The High-Energy Limit of 2-to-2 Scattering Amplitudes

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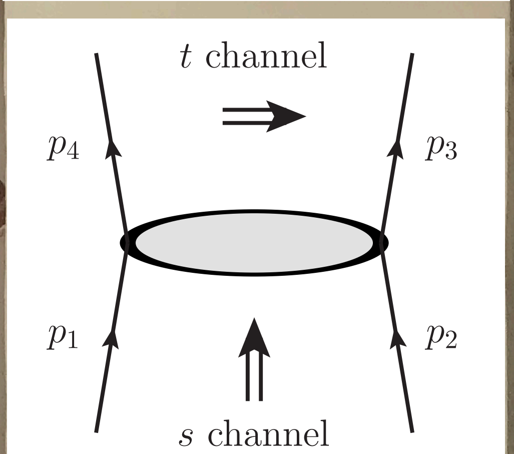
Einan Gardi (Higgs Centre, Edinburgh)

in collaboration with
Simon Caron-Huot
Joscha Reichel
Leonardo Vernazza

Refs.: JHEP 1803 (2018) 098
and a paper soon to appear



The High-Energy Limit of 2-to-2 Partonic Scattering Amplitudes

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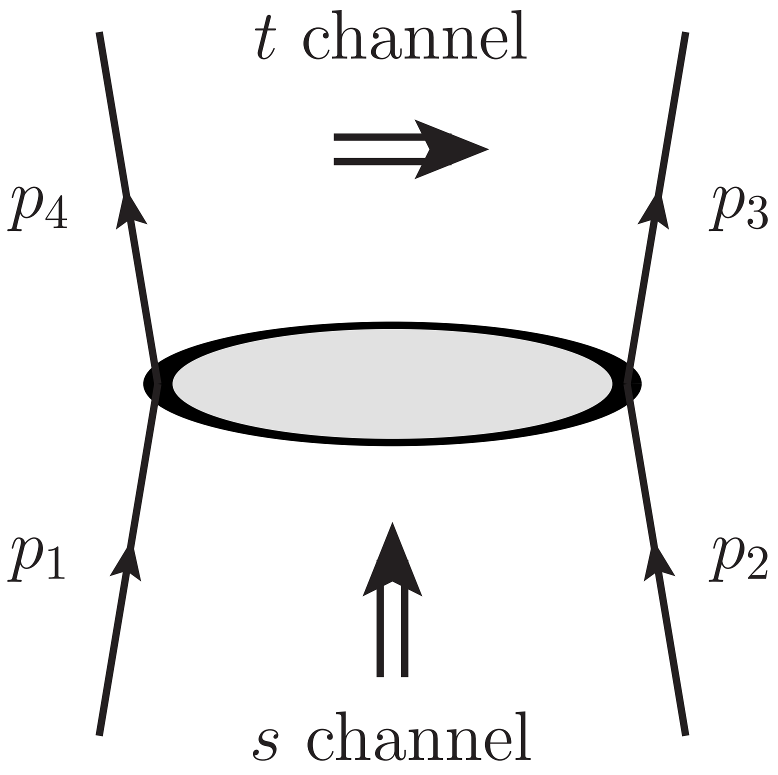
Motivation



✓ The High-energy limit is a rich source of experimentally-accessible physical phenomena, e.g.

- total cross section;
- jets at high rapidities;
- high gluon densities

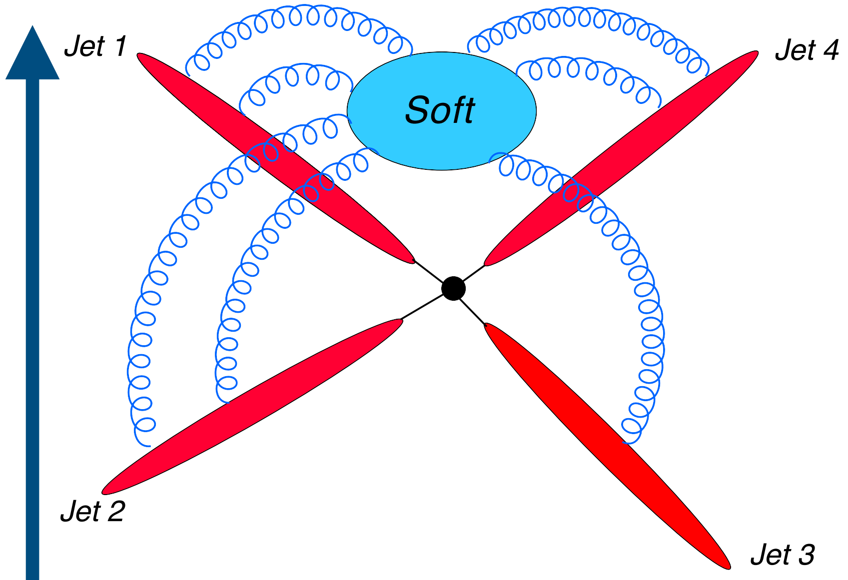
✓ QCD dynamics simplifies, allowing systematic theoretical study using Wilson lines and evolution equations

✓ Unique access to all-order properties of scattering amplitudes — complementing the study of IR divergences and summation of perturbation theory

The Regge limit overlap with infrared singularities

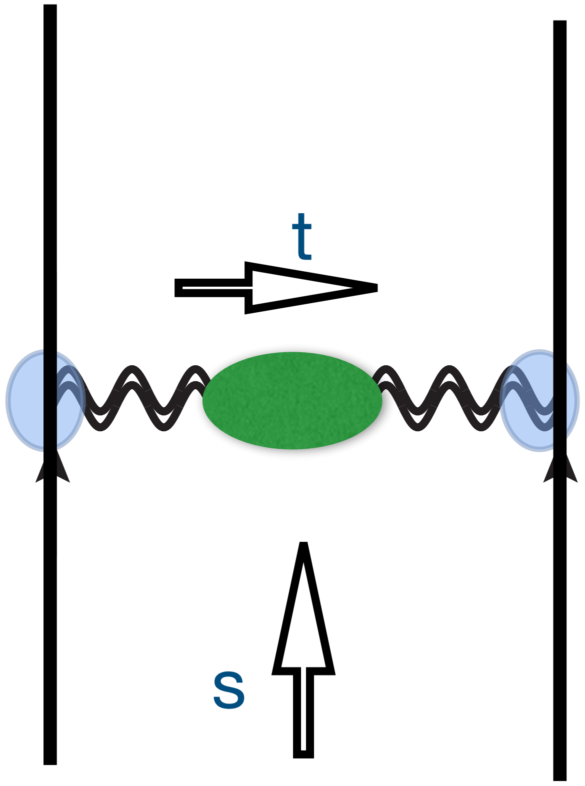
Exponentiation of infrared (soft) singularities $1/\epsilon$
in fixed-angle amplitude using factorisation into
Soft-Collinear-Hard subprocesses

IR



High-energy limit
of the soft
anomalous dimension

= Soft limit of BFKL



Regge limit

High-Energy limit:
BFKL resummation of log(s/t)
Regge factorization

IR Singularities for amplitudes with massless legs

Exponentiation of IR singularities in fixed-angle scattering:

$$\mathcal{M}\left(\frac{p_i}{\mu},\alpha_s,\epsilon\right) = \mathrm{P}\exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\Gamma\left(\lambda,\alpha_s(\lambda^2,\epsilon)\right)\right\}\mathcal{H}\left(\frac{p_i}{\mu},\alpha_s\right)$$

The Dipole Formula:

$$\Gamma_{\mathrm{Dip.}}(\lambda,\alpha_s) = \frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right)\sum_{(i,j)}\ln\left(\frac{\lambda^2}{-s_{ij}}\right)\mathbf{T}_i\cdot\mathbf{T}_j + \sum_{i=1}^{n}\gamma_{J_i}\left(\alpha_s\right)$$

Lightlike Cusp anomalous dimension
(now known to 4 loops)

Catani (1998)
Dixon, Mert-Aybat and Sterman (2006)
Becher & Neubert, EG & Magnea (2009)

There are two types of **corrections to the dipole formula**:

1. Corrections induced by higher Casimir contributions to the cusp anomalous dimension — starting at 4 loops.
2. Functions of **conformally-invariant cross ratios** — starting at 3-loops:

$$\Gamma = \Gamma_{\mathrm{Dip.}} + \Delta(\rho_{ijkl})$$

$$\rho_{ijkl} = \frac{(p_i\cdot p_j)(p_k\cdot p_l)}{(p_i\cdot p_k)(p_j\cdot p_l)}$$

The three-loop correction to the soft anomalous dimension

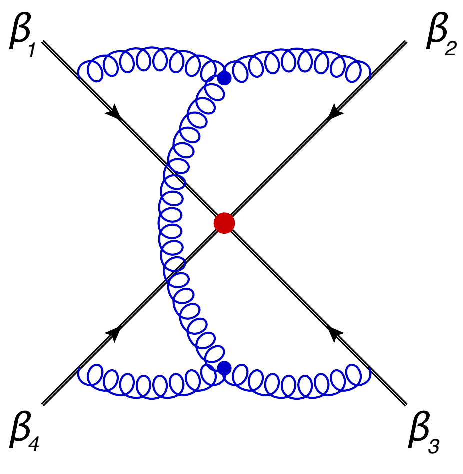
Ø. Almelid, C. Duhr, EG
Phys. Rev. Lett. **117**, 172002

$$\Delta_n^{(3)}(z,\bar{z}) = 16\left(\frac{\alpha_s}{4\pi}\right)^3 f_{abe}f_{cde}\Bigg\{\sum_{1\le i<j<k<l\le n}\Big[\,\mathbf{T}_i^a\mathbf{T}_j^b\mathbf{T}_k^c\mathbf{T}_l^d\left(F\left(1-1/z\right)-F\left(1/z\right)\right)$$

$$+\,\mathbf{T}_i^a\mathbf{T}_k^b\mathbf{T}_j^c\mathbf{T}_l^d\left(F(1-z)-F(z)\right)$$

$$+\,\mathbf{T}_i^a\mathbf{T}_l^b\mathbf{T}_j^c\mathbf{T}_k^d\left(F\left(1/(1-z)\right)-F\left(1-1/(1-z)\right)\right)\Big]$$

$$-\sum_{i=1}^{n}\sum_{\substack{1\le j<k\le n\\ j,k\neq i}}\left\{\mathbf{T}_i^a,\mathbf{T}_i^d\right\}\mathbf{T}_j^b\mathbf{T}_k^c\left(\zeta_5+2\zeta_2\zeta_3\right)\Bigg\}$$



$$\boxed{F(z)=\mathcal{L}_{10101}(z)+2\zeta_2\left(\mathcal{L}_{100}(z)+\mathcal{L}_{001}(z)\right)}$$

$$\rho_{1234} = z\bar{z}$$
$$\rho_{1432} = (1-z)(1-\bar{z})$$

$\mathcal{L}_{10\ldots}(z)$ are the single-valued harmonic polylogarithms (SVHPLs) introduced by Francis Brown in 2009. They are single-valued in the region where $\bar{z} = z^*$.

The result was re-derived it by boostrap — a crucial input was the Regge limit!

Ø. Almelid, C. Duhr, EG, A. McLeod, C.D. White, JHEP 09 (2017) 073

State-of-the-art knowledge of the soft anomalous dimension in the high-energy limit

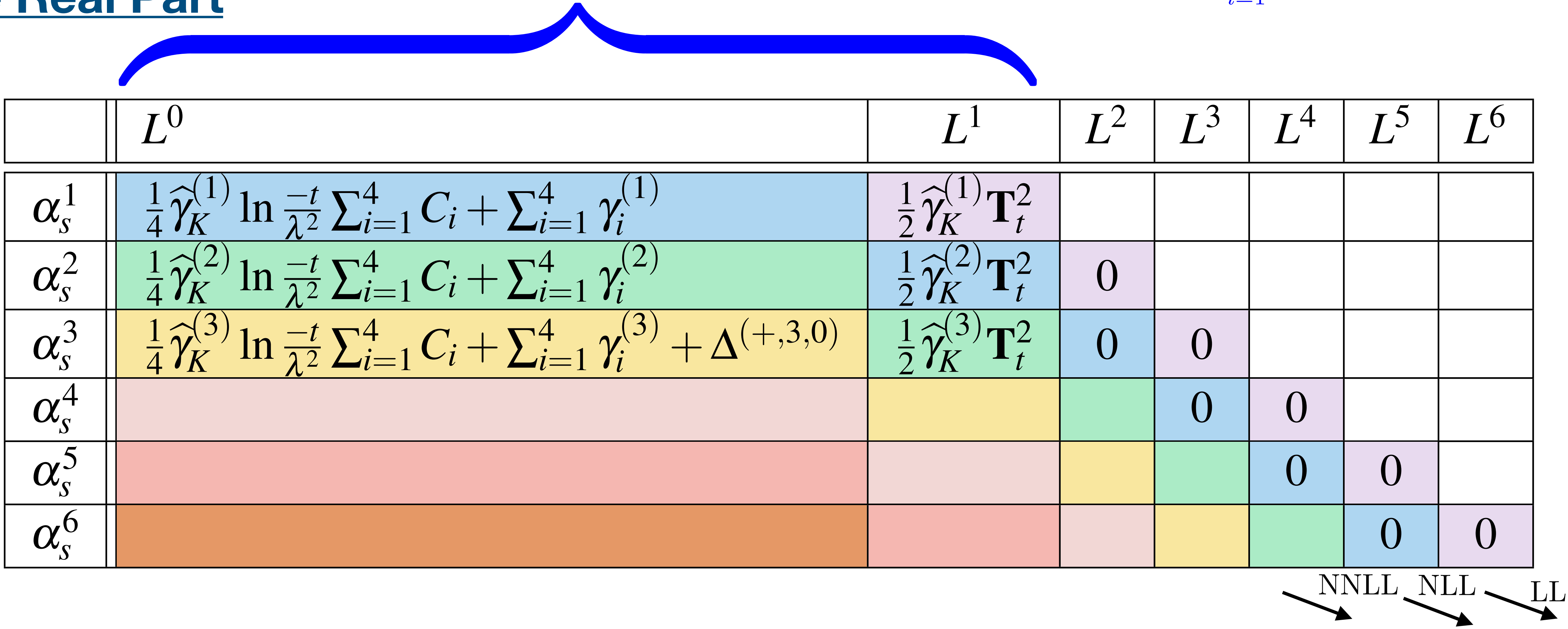
In the high-energy limit the soft anomalous dimension $\Gamma = \Gamma_{\text{Dip.}} + \Delta(\rho_{ijkl})$ may be arranged in towers of logarithms:

$$L \equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) = \log\left|\frac{s}{t}\right| - i\frac{\pi}{2}$$

