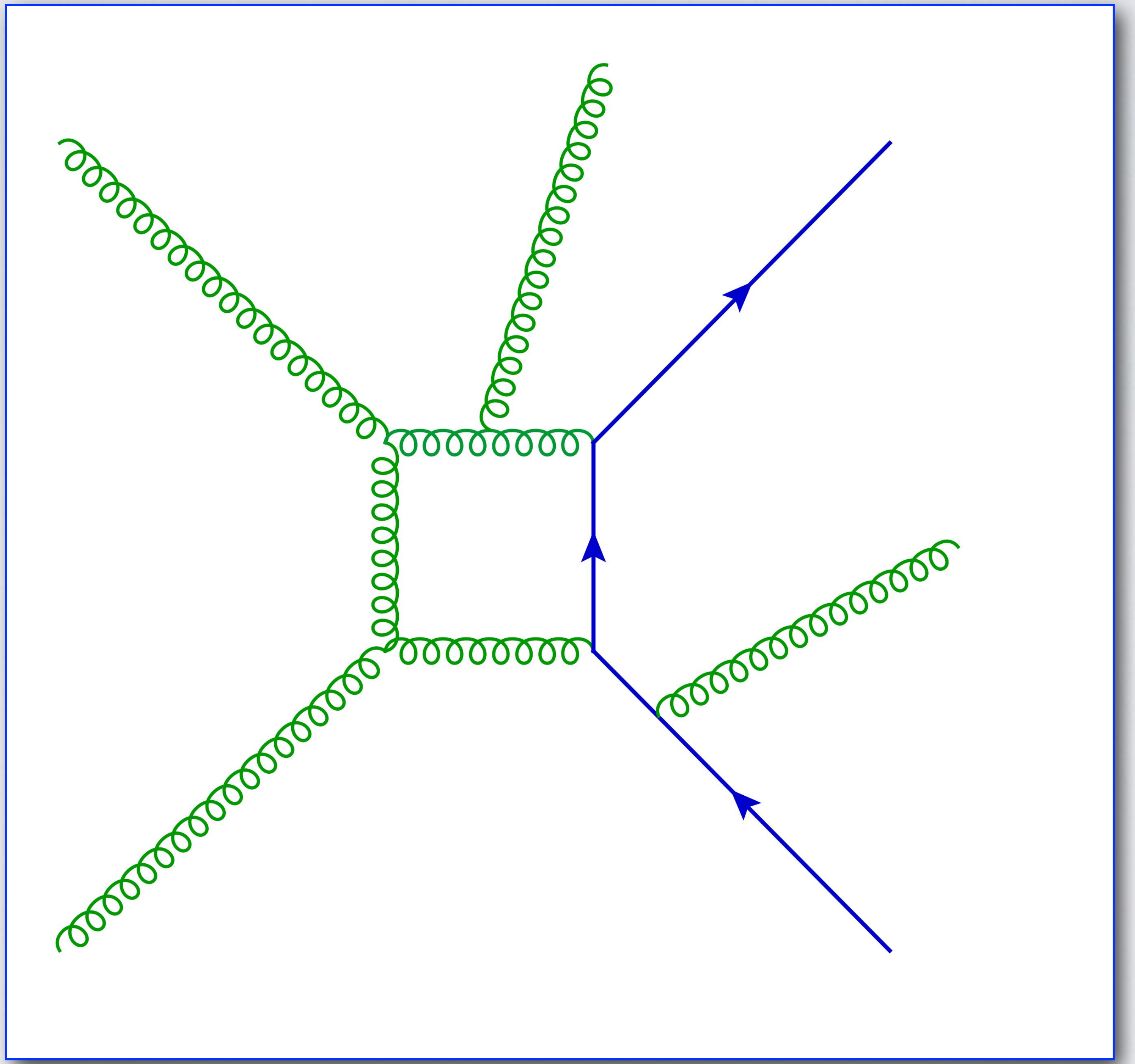

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$$\frac{dE}{E}\,\frac{dk_\perp}{k_\perp}$$

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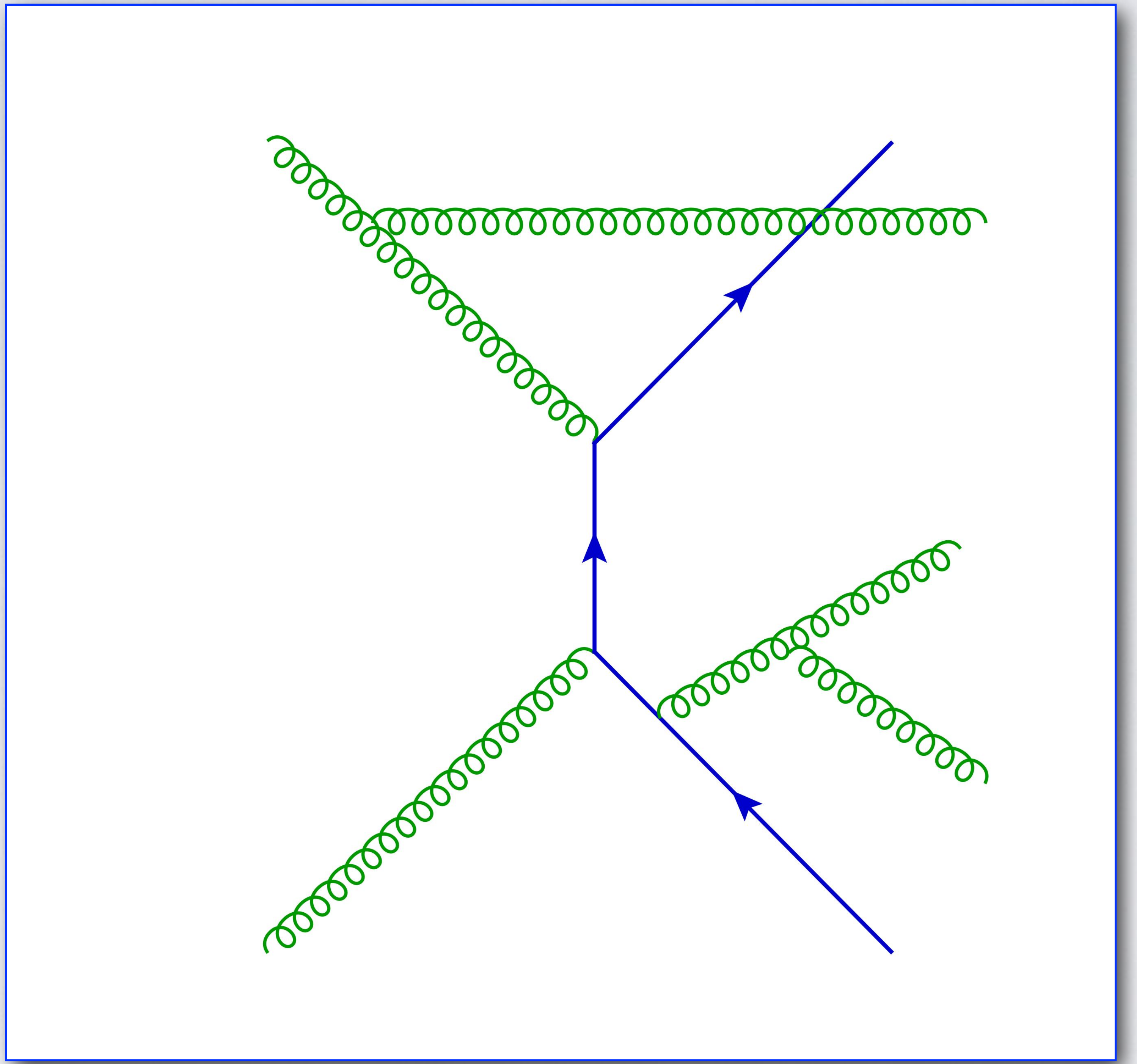

A diagram contributing a double-real NNLO correction to t-tbar-jet production

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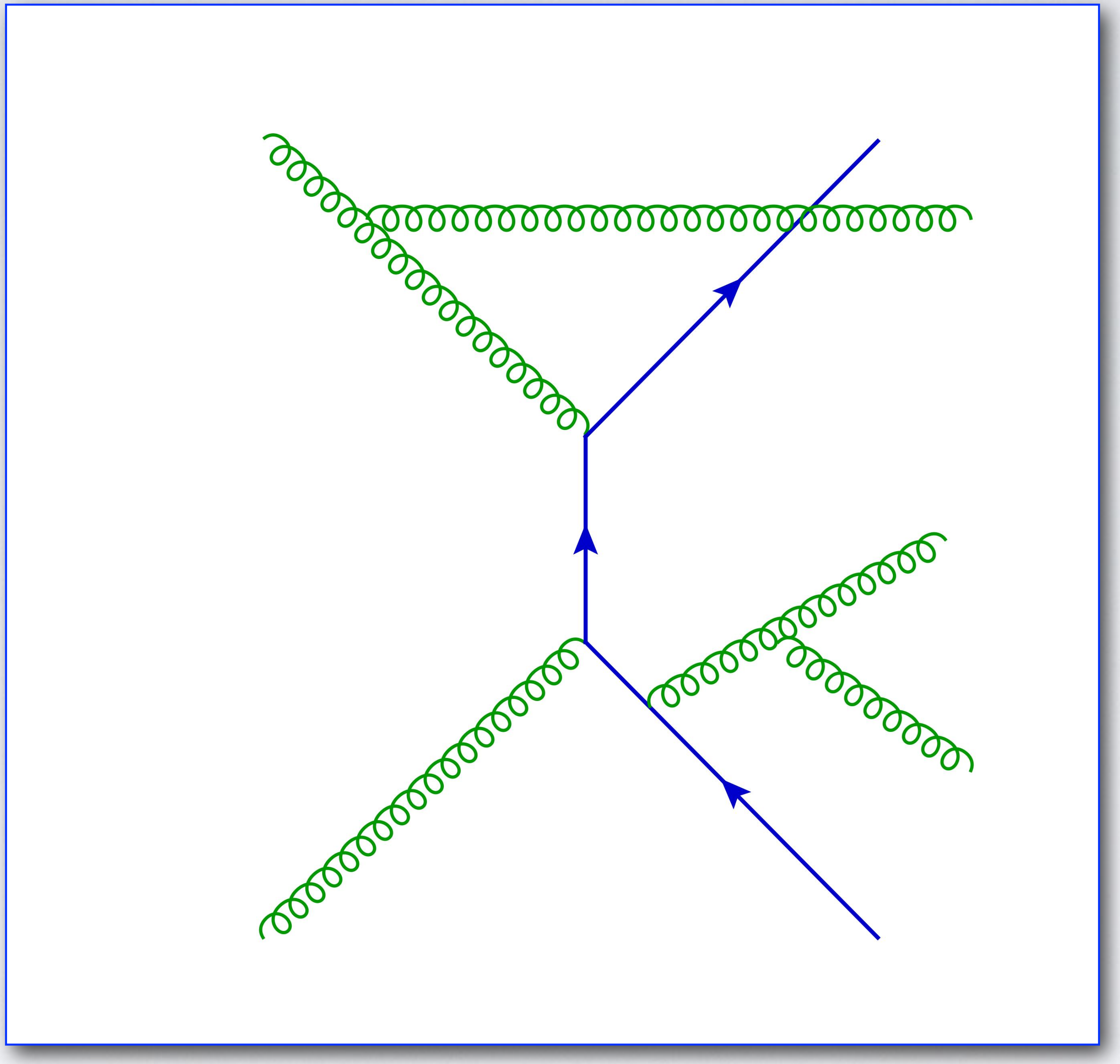

