# Helicity-dependent generalization of the JIMWLK-evolution

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# Overview

**Part 1: Proton spin puzzle**

- 19 May 1988
- Theoretical prediction in the 70's
- Small-x asymptotics

**Part 2: Helicity JIMWLK**

- Getting started: unpolarized case.
- Toward "helicity-JIMWLK"
- *(Lifetime ordering)*
- *(Target weight functional properties)*

Based on [arXiv:1910.04268]

# Part 1: Proton spin puzzle

# Proton spin puzzle / crisis.

Volume 206, number 2 PHYSICS LETTERS B 19 May 1988

The spin asymmetry in deep inelastic scattering of longitudinally polarised muons by longitudinally polarised protons has been measured over a large $x$ range $(0.01<x<0.7)$. The spin-dependent structure function $g_1(x)$ for the proton has been determined and its integral over $x$ found to be $0.114 \pm 0.012 \pm 0.026$, in disagreement with the Ellis–Jaffe sum rule. Assuming the validity of the Bjorken sum rule, this result implies a significant negative value for the integral of $g_1$ for the neutron. These values for the integrals of $g_1$ lead to the conclusion that the total quark spin constitutes a rather small fraction of the spin of the nucleon.

**Reminder**

$$g_1^\gamma = \frac{1}{2} \sum_q e_q^2 \; (\Delta q + \Delta \bar{q}) \,, \qquad \Delta q = q^\uparrow - q^\downarrow \text{ w.r.t. the proton spin} \tag{1}$$

and they observed for the proton

$$\int_{0.01}^{0.7} g_1(x) \, \mathrm{d}x = 0.114 \pm 0.012(\text{stat.}) \pm 0.026(\text{syst.}) \tag{2}$$

**Remarks**

- In blue: finite range of integration. "*... the small $x$ region is expected to make a large contribution to the integrals.*"
- In red: Ellis-Jaffe sum rule. → Theoretical understanding of the 70's.

→ How do we understand this value?

# Theoretical prediction in the 70's

$\rightarrow$ How do we understand this value? $0.114 \pm 0.012(\text{stat.}) \pm 0.026(\text{syst.})$

**Ellis-Jaffe sum rule, assumptions**

- Sea $q\bar{q}$: $\lambda^+(x) \simeq \lambda^-(x) \simeq \bar{\lambda}^+(x) \simeq \bar{\lambda}^-(x)$
- Ansatz $\Delta s \sim 0$ (no intrinsic strangeness)
- Belief that valence quarks carry the proton spin.

we obtain[11]

$$\int_0^1 d\xi\, g_1^{ep}(\xi) = \frac{g_A}{12}(1.78), \tag{6}$$

$$\int_0^1 d\xi\, g_1^{en}(\xi) = \frac{g_A}{12}(-0.22), \tag{7}$$

where $g_A = 1.248 \pm .010$.

**Ellis-Jaffe sum rule prediction (70's):** $0.185 \pm 0.0015 \longrightarrow$ Not compatible with 0.114

*Where is the missing spin ?*

Old fundamental problem ($\sim 30y$) $\rightarrow$ looking at small number adding up to $1/2$.

- There are progresses $\rightarrow$ Still don't understand the spin of the proton in term of QCD dof.

A more recent picture of the proton spin.

Spin sum rule (Jaffe Manohar decomposition [Nucl. Phys. B337, 509 (1990)])

Quark OAM

Gluon OAM

$$\frac{1}{2} = \frac{1}{2}\Sigma_q + \Sigma_g + L_q + L_g$$

Gluon spin

Quark spin

Proton spin

*Possibilities:*

- Gluon spin
- Quark/Gluon angular orbital momentum

*Problems*

- Gauge-dependence and non-unicity of the decomposition.
  (e.g. Ji-decomposition has no identification of gluon spin/OAM)

**Large and low x region.** Experiments only access a finite range of $x$...

$$\Sigma_q = \int_0^1 \mathrm{d}x \left(q^{\uparrow}(x) - q^{\downarrow}(x)\right) \quad (3)$$

*Possibilities*

- Large-x?
- Small-x?

# Potential solution, small-x tail (1/3)

**Question**: What is the value of the following?

$$\int_0^{x_{min}} \mathrm{d}x \, \left(q^{\uparrow}(x) - q^{\downarrow}(x)\right)\Big|_{\text{Global fit}} - \int_0^{x_{min}} \mathrm{d}x \, \left(q^{\uparrow}(x) - q^{\downarrow}(x)\right)\Big|_{\text{Small-x}} = ?? \qquad (4)$$