$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\, \mathbf{T}_t^2 + \mathbf{\Gamma}_{\text{NLL}}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\text{NNLL}}\left(\alpha_s, L\right) + \ldots .$$

The Real Part

$$\Gamma^{\text{dip.}}\left(\{p_i\}, \lambda, \alpha_s(\lambda^2)\right) \xrightarrow{\text{Regge}} \frac{\gamma_K(\alpha_s)}{2}\left[L\, \mathbf{T}_t^2 + i\pi\, \mathbf{T}_{s-u}^2 + \frac{C_{\text{tot}}}{2}\log\frac{-t}{\lambda^2}\right] + \sum_{i=1}^{4}\gamma_i(\alpha_s) + \mathcal{O}\left(\frac{t}{s}\right),$$



	L^0	L^1	L^2	L^3	L^4	L^5	L^6
α_s^1	$\frac{1}{4}\widehat{\gamma}_K^{(1)} \ln\frac{-t}{\lambda^2}\sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(1)}$	$\frac{1}{2}\widehat{\gamma}_K^{(1)}\mathbf{T}_t^2$					
α_s^2	$\frac{1}{4}\widehat{\gamma}_K^{(2)} \ln\frac{-t}{\lambda^2}\sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(2)}$	$\frac{1}{2}\widehat{\gamma}_K^{(2)}\mathbf{T}_t^2$	0				
α_s^3	$\frac{1}{4}\widehat{\gamma}_K^{(3)} \ln\frac{-t}{\lambda^2}\sum_{i=1}^4 C_i + \sum_{i=1}^4 \gamma_i^{(3)} + \Delta^{(+,3,0)}$	$\frac{1}{2}\widehat{\gamma}_K^{(3)}\mathbf{T}_t^2$	0	0			
α_s^4				0	0		
α_s^5					0	0	
α_s^6						0	0

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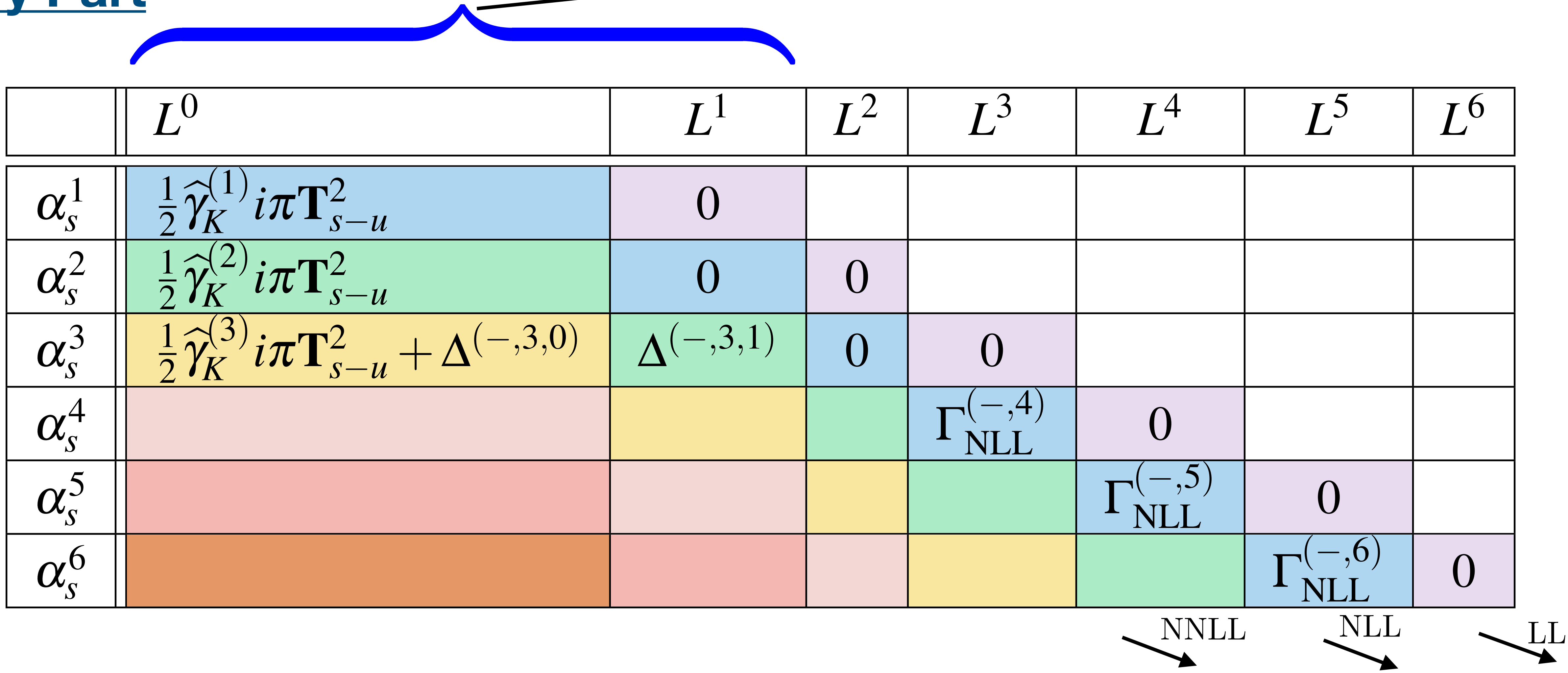
$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\,\mathbf{T}_t^2 + \mathbf{\Gamma}_{\text{NLL}}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\text{NNLL}}\left(\alpha_s, L\right) + \ldots\,.$$

$$\Gamma^{\text{dip.}}\left(\{p_i\},\lambda,\alpha_s(\lambda^2)\right) \xrightarrow{\text{Regge}} \frac{\gamma_K(\alpha_s)}{2}\left[L\,\mathbf{T}_t^2 + \boxed{i\pi\,\mathbf{T}_{s-u}^2} + \frac{C_{\text{tot}}}{2}\log\frac{-t}{\lambda^2}\right] + \sum_{i=1}^{4}\gamma_i(\alpha_s) + \mathcal{O}\left(\frac{t}{s}\right),$$

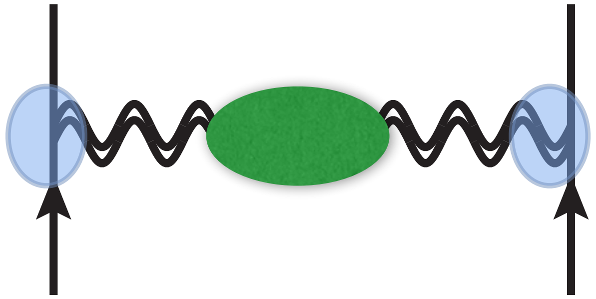
Imaginary Part



	L^0	L^1	L^2	L^3	L^4	L^5	L^6
α_s^1	$\frac{1}{2}\widehat{\gamma}_K^{(1)} i\pi\mathbf{T}_{s-u}^2$	0					
α_s^2	$\frac{1}{2}\widehat{\gamma}_K^{(2)} i\pi\mathbf{T}_{s-u}^2$	0	0				
α_s^3	$\frac{1}{2}\widehat{\gamma}_K^{(3)} i\pi\mathbf{T}_{s-u}^2 + \Delta^{(-,3,0)}$	$\Delta^{(-,3,1)}$	0	0			
α_s^4				$\Gamma_{\text{NLL}}^{(-,4)}$	0		
α_s^5					$\Gamma_{\text{NLL}}^{(-,5)}$	0	
α_s^6						$\Gamma_{\text{NLL}}^{(-,6)}$	0

The High-energy limit: a Reggeized gluon

Leading logs of (-t/s) are summed through gluon Reggeization:



$$\frac{1}{t} \longrightarrow \frac{1}{t}\left(\frac{s}{-t}\right)^{\alpha(t)}$$

Exponentiation of high-energy logarithms is **fully consistent with the dipole formula for IR singularities:**

$$\alpha(t) = \frac{1}{4}\mathbf{T}_t^2 \int_0^{-t} \frac{d\lambda^2}{\lambda^2}\widehat{\gamma}_K(\alpha_s(\lambda^2,\epsilon))$$

Korchemsky (1993)
Korchemskaya and Korchemsky (1996)
Del Duca, Duhr, EG, Magnea & White (2011)

For the **Real part** of the amplitude, this “Regge pole” factorization can be improved to NLL by introducing impact factors and corrections to the trajectory. — **but** beyond this, the exchange of **multiple** Reggeized gluons kick in!

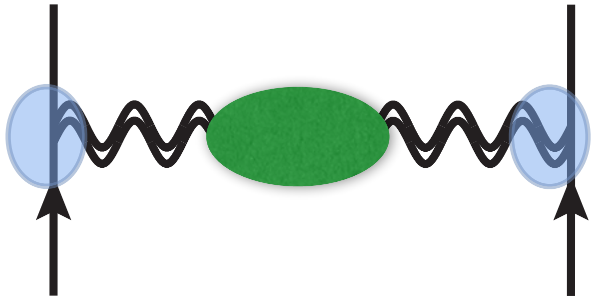
Del Duca, Glover (2001), Del Duca, Falcioni, Magnea, Vernazza (2013)

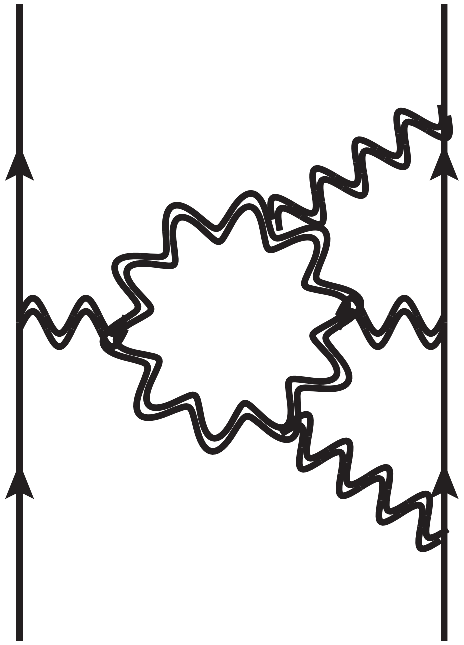
High-energy limit: exchange of multiple Reggeized gluons

Recent progress: we now know to use JIMWLK/BFKL rapidity evolution to **compute** multiple Reggeized gluon contributions to 2-to-2 amplitudes.