$$\left(\frac{dE}{E}\,\frac{dk_\perp}{k_\perp}\right)^2$$

A diagram contributing a double-real NNLO correction to t-tbar-jet production

Our viewpoint

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- We are interested in subtraction for complicated process at very high orders.
- The factorisation of virtual corrections contains all-order information, not fully exploited.
  - Exponentiation ties together high orders to low orders.
  - Classes of possible virtual poles are absent, with implications for real radiation.
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- Can one use the structure of virtual singularities as an organising principle for subtraction?
- Can the simplifying features of virtual corrections be exported to real radiation?

A multi-year effort

The subtraction problem at NLO is completely solved, with efficient algorithms applicable to any process for which matrix elements are known.

At NNLO after fifteen years of efforts several groups have working algorithms, successfully applied to `simple' process with up to four legs. Heavy computational costs.

- Antenna Subtraction.
- Stripper
- Nested Soft-Collinear Subtractions.
- ColourfulNNLO.
- N-Jettiness Slicing.
- Q_T Slicing.
- Projection to Born.
- Unsubtraction.
- Geometric Slicing …

ALGORITHMS

NLO Subtraction

The computation of a generic IRC-safe observable at NLO requires the combination

$$\langle O \rangle_{\rm NLO} = \lim_{d \to 4} \left\{ \int d\Phi_n \left[B_n + V_n \right] O_n + \int d\Phi_{n+1} R_{n+1} \, O_{n+1} \right\}$$

The necessary numerical integrations require finite ingredients in d=4. Define counterterms

$$\langle O \rangle_{\rm ct} = \int d\Phi_n \, d\hat{\Phi}_1 \, K_{n+1} \, O_n \, .$$

$$I_n = \int d\hat{\Phi}_1 \, K_{n+1} \, ,$$

Add and subtract the same quantity to the observable: each contribution is now finite.

$$\langle O \rangle_{\rm NLO} = \int d\Phi_n \left[B_n^{(4)} + \left(V_n + I_n \right)^{(4)} \right] O_n + \int d\Phi_n \left[\int d\Phi_1^{(4)} R_{n+1}^{(4)} O_{n+1} - \int d\hat{\Phi}_1^{(4)} K_{n+1}^{(4)} O_n \right]$$

Search for the simplest fully local integrand K_{n+1} with the correct singular limits.

NNLO Subtraction

The pattern of cancellations is more intricate at higher orders

$$\langle O \rangle_{\text{NNLO}} = \lim_{d\to 4} \left\{ \int d\Phi_n \left[B_n + V_n + VV_n \right] O_n + \int d\Phi_{n+1} \left[R_{n+1} + RV_{n+1} \right] O_{n+1} + \int d\Phi_{n+2}\, RR_{n+2}\, O_{n+2} \right\} .$$

More counterterm functions need to be defined

$$I^{(\mathbf{1})}_{n+1} = \int d\hat{\Phi}'_1\, K^{(\mathbf{1})}_{n+2} ,$$
$$I^{(\mathbf{2})}_{n} = \int d\hat{\Phi}_2\, K^{(\mathbf{2})}_{n+2} = \int d\hat{\Phi}_1\, d\hat{\Phi}'_1\, K^{(\mathbf{2})}_{n+2} ,$$

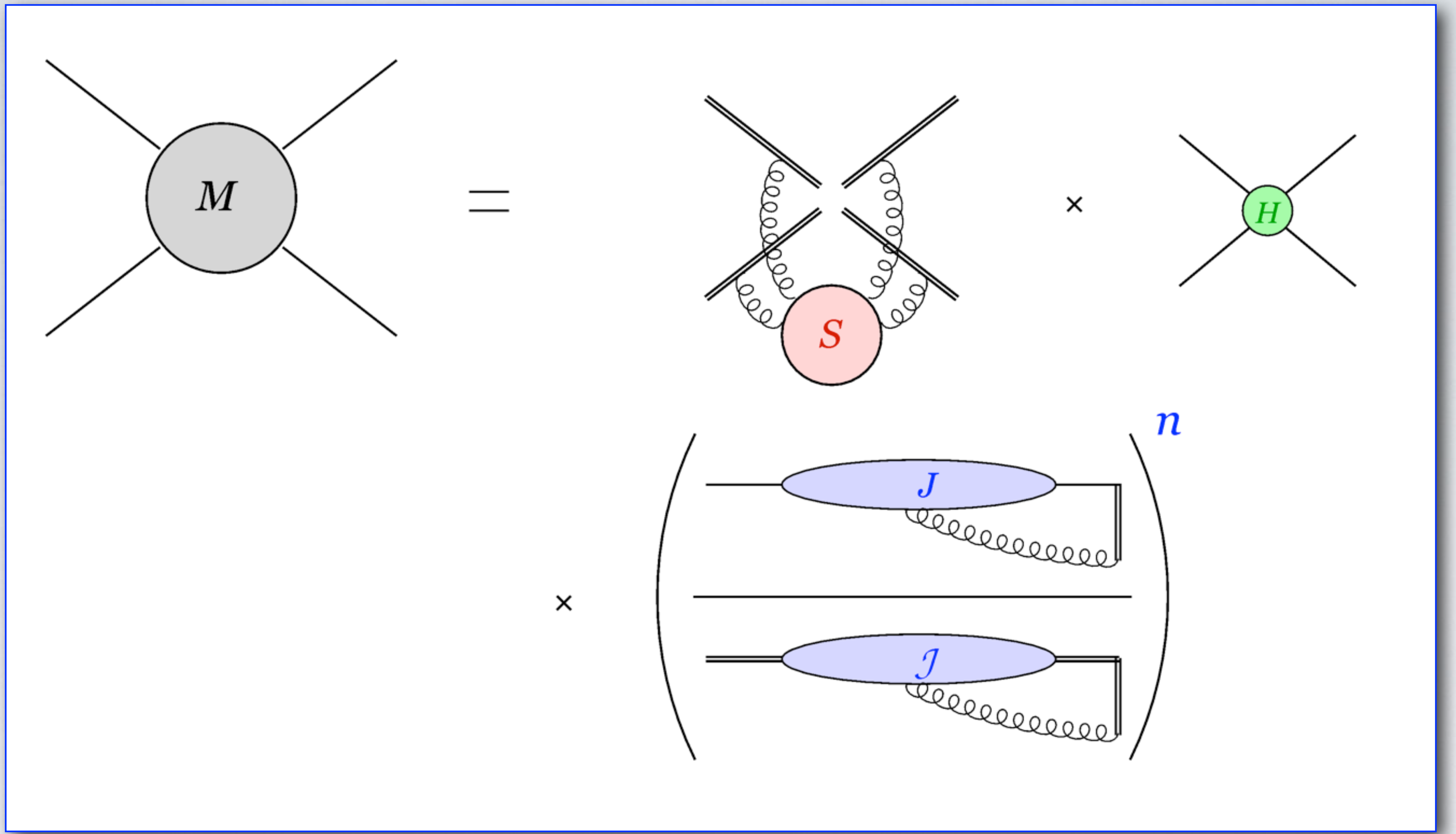
$$I^{(\mathbf{RV})}_{n} = \int d\hat{\Phi}_1\, K^{(\mathbf{RV})}_{n+1} .$$

A finite expression for the observable in d=4 must combine several ingredients

$$\frac{d\sigma_{\text{NNLO}}}{dX} = \int d\Phi_n \left(VV + I^{(2)} + I^{(\mathbf{RV})} \right) \delta_n + \int \left[\left(d\Phi_{n+1}\, RV + d\widehat{\Phi}_{n+1}\, I^{(1)} \right) \delta_{n+1} - d\widehat{\Phi}_{n+1} \left(\overline{K}^{(\mathbf{RV})} - I^{(12)} \right) \delta_n \right] + \int \left[d\Phi_{n+2}\, RR\, \delta_{n+2} - d\widehat{\Phi}_{n+2}\, \overline{K}^{(1)}\, \delta_{n+1} - d\widehat{\Phi}_{n+2} \left(\overline{K}^{(2)} + \overline{K}^{(12)} \right) \delta_n \right] .$$

FACTORISATION

Virtual factorisation: pictorial



A pictorial representation of soft-collinear factorisation for fixed-angle scattering amplitudes