Parton disribution functions at small-x are given by

$$PDF(x) \sim \left(\frac{1}{x}\right)^{\alpha}, \qquad \text{where } \alpha \text{ is the intercept.} \qquad (5)$$

| Observable | Evolution | Intercept | $Q^2 = 3$ GeV$^2$ $\alpha_s = 0.343$ | $Q^2 = 10$ GeV$^2$ $\alpha_s = 0.249$ | $Q^2 = 87$ GeV$^2$ $\alpha_s = 0.18$ |
|---|---|---|---|---|---|
| Unpolarized flavor singlet structure function $F_2$ | LO BFKL Pomeron | $1 + \frac{\alpha_s N_c}{\pi} 4 \ln 2$ | 1.908 | 1.659 | 1.477 |
| Unpolarized flavor non-singlet structure function $F_2$ | Reggeon | $\sqrt{\frac{2\,\alpha_s C_F}{\pi}}$ | 0.540 | 0.460 | 0.391 |
| Flavor singlet structure function $g_1^S$ | us (Pure Glue) | $2.31\sqrt{\frac{\alpha_s N_c}{2\pi}}$ | 0.936 | 0.797 | 0.678 |
| | BER (Pure Glue) | $3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$ | 1.481 | 1.262 | 1.073 |
| | BER ($N_f = 4$) | $3.45\sqrt{\frac{\alpha_s N_c}{2\pi}}$ | 1.400 | 1.190 | 1.011 |
| Flavor non-singlet structure function $g_1^{NS}$ | BER and us (large-$N_c$) | $\sqrt{\frac{\alpha_s N_c}{\pi}}$ | 0.572 | 0.488 | 0.415 |

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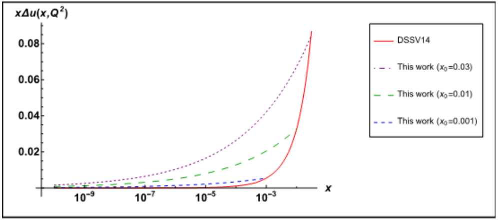
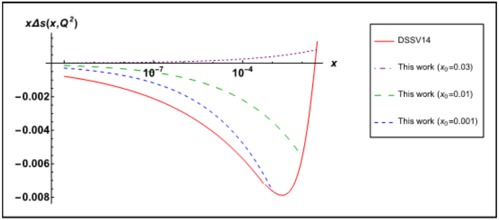
**Ballpark Phenomenology**
Attach the small-x tail to existing helicity PDF given by the intercept $\alpha$.

$$hPDF \sim \left(\frac{1}{x}\right)^{\alpha}$$

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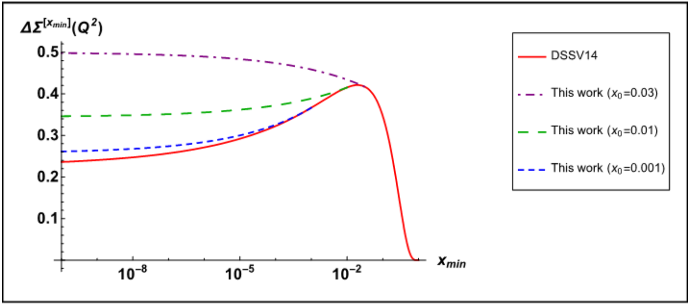
$x_0$ is the small value where the asymptotic is attached to the existing helicity PDFs

Define

$$\Delta\Sigma^{[x_{min}]}(Q^2) \equiv \int_{x_{min}}^{1} dx \Delta\Sigma(x, Q^2) \quad (6)$$

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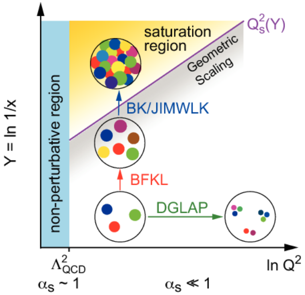
# Part 2: Helicity JIMWLK




**Unpolarized case, small-x evolution**

BK (large $N_c$) $\longleftrightarrow$ JIMWLK (all $N_c$)

**Helicity-dependent small-x evolution**

KPS (large $N_c$ or large $N_c \& N_f$) $\longleftrightarrow$ " *h-JIMWLK* " (all $N_c$)

**Some remarks on KPS**

- KPS $\sim$ "helicity BK"
- KPS equations, written in the shock wave formalism.
- Found by sub-eikonal correction to the eikonal shock-wave.
- Equations written using Wilson Lines and *polarized Wilson Lines*
- Closes in two limits: large $Nc$ / large $N_c \& N_f$.

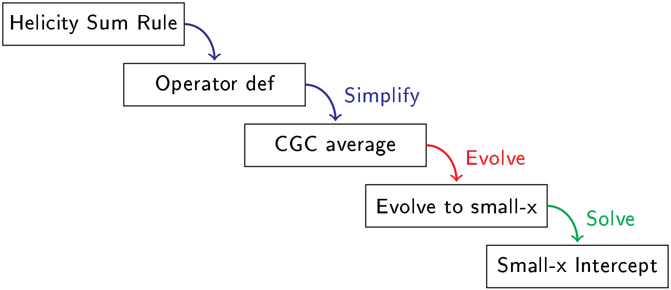
$\Rightarrow$ Does not close for arbitrary $N_c$ and $N_f$