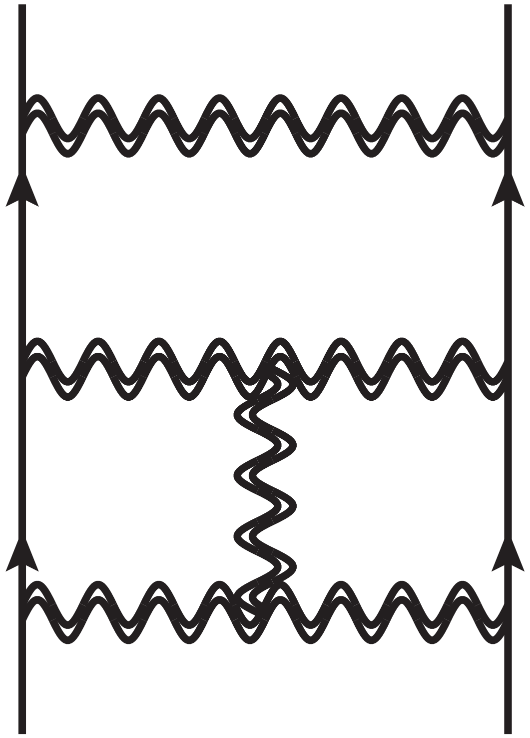
Signature odd (Real) part of the amplitude

LL, NLL





NNLL

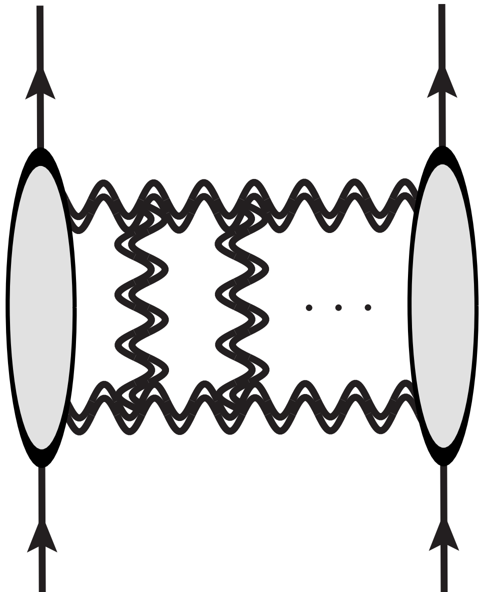


Caron-Huot, EG, Vernazza - JHEP 06 (2017) 016 — checked against Henn & Mistlberger (2017)

Signature even (Imaginary) amplitude

Caron-Huot JHEP 05 (2015) 093

Caron-Huot, EG, Reichel, Vernazza - JHEP 1803 (2018) 098
(and a paper soon to appear)



NLL

The BFKL equation in dimensional regularization — an iterative solution

The BFKL equation for the **even amplitude** takes the form:

$$\frac{d}{dL}\Omega(p,k) = \frac{\alpha_s B_0(\epsilon)}{\pi}\hat{H}\Omega(p,k)$$

The Hamiltonian is non-trivial, and we do not know to directly diagonalise it, but we can always use an iterative (perturbative) solution:

Substituting:

$$\Omega(p,k) = \sum_{\ell=1}^{\infty}\left(\frac{\alpha_s}{\pi}B_0\right)^{\ell}\frac{L^{\ell-1}}{(\ell-1)!}\,\Omega^{(\ell-1)}(p,k)$$

It follows that the wavefunction is defined by iterating the Hamiltonian:

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\Omega^{(\ell-2)}(p,k)$$

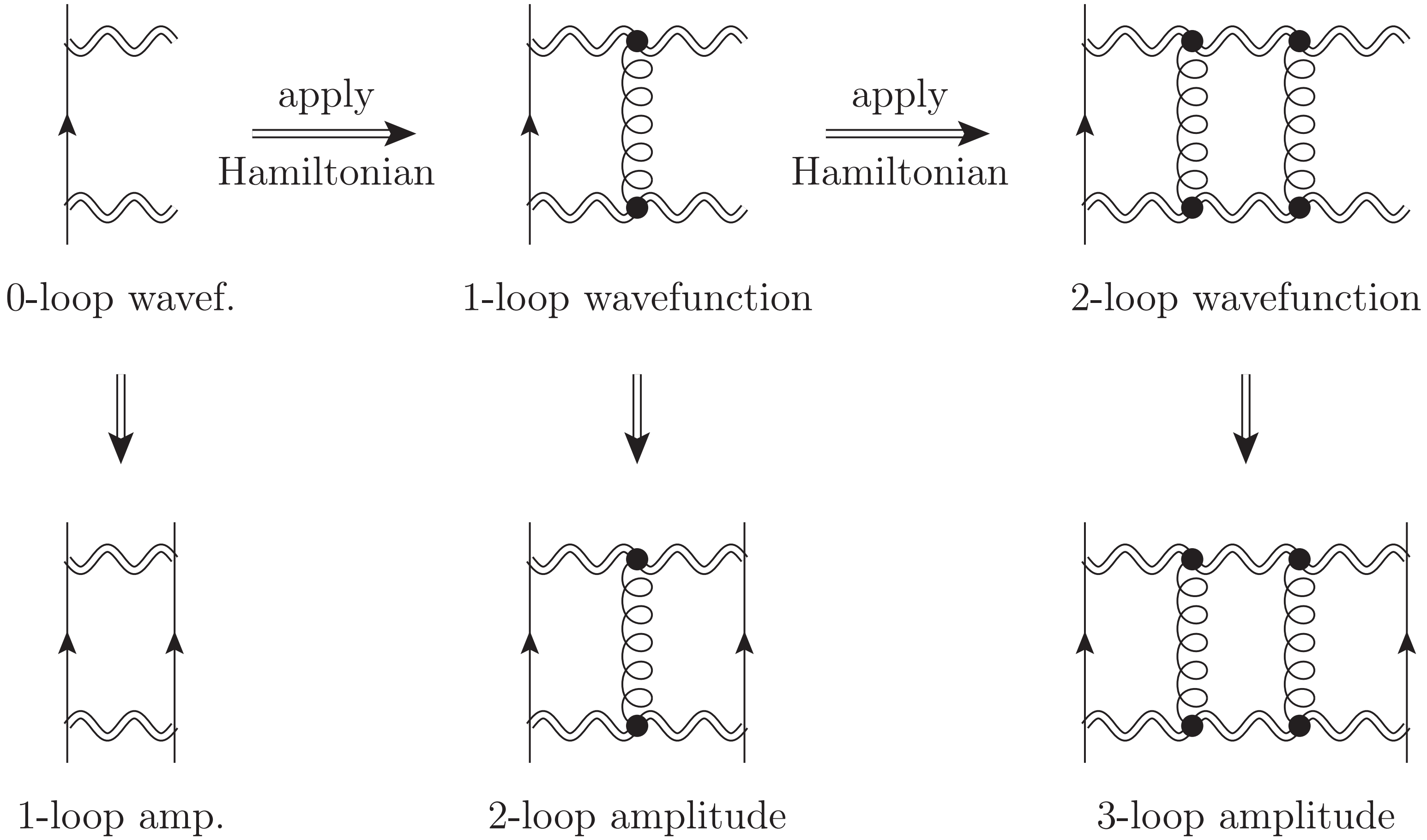
The initial condition: $\Omega^{(0)} = 1$

The BFKL equation in dimensional regularization

The even amplitude is determined by the exchange of a pair of Reggized gluons.

Applying the Hamiltonian is equivalent to adding a rang in the ladder:

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k)$$

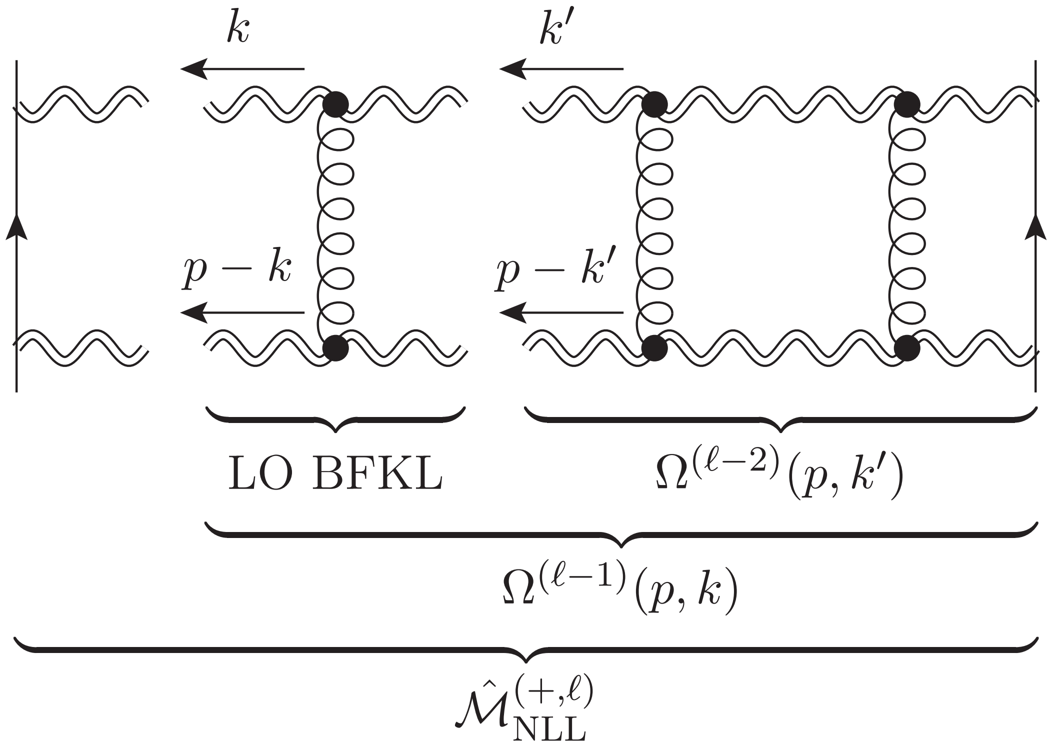


At each order the amplitude is $\hat{\mathcal{M}}^{(+,\ell)}_{\text{NLL}} = -i\pi \frac{B_0^{\ell}}{(\ell-1)!} \int [\mathrm{D}k] \frac{p^2}{k^2(p-k)^2} \Omega^{(\ell-1)}(p,k) \mathbf{T}^2_{s-u} \mathcal{M}^{(\text{tree})}$

$$[\mathrm{D}k] \equiv \frac{\pi}{B_0} \left(\frac{\mu^2}{4\pi e^{-\gamma_E}} \right)^{\epsilon} \frac{d^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}}$$

$$\hat{\mathcal{M}}^{(+)}_{\text{NLL}}\left(\frac{s}{-t}\right) = \sum_{\ell=1}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^{\ell} L^{\ell-1} \hat{\mathcal{M}}^{(+,\ell)}_{\text{NLL}}$$

The BFKL equation in more detail



Let us look at the dimensionally-regularised BFKL equation in more detail:

$$\Omega^{(\ell-1)}(p,k) = \hat H\,\Omega^{(\ell-2)}(p,k), \qquad \hat H = (2C_A - \mathbf{T}_t^2)\,\underset{\text{Integrate}}{\underset{\uparrow}{\hat H_{\rm i}}} + (C_A - \mathbf{T}_t^2)\,\underset{\text{Multiply}}{\underset{\uparrow}{\hat H_{\rm m}}}$$

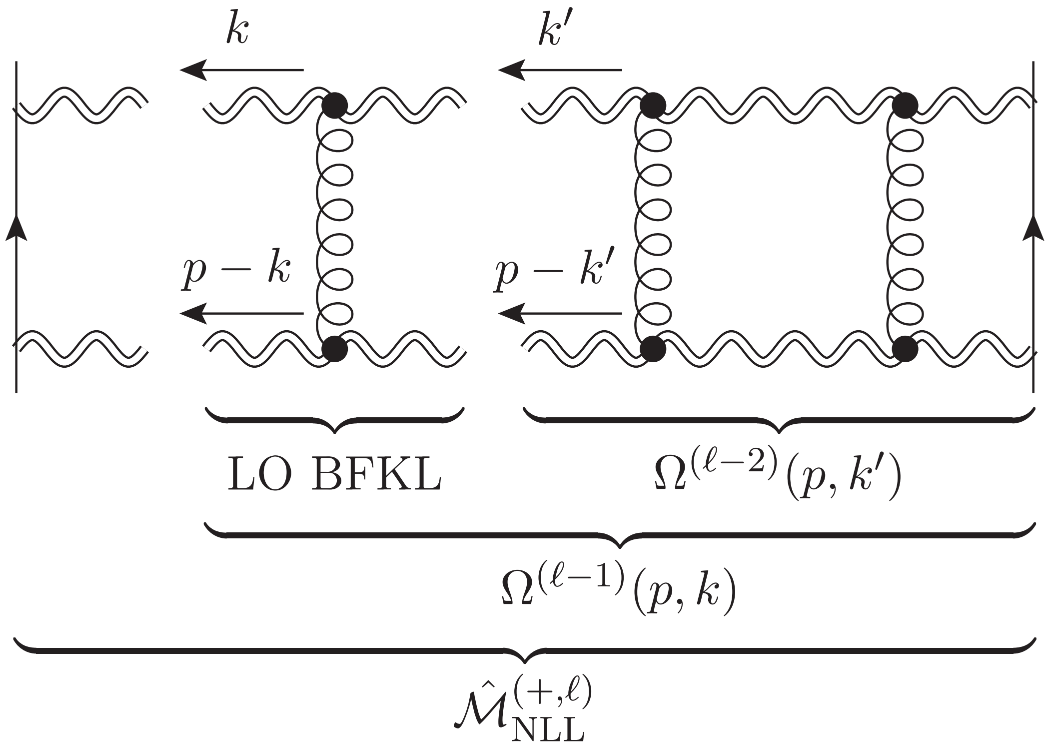
$$\hat H_{\rm i}\,\Psi(p,k) = \int [{\rm D}k']\,f(p,k,k')\Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat H_{\rm m}\,\Psi(p,k) = J(p,k)\,\Psi(p,k)$$

$$J(p,k) = \frac{1}{2\epsilon} + \int [{\rm D}k']\,f(p,k,k')$$
$$= \frac{1}{2\epsilon}\left[2 - \left(\frac{p^2}{k^2}\right)^{\epsilon} - \left(\frac{p^2}{(p-k)^2}\right)^{\epsilon}\right].$$

$$f(p,k,k') \equiv \frac{k^2}{k'^2(k-k')^2} + \frac{(p-k)^2}{(p-k')^2(k-k')^2} - \frac{p^2}{k'^2(p-k')^2}$$