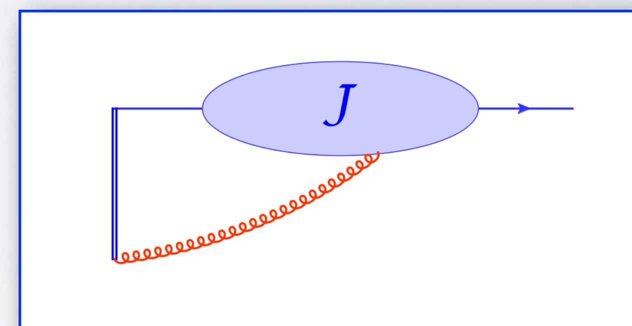
Operator Definitions

The precise functional form of this graphical factorisation is

$$\mathcal{A}_n\left(\frac{p_i}{\mu}\right) = \prod_{i=1}^{n}\left[\frac{\mathcal{J}_i\Big((p_i\cdot n_i)^2/(n_i^2\mu^2)\Big)}{\mathcal{J}_{E,\,i}\Big((\beta_i\cdot n_i)^2/n_i^2\Big)}\right]\,\mathcal{S}_n\left(\beta_i\cdot\beta_j\right)\;\mathcal{H}_n\left(\frac{p_i\cdot p_j}{\mu^2},\frac{(p_i\cdot n_i)^2}{n_i^2\mu^2}\right)$$

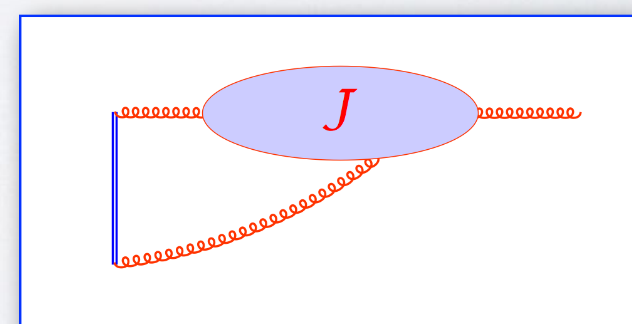
Here we introduced dimensionless four-velocities $\beta_i = p_i/Q$, and factorisation vectors n_i^μ, $n_i^2 \neq 0$ to define the jets in a gauge-invariant way. For outgoing quarks

$$\overline{u}_s(p)\,\mathcal{J}_q\left(\frac{(p\cdot n)^2}{n^2\mu^2}\right) = \langle p,s\,|\,\overline{\psi}(0)\,\Phi_n(0,\infty)\,|0\rangle$$



where Φ_n is the Wilson line operator along the direction n. For outgoing gluons

$$g_s\,\varepsilon_\mu^{*\,(\lambda)}(k)\mathcal{J}_g^{\mu\nu}\left(\frac{(k\cdot n)^2}{n^2\mu^2}\right) \equiv \langle k,\lambda|\,\Big[\Phi_n(\infty,0)\,iD^\nu\,\Phi_n(0,\infty)\Big]\,|0\rangle\ ,$$

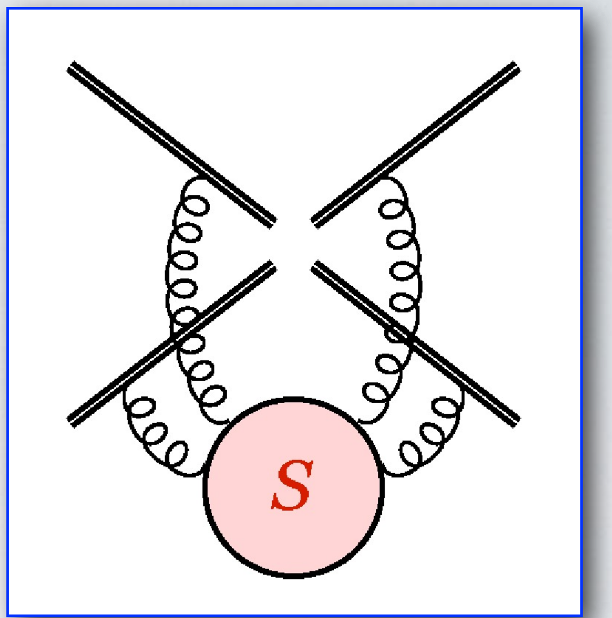


Becher, Bell 2010

Wilson line correlators

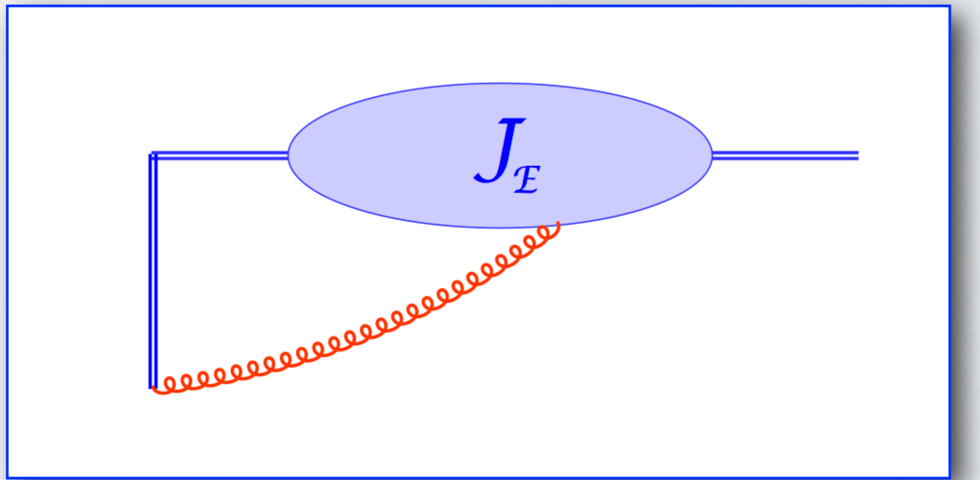
The soft function S is a color operator, mixing the available color tensors. It is defined by a correlator of Wilson lines.

$$\mathcal{S}_n\left(\beta_i \cdot \beta_j\right) = \langle 0 | \prod_{k=1}^{n} \Phi_{\beta_k}(\infty, 0) \, | 0 \rangle$$



The soft jet function $J_{\mathcal{E}}$ contains soft-collinear poles: it is defined by replacing the field in the ordinary jet J with a Wilson line in the appropriate color representation.

$$\mathcal{J}_{\rm E}\left(\frac{(\beta \cdot n)^2}{n^2}\right) = \langle 0 | \, \Phi_\beta(\infty, 0) \, \Phi_n(0, \infty) \, | 0 \rangle$$



Wilson-line matrix elements exponentiate non-trivially and have tightly constrained functional dependence on their arguments. They are known to three loops.

COUNTERTERMS

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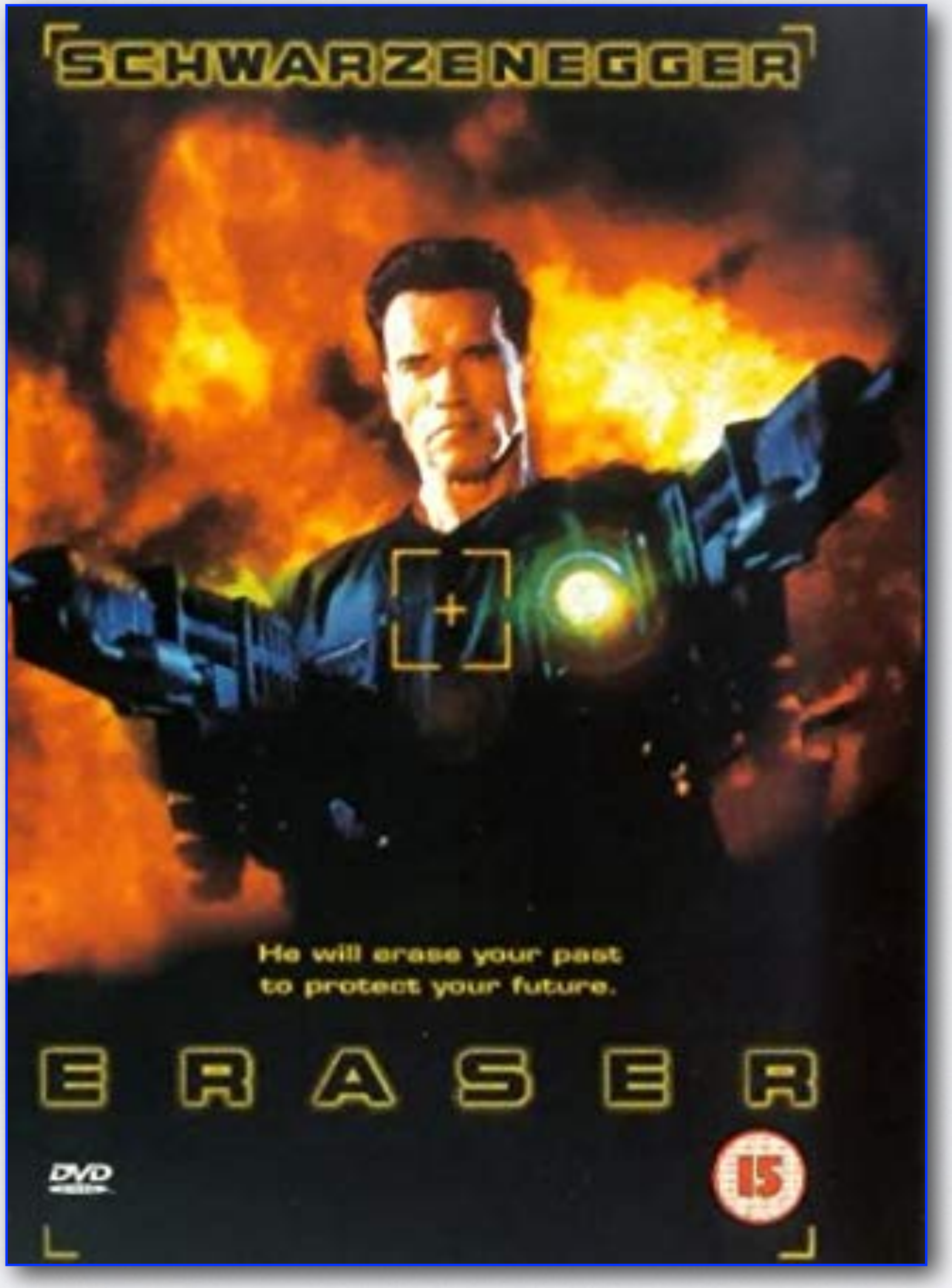

Soft cross sections: pictorial

Consider first the (academic) case of purely soft final state divergences.