**Rmk**: $\exists$ other frameworks for $g_1$ at small-x, such as Bartel, Ermolaev, and Ryskin (1996).

# The small-x region (2/2)

Yuri's (and *al.*) approach "Simplify, Evolve, and Solve"

[Y.V Kovchegov, D. Pitonyak, and M. D. Sievert 2016 2017] [Y.V Kovchegov, and M. D. Sievert 2018]




**Helicity distributions** (flavor-singlet)

$$g_{1L}^{S}(x,k_T^2) = \frac{8N_c}{(2\pi)^6} \int d^2\zeta\, d^2w\, d^2y\, e^{-i\underline{k}\cdot(\underline{\zeta}-\underline{y})} \int_{\Lambda^2/s}^{1} \frac{dz}{z} \, \frac{\underline{\zeta}-\underline{w}}{|\underline{\zeta}-\underline{w}|^2} \cdot \frac{\underline{y}-\underline{w}}{|\underline{y}-\underline{w}|^2}\, G_{\underline{w},\underline{\zeta}}(zs) \qquad (7)$$

where

$$G_{\underline{w},\underline{\zeta}}(zs) = \frac{k_1^- \, p^+}{N_c} \, \mathrm{Re}\, \left\langle \mathrm{T\,tr}\left[V_{\underline{\zeta}}\, V_{\underline{w}}^{pol\,\dagger}\right] + \mathrm{T\,tr}\left[V_{\underline{w}}^{pol}\, V_{\underline{\zeta}}^{\dagger}\right]\right\rangle \qquad (8)$$

*Think of it as a regular dipole amplitude (for the moment)*

# Derivation of JIMWLK - Following [A.H. Mueller, Phys.Lett. **B523**, 243 (2001)]

Rapidity-dependent expectation value of an arbitrary operator $\hat{\mathcal{O}}$ is

$$\langle \hat{\mathcal{O}}_\alpha \rangle_Y = \frac{\int \mathcal{D}\alpha \; \hat{\mathcal{O}} \; \mathcal{W}_Y[\alpha]}{\int \mathcal{D}\alpha \; \mathcal{W}_Y[\alpha]} \qquad \int \mathcal{D}\alpha \; \mathcal{W}_Y[\alpha] = 1$$

Rapidity $Y$

$\alpha \equiv A^+$ with $A^+$ in the Lorentz gauge

target weight functional

target average

**Step 1:** Increase $Y \longrightarrow Y + \delta Y$. Two equivalent pictures:

(a) Test operator evolves $\rightarrow$ additional gluons by a kernel $\mathcal{K}$.

(b) Evolve the target weight functional.

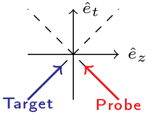
**Step 2:** Equate the two to find:

$$\partial_Y \langle \hat{\mathcal{O}}_\alpha \rangle_Y = \int \mathcal{D}\alpha \left( \mathcal{K} \cdot \hat{\mathcal{O}}_\alpha \right) \mathcal{W}_Y[\alpha] = \int \mathcal{D}\alpha \; \hat{\mathcal{O}} \; \partial_Y \mathcal{W}_Y[\alpha] \tag{9}$$

**Step 3:** Integrate by parts:

$$\mathcal{K} \cdot \mathcal{W}_Y[\alpha] = \partial_Y \mathcal{W}_Y[\alpha] \tag{10}$$

Using lc coordinates $a^{\pm} = \frac{1}{\sqrt{2}}(a^0 \pm a^3)$
Frame choice: $\rightarrow$ Probe minus-moving, target plus-moving.




**Def:** Wilson Lines are

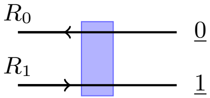
$$W_{\underline{x}}^{(R)}[b^-, a^-] \equiv \mathsf{P}\ \exp\left\{ ig \int_{a^-}^{b^-} dx^-\ t_R^a\ \alpha^a(x^+ = 0, x^-, \underline{x}) \right\} \tag{11}$$

$\Rightarrow$ Depends only on the background field $\alpha = A^+$ (Lorentz Gauge).

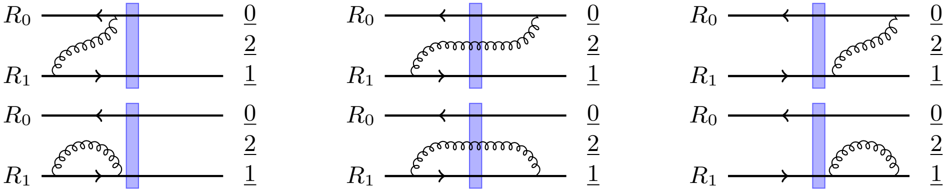
Shorthand notation: $W_{\underline{x}}^{(R)} \equiv W_{\underline{x}}^{(R)}[\infty, -\infty]$.

**Consider** a test operator of two Wilson Lines (WL)

$$\hat{\mathcal{O}}_{\underline{1},\underline{0}} = W_{\underline{0}}^{(R_0)\dagger} \otimes W_{\underline{1}}^{(R_1)} = \tag{12}$$



**Step 1a:** Diagrams to consider (plus horizontal mirroring)




⇒ How does one generate all those diagrams?