BFKL iteration through two loops



$$\hat{H}_{\rm i}\,\Psi(p,k) = \int [{\rm D}k']\,f(p,k,k')\Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat{H}_{\rm m}\,\Psi(p,k) = J(p,k)\,\Psi(p,k)$$

Integrate ↓

Multiply ↓

$$\Omega^{(\ell-1)}(p,k) = \hat{H}\,\Omega^{(\ell-2)}(p,k), \qquad \hat{H} = (2C_A - \mathbf{T}_t^2)\,\hat{H}_{\rm i} + (C_A - \mathbf{T}_t^2)\,\hat{H}_{\rm m}$$

$$\Omega^{(0)}(p,k) = 1$$

$$\Omega^{(1)}(p,k) = (C_A - \mathbf{T}_t^2)J(p,k)$$

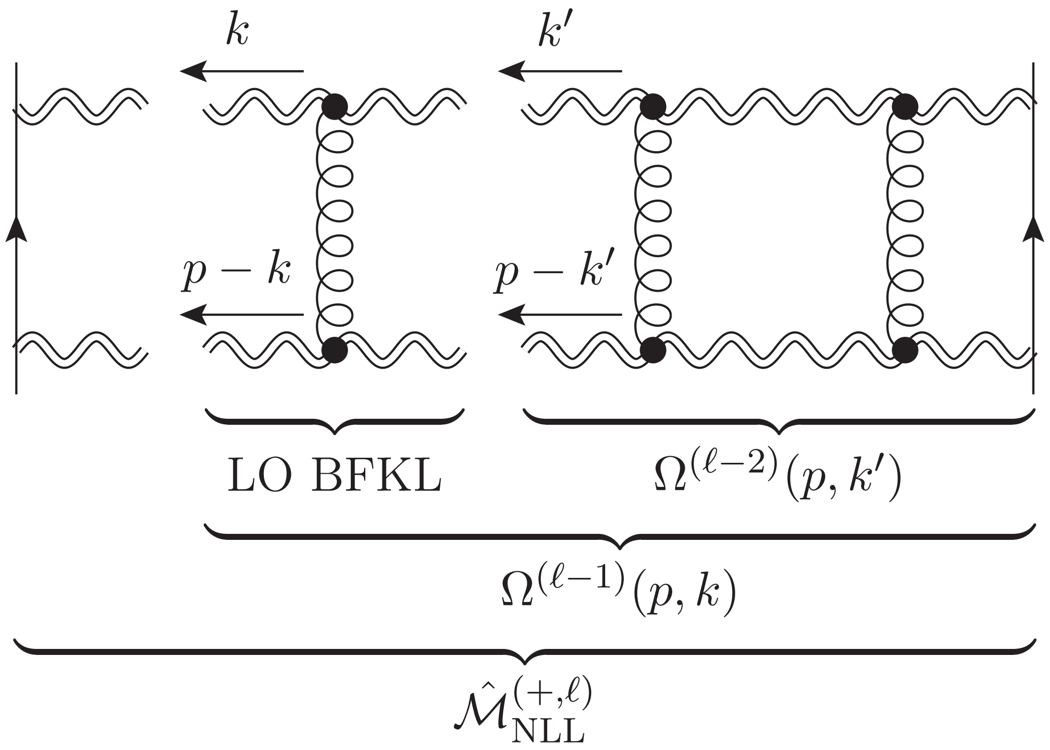
$$\Omega^{(2)}(p,k) = (C_A - \mathbf{T}_t^2)^2 J^2(p,k) + (C_A - 2\mathbf{T}_t^2)(C_A - \mathbf{T}_t^2)\int [{\rm D}k']f(p,k,k')\,[J(p,k') - J(p,k)]$$

⋮

At higher orders this yields increasingly difficult integrals…

The soft approximation

We observe: the wavefunction, at any loop order, is finite!



$$\hat{H}_{\text{i}}\,\Psi(p,k) = \int [\mathrm{D}k']\, f(p,k,k') \Big[\Psi(p,k') - \Psi(p,k)\Big],$$

$$\hat{H}_{\text{m}}\,\Psi(p,k) = J(p,k)\,\Psi(p,k)$$

$$f(p,k,k') \equiv \frac{k^2}{k'^2(k-k')^2} + \frac{(p-k)^2}{(p-k')^2(k-k')^2} - \frac{p^2}{k'^2(p-k')^2}$$

Taking the soft limit $k \ll p$:

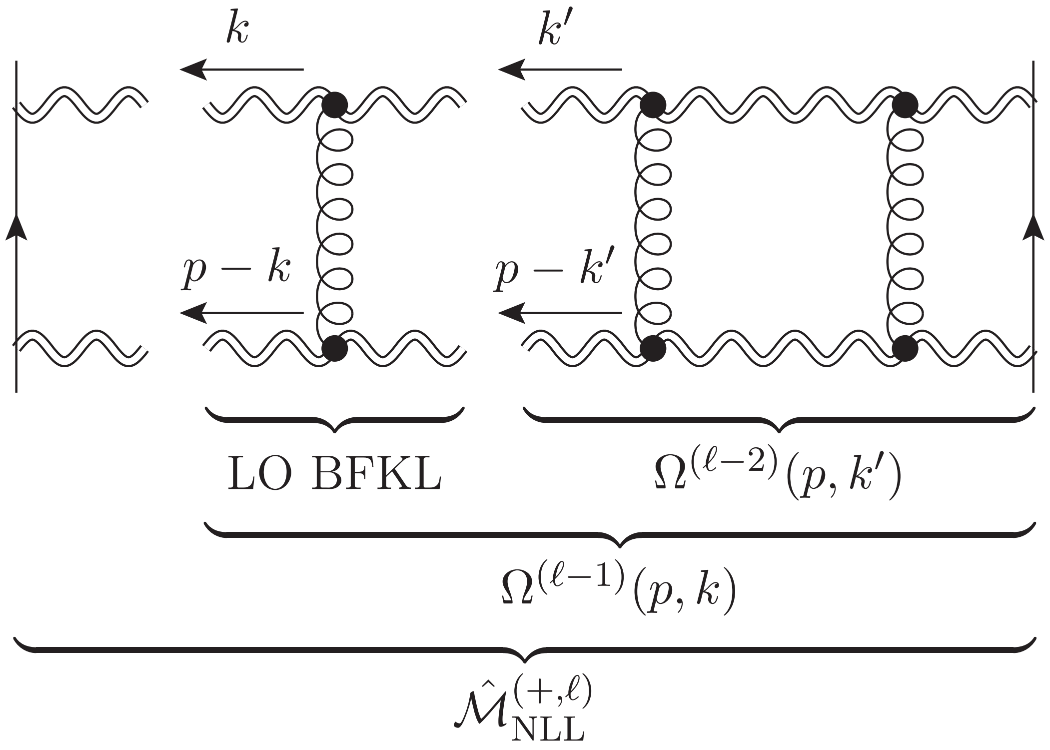
$$f(p,k,k')|_{k\ll k'\sim p} \longrightarrow \qquad 0 + \frac{p^2}{(p-k')^2k'^2} - \frac{p^2}{k'^2(p-k')^2} = 0,$$

$$f(p,k,k')|_{k\sim k'\ll p} \longrightarrow \qquad \frac{k^2}{k'^2(k-k')^2} + \frac{1}{(k-k')^2} - \frac{1}{k'^2} = \frac{2(k\cdot k')}{k'^2(k-k')^2}.$$

Indeed, for small k the integral over k' is dominated by $k' \simeq k$

Conclusion: The soft limit closes under BFKL evolution! The soft limit corresponds to the entire rail, one of the two Reggeons, being soft.

All orders solution for the soft approximation



Solving for the wavefunction in the soft approximation:

$$J_s(p,k) = \frac{1}{2\epsilon}\left[1-\left(\frac{p^2}{k^2}\right)^{\epsilon}\right] \qquad \xi \equiv \left(p^2/k^2\right)^{\epsilon}$$

$$\int [{\rm D}k']\,\frac{2(k\cdot k')}{k'^2(k-k')^2}\left(\frac{p^2}{k'^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\frac{B_n(\epsilon)}{B_0(\epsilon)}\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}$$

with $\quad B_n(\epsilon) = e^{\epsilon\gamma_{\rm E}}\frac{\Gamma(1-\epsilon)}{\Gamma(1+n\epsilon)}\frac{\Gamma(1+\epsilon+n\epsilon)\Gamma(1-\epsilon-n\epsilon)}{\Gamma(1-2\epsilon-n\epsilon)}\,.$

$$\hat{B}_n(\epsilon) = 1-\frac{B_n(\epsilon)}{B_0(\epsilon)} = 2n(2+n)\zeta_3\epsilon^3+3n(2+n)\zeta_4\epsilon^4+\ldots.$$

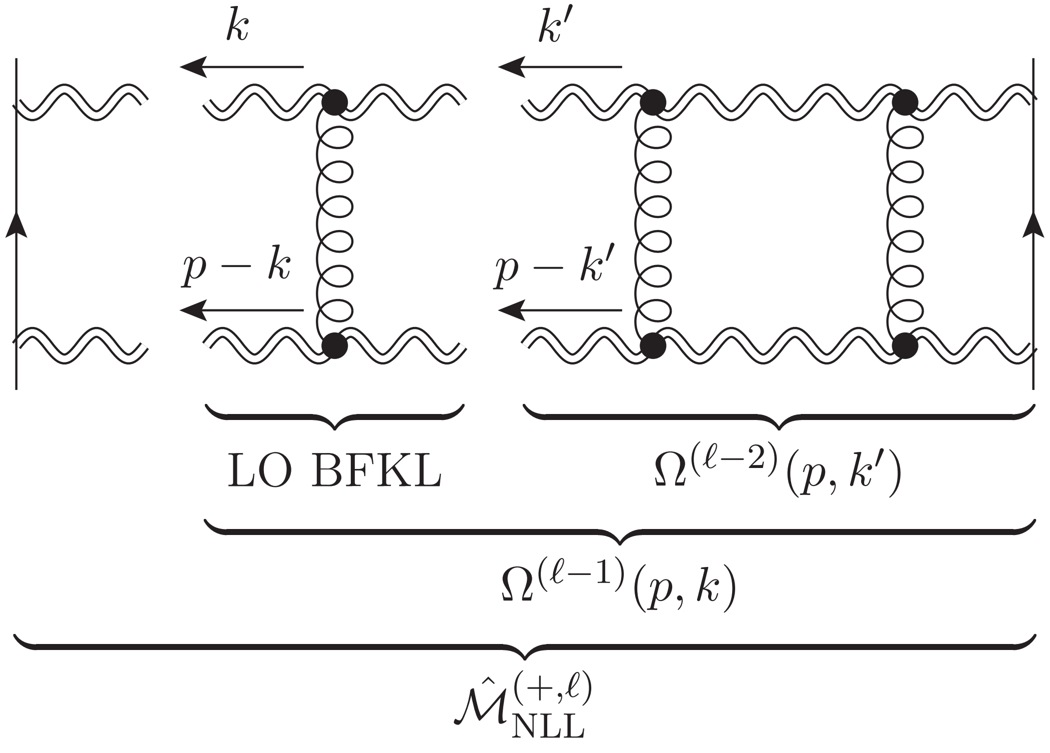
$$\begin{aligned}
\Omega^{(0)}(\xi) =& 1,\\
\Omega^{(1)}(\xi) =& \frac{(C_A-\mathbf{T}_t)}{2\epsilon}\Big(1-\xi\Big),\\
\Omega^{(2)}(\xi) =& \frac{(C_A-\mathbf{T}_t)^2}{(2\epsilon)^2}\left\{1-2\xi+\xi^2\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right\},\\
\Omega^{(3)}(\xi) =& \frac{(C_A-\mathbf{T}_t)^3}{(2\epsilon)^3}\left\{1-3\xi+3\xi^2\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right.\\
&\left.-\,\xi^3\left[1-\hat{B}_1(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\left[1-\hat{B}_2(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right]\right\}
\end{aligned}$$

All-order result:

$$\Omega^{(\ell-1)}(p,k) = \frac{(C_A-\mathbf{T}_t)^{\ell-1}}{(2\epsilon)^{\ell-1}}\sum_{n=0}^{\ell-1}(-1)^n\left(\begin{array}{c}\ell-1\\ n\end{array}\right)\left(\frac{p^2}{k^2}\right)^{n\epsilon}\prod_{m=0}^{n-1}\left\{1-\hat{B}_m(\epsilon)\frac{2C_A-\mathbf{T}_t}{C_A-\mathbf{T}_t}\right\}$$

The amplitude in the soft approximation

Having solved for the wavefunction we can compute the amplitude.