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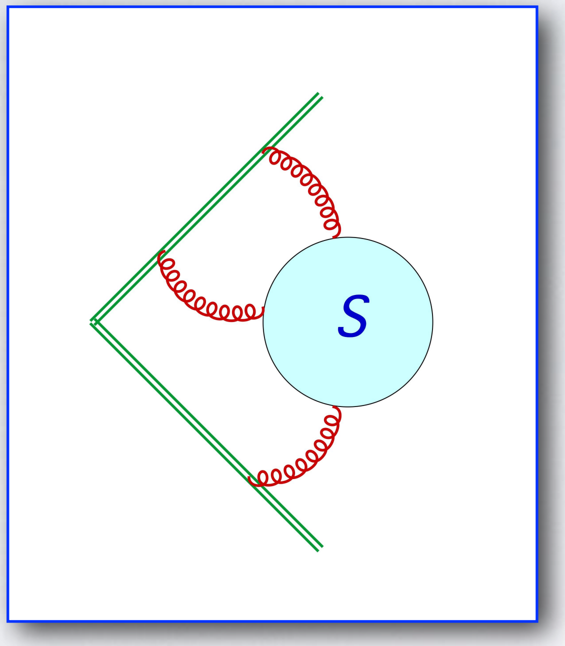
At amplitude level poles factorise and exponentiate.



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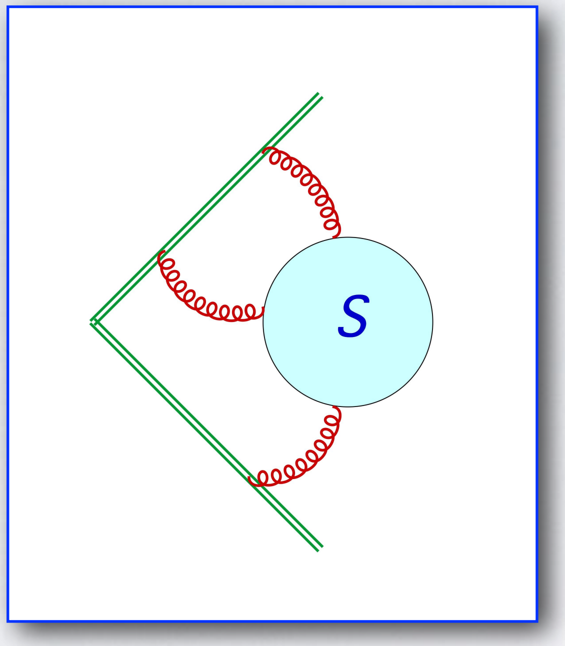
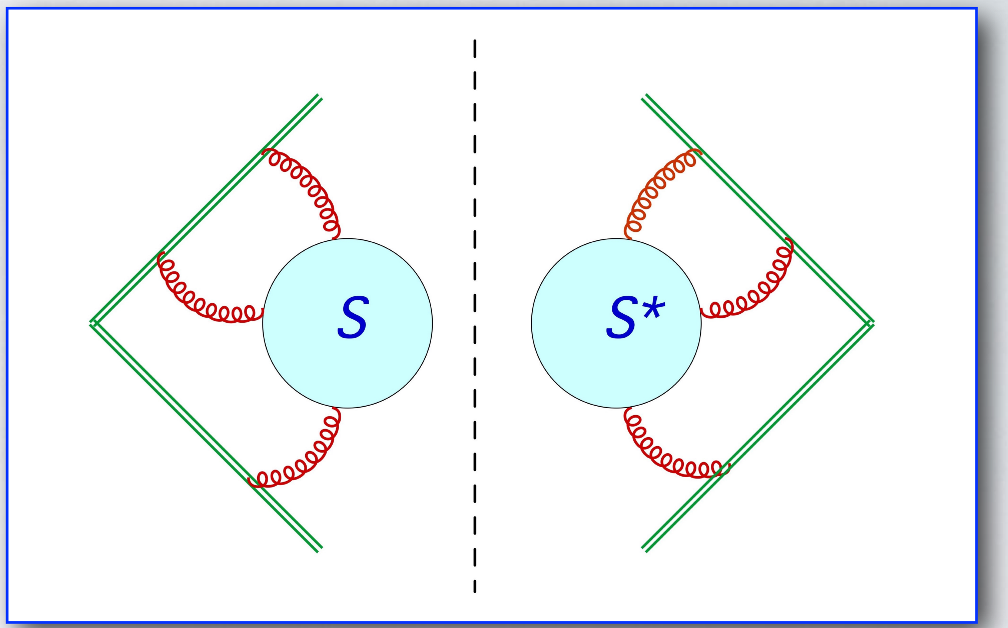


We need to build cross-section level quantities.



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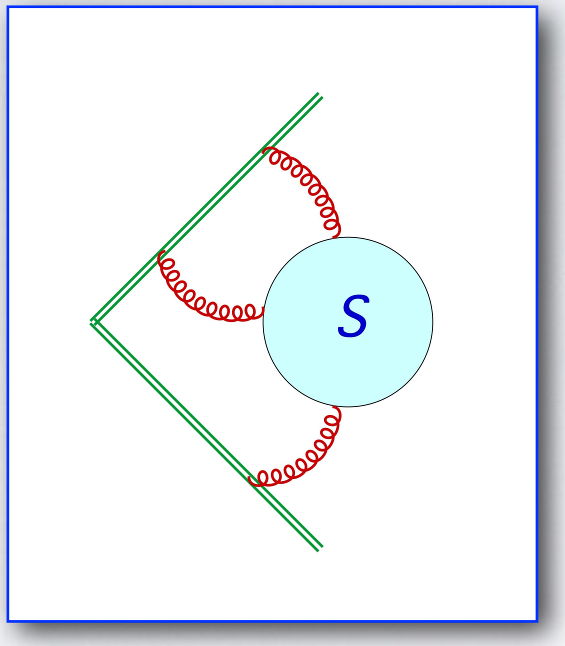

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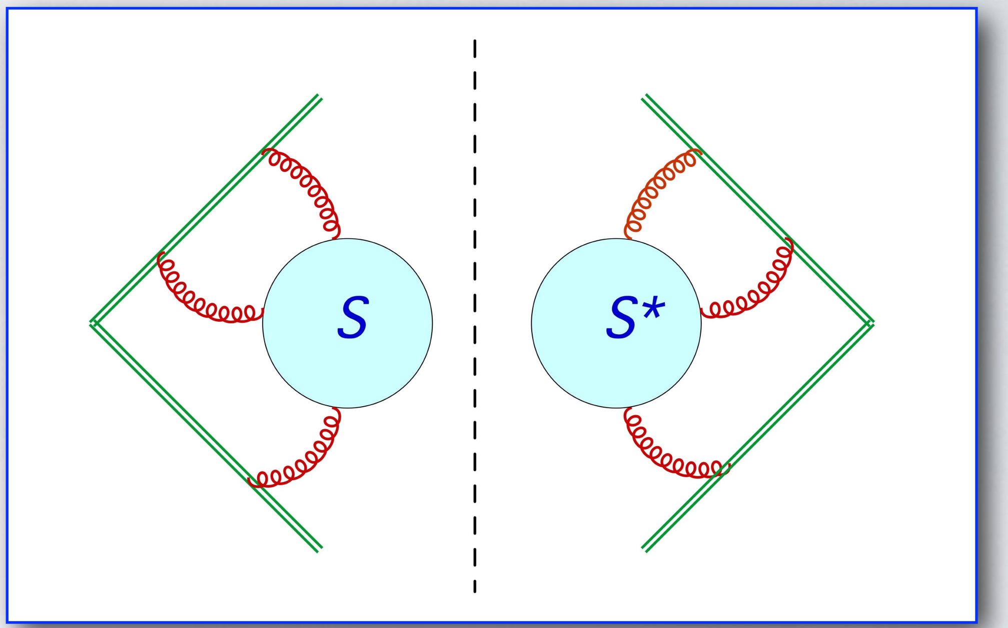
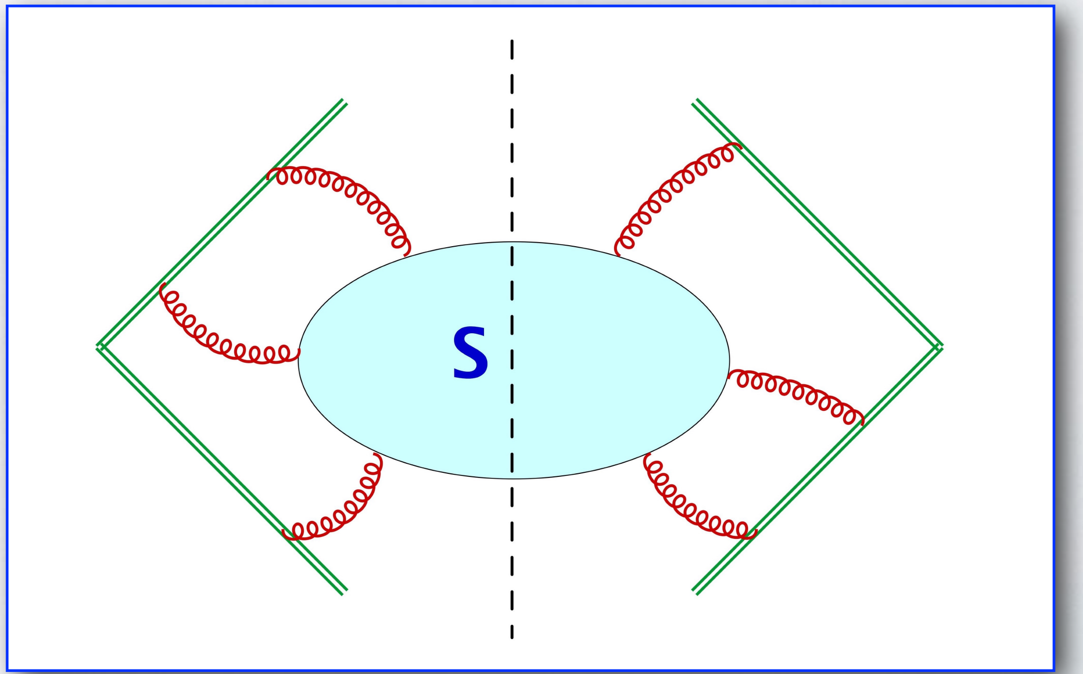


- Inclusive eikonal cross sections are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local IR counterterms.