Consider the second diagram in the first row:

$$-\alpha_s \int \mathrm{d}^2\underline{x}_2 \frac{\underline{x}_{21}\cdot\underline{x}_{20}}{\underline{x}_{21}^2\underline{x}_{20}^2} U_{\underline{2}}^{ba} \left( W_{\underline{0}}^{(R_0)\dagger} t^b_{R_0} \otimes W_{\underline{1}}^{(R_1)} t^a_{R_1} \right) \tag{13}$$

Difficulties for a generic test operator $\hat{\mathcal{O}}$

(1) Color generator → depends on the WL irrep.

(2) Position of the emission/absorption vertex → differs in every topology.

(3) Grows of the number of diagrams with the number of WL involved.

## Derivation of JIMWLK - In practice and tricks to learn (3/4)

**Main trick:** take functional derivative w.r.t. the $\alpha$-field.

$$\frac{\delta\, W_{\underline{x}}^{(R)}}{\delta\, \alpha^a(y^-, \underline{y})} = ig\, \delta^{(2)}(\underline{y} - \underline{x})\; W_{\underline{x}}^{(R)}[\infty, y^-]\, t_R^a\, W_{\underline{x}}^{(R)}[y^-, -\infty] \tag{14}$$

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$\Rightarrow$ Valid for any irrep $R$.
(1) ✓

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**Remarks:**

- Semi-infinite WL which does **not** cross the shock-wave $\rightarrow$ Identity in color space.
- Semi-infinite WL which does cross the shock-wave $\rightarrow$ Complete to infinite WL.

$$\frac{\delta\, W_{\underline{x}}^{(R)}}{\delta\, \alpha^a(y^-, \underline{y})} = ig\, \delta^{(2)}(\underline{y} - \underline{x}) \begin{cases} t_R^a\, W_{\underline{x}}^{(R)} & \text{for } y^- > 0 \\ W_{\underline{x}}^{(R)}\, t_R^a & \text{for } y^- < 0 \end{cases} \tag{15}$$

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$\Rightarrow$ Sign of $y^-$ gives the position of $t_R$ w.r.t. the WL.
(2) ✓

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**Remarks:**

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- Semi-infinite WL which does cross the shock-wave $\rightarrow$ Complete to infinite WL.

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$\Rightarrow$ Sign of $y^-$ gives the position of $t_R$ w.r.t. the WL.
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- Leibniz rule for the functional derivative $\rightarrow$ All vertex insertions in the test operator

**Main trick:** take functional derivative w.r.t. the $\alpha$-field.

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$\Rightarrow$ Valid for any irrep $R$.
(1) ✓

**Remarks:**

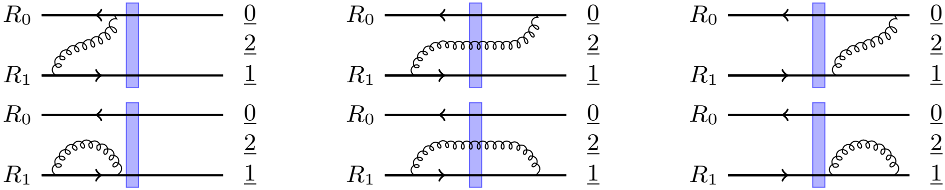
- Semi-infinite WL which does **not** cross the shock-wave $\to$ Identity in color space.
- Semi-infinite WL which does cross the shock-wave $\to$ Complete to infinite WL.

$$\frac{\delta\, W_{\underline{x}}^{(R)}}{\delta\, \alpha^a(y^-, \underline{y})} = ig\, \delta^{(2)}(\underline{y} - \underline{x})\; \begin{cases} t_R^a\, W_{\underline{x}}^{(R)} & \text{for } y^- > 0 \\ W_{\underline{x}}^{(R)}\, t_R^a & \text{for } y^- < 0 \end{cases} \tag{15}$$

$\Rightarrow$ Sign of $y^-$ gives the position of $t_R$ w.r.t. the WL.
(2) ✓

- Leibniz rule for the functional derivative $\to$ All vertex insertions in the test operator

$\Rightarrow$ Valid for any number of Wilson Lines.
(3) ✓




⇒ How does one generate all those diagrams? ✓

$$\mathcal{K}_{JIMWLK} \equiv \frac{\alpha_s}{\pi^2} \int \mathrm{d}^2 x_\perp\, \mathrm{d}^2 y_\perp\, \mathrm{d}^2 w_\perp\, \frac{(\underline{x}-\underline{w})\cdot(\underline{y}-\underline{w})}{|\underline{x}-\underline{w}|^2\, |\underline{y}-\underline{w}|^2} \left(U_{\underline{w}} - \frac{U_{\underline{x}}+U_{\underline{y}}}{2}\right)^{ba} \frac{(ig)^{-2}\, \delta^2}{\delta\, \alpha^a(x^- < 0, \underline{x})\, \delta\, \alpha^b(y^- > 0, \underline{y})}. \tag{16}$$