Summing over the two soft limits, we get (at any given order):

$$\hat{\mathcal{M}}^{(+,\ell)}_{\rm NLL} = -i\pi\frac{(B_0)^\ell}{(\ell-1)!}\int[{\rm D}k]\,\frac{p^2}{k^2(p-k)^2}\,\Omega^{(\ell-1)}(p,k)\,\mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)}$$

All IR divergences can be resumed into a closed form expression:

$$\boxed{\hat{\mathcal{M}}^{(+)}_{\rm NLL}\Big|_{\rm IR} = \frac{i\pi}{L(C_A-\mathbf{T}_t^2)}\left(1-R(\epsilon)\frac{C_A}{C_A-\mathbf{T}_t^2}\right)^{-1}\left[\exp\left\{\frac{B_0(\epsilon)}{2\epsilon}\frac{\alpha_s}{\pi}L(C_A-\mathbf{T}_t)\right\}-1\right]\mathbf{T}^2_{s-u}\,\mathcal{M}^{(\rm tree)}+\mathcal{O}(\epsilon^0).}$$

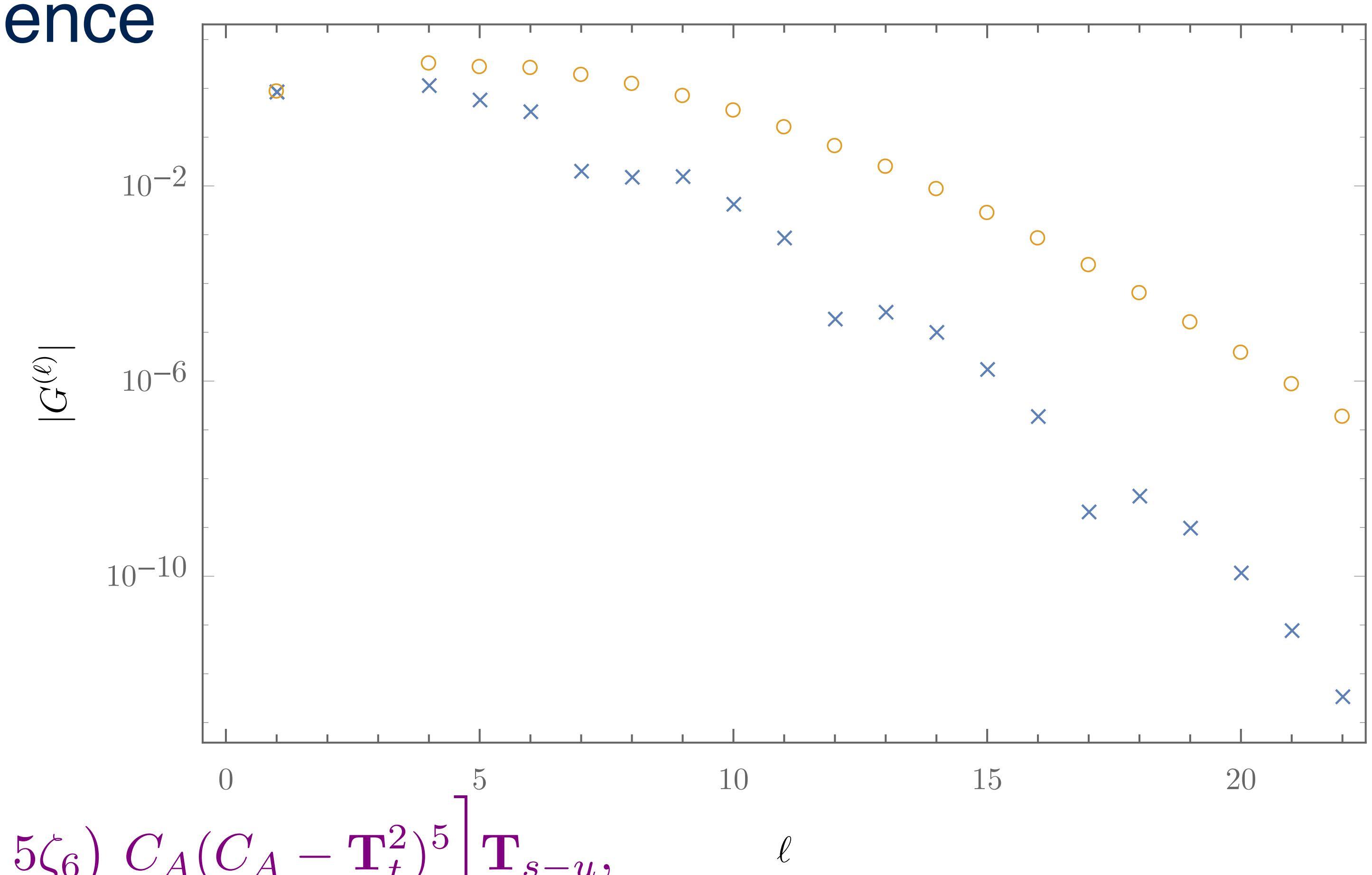
$$\begin{aligned} R(\epsilon) \equiv \frac{B_0(\epsilon)}{B_{-1}(\epsilon)}-1 &= \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}-1 \\ &= -2\zeta_3\,\epsilon^3-3\zeta_4\,\epsilon^4-6\zeta_5\epsilon^5-\left(10\zeta_6-2\zeta_3^2\right)\epsilon^6+\mathcal{O}(\epsilon^7). \end{aligned}$$

The Soft Anomalous dimension in the High-energy limit (NLL)

$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\, \mathbf{T}_t^2 + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s, L\right) + \dots .$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-)} = i\pi \frac{\alpha_s}{\pi} G\left(\frac{\alpha_s}{\pi} L\right) \frac{1}{2}(\mathbf{T}_s^2 - \mathbf{T}_u^2)$$

$G(x)$ is an entire function! Its **inverse** Borel transform has a finite radius of convergence



$$\mathbf{\Gamma}_{\rm NLL}^{(-,1)} = i\pi\, \mathbf{T}_{s-u}$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,2)} = 0$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,3)} = 0,$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,4)} = -i\pi \frac{\zeta_3}{24} C_A (C_A - \mathbf{T}_t^2)^2\, \mathbf{T}_{s-u},$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,5)} = -i\pi \frac{\zeta_4}{128} C_A (C_A - \mathbf{T}_t^2)^3\, \mathbf{T}_{s-u},$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,6)} = -i\pi \frac{\zeta_5}{640} C_A (C_A - \mathbf{T}_t^2)^4\, \mathbf{T}_{s-u},$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,7)} = i\pi \frac{1}{720}\left[\frac{\zeta_3^2}{16} C_A^2 (C_A - \mathbf{T}_t^2)^4 + \frac{1}{32}\left(\zeta_3^2 - 5\zeta_6\right) C_A (C_A - \mathbf{T}_t^2)^5\right] \mathbf{T}_{s-u},$$

$$\mathbf{\Gamma}_{\rm NLL}^{(-,8)} = i\pi \frac{1}{5040}\left[\frac{3\zeta_3\zeta_4}{32} C_A^2 (C_A - \mathbf{T}_t^2)^5 + \frac{3}{64}\left(\zeta_3\zeta_4 - 3\zeta_7\right) C_A (C_A - \mathbf{T}_t^2)^6\right] \mathbf{T}_{s-u}.$$

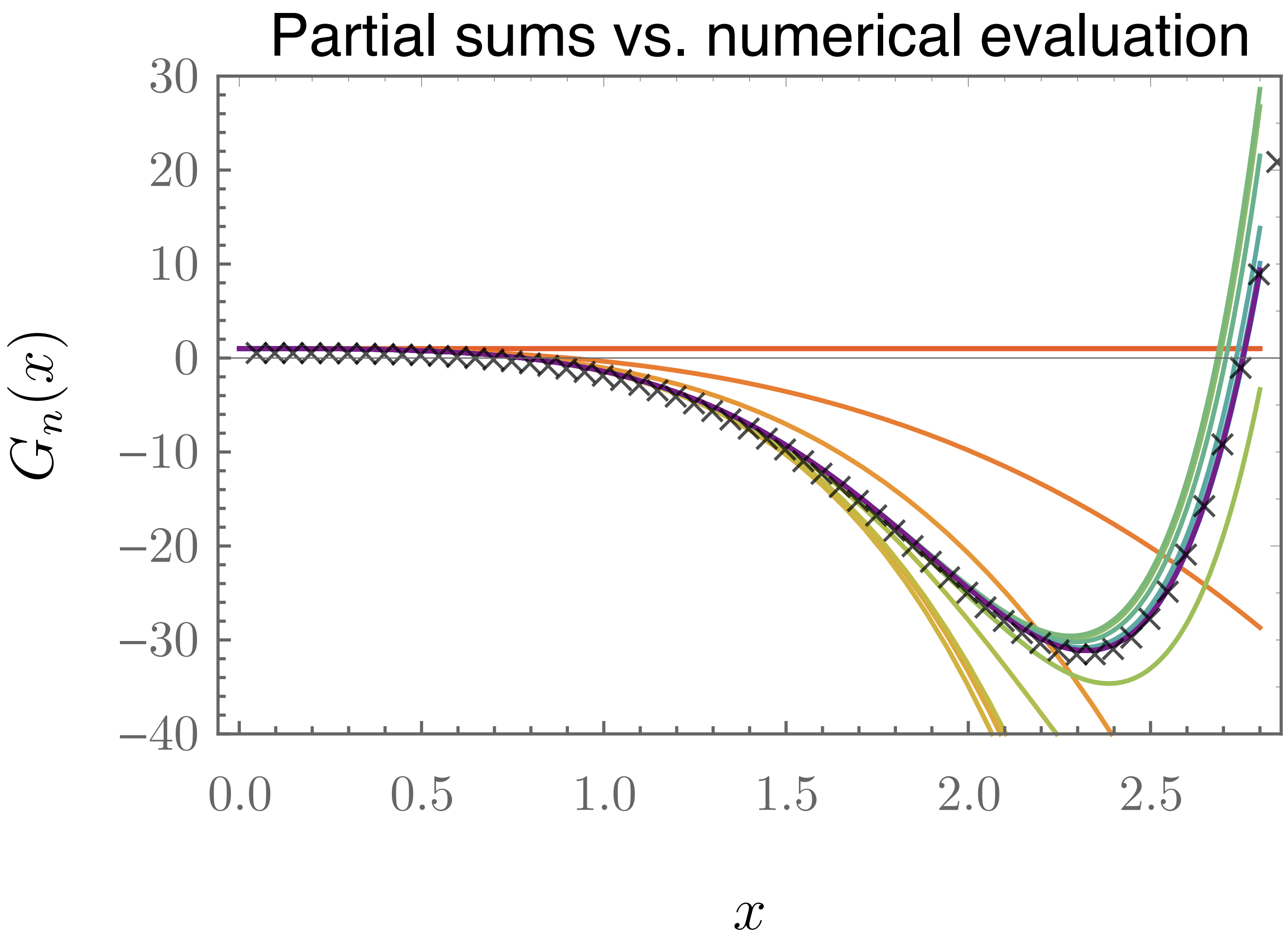
The Soft Anomalous dimension in the High-energy limit (NLL)

$$\mathbf{\Gamma}^{(-)}_{\mathrm{NLL}} = i\pi \frac{\alpha_s}{\pi} G\left(\frac{\alpha_s}{\pi} L\right) \frac{1}{2}(\mathbf{T}_s^2 - \mathbf{T}_u^2)$$

$G(x)$ is an entire function! Its **inverse** Borel transform has a finite radius of convergence.

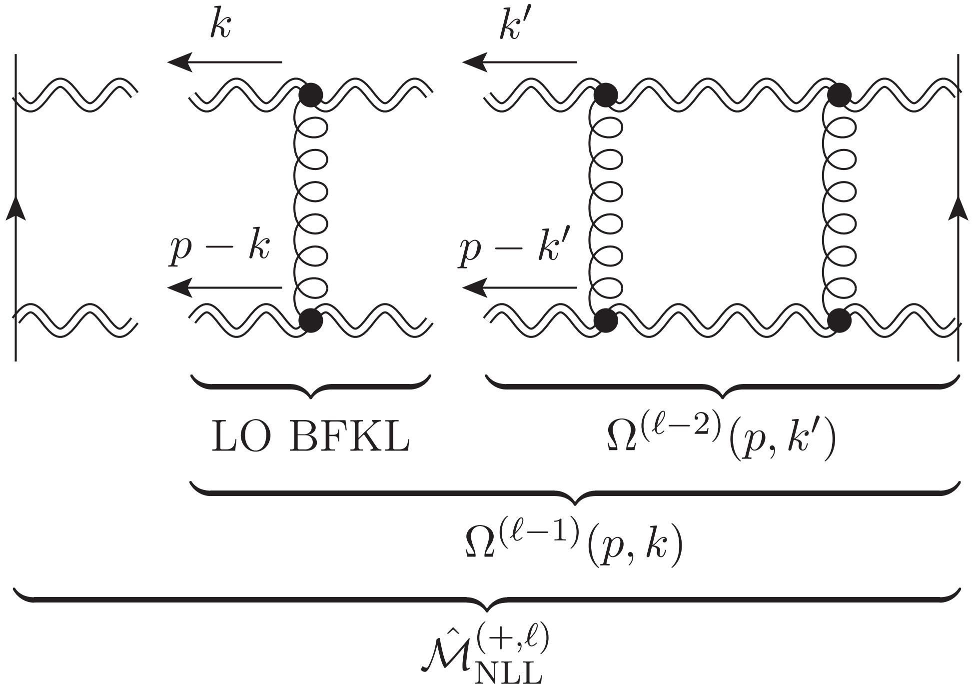
The inverse Borel transform provides a practical way to evaluate G(x) numerically.