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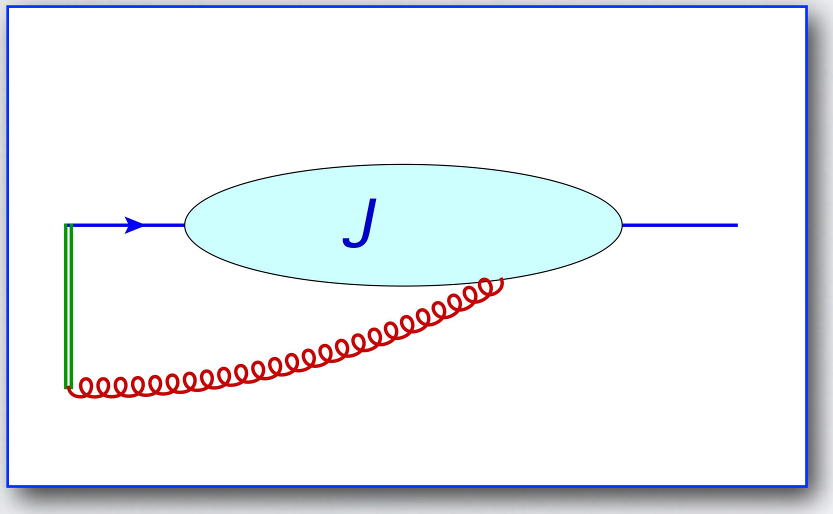

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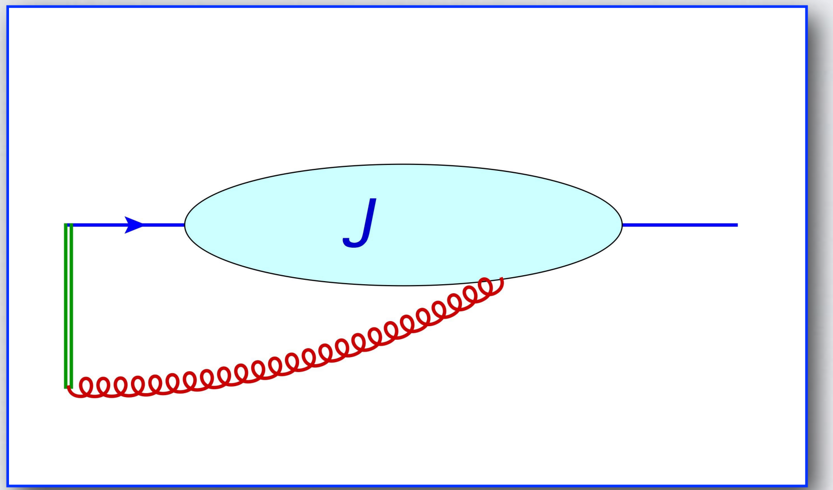
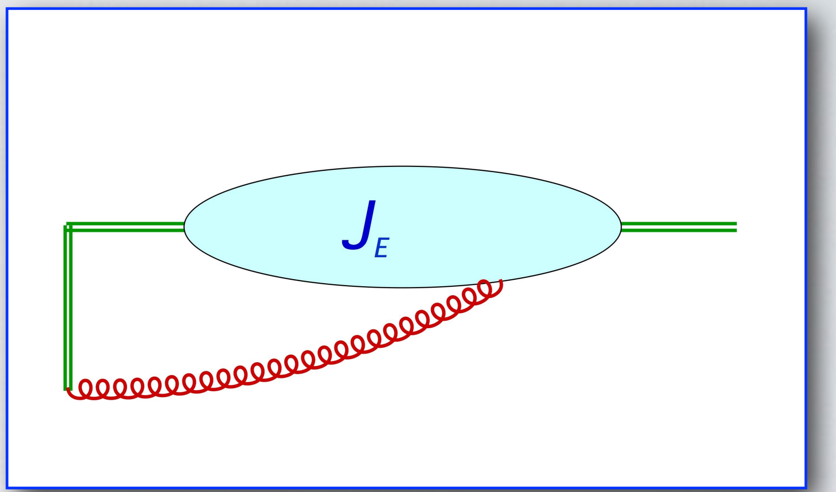


Soft-collinear poles can be subtracted



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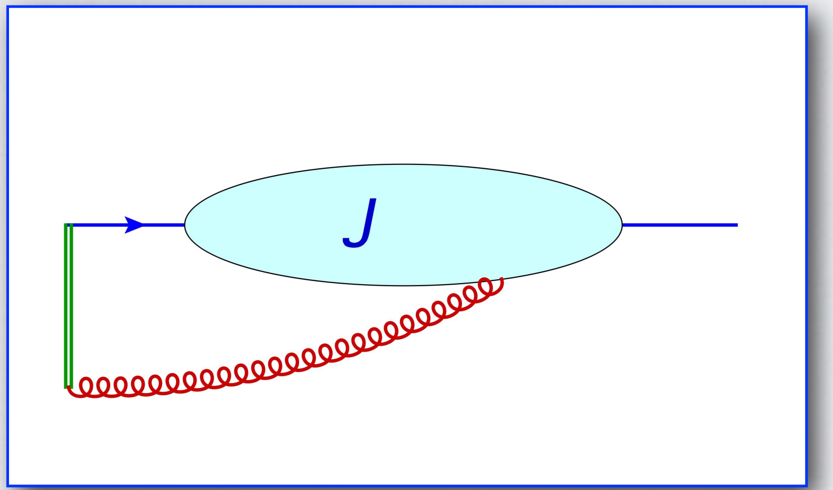

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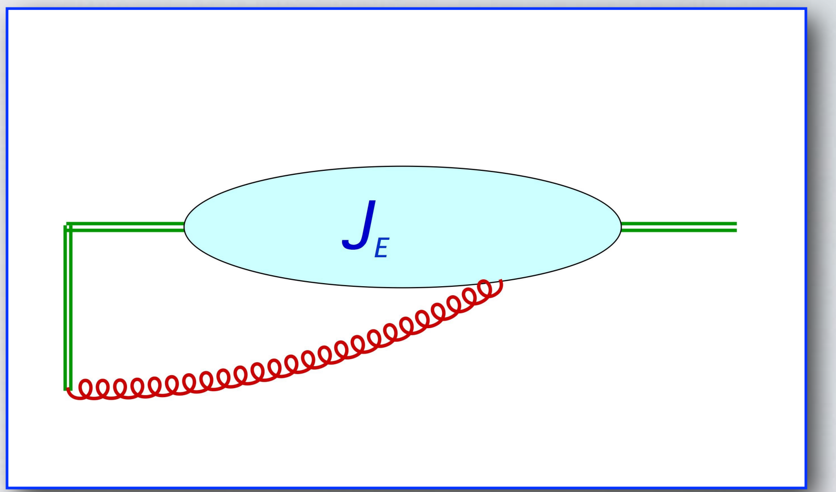
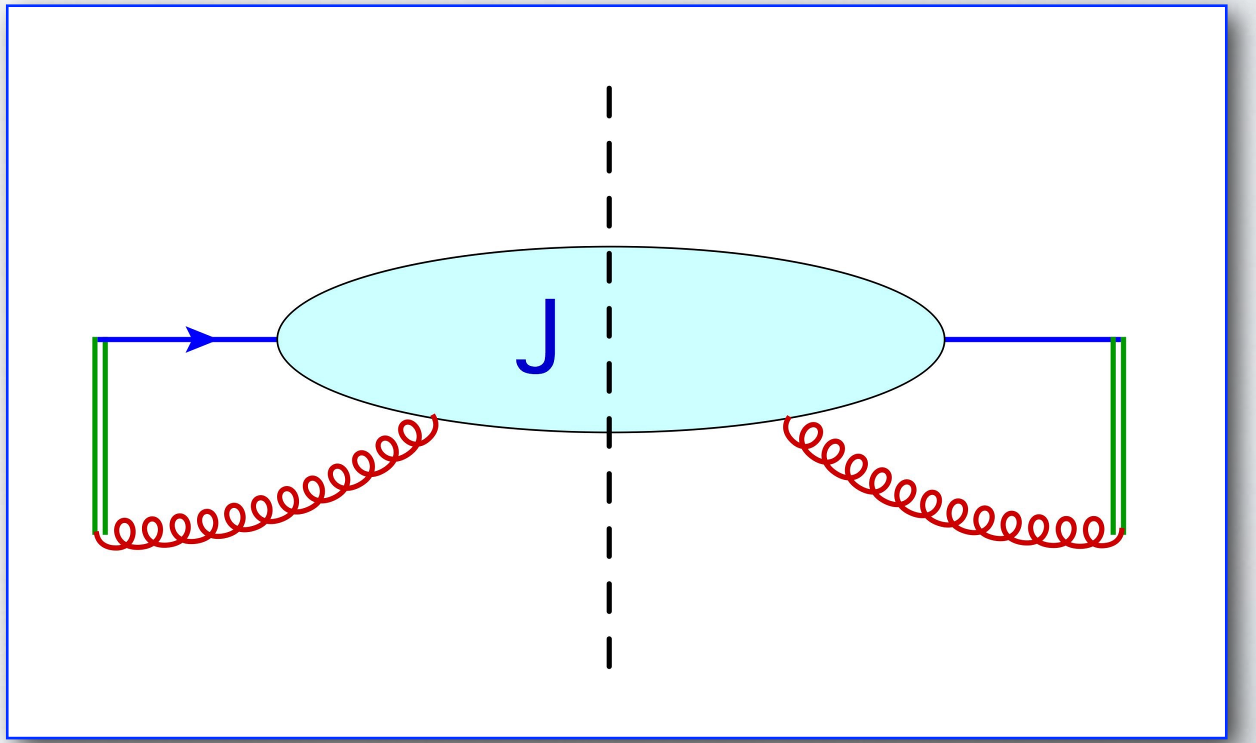


- Inclusive `jet cross sections' are finite.
- They are building blocks for threshold and Q_T resummations.
- They are defined by gauge-invariant operator matrix elements.
- Fixing the quantum numbers of particles crossing the cut one obtains local collinear counterterms.
- Eikonal jet cross sections subtract the soft-collinear double counting.