Equating the two picture and integrating by parts yields

$$\partial_Y W_Y[\alpha] = \mathcal{K}_{JIMWLK} \cdot W_Y[\alpha] \tag{17}$$

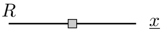
Based on [arXiv:1910.04268]

Polarized Wilson Lines:

$$W_{\underline{x}}^{(R),pol}[b^-,a^-] = W_{\underline{x}}^{(R),pol,q}[b^-,a^-] + W_{\underline{x}}^{(R),pol,g}[b^-,a^-] \tag{18}$$

⇒ Depends on $\alpha, \beta, \psi, \bar{\psi}$, where $\beta = \epsilon_\perp^{ij}\partial_i A_j$.

⇒ Two components: $q\bar{q}$ t-channel exchange $(\psi, \bar{\psi})$ and $g$ t-channel exchange $(\beta)$

$\left(W_{\underline{x}}^{(R)}\right)^{pol} \equiv$ 

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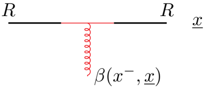
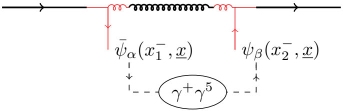
$\left(W_{\underline{x}}^{(R)}\right)^{pol,g} \propto$ 


$V_{\underline{x}}^{pol,q} \propto$ 

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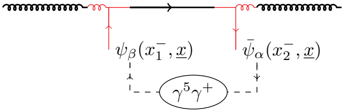
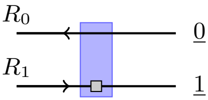
$U_{\underline{x}}^{pol,q} \propto$ 

+ c.c. .

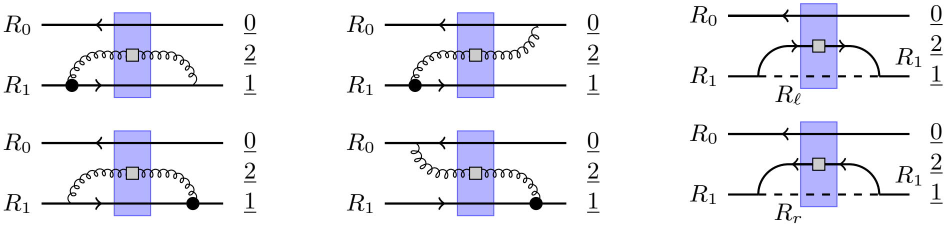
# Toward “helicity-JIMWLK” - Diagrammatic content

**Consider** the test operator

$$\hat{\mathcal{O}}_{\underline{1},\underline{0}} = s\left(W_{\underline{0}}^{(R_0)\dagger} \otimes W_{\underline{1}}^{(R_1),pol} + c.c.\right) = \tag{19}$$




**Sub-eikonal helicity-dependent diagrams.** Flavor-singlet case only here!




- The polarization information is carried by the WL at $\underline{x}_2$.
- The latter two diagrams change the irrep of the WL crossing the shock-wave at $\underline{x}_1$.

# Toward "helicity-JIMWLK" - Functional derivatives (1/2)

$$\frac{\delta W_{\underline{x}}^{(R)\,pol,q}}{\delta \alpha^a(y^-,\underline{y})} = ig\,\delta^{(2)}(\underline{y}-\underline{x}) \begin{cases} W_{\underline{x}}^{(R)\,pol}\, t_R^a & \text{for } y^- < 0 \\ t_R^a\, W_{\underline{x}}^{(R)\,pol} & \text{for } y^- > 0 \end{cases} \tag{20}$$

$$\frac{\delta W_{\underline{x}}^{(R)\,pol}}{\delta \beta^a(y^-,\underline{y})} = \eta_R \frac{igp_1^+}{s}\;\delta^{(2)}(\underline{y}-\underline{x}) \begin{cases} W_{\underline{x}}^{(R)}\, t_R^a & \text{for } y^- < 0 \\ t_R^a\, W_{\underline{x}}^{(R)} & \text{for } y^- > 0 \end{cases} \tag{21}$$

with $\eta_R = \delta_{R=F} + \delta_{R=\bar{F}} + 2\delta_{R=A}$

$\Rightarrow \frac{\delta}{\delta\,\beta}$ acting on a polarized WL gives an unpolarized WL!

$$\frac{\delta^2\; (V_{\underline{x}}^{pol,q})_{j'i'}}{\delta\,\bar{\psi}_{\alpha,i}(y_1^-,\underline{y}_1)\;\delta\,\psi_{\beta,j}(y_2^-,\underline{y}_2)} = \delta^{(2)}(\underline{y}_1-\underline{x})\,\delta^{(2)}(\underline{y}_2-\underline{x})\left(ig\sqrt{p_1^+/s}\right)^2 \begin{cases} (t^b)_{j'j}\, U_{\underline{x}}^{ba}\, (t^a)_{ii'} & \text{for } y_1^- < 0 < y_2^- \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