The calculation is valid even for $\alpha_s \log(s/(-t)) \gg 1$



The BFKL equation in two transverse dimensions

Taking the two-dimensional limit we can work with a pair of complex-conjugated variables:



$$k = k_x + ik_y, \quad k' = k'_x + ik'_y \quad \text{and} \quad p = p_x + ip_y$$

$$\frac{k_x + ik_y}{p_x + ip_y} = \frac{z}{z-1} \qquad \text{and} \qquad \frac{k'_x + ik'_y}{p_x + ip_y} = \frac{w}{w-1}$$

$$\boxed{\Omega^{(\ell-1)}_{2d}(z,\bar{z}) = \hat{H}_{2d}\Omega^{(\ell-2)}_{2d}(z,\bar{z})}$$

Defining $C_1 = 2C_A - \mathbf{T}_t^2 \qquad C_2 = C_A - \mathbf{T}_t^2$

$$\hat{H}_{2d}\psi(z,\bar{z}) = C_1\hat{H}_{2d,\,i}\psi(z,\bar{z}) + C_2\hat{H}_{2d,\,m}\psi(z,\bar{z})$$

Integrate $\hat{H}_{2d,\,i}\psi(z,\bar{z}) = \dfrac{1}{4\pi}\displaystyle\int d^2w K(w,\bar{w},z,\bar{z})\left[\psi(w,\bar{w}) - \psi(z,\bar{z})\right]$

Multiply $\hat{H}_{2d,\,m}\psi(z,\bar{z}) = j(z,\bar{z})\psi(z,\bar{z})$

With the kernel: $K(w,\bar{w},z,\bar{z}) = \dfrac{z\bar{w} + w\bar{z}}{w\bar{w}(z-w)(\bar{z}-\bar{w})} = \dfrac{1}{\bar{w}(z-w)} + \dfrac{2}{(z-w)(\bar{z}-\bar{w})} + \dfrac{1}{w(\bar{z}-\bar{w})}$

$$j(z,\bar{z}) = \frac{1}{2}\log\left[\frac{z}{(1-z)^2}\frac{\bar{z}}{(1-\bar{z})^2}\right] = \frac{1}{2}\mathcal{L}_0(z,\bar{z}) + \mathcal{L}_1(z,\bar{z})$$

We observe:

- Two symmetries: $z \longleftrightarrow 1/z$ and $z \longleftrightarrow \bar{z}$
- The 2d wavefunction (at any order) can be expressed in terms of pure **Single-Valued Harmonic Polylogarithms (SVHPLs)** of uniform weight.

Iterating the BFKL Hamiltonian in two dimensions

The 2d wavefunction computed in terms of pure **Single-Valued Harmonic Polylogarithms (SVHPLs)**

$$\Omega_{2d}^{(\ell-1)}(z,\bar{z}) = \hat{H}_{2d}\Omega_{2d}^{(\ell-2)}(z,\bar{z})$$

The action of the Hamiltonian on a generic SVHPL can be traded for the following DEs:

$$\begin{aligned}
\frac{d}{dz}\hat{H}_{\mathrm{2d,i}}\mathcal{L}_{0,\sigma}(z,\bar{z}) &= \frac{\hat{H}_{\mathrm{2d,i}}\mathcal{L}_{\sigma}(z,\bar{z})}{z} \\
\frac{d}{dz}\hat{H}_{\mathrm{2d,i}}\mathcal{L}_{1,\sigma}(z,\bar{z}) &= \frac{\hat{H}_{\mathrm{2d,i}}\mathcal{L}_{\sigma}(z,\bar{z})}{1-z} - \frac{1}{4}\frac{\mathcal{L}_{1,\sigma}(z,\bar{z})}{z} \\
&\quad - \frac{1}{4}\frac{\mathcal{L}_{0,\sigma}(z,\bar{z}) + 2\mathcal{L}_{1,\sigma}(z,\bar{z}) - [\mathcal{L}_{0,\sigma}(w,\bar{w}) + \mathcal{L}_{1,\sigma}(w,\bar{w})]_{w,\bar{w}\to\infty}}{1-z}
\end{aligned}$$

- An algorithm is set up to iteratively determine the wavefunction to any loop order. The first few:

$$\begin{aligned}
\Omega_{\mathrm{2d}}^{(1)} &= \frac{1}{2}C_2\left(\mathcal{L}_0 + 2\mathcal{L}_1\right) \\
\Omega_{\mathrm{2d}}^{(2)} &= \frac{1}{2}C_2^2\left(\mathcal{L}_{0,0} + 2\mathcal{L}_{0,1} + 2\mathcal{L}_{1,0} + 4\mathcal{L}_{1,1}\right) + \frac{1}{4}C_1C_2\left(-\mathcal{L}_{0,1} - \mathcal{L}_{1,0} - 2\mathcal{L}_{1,1}\right) \\
\Omega_{\mathrm{2d}}^{(3)} &= \frac{3}{4}C_2^3\left(\mathcal{L}_{0,0,0} + 2\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1} + 2\mathcal{L}_{1,0,0} + 4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right) \\
&\quad + \frac{1}{4}C_1C_2^2\left(2\zeta_3 - 2\mathcal{L}_{0,0,1} - 3\mathcal{L}_{0,1,0} - 7\mathcal{L}_{0,1,1} - 2\mathcal{L}_{1,0,0} - 7\mathcal{L}_{1,0,1} - 7\mathcal{L}_{1,1,0} - 14\mathcal{L}_{1,1,1}\right) \\
&\quad + \frac{1}{16}C_1^2C_2\left(\mathcal{L}_{0,0,1} + 2\mathcal{L}_{0,1,0} + 4\mathcal{L}_{0,1,1} + \mathcal{L}_{1,0,0} + 4\mathcal{L}_{1,0,1} + 4\mathcal{L}_{1,1,0} + 8\mathcal{L}_{1,1,1}\right)
\end{aligned}$$

- A closed-form resummed expression is yet unknown.

The full amplitude: combining the soft and 2d calculations

The soft wavefunction generates all IR singularities in the amplitude.

We can therefore split the full wavefunction
into soft and hard:
and use dim. reg. only for the soft:

$$\Omega(p,k) = \Omega_{\mathrm{hard}}(p,k) + \Omega_{\mathrm{soft}}(p,k)$$

$$\Omega^{(2\mathrm{d})}_{\mathrm{hard}}(z,\bar{z}) \equiv \lim_{\epsilon\to 0}\Omega_{\mathrm{hard}} = \Omega^{(2\mathrm{d})}(z,\bar{z}) - \Omega^{(2\mathrm{d})}_{\mathrm{soft}}(z,\bar{z})$$

The full amplitude is therefore recovered by summing two integrals:

$$\hat{\mathcal{M}}^{(+,\,\mathrm{NLL})}_{ij\to ij}\left(\frac{s}{-t}\right) = -i\pi\left[\int [\mathrm{D}k]\frac{p^2}{k^2(p-k)^2}\Omega_{\mathrm{soft}}(p,k) + \frac{1}{4\pi}\int\frac{d^2z}{z\bar{z}}\Omega^{(2\mathrm{d})}_{\mathrm{hard}}(z,\bar{z})\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\mathrm{tree})}_{ij\to ij}$$

In principle the algorithm can be run to any order, subject to computing resources.
In practice we stopped at **13 loops**. The first few orders are:

$$\hat{\mathcal{M}}^{(1)}_{ij\to ij} = i\pi\frac{1}{2\epsilon}\mathbf{T}^2_{s-u}\mathcal{M}^{(\mathrm{tree})}_{ij\to ij}$$

$$\hat{\mathcal{M}}^{(2)}_{ij\to ij} = i\pi C_2\left[\frac{1}{8\epsilon^2} - \frac{\zeta(2)}{8}\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\mathrm{tree})}_{ij\to ij}$$

$$\hat{\mathcal{M}}^{(3)}_{ij\to ij} = i\pi C_2^2\left[\frac{1}{48\epsilon^3} - \frac{\zeta(2)}{32\epsilon} - \frac{29}{48}\zeta(3)\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\mathrm{tree})}_{ij\to ij}$$

$$\hat{\mathcal{M}}^{(4)}_{ij\to ij} = i\pi\left[\frac{C_2^3}{384\epsilon^4} - \frac{C_2^3\zeta(2)}{192\epsilon^2} - \left(\frac{7}{288}C_2^3\zeta(3) + \frac{1}{192}C_2^2C_A\zeta(3)\right)\frac{1}{\epsilon} - \frac{C_2^3\zeta(4)}{48} - \frac{C_2^2C_A\zeta(4)}{128}\right]\mathbf{T}^2_{s-u}\mathcal{M}^{(\mathrm{tree})}_{ij\to ij}$$

$$\begin{aligned}\hat{\mathcal{M}}^{(5)}_{ij\to ij} = i\pi\Bigg[&\frac{C_2^4}{3840\epsilon^5} - \frac{C_2^4\zeta(2)}{1536\epsilon^3} + \left(-\frac{7C_2^4\zeta(3)}{2304} - \frac{C_2^3C_A\zeta(3)}{1920}\right)\frac{1}{\epsilon^2} + \left(-\frac{9C_2^4\zeta(4)}{4096} - \frac{C_2^3C_A\zeta(4)}{1280}\right)\frac{1}{\epsilon}\\ &+ C_2^4\left(\frac{35\zeta(2)\zeta(3)}{4608} - \frac{293\zeta(5)}{1280}\right) + C_2^3C_A\left(\frac{1}{768}\zeta(2)\zeta(3) + \frac{253\zeta(5)}{1920}\right) - \frac{1}{48}C_2^2C_A^2\zeta(5)\Bigg]\mathbf{T}^2_{s-u}\mathcal{M}^{(\mathrm{tree})}_{ij\to ij}\end{aligned}$$

To this order (five loops) all integrals have also been computed directly in dim. reg.

Conclusions

- The high-energy limit and infrared factorization are complementary avenues in studying amplitudes.
- Rapidity evolution equations can be efficiently used to compute partonic scattering amplitudes to high loop orders.
- Number-theory findings:
  - Soft amplitude can be resummed using Gamma functions.
  - Hard amplitude is expressed in terms of SV Zeta values (e.g. no even Zetas; first multi-Zeta occurs at 11 loops). It cannot be resummed into Gamma functions.
- Large order behaviour aspects:
  - The soft anomalous dimension (at NLL in the Regge limit) is an entire function. Its calculation extends to $\alpha_s \log(s/(-t)) \gg 1$
  - The finite part of the NLL amplitude has a finite radius of convergence, with asymptotically sign-oscillating coefficients.