Soft counterterms: all orders

Introduce eikonal form factors for the emission of m soft partons from n hard ones.

$$\mathcal{S}_{n,m}\left(k_1,\ldots,k_m;\beta_i\right) \equiv \langle k_1,\lambda_1;\ldots;k_m,\lambda_m|\prod_{i=1}^{n}\Phi_{\beta_i}(\infty,0)\;|0\rangle$$
$$\equiv \epsilon_{\mu_1}^{*\,(\lambda_1)}(k_1)\ldots\epsilon_{\mu_m}^{*\,(\lambda_m)}(k_m)\;J_{\mathcal{S}}^{\mu_1\ldots\mu_m}\left(k_1,\ldots,k_m;\beta_i\right)$$
$$\equiv \sum_{p=0}^{\infty}\mathcal{S}_{n,m}^{(p)}\left(k_1,\ldots,k_m;\beta_i\right)$$

These matrix elements define soft gluon multiple emission currents. They are gauge invariant and they contain loop corrections to all orders.

Existing finite order calculations and all-order arguments are consistent with the factorisation

$$\mathcal{A}_{n,m}\left(k_1,\ldots,k_m;p_i\right) = \mathcal{S}_{n,m}\left(k_1,\ldots,k_m;\beta_i\right)\mathcal{H}_n(p_i) + \mathcal{R}_{n,m}\left(k_1,\ldots,k_m;p_i\right)$$

with corrections that are finite in dimensional regularisation, and integrable in the soft gluon phase space. It is a working assumption: a formal all-order proof is still lacking.

Soft counterterms: all orders

The factorisation is reflected at cross-section level, for fixed final state quantum numbers.

$$\sum_{\lambda_i} |\mathcal{A}_{n,\,m}\,(k_1, \ldots, k_m; p_i)|^2 \;\simeq\; \mathcal{H}_n^\dagger(p_i)\, S_{n,\,m}\,(k_1, \ldots, k_m; \beta_i)\; \mathcal{H}_n(p_i)$$

The *cross-section level* "radiative soft functions" are Wilson-line squared matrix elements

$$S_{n,\,m}\,(k_1, \ldots, k_m; \beta_i) \equiv \sum_{p=0}^{\infty} S^{(p)}_{n,\,m}\,(k_1, \ldots, k_m; \beta_i)$$
$$\equiv \sum_{\lambda_i} \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(0, \infty)\, |k_1, \lambda_1; \ldots; k_m, \lambda_m\rangle \, \langle k_1, \lambda_1; \ldots; k_m, \lambda_m| \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0)\, |0\rangle \;.$$

These functions provide a complete list of local soft subtraction counterterms, to all orders. Indeed, summing over particle numbers and integrating over the soft phase space one finds

$$\sum_{m=0}^{\infty} \int d\Phi_m\, S_{n,\,m}\,(k_1, \ldots, k_m; \beta_i) \;=\; \langle 0| \prod_{i=1}^{n} \Phi_{\beta_i}(0, \infty) \prod_{i=1}^{n} \Phi_{\beta_i}(\infty, 0)\, |0\rangle$$

This is a finite fully inclusive soft cross section, order by order in perturbation theory.

Soft current for NLO

At NLO, only the tree-level single-emission current is required, simply defined by

$$\epsilon^{*\,(\lambda)}(k) \cdot J_{\mathcal{S}}^{(0)}(k,\beta_i) \;=\; \mathcal{S}_{n,\,1}^{(0)}\,(k;\beta_i) \;=\; \langle k,\lambda|\,\prod_{i=1}^{n} \Phi_{\beta_i}(\infty,0)\;|0\rangle\Big|_{\text{tree}}$$

One obviously recovers all the well-known results for the leading-order soft gluon current

$$\mathcal{A}_{n,\,1}^{(0)}(k,p_i) \;=\; \epsilon^{*\,(\lambda)}(k) \cdot J_{\mathcal{S}}^{(0)}(k,\beta_i)\,\mathcal{H}_n^{(0)}(p_i) + \mathcal{O}(k^0)$$

$$J_{\mathcal{S}}^{\mu\,(0)}(k,\beta_i) \;=\; g\,\sum_{i=1}^{n} \frac{\beta_i^\mu}{\beta_i \cdot k}\,\mathbf{T}_i\;.$$

For the cross-section, the tree-level single-radiation soft function acts as a local counterterm.

$$\sum_\lambda \left|\mathcal{A}_{n,\,1}^{(0)}(k,p_i)\right|^2 \simeq \mathcal{H}^{(0)\,\dagger}(p_i)\,S_{n,\,1}^{(0)}\,(k;\beta_i)\;\mathcal{H}_n^{(0)}(p_i)$$

$$= -\,4\pi\alpha_s \sum_{i,j=1}^{n} \frac{\beta_i \cdot \beta_j}{\beta_i \cdot k\,\beta_j \cdot k}\,\mathcal{A}_n^{(0)\dagger}(p_i)\,\mathbf{T}_i \cdot \mathbf{T}_j\,\mathcal{A}_n^{(0)}(p_i)$$

- The single-radiative soft function acts as a color operator on the color-correlated Born.
- Beyond NLO, tree-level multiple gluon emission currents also follow from this definition.

Soft currents for NNLO

At one loop, for single radiation, our definition of the soft currents gives

$$\mathcal{A}_{n,1}(k;p_i) \simeq \mathcal{S}_{n,1}(k;\beta_i)\,\mathcal{H}_n(p_i) = \mathcal{S}^{(0)}_{n,1}(k;\beta_i)\,\mathcal{H}^{(1)}_n(p_i) + \mathcal{S}^{(1)}_{n,1}(k;\beta_i)\,\mathcal{H}^{(0)}_n(p_i)$$

The factorisation proposed in the classic work by Catani-Grazzini appears different

$$\mathcal{A}_{n,1}(k;p_i) \simeq \epsilon^{*\,(\lambda)}(k)\cdot J_{\rm CG}(k,\beta_i)\,\mathcal{A}_n(p_i)$$

but it is easily matched using the factorisation of the non-radiative amplitude

$$\mathcal{A}_n(p_i) \simeq \mathcal{S}_n(\beta_i)\,\mathcal{H}_n(p_i) \quad \longrightarrow \quad \mathcal{H}^{(1)}_n(p_i) = \mathcal{A}^{(1)}_n(p_i) - \mathcal{S}^{(1)}_n(\beta_i)\,\mathcal{A}^{(0)}_n(p_i)$$

Recombining, we get an explicit eikonal expression for the CG one-loop soft current

$$\epsilon^{*\,(\lambda)}(k)\cdot J^{(1)}_{\rm CG}(k,\beta_i) = \mathcal{S}^{(1)}_{n,1}(k;\beta_i) - \mathcal{S}^{(0)}_{n,1}(k;\beta_i)\,\mathcal{S}^{(1)}_n(\beta_i)$$

The two calculations are easily matched: same diagrammatic content, cancellations and result.

Soft currents for N3LO

The procedure is easily generalised to generic higher orders. At two loops one finds

$$\mathcal{A}^{(2)}_{n,1}(k;p_i) \simeq \mathcal{S}^{(0)}_{n,1}(k;\beta_i)\,\mathcal{H}^{(2)}_n(p_i) + \mathcal{S}^{(1)}_{n,1}(k;\beta_i)\,\mathcal{H}^{(1)}_n(p_i) + \mathcal{S}^{(2)}_{n,1}(k;\beta_i)\,\mathcal{H}^{(0)}_n(p_i)$$

To map to the CG definition, express the two-loop hard part in terms of the amplitude

$$\mathcal{H}^{(2)}_n(p_i) = \mathcal{A}^{(2)}_n(p_i) - \mathcal{S}^{(1)}_n(\beta_i)\,\mathcal{A}^{(1)}_n(p_i) + \left[\mathcal{S}^{(1)}_n(\beta_i)\right]^2 \mathcal{A}^{(0)}_n(p_i) - \mathcal{S}^{(2)}_n(\beta_i)\,\mathcal{A}^{(0)}_n(p_i)$$

Recombining, we get an explicit eikonal expression for the two-loop single-gluon soft current

$$\epsilon^{*\,(\lambda)}(k)\cdot J^{(2)}_{\rm CG}(k,\beta_i) = \mathcal{S}^{(2)}_{n,1}(k;\beta_i) - \mathcal{S}^{(1)}_{n,1}(k;\beta_i)\,\mathcal{S}^{(1)}_n(\beta_i) - \mathcal{S}^{(0)}_{n,1}(k;\beta_i)\left[\mathcal{S}^{(2)}_n(\beta_i) - \left(\mathcal{S}^{(1)}_n(\beta_i)\right)^2\right]$$

For the two-leg case, this was computed in (Badger, Glover 2004) to $O(\epsilon^0)$ and by (Duhr, Gehrmann 2013) to $O(\epsilon^2)$, by taking soft limits of full matrix elements. This definition allows to extend the calculation to the general case.

A similar definition emerges for the double-gluon soft current at one and two loops. Based on eikonal Feynman rules, one can begin the process of systematising these calculations.