$\Rightarrow$ Change the representation of the WL crossing the shock-wave.

$$\begin{aligned} \frac{\delta^2\; (U_{\underline{x}}^{pol,q})^{ab}}{\delta\,\bar{\psi}_{\alpha,i}(y_1^-,\underline{y}_1)\;\delta\,\psi_{\beta,j}(y_2^-,\underline{y}_2)} = &-\;\delta^{(2)}(\underline{y}_1-\underline{x})\,\delta^{(2)}(\underline{y}_2-\underline{x})\left(ig\sqrt{p_1^+/s}\right)^2\left(\frac{\gamma^+\gamma^5}{2}\right)_{\alpha\beta} \\ &\times\Big[\theta(y_1^-)\,\theta(-y_2^-)\;U_{\underline{x}}^{aa'}[\infty,y_1^-]\;\left(t^{a'}\;V_{\underline{x}}[y_1^-,y_2^-]\,t^{b'}\right)_{ij}\;U_{\underline{x}}^{b'b}[y_2^-,-\infty] \\ &+\theta(y_2^-)\,\theta(-y_1^-)\;U_{\underline{x}}^{aa'}[\infty,y_2^-]\;\left(t^{b'}\;V_{\underline{x}}[y_1^-,y_2^-]\,t^{a'}\right)_{ij}\;U_{\underline{x}}^{b'b}[y_1^-,-\infty]\Big]. \end{aligned} \tag{23}$$

**Shorthand notations** (to absorb some of the trivial factors)

For the field $\alpha(x)$ and the field $\beta(x)$

$$D^{+}_{\underline{y},a,\lessgtr} \equiv (ig)^{-1} \frac{\delta}{\delta\,\alpha^a(y^- \lessgtr 0, \underline{y})}. \tag{24}$$

$$D^{\perp}_{\underline{y},a,\lessgtr} \equiv \left(\frac{ig\,p_1^+}{s}\right)^{-1} \frac{\delta}{\delta\,\beta^a(y^- \lessgtr 0, \underline{y})}. \tag{25}$$

For the quark fields we have

$$D^{\psi}_{\underline{y},\alpha,i,\lessgtr} \equiv \left(ig\sqrt{p_1^+/s}\right)^{-1} \frac{\delta}{\delta\,\psi_{\alpha,i}(y^- \lessgtr 0, \underline{y})}, \tag{26}$$

$$D^{\bar{\psi}}_{\underline{y},\alpha,i,\lessgtr} \equiv \left(ig\sqrt{p_1^+/s}\right)^{-1} \frac{\delta}{\delta\,\bar{\psi}_{\alpha,i}(y^- \lessgtr 0, \underline{y})}. \tag{27}$$

**Evolution Kernel** for the helicity-dependent evolution:

$$\begin{aligned}\mathcal{K}_h[\tau,\tau'] &\equiv \left(\mathcal{K}^{eik} + \mathcal{K}^{\beta} + \mathcal{K}^{\psi,\bar{\psi}}\right)[\tau,\tau'] \\ &= \frac{\alpha_s}{\pi^2}\int \mathrm{d}^2 w_\perp \frac{\underline{X}'\cdot\underline{Y}'}{X'^2\,Y'^2}\;\theta^{(3)}(\tau-\tau')\theta\left(z'-\frac{\Lambda^2}{s}\right)\theta\left(X'^2-\tfrac{1}{z'\,s}\right)\theta\left(Y'^2-\tfrac{1}{z'\,s}\right) \\ &\times \Big\{U_{\underline{w}}^{ba} D^{+}_{\underline{x},a,<} D^{+}_{\underline{y},b,>} - \tfrac{1}{2}\left(D^{+}_{\underline{x},a,<} D^{+}_{\underline{y},a,<} + D^{+}_{\underline{x},a,>} D^{+}_{\underline{y},a,>}\right) \\ &+ \tfrac{1}{2} U_{\underline{w}}^{pol,ba}(D^{+}_{\underline{x},a,<} D^{\perp}_{\underline{y},b,>} + D^{\perp}_{\underline{x},a,<} D^{+}_{\underline{y},b,>}) \\ &+ \left(\tfrac{1}{2}\gamma^5\gamma^-\right)_{\beta\alpha} \tfrac{1}{2}\left((V_{\underline{w}}^{pol})_{ij} D^{\bar{\psi}}_{\underline{x},j,\alpha,<} D^{\psi}_{\underline{y},i,\beta,>} + (V_{\underline{w}}^{pol\,\dagger})_{ij} D^{\bar{\psi}}_{\underline{x},j,\alpha,>} D^{\psi}_{\underline{y},i,\beta,<}\right)\Big\}.\end{aligned} \tag{28}$$

where $\underline{X} \equiv \underline{x} - \underline{w}$, $\underline{Y} \equiv \underline{y} - \underline{w}$, $\tau \equiv \{z, zX^2, zY^2\}$, and $\tau' \equiv \{z', z'X'^2, z'Y'^2\}$.