The High-Energy Limit in 2-to-2 Scattering

From the dispersion representation of the amplitude

$$\mathcal{M}(s,t) = \frac{1}{\pi}\int_0^\infty \frac{d\hat{s}}{\hat{s}-s-i0}\,D_s(\hat{s},t) + \frac{1}{\pi}\int_0^\infty \frac{d\hat{u}}{\hat{u}+s+t-i0}\,D_u(\hat{u},t)$$

with the reality property of the discontinuities, it follows:

Amplitudes of a given signature $\mathcal{M}^{(\pm)}(s,t) = \frac{1}{2}\Big(\mathcal{M}(s,t) \pm \mathcal{M}(-s-t,t)\Big)$

are, respectively:

$$\mathcal{M}^{(-)}(s,t) \qquad\qquad \text{real}$$

$$\mathcal{M}^{(+)}(s,t) \qquad\qquad \text{imaginary}$$

when expressed in terms of the signature-even logarithm:

$$\begin{aligned} L &\equiv \frac{1}{2}\left(\log\frac{-s-i0}{-t} + \log\frac{-u-i0}{-t}\right) \\ &= \log\left|\frac{s}{t}\right| - i\frac{\pi}{2} \end{aligned}$$

Finite corrections to the amplitude: radius of convergence

For the singlet the 27 colour representations in the t-channel,

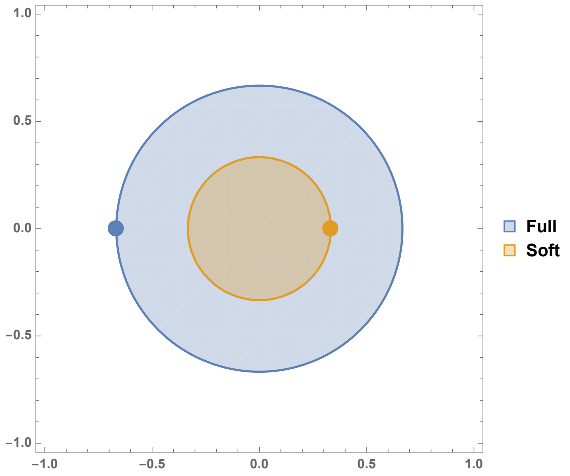
$$\hat{\mathcal{M}}^{(+)}_{\rm NLL} = \frac{i\pi}{L}\,\Xi^{(+)}_{\rm NLL}\,\mathbf{T}^2_{s-u}\mathcal{M}_{\rm tree} \qquad\qquad x \equiv \frac{\alpha_s(-t)}{\pi}\ln\left(\frac{s}{-t}\right)$$

$$\begin{aligned}\Xi^{(+)[1]}_{\rm NLL} = &-0.6169\,x^2 - 6.536\,x^3 - 0.8371\,x^4 - 8.483\,x^5 - 1.529\,x^6 - 12.67\,x^7 + 1.610\,x^8\\ &- 20.62\,x^9 + 16.48\,x^{10} - 35.98\,x^{11} + 46.07\,x^{12} + \mathcal{O}(x^{13}),\end{aligned}$$

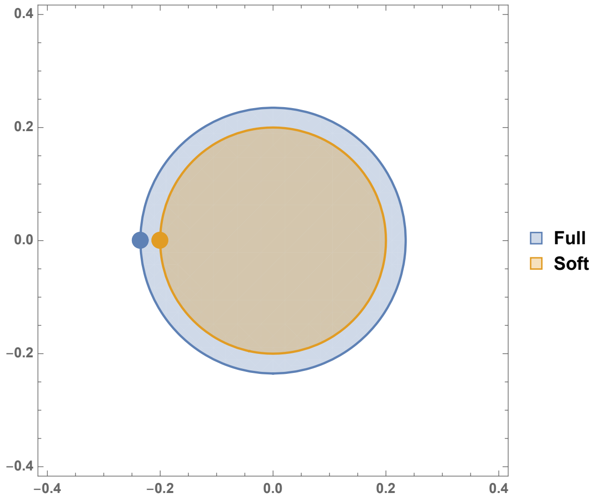
$$\begin{aligned}\Xi^{(+)[27]}_{\rm NLL} = &\,1.028\,x^2 - 18.16\,x^3 + 2.184\,x^4 - 196.0\,x^5 + 372.3\,x^6 - 2821\,x^7 + 9382\,x^8\\ &- 46494\,x^9 + 180397\,x^{10} - 797524\,x^{11} + 3.239\times 10^6\,x^{12} + \mathcal{O}(x^{13}).\end{aligned}$$

Applying **Padé Approximants** we extract the position of the nearest singularity:

Singlet Rep.



27 Rep.



The soft amplitude:

$$\hat{\mathcal{M}}_{\rm NLL,\,soft} = \frac{i\pi}{LC_2}\left\{\left(e^{\frac{B_0}{2\epsilon}C_2 x} - 1\right)\frac{B_{-1}(\epsilon)}{B_0(\epsilon)}\left(1 - B_{-1}(\epsilon)\frac{C_1}{C_2}\right)^{-1} + \left(1 - e^{\gamma_E C_1\,x}\frac{\Gamma(1-C_2 x)}{\Gamma(1+C_2 x)}\frac{\Gamma^{2-\frac{C_1}{C_2}}\left(1+C_2\frac{x}{2}\right)}{\Gamma^{2-\frac{C_1}{C_2}}\left(1-C_2\frac{x}{2}\right)}\right)\right\}\mathbf{T}_{s-u}\,\mathcal{M}^{\rm (tree)}$$

- Confirms Padé analysis
- Singularity at $C_2 x = 1$ cancels in the full amplitude

The High-Energy Limit in 2-to-2 Scattering: colour and signature

The high-energy limit is dominated by **t-channel exchange, helicity-conserving** configuration.

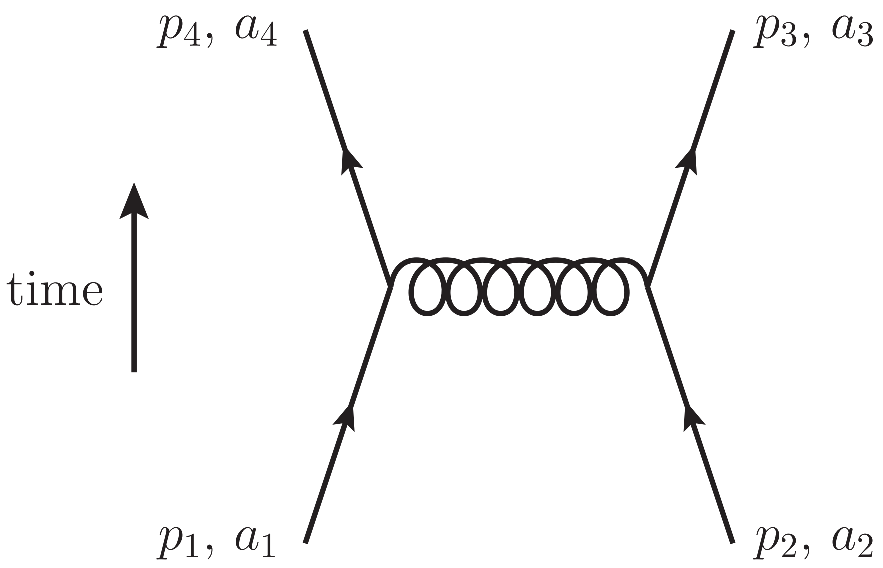
The leading-order amplitude is a **t-channel** gluon exchange, corresponding to an antisymmetric octet representation. It has **odd signature**:

odd under s ⟷ u

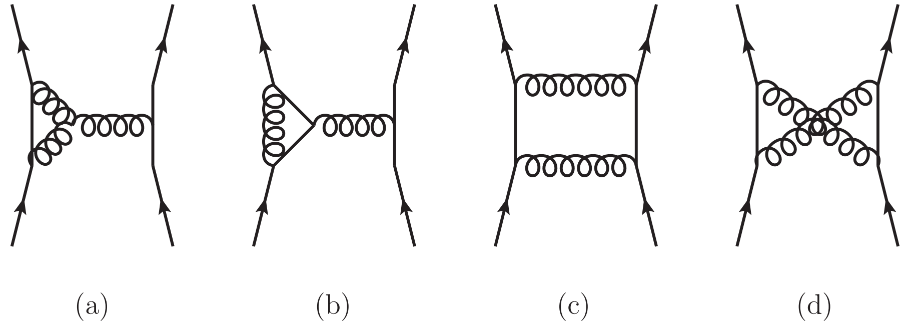
helicity is conserved

$$\mathcal{M}^{(\mathrm{tree})}_{ij\to ij} = \mathcal{M}^{(\mathrm{tree})^{(-)}}_{ij\to ij} = g_s^2 \underset{\uparrow}{\frac{2s}{t}} (\mathbf{T}_i^b)_{a_1 a_4} (\mathbf{T}_j^b)_{a_2 a_3} \delta_{\lambda_1\lambda_4} \underset{\downarrow}{\delta_{\lambda_2\lambda_3}}$$

$$\mathcal{M}^{(\mathrm{tree})^{(+)}}_{ij\to ij} = 0$$



At higher orders (beyond LL) it's useful to decompose the amplitude using a **t-channel colour basis**:



$$\mathcal{M}(s,t) = \sum_i c^{[i]}\, \mathcal{M}^{[i]}(s,t)$$

qq, qg scattering: odd: $\mathcal{M}^{[8_a]}$, even: $\mathcal{M}^{[1]}$, $\mathcal{M}^{[8_s]}$

gg scattering: odd: $\mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}$, even: $\mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]}, \mathcal{M}^{[0]}$

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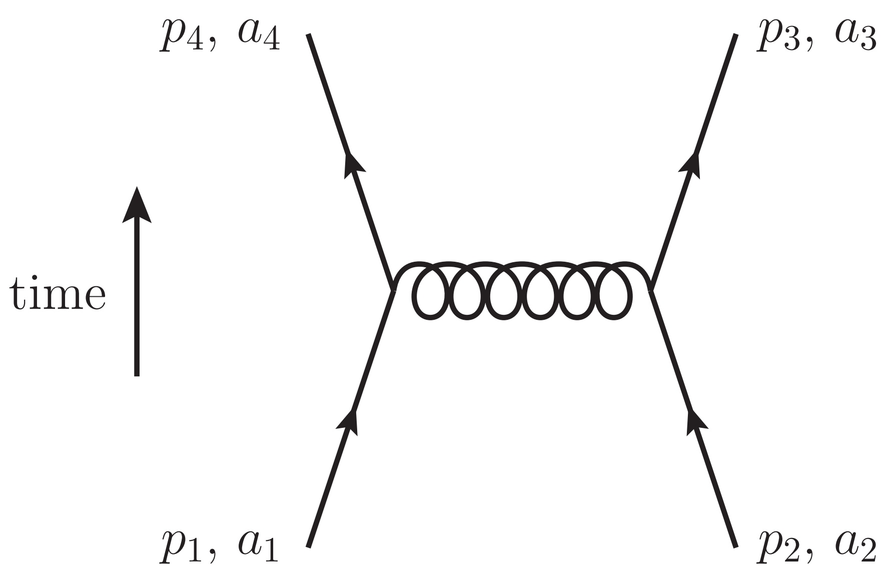

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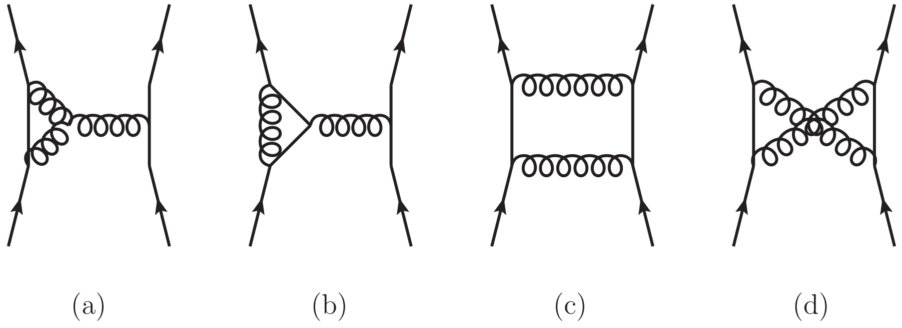

(a) (b) (c) (d)

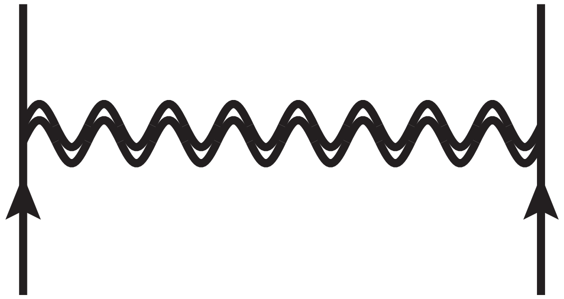
qq, qg scattering: odd: $\mathcal{M}^{[8_a]}$, even: $\mathcal{M}^{[1]}$, $\mathcal{M}^{[8_s]}$

gg scattering: odd: $\mathcal{M}^{[8_a]}, \mathcal{M}^{[10+\overline{10}]}$, even: $\mathcal{M}^{[1]}, \mathcal{M}^{[8s]}, \mathcal{M}^{[27]}, \mathcal{M}^{[0]}$

Leading-logarithmic gluon Reggeization in dimensional regularization

Reggeization can be seen to be a consequence of an evolution equation corresponding to rapidity divergence.