Collinear counterterms: all orders

For collinear poles, introduce jet matrix elements for the emission of m partons. For quarks

$$\overline{u}_s(p)\,\mathcal{J}_{q,\,m}\,(k_1, \ldots, k_m; p, n) \;\equiv\; \langle p, s; k_1, \lambda_1; \ldots; k_m, \lambda_m |\, \overline{\psi}(0)\, \Phi_n(0, \infty)\, |0\rangle$$

At *cross-section level*, "radiative jet functions" can be defined as Fourier transforms of squared matrix elements, to account for the non-trivial momentum flow. We propose

$$\begin{aligned} J_{q,\,m}\,(k_1, \ldots k_m; l, p, n) \;&\equiv\; \sum_{p=0}^{\infty} J^{(p)}_{q,\,m}\,(k_1, \ldots k_m; l, p, n) \\ &\equiv \int d^d x\, \mathrm{e}^{\mathrm{i}l\cdot x} \sum_{\{\lambda_j\}} \langle 0|\, \Phi_n(\infty, x)\, \psi(x)\, |p, s; k_j, \lambda_j\rangle\, \langle p, s; k_j, \lambda_j|\, \overline{\psi}(0)\, \Phi_n(0, \infty)\, |0\rangle \end{aligned}$$

These functions provide a complete list of local collinear counterterms, to all orders. Summing over particle numbers and integrating over the collinear phase space one finds

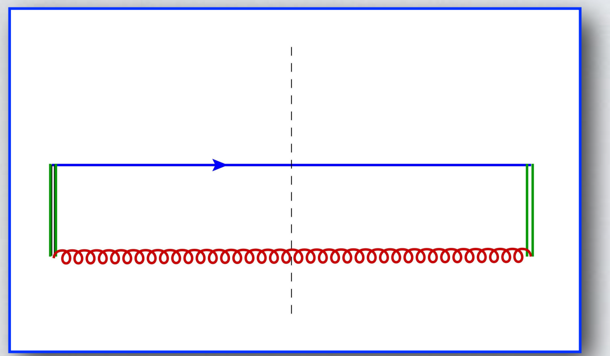
$$\sum_{m=0}^{\infty} \int d\Phi_{m+1}\, J_{q,\,m}\,(k_1, \ldots, k_m; l, p, n) \;=\; \mathrm{Disc}\left[\int d^d x\, \mathrm{e}^{\mathrm{i}l\cdot x}\; \langle 0|\, \Phi_n(\infty, x)\psi(x)\overline{\psi}(0)\Phi_n(0, \infty)\, |0\rangle\right]$$

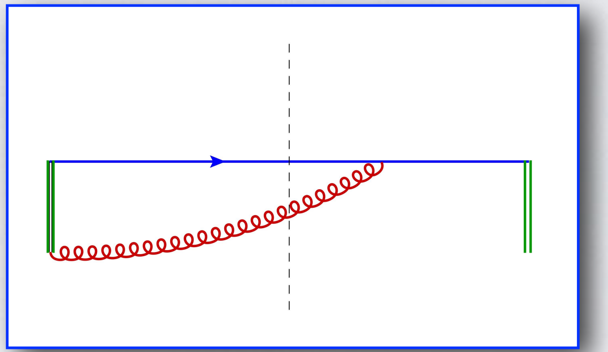
A "two-point function", finite order by order in perturbation theory. Note however

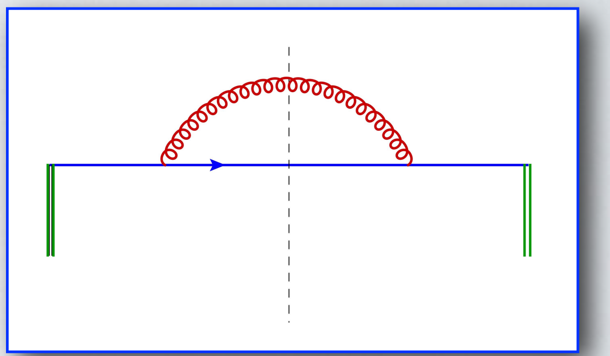
- The collinear limit must still be taken (as $l^2 \to 0$), unlike the case of radiative soft functions.
- Working with $n^2 \neq 0$ eliminates spurious collinear poles, but is cumbersome in practice.

Collinear counterterms: NLO

At NLO, only tree-level single-emission contributes, resulting (for quarks) in three diagrams







Summing over helicities, and taking the $n^2 \to 0$ limit, one finds a spin-dependent kernel

$$\sum_s J_{q,1}(k;l,p,n) = \frac{4\pi\alpha_s C_F}{l^2}(2\pi)^d\,\delta^d(l-p-k)\left[-\not{l}\gamma_\mu\not{p}\gamma^\mu\not{l} + \frac{1}{k\cdot n}\left(\not{l}\not{n}\not{p}+\not{p}\not{n}\not{l}\right)\right]$$

With a Sudakov decomposition

$$p^\mu = z l^\mu + \mathcal{O}(l_\perp)\,, \qquad k^\mu = (1-z) l^\mu + \mathcal{O}(l_\perp)\,, \qquad n^2 = 0$$

and taking $l_\perp \to 0$, one recovers the full unpolarised DGLAP LO splitting kernel.

$$\sum_s J_{q,1}(k;l,p,n) = \frac{8\pi\alpha_s C_F}{l^2}(2\pi)^d\,\delta^d(l-p-k)\left[\frac{1+z^2}{1-z} - \epsilon(1-z) + \mathcal{O}(l_\perp)\right]$$

- The three diagrams map precisely to the axial gauge calculation by Catani, Grazzini.
- All LO DGLAP kernels are easily reproduced, triple collinear limits are under way.

NLO subtraction

The outlines of a subtraction procedure emerge. Begin by expanding the virtual matrix element

$$\mathcal{A}_n(p_i) = \Big[\mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) + \sum_{i=1}^{n}\Big(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\Big)\,\mathcal{S}_n^{(0)}(\beta_i)\,\mathcal{H}_n^{(0)}(p_i)\Big]\Big(1 + \mathcal{O}\left(\alpha_s^2\right)\Big)$$

From the master formula, get the virtual poles of the cross section in terms of virtual kernels

$$V_n \equiv 2\,\mathbf{Re}\Big[\mathcal{A}_n^{(0)*}\mathcal{A}_n^{(1)}\Big] \simeq \mathcal{H}_n^{(0)\,\dagger}(p_i)S_{n,\,0}^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \sum_i\Big(J_{i,\,0}^{(1)}(p_i) - J_{\mathrm{E},\,i,\,0}^{(1)}(\beta_i)\Big)\Big|\mathcal{A}_n^{(0)}(p_i)\Big|^2$$

Go through the list of proposed soft and collinear counterterms to collect the relevant ones

$$S_{n,0}^{(1)}(\beta_i) + \int d\Phi_1\, S_{n,\,1}^{(0)}(k,\beta_i) = \text{finite}$$

$$J_{i,\,0}^{(1)}(l,p,n) + \int d\Phi_1\, J_{i,\,1}^{(0)}(k;l,p,n) = \text{finite}$$

Construct the appropriate local functions.

$$K_{n+1}^{\mathrm{NLO,\,s}} = \mathcal{H}_n^{(0)\,\dagger}(p_i)\,S_{n,\,1}^{(0)}(k,\beta_i)\,\mathcal{H}_n^{(0)}(p_i)$$

$$K_{n+1}^{\mathrm{NLO,\,c}} = \sum_{i=1}^{n} J_{i,\,1}^{(0)}(k_i;l,p_i,n_i)\,\Big|\mathcal{A}_n^{(0)}\,(p_1,\ldots,p_{i-1},l,p_{i+1},\ldots,p_n)\Big|^2$$

with a similar expression for the anti-subtraction of the soft-collinear region in terms of $J_{\mathcal{E}}$.

NNLO subtraction

Let us follow the same procedure at NNLO. Collect the poles of the virtual amplitude

$$\begin{aligned}\mathcal{A}_n^{(2)}(p_i) &= \mathcal{S}_n^{(2)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(2)}(p_i) + \mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(1)}(p_i) \\ &+ \sum_{i=1}^{n}\left[\mathcal{J}_i^{(2)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(2)}(\beta_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\right)\right]\mathcal{A}_n^{(0)}(p_i) \\ &+ \sum_{i<j=1}^{n}\left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\right)\left(\mathcal{J}_j^{(1)}(p_j) - \mathcal{J}_{\mathrm{E},\,j}^{(1)}(\beta_j)\right)\mathcal{A}_n^{(0)}(p_i) \\ &+ \sum_{i=1}^{n}\left(\mathcal{J}_i^{(1)}(p_i) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}(\beta_i)\right)\left[\mathcal{S}_n^{(1)}(\beta_i)\mathcal{H}_n^{(0)}(p_i) + \mathcal{S}_n^{(0)}(\beta_i)\mathcal{H}_n^{(1)}(p_i)\right]\end{aligned}$$

Cross-section level soft and jet functions have non-trivial structure starting at NNLO

$$S_n^{(2)} = \mathcal{S}_n^{(0)\,\dagger}\mathcal{S}_n^{(2)} + \mathcal{S}_n^{(2)\,\dagger}\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)\,\dagger}\mathcal{S}_n^{(1)}$$

$$J_{q,\,m}^{(2)} = \int d^d x\, \mathrm{e}^{\mathrm{i}l\cdot x}\sum_{\{\lambda_j\}}\left[\mathcal{J}_{q,\,m}^{(1)\,\dagger}(x)\,\not{n}\,\mathcal{J}_{q,\,m}^{(1)}(0) + \mathcal{J}_{q,\,m}^{(0)\,\dagger}(x)\,\not{n}\,\mathcal{J}_{q,\,m}^{(2)}(0) + \mathcal{J}_{q,\,m}^{(0)}(x)\,\not{n}\,\mathcal{J}_{q,\,m}^{(2)\,\dagger}(0)\right]$$

All poles of the squared virtual amplitude can nonetheless be expressed in terms of squared jets and eikonal correlators, which leads to the identification of local NNLO counterterms.