**Evolution Equation**

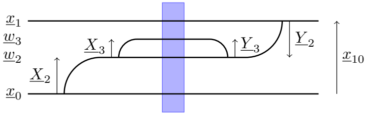
$$\mathcal{W}_\tau[\alpha,\beta,\psi,\bar{\psi}] = \mathcal{W}^{(0)}_\tau[\alpha,\beta,\psi,\bar{\psi}] + \int d^3\tau'\; \mathcal{K}_h[\tau,\tau']\cdot\mathcal{W}_{\tau'}[\alpha,\beta,\psi,\bar{\psi}] \tag{29}$$

where $d^3\tau' \equiv \frac{dz'}{z'}\mathrm{d}^2\underline{X}'\mathrm{d}^2\underline{Y}'$.

$\Rightarrow$ Need an initial condition $\mathcal{W}^{(0)}_\tau$.

# Life-time ordering

Two steps in the evolution of $\hat{\mathcal{O}}_{\underline{1},\underline{0}}$




To generate DLA-evolution:

$$\tau = \{z, z\,\underline{x}_{10}^2, z\,\underline{x}_{10}^2\}$$
$$\tau' = \{z_2, z_2\,\underline{x}_{20}^2, z_2\,\underline{x}_{21}^2\}$$
$$\tau'' = \{z_3, z_3\,\underline{x}_{32}^2, z_3\,\underline{x}_{32}^2\}$$

We previously had:

$$\mathcal{K}_k = \mathcal{K}^{eik} + \mathcal{K}^{\beta} + \mathcal{K}^{\psi,\bar{\psi}} \tag{30}$$

Comparing to JIMWLK-kernel

$$\mathcal{K}_{JIMWLK} = \int \mathrm{d}^2\underline{X}\mathrm{d}^2\underline{Y}\ \widetilde{\mathcal{K}^{eik}} \tag{31}$$

where $\widetilde{\mathcal{K}^{eik}}$ does not have the $\theta(\tau - \tau')$ imposing lifetime ordering.

- For LLA JIMWLK this is unnecessary since UV and IR integrals converge.
- Beyond LLA, solutions are known to be unstable, but lifetime ordering condition helps!
  [E. Iancu, J. D. Madrigal, A. H. Mueller, G. Soyez, D. N. Triantafyllopoulos, Phys. Lett. **B744**, 293 (2015)] and [B. Ducloué, E. Iancu, A. H. Mueller, G. Soyez, and D. N. Triantafyllopoulos, JHEP **04**, 081 (2019)]
- Curiosity: this lifetime ordering appears here for $\mathcal{K}^{eik}$ as a formal requirement for the DLA helicity-evolution.

## Target weight functional properties (1/3)

*Did I cheat you ?*

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*Did I cheat you ?* **Yes** $\rightarrow$ On the last step: integration by parts.

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What I shown up to now

$$\langle \hat{\mathcal{O}}_{pol} \rangle_\tau = \langle \hat{\mathcal{O}}_{pol} \rangle_0 + \int \mathcal{D}\alpha \mathcal{D}\beta \mathcal{D}\psi \mathcal{D}\bar{\psi} \int \mathrm{d}^3\tau' \left( \mathcal{K}_k[\tau, \tau'] \cdot \hat{\mathcal{O}}_{pol} \right) \mathcal{W}_{\tau'} \tag{32}$$

and what we want

$$\langle \hat{\mathcal{O}}_{pol} \rangle_\tau = \langle \hat{\mathcal{O}}_{pol} \rangle_0 + \int \mathcal{D}\alpha \mathcal{D}\beta \mathcal{D}\psi \mathcal{D}\bar{\psi} \int \mathrm{d}^3\tau' \; \hat{\mathcal{O}}_{pol} \left( \mathcal{K}_k[\tau, \tau'] \cdot \mathcal{W}_{\tau'} \right) \tag{33}$$

One needs (and similar for $\mathcal{W}_0$)

$$\begin{cases} \lim\limits_{\alpha(x) \to \pm\infty} \mathcal{W}_\tau[\alpha, \beta, \psi, \bar{\psi}] = 0, & \forall x, \\ \lim\limits_{\beta(x) \to \pm\infty} \mathcal{W}_\tau[\alpha, \beta, \psi, \bar{\psi}] = 0, & \forall x. \end{cases} \tag{34}$$

# Target weight functional properties (2/3)

**Assume** the following ansatz

$$\mathcal{W}_\tau = \mathcal{W}_\tau^{unpol} + \Sigma\, \mathcal{W}_\tau^{pol} \tag{35}$$

Since $\mathcal{K}_h$ contains $\mathcal{K}_{JIMWLK}$, it would be natural to recover the latter after integrating out sub-eikonal fields in the former:

$$\int \mathcal{D}\beta\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathcal{W}_\tau^{unpol}[\alpha,\beta,\psi,\bar{\psi}] = \mathcal{W}_\tau^{JIMWLK}[\alpha] \tag{36}$$

$$\int \mathcal{D}\beta\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathcal{W}_\tau^{pol}[\alpha,\beta,\psi,\bar{\psi}] = 0 \tag{37}$$