At leading logarithmic accuracy, in dimensional regularization:



$$\frac{d}{dL}\mathcal{M}^{(-)}_{\rm LL} = \alpha(t)\mathcal{M}^{(-)}_{\rm LL} \qquad L = \ln(s/(-t))$$

with
$$\alpha(-p^2) = \alpha_s \mathbf{T}_t^2 \left(\frac{\mu^2}{4\pi e^{-\gamma_E}}\right)^{\epsilon} \int \frac{d^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}} \frac{p^2}{k^2(p-k)^2}$$
$$= \frac{\alpha_s}{\pi}\mathbf{T}_t^2 \left(\frac{\mu^2}{p^2}\right)^{\epsilon} \frac{B_0(\epsilon)}{2\epsilon} + \mathcal{O}(\alpha_s^2)$$

$$B_0(\epsilon) = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \frac{\zeta_2}{2}\epsilon^2 - \frac{7\zeta_3}{3}\epsilon^3 + \ldots$$

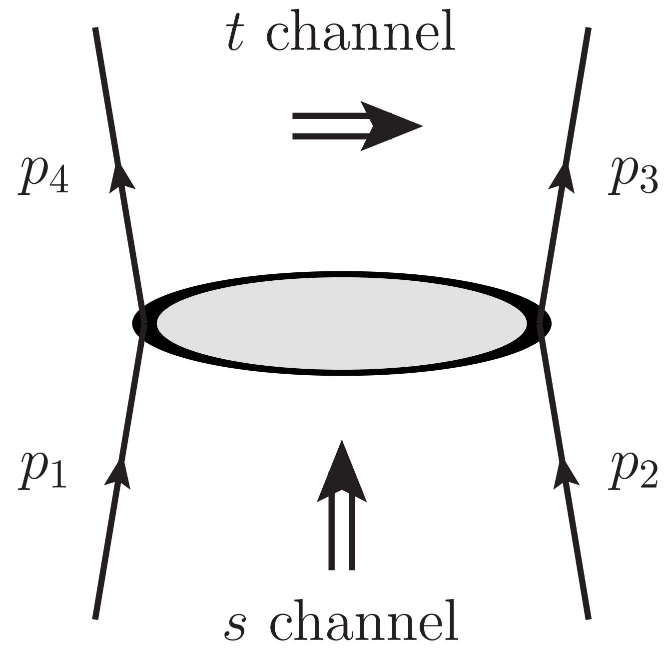
$$\mathcal{M}^{(-)}_{\rm LL} = (s/(-t))^{\alpha(t)} \times \mathcal{M}^{\rm tree}$$

From now on we consider the ***reduced amplitude*** $\hat{\mathcal{M}}_{ij\to ij} \equiv e^{-\mathbf{T}_t^2\,\alpha(t)\,L}\,\mathcal{M}_{ij\to ij}$

The High-Energy Limit in 2-to-2 Scattering: colour and signature

Here, instead of using a particular colour-flow basis,
we use colour operators, acting as generators on a given parton:

$$\begin{cases} \mathbf{T}_s = \mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{T}_3 - \mathbf{T}_4 \\ \mathbf{T}_u = \mathbf{T}_1 + \mathbf{T}_3 = -\mathbf{T}_2 - \mathbf{T}_4 \\ \mathbf{T}_t = \mathbf{T}_1 + \mathbf{T}_4 = -\mathbf{T}_2 - \mathbf{T}_3 \end{cases}$$



Using the colour conservation: $(\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{T}_4)\,\mathcal{M} = 0$

One obtains $\mathbf{T}_s^2 + \mathbf{T}_u^2 + \mathbf{T}_t^2 = \sum_{i=1}^{4} C_i$ = sum over the quadratic Casimirs

This leaves just two independent quadratic operators: $\mathbf{T}_t^2$ is even,

$$\mathbf{T}_{s-u}^2 \equiv \frac{\mathbf{T}_s^2 - \mathbf{T}_u^2}{2} \quad \text{is odd}$$

The signature-even amplitude is characterised by
the odd colour operator, acting on the tree amplitude:

$$\mathcal{M}_{\text{NLL}}^{(+)} \simeq i\pi \left[\frac{1}{2\epsilon} \frac{\alpha_s}{\pi} + \mathcal{O}\left(\alpha_s^2 L\right) \right] \mathbf{T}_{s-u}^2 \mathcal{M}^{(\text{tree})}$$

The Soft Anomalous dimension in the High-energy limit (NLL)

Results (based on rapidity evolution — see below):

$$\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\, \mathbf{T}_t^2 + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s, L\right) + \ldots \,.$$

In the high-energy limit the soft anomalous dimension for 2-to-2 scattering is now known **to all orders** at NLL accuracy**:**

Odd Amplitude (Real part)

$$\mathbf{\Gamma}_{\rm NLL}^{(+)} = \left(\frac{\alpha_s(\lambda)}{\pi}\right)^2 \frac{\gamma_K^{(2)}}{2} L\, \mathbf{T}_t^2 + \left(\frac{\alpha_s(\lambda)}{\pi}\right) \sum_{i=1}^{2} \left(\frac{\gamma_K^{(1)}}{2} C_i \log\frac{-t}{\lambda^2} + 2\gamma_i^{(1)}\right)$$

Even Amplitude (Imaginary part)

Caron-Huot, EG, Reichel, Vernazza - JHEP 1803 (2018) 098

$$\mathbf{\Gamma}_{\rm NLL}^{(-)} = i\pi \frac{\alpha_s}{\pi}\, G\left(\frac{\alpha_s}{\pi} L\right) \frac{1}{2}\left(\mathbf{T}_s^2 - \mathbf{T}_u^2\right)$$

$$G^{(l)} = \frac{1}{(l-1)!}\left[\frac{(C_A - \mathbf{T}_t^2)}{2}\right]^{l-1} \left(1 - \frac{C_A}{C_A - \mathbf{T}_t^2} R(\epsilon)\right)^{-1}\Bigg|_{\epsilon^{l-1}}$$

$$R(\epsilon) = \frac{\Gamma^3(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} - 1 = -2\zeta_3\,\epsilon^3 - 3\zeta_4\,\epsilon^4 - 6\zeta_5\epsilon^5 - \left(10\zeta_6 - 2\zeta_3^2\right)\epsilon^6 + \mathcal{O}(\epsilon^7)$$

The Soft Anomalous Dimension in the High-energy limit (beyond NLL)

Results beyond NLL accuracy: $\mathbf{\Gamma}\left(\alpha_s\right) = \frac{\alpha_s}{\pi} L\, \mathbf{T}_t^2 + \mathbf{\Gamma}_{\rm NLL}\left(\alpha_s, L\right) + \mathbf{\Gamma}_{\rm NNLL}\left(\alpha_s, L\right) + \ldots$

Based on rapidity evolution $\boxed{\mathbf{\Gamma}_{\rm NNLL}^{(+)} = \mathcal{O}(\alpha_s^4)}$ Caron-Huot, EG, Vernazza - JHEP 06 (2017) 016

— consistent with the Soft Anomalous Dimension 3-loop result.

The absence of $\alpha_s^3 L^k$ for $k \geq 1$ in the Real part and for $k \geq 2$ in the Imaginary part, is a non-trivial prediction from rapidity evolution, which underpins the structure of corrections to the dipole formula.

Based on the Soft Anomalous Dimension 3-loop result we also know:

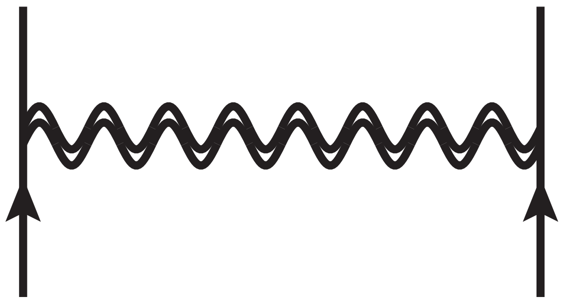
$$\mathbf{\Gamma}_{\rm NNLL}^{(-)} = i\pi \left[\frac{\zeta_3}{4}(C_A - \mathbf{T}_t^2)^2 \left(\frac{\alpha_s}{\pi}\right)^3 L + \mathcal{O}(\alpha_s^4)\right] \mathbf{T}_{s-u}^2$$

$$\mathbf{\Gamma}_{\rm N^3LL}^{(-)} = i\pi \left[\frac{11\zeta_4}{4}(C_A - \mathbf{T}_t^2)^2 \left(\frac{\alpha_s}{\pi}\right)^3 + \mathcal{O}(\alpha_s^4)\right] \mathbf{T}_{s-u}^2 \qquad \mathbf{\Gamma}_{\rm N^3LL}^{(+)} = \mathcal{O}(\alpha_s^3)$$

Leading-logarithmic gluon Reggeization in dimensional regularization

Reggeization can be seen to be a consequence of an evolution equation:

At leading logarithmic accuracy, in dimensional regularization:



$$\frac{d}{dL}\mathcal{M}_{\rm LL}^{(-)} = \alpha(t)\mathcal{M}_{\rm LL}^{(-)} \qquad L = \ln(s/(-t))$$

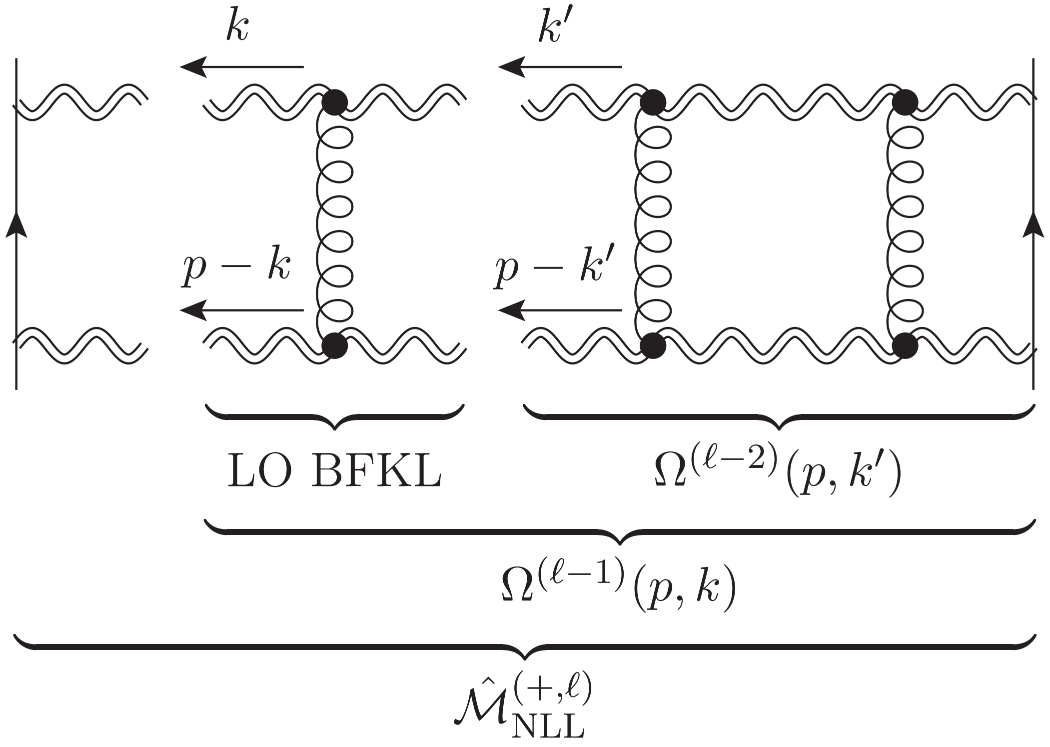
with $$\alpha(-p^2) = \alpha_s \mathbf{T}_t^2 \left(\frac{\mu^2}{4\pi e^{-\gamma_E}}\right)^{\epsilon} \int \frac{d^{2-2\epsilon}k}{(2\pi)^{2-2\epsilon}} \frac{p^2}{k^2(p-k)^2}$$
$$= \frac{\alpha_s}{\pi}\mathbf{T}_t^2 \left(\frac{\mu^2}{p^2}\right)^{\epsilon} \frac{B_0(\epsilon)}{2\epsilon} + \mathcal{O}(\alpha_s^2)$$

$$B_0(\epsilon) = e^{\epsilon\gamma_E}\frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)} = 1 - \frac{\zeta_2}{2}\epsilon^2 - \frac{7\zeta_3}{3}\epsilon^3 + \ldots$$

$$\mathcal{M}_{\rm LL}^{(-)} = (s/(-t))^{\alpha(t)} \times \mathcal{M}^{\rm tree}$$

From now on we consider the ***reduced amplitude*** $\hat{\mathcal{M}}_{ij\to ij} \equiv e^{-\,\mathbf{T}_t^2\,\alpha(t)\,L}\,\mathcal{M}_{ij\to ij}$

Iterative solution within the soft approximation



Let us solve for the wavefunction order-by-order within the soft approximation:

$$J_s(p,k) = \frac{1}{2\epsilon}\left[1-\left(\frac{p^2}{k^2}\right)^{\epsilon}\right]$$

$$\Omega_s^{(\ell-1)}(p,k) = \hat{H}_s\,\Omega_s^{(\ell-2)}(p,k)$$

$$\hat{H}_s\Psi(p,k) = (2C_A-\mathbf{T}_t)\int[\mathrm{D}k']\frac{2(k\cdot k')}{k'^2(k-k')^2}\Big[\Psi(p,k')-\Psi(p,k)\Big] + (C_A-\mathbf{T})\,J_s(p,k)\,\Psi(p,k)$$

By the action of the Hamiltonian, powers of $\xi \equiv \left(p^2/k^2\right)^{\epsilon}$ transform into such powers:

$$\int[\mathrm{D}k']\,\frac{2(k\cdot k')}{k'^2(k-k')^2}\,\left(\frac{p^2}{k'^2}\right)^{n\epsilon} = -\frac{1}{2\epsilon}\,\frac{B_n(\epsilon)}{B_0(\epsilon)}\,\left(\frac{p^2}{k^2}\right)^{(n+1)\epsilon}$$

with $$B_n(\epsilon) = e^{\epsilon\gamma_{\rm E}}\,\frac{\Gamma(1-\epsilon)}{\Gamma(1+n\epsilon)}\,\frac{\Gamma(1+\epsilon+n\epsilon)\Gamma(1-\epsilon-n\epsilon)}{\Gamma(1-2\epsilon-n\epsilon)}\,.$$

Conclusion: the soft wavefunction is a polynomial in $\xi \equiv \left(p^2/k^2\right)^{\epsilon}$