NNLO subtraction: double collinear

Cross-section level double-virtual poles originate from a number of different configurations

$$(VV)_n \equiv (VV)_n^{(2s)} + (VV)_n^{(1s)} + \sum_{i=1}^{n} (VV)_{n,i}^{(2hc)} + \sum_{i<j=1}^{n} (VV)_{n,ij}^{(2hc)} + \sum_{i=1}^{n} (VV)_{n,i}^{(1hc,1s)} + \sum_{i=1}^{n} (VV)_{n,i}^{(1hc)}$$

Focus on double collinear radiation along the direction of a selected hard particle. One finds

$$(VV)_{n,i}^{(2hc)} = \left[J_{i,0}^{(2)} - J_{E,i,0}^{(2)} - J_{E,i,0}^{(1)} \left(J_{i,0}^{(1)} - J_{E,i,0}^{(1)} \right) \right] \left| \mathcal{A}_n^{(0)} \right|^2$$

It is easy to identify finite combinations of virtual and real (hard) collinear radiation

$$J_{i,0}^{(2)} + \int d\Phi_1 \, J_{i,1}^{(1)} + \int d\Phi_2 \, J_{i,2}^{(0)} = \text{finite}$$

$$\left[J_{E,i,0}^{(1)} + \int d\Phi_1 \, J_{E,i,1}^{(0)} \right] \left[J_{i,0}^{(1)} - J_{E,i,0}^{(1)} + \int d\Phi_1' \left(J_{i,1}^{(0)} - J_{E,i,1}^{(0)} \right) \right] = \text{finite}$$

Real radiation naturally organises into single and double unresolved, and real-virtual terms

$$K_{n+2,i}^{\text{NNLO},(\mathbf{2},\text{hc})} = \left[J_{i,2}^{(0)} - J_{E,i,2}^{(0)} - J_{E,i,1}^{(0)} \left(J_{i,1}^{(0)} - \mathcal{J}_{E,i,1}^{(0)} \right) \right] \left| \mathcal{A}_n^{(0)} \right|^2$$

$$K_{n+2,i}^{\text{NNLO},(\mathbf{1},\text{hc})} = \left(J_{i,1}^{(0)} - \mathcal{J}_{E,i,1}^{(0)} \right) \left| \mathcal{A}_{n+1}^{(0)} \right|^2$$

$$K_{n+1,i}^{\text{NNLO},(\mathbf{RV},\text{hc})} = \left[J_{i,1}^{(1)} - \mathcal{J}_{E,i,1}^{(1)} - J_{i,0}^{(1)} J_{E,i,1}^{(0)} - J_{E,i,0}^{(1)} J_{i,1}^{(0)} + 2 J_{E,i,0}^{(1)} J_{E,i,1}^{(0)} \right] \left| \mathcal{A}_n^{(0)} \right|^2 .$$

N^{3}LO subtraction

At three loops, the organisation of virtual poles becomes more intricate …

$$
\begin{aligned}
\mathcal{A}_n^{(3)} \; = \; & \mathcal{S}_n^{(0)}\mathcal{H}_n^{(3)} + \mathcal{S}_n^{(1)}\mathcal{H}_n^{(2)} + \mathcal{S}_n^{(2)}\mathcal{H}_n^{(1)} + \mathcal{S}_n^{(3)}\mathcal{H}_n^{(0)} \\
& + \sum_i \Big[\mathcal{J}_i^{(2)} - \mathcal{J}_{\mathrm{E},\,i}^{(2)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\right)\Big]\left(\mathcal{S}_n^{(1)}\mathcal{H}_n^{(0)} + \mathcal{S}_n^{(0)}\mathcal{H}_n^{(1)}\right) \\
& + \sum_{i,j>i}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\right)\left(\mathcal{J}_j^{(1)} - \mathcal{J}_{\mathrm{E},\,j}^{(1)}\right)\left(\mathcal{S}_n^{(1)}\mathcal{H}_n^{(0)} + \mathcal{S}_n^{(0)}\mathcal{H}_n^{(1)}\right) \\
& + \sum_i\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\right)\Big[\mathcal{S}_n^{(1)}\mathcal{H}_n^{(1)} + \mathcal{S}_n^{(2)}\mathcal{H}_n^{(0)} + \mathcal{S}_n^{(0)}\mathcal{H}_n^{(2)}\Big] \\
& + \bigg\{\sum_i\Big[\mathcal{J}_i^{(3)} - \mathcal{J}_{\mathrm{E},\,i}^{(3)} - \mathcal{J}_{\mathrm{E},\,i}^{(2)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\right) - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\Big[\mathcal{J}_i^{(2)} - \mathcal{J}_{\mathrm{E},\,i}^{(2)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\right)\Big]\Big] \\
& \quad + \sum_{i,\,j\neq i}\Big[\mathcal{J}_i^{(2)} - \mathcal{J}_{\mathrm{E},\,i}^{(2)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\right)\Big]\left(\mathcal{J}_j^{(1)} - \mathcal{J}_{\mathrm{E},\,j}^{(1)}\right) \\
& \quad + \sum_{i,\,j>i}\sum_{k>j}\left(\mathcal{J}_i^{(1)} - \mathcal{J}_{\mathrm{E},\,i}^{(1)}\right)\left(\mathcal{J}_j^{(1)} - \mathcal{J}_{\mathrm{E},\,j}^{(1)}\right)\left(\mathcal{J}_k^{(1)} - \mathcal{J}_{\mathrm{E},\,k}^{(1)}\right)\bigg\}\mathcal{S}_n^{(0)}\mathcal{H}_n^{(0)} \, .
\end{aligned}
$$

… and constructing the cross section generates further complexity.

$$
VVV_n = 2\,\mathbf{Re}\left[\mathcal{A}_n^{(0)\,\dagger}(p_i)\,\mathcal{A}_n^{(3)}(p_i)\right] + 2\mathbf{Re}\left[\mathcal{A}_n^{(1)\,\dagger}(p_i)\,\mathcal{A}_n^{(2)}(p_i)\right] ,
$$

Nevertheless, all poles must result from cross-section-level functions. For the soft region

$$
\begin{aligned}
S_n^{(1)} &= \mathcal{S}_n^{(0)\,\dagger}\mathcal{S}_n^{(1)} + \mathcal{S}_n^{(1)\,\dagger}\mathcal{S}_n^{(0)} \, , \\
S_n^{(2)} &= \mathcal{S}_n^{(0)\,\dagger}\mathcal{S}_n^{(2)} + \mathcal{S}_n^{(2)\,\dagger}\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(1)\,\dagger}\mathcal{S}_n^{(1)} \, , \\
S_n^{(3)} &= \mathcal{S}_n^{(1)\,\dagger}\mathcal{S}_n^{(2)} + \mathcal{S}_n^{(2)\,\dagger}\mathcal{S}_n^{(1)} + \mathcal{S}_n^{(3)\,\dagger}\mathcal{S}_n^{(0)} + \mathcal{S}_n^{(0)\,\dagger}\mathcal{S}_n^{(3)} \, .
\end{aligned}
$$

N^{3}LO subtraction

Three-loop soft poles naturally arrange into triple-, double- and single-soft contributions.

$$
\begin{aligned}
(VVV)_n^{(3s)} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,0}^{(3)}\, \mathcal{H}_n^{(0)}\,, \\
(VVV)_n^{(2s)} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,0}^{(2)}\, \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,0}^{(2)}\, \mathcal{H}_n^{(0)}\,, \\
(VVV)_n^{(1s)} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,0}^{(1)}\, \mathcal{H}_n^{(2)} + \mathcal{H}_n^{(2)\,\dagger}\, S_{n,\,0}^{(1)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,0}^{(1)}\, \mathcal{H}_n^{(1)}\,.
\end{aligned}
$$

Applying the completeness relation for the soft sector identifies real-radiation counterterms

$$
\begin{aligned}
K_{n+3}^{(\mathbf{3}\,\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,3}^{(0)}\, \mathcal{H}_n^{(0)} \\
K_{n+2}^{(\mathbf{RRV},\,\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,2}^{(1)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,2}^{(0)}\, \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,2}^{(0)}\, \mathcal{H}_n^{(0)} \\
K_{n+1}^{(\mathbf{RVV},\,\mathrm{s})} &= \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(2)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(1)}\, \mathcal{H}_n^{(1)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,1}^{(1)}\, \mathcal{H}_n^{(0)} \\
&\quad + \mathcal{H}_n^{(0)\,\dagger}\, S_{n,\,1}^{(0)}\, \mathcal{H}_n^{(2)} + \mathcal{H}_n^{(2)\,\dagger}\, S_{n,\,1}^{(0)}\, \mathcal{H}_n^{(0)} + \mathcal{H}_n^{(1)\,\dagger}\, S_{n,\,1}^{(0)}\, \mathcal{H}_n^{(1)}\,.
\end{aligned}
$$

The cancellation of soft poles between virtual and real contributions is then guaranteed.

$$
(VVV)_n^{(\mathrm{s})} + \int d\Phi_{\mathrm{rad},\,3}\, K_{n+3}^{(\mathbf{3}\,\mathrm{s})} + \int d\Phi_{\mathrm{rad},\,2}\, K_{n+2}^{(\mathbf{RRV},\,\mathrm{s})} + \int d\Phi_{\mathrm{rad}}\, K_{n+1}^{(\mathbf{RVV},\,\mathrm{s})} = \mathrm{finite}\,.
$$

OUTLOOK

Outlook

- A number of successful NNLO subtraction algorithms are available.
- They are computationally expensive, either analytically, or numerically, or both.
- Extensions to multi-leg processes or higher orders is expected to be useful but hard.
- Work on refining existing tools to find the `minimal toolbox' is necessary and under way.
- The factorisation of soft and collinear virtual amplitudes contains important information.
- A general all-order definition of soft and/or collinear counterterms has been proposed.
- Existing results at NLO and beyond are reproduced and systematised.
- Tracing the real emission counterterms starting from virtual poles is a useful strategy.
- A parallel effort to construct a detailed analytic subtraction algorithm is under way.
- The organisation of counterterms for massless final states at N^3LO is completed.
- What we have is promising preliminary evidence: a lot of work remains to be done.

THANK YOU!