Consider an operator $\mathcal{O}[\alpha,\beta,\psi,\bar{\psi}] \equiv \mathcal{O}_{unpol}[\alpha] + \sigma\, \mathcal{O}_{pol}[\alpha,\beta,\psi,\bar{\psi}]$

$$\langle\mathcal{O}\rangle_\tau = \int \mathcal{D}\alpha\mathcal{D}\beta\mathcal{D}\psi\mathcal{D}\bar{\psi}\,[\mathcal{O}_{unpol} + \sigma\mathcal{O}_{pol}]\left[\mathcal{W}_\tau^{unpol} + \Sigma\, \mathcal{W}_\tau^{pol}\right] \tag{38}$$

**PT-symmetry** $\rightarrow$ keep $\{1,\, \sigma\Sigma\}$-terms only

$$\int \mathcal{D}\alpha\mathcal{D}\beta\mathcal{D}\psi\mathcal{D}\bar{\psi}\ \mathcal{O}_{pol}\,\mathcal{W}_\tau^{unpol} = 0 \tag{39}$$

# Target weight functional properties (3/3)

Implications for the weight functional evolution

$$\mathcal{W}_\tau^{JIMWLK}[\alpha] = \mathcal{W}_0 + \int \mathrm{d}^3\tau' \; \mathcal{K}^{eik}[\tau,\tau'] \; \mathcal{W}_{\tau'}^{JIMWLK}[\alpha] \tag{40}$$

$\Rightarrow$ Recover the JIMWLK evolution as expected

$$\mathcal{W}_\tau^{pol}[\alpha,\beta,\psi,\bar{\psi}] = \mathcal{W}_0^{pol} + \int \mathrm{d}^3\tau' \; \mathcal{K}_h[\tau,\tau'] \; \mathcal{W}_{\tau'}^{pol}[\alpha,\beta,\psi,\bar{\psi}] \tag{41}$$

$\Rightarrow$ Need the full kernel for the evolution

**Summary**

- Helicity-dependent generalization of the JIMWLK evolution. Allowing all $N_c$ and $N_f$ numerical study of small-x asymptotics of helicity distributions.
- Kernel containing the eikonal LO JIMWLK kernel and terms involving sub-eikonal interactions with quark-field and gluon-field.
- Helicity dependent Kernel in DLA requires lifetime and longitudinal momentum -orderings. $\longrightarrow$ Independent argument in favor to keep-lifetime ordering to all orders in JIMWLK and BK kernels

**Prospects**

- Initial condition for $\mathcal{W}^{(0)}$ based on an helicity dependent MV-model (almost out).
- Flavor non-singlet helicity-dependent evolution kernel.
- Both require operators that are non diagonal in flavor and irrep (e.g. $q \to g$ by exchanging polarization information with the shock wave, or sources for fermion fields)

# Backup : Polarized Wilson Lines

$$W_{\underline{x}}^{(R)\,pol,g}[b^-,a^-] = \eta_R \frac{igp_1^+}{s}\,\theta(b^-)\,\theta(-a^-)\int_{a^-}^{b^-} \mathrm{d}x^-\; W_{\underline{x}}^{(R)}[b^-,x^-]\,\beta(x^-,\underline{x})\,W_{\underline{x}}^{(R)}[x^-,a^-] \tag{42}$$

with $\eta_R = \delta_{R=F} + \delta_{R=\bar{F}} + 2\delta_{R=A}$

$$V_{\underline{x}}^{pol,q}[b^-,a^-] = -\frac{g^2 p_1^+}{s}\,\theta(b^-)\,\theta(-a^-) \tag{43}$$

$$\int_{a^-}^{b^-} \mathrm{d}x_1^- \int_{x_1^-}^{b^-} \mathrm{d}x_2^-\, V_{\underline{x}}[b^-,x_2^-]t^b\psi_\beta(x_2^-,\underline{x})U_{\underline{x}}^{ba}[x_2^-,x_1^-]\left[\tfrac{1}{2}\gamma^+\gamma^5\right]_{\alpha\beta}\bar{\psi}_\alpha(x_1^-,\underline{x})t^a V_{\underline{x}}[x_1^-,a^-].$$

$$\left(U_{\underline{x}}^{pol,q}[b^-,a^-]\right)^{ab} = -\frac{g^2 p_1^+}{s}\,\theta(b^-)\,\theta(-a^-) \tag{44}$$

$$\int_{a^-}^{b^-} \mathrm{d}x_1^- \int_{x_1^-}^{b^-} \mathrm{d}x_2^-\, U^{aa'}[b^-,x_2^-]\bar{\psi}(x_2^-,\underline{x})t^{a'}V_{\underline{x}}[x_2^-,x_1^-]\left[\tfrac{1}{2}\gamma^+\gamma^5\right]t^{b'}\psi(x_1^-,\underline{x})U_{\underline{x}}^{b'b}[x_1^-,a^-]$$

$+$ c.c..