Generalized generating functional for mixed representation Green's functions

Luca Smaldone
in collaboration with
Massimo Blasone and Petr Jizba

Dipartimento di Fisica "E.R.Caianiello"
and INFN Sezione di Napoli, Gruppo Collegato di Salerno

Genova, 21 November 2018

Contents

Motivations

- In QFT the Stone–von Neumann theorem does not apply. This feature has been systematically studied only recently with functional integral methods[1]
- This feature reveals to be central in the study of flavor mixing in QFT, where the vacuum presents an exotic structure. Standard functional integral methods seem to be inappropriate in dealing with corresponding Green's functions
- A generalization of generating functional of Green's functions has been developed in QM[2], to deal with these exotic Green's functions

[1] M. Blasone, P. Jizba, L. S., Ann. Phys. **383**, 205 (2017)
[2] M. Blasone, P. Jizba and L.S., Phys. Rev. A **96**, 052107 (2017)

Canonical transformations in classical mechanics

Hamiltonian dynamics

A classical system is described by the Hamilton equations

$$\dot{q}_j \;=\; \frac{\partial H\left(\{q\},\{p\}\right)}{\partial p_j}\,, \qquad \dot{p}_j \;=\; -\frac{\partial H\left(\{q\}\,\{p\}\right)}{\partial q_j}$$

The time evolution of an observable f is given by

$$\dot{f} \;=\; \{f, H\} + \frac{\partial f}{\partial t}$$

The Poisson bracket is defined as

$$\{f, g\} \;=\; \sum_{j=1}^{N} \left(\frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j} \;-\; \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} \right)$$

Poisson bracket

Properties of Poisson bracket:

- Antisymmetry

$$\{f, g\} \;=\; -\{g, f\}$$

- Bilinearity

$$\{f, c_1\, g_1 + c_2\, g_2\} \;=\; c_1\, \{f, g_1\} \;+\; c_2\, \{f, g_2\}\,, \quad c_1, c_2 \in \mathbb{R}$$

- Jacobi identity

$$\{f, \{g, h\}\} \;+\; \{h, \{f, g\}\} \;+\; \{g, \{h, f\}\} \;=\; 0$$

Weyl–Heisenberg algebra

Let us now consider the space $\mathcal{M} \oplus \mathbb{R}$ of linear functions on the phase space (including constant functions). The pair $(\mathcal{M} \oplus \mathbb{R}, \{\ ,\ \})$ is a Lie algebra: the Weyl–Heisenberg algebra w_N.

$$\{q_j, p_k\} = 1 \cdot \delta_{jk} \qquad \{q_j, q_k\} = \{p_j, p_k\} = 0$$

$$\{q_j, 1\} = \{p_j, 1\} = 0$$

are known as canonical commutation relations (CCR)

Canonical transformations

A canonical transformation is a relation

$$Q_j \equiv Q_j(\{q\},\{p\},t)\ , \qquad P_j \equiv P_j(\{q\},\{p\},t)$$

which preserve the form of the Hamilton equations:

$$\dot{Q}_j = \frac{\partial K(\{Q\},\{P\})}{\partial P_j}\,, \qquad \dot{P}_j = -\frac{\partial K(\{Q\},\{P\})}{\partial Q_j}$$

K plays the rôle of the new Hamiltonian.

Canonical transformations (2)

Old and new variables are related as

$$\sum_{j=1}^{N} p_j \, \dot{q}_j \, - \, H \; = \; \sum_{j=1}^{N} P_j \, \dot{Q}_j \, - \, K \, + \, \dot{F}$$

where F is the generating function of the canonical transformation.

One can prove that

$$\begin{aligned} \{Q_j, P_k\} \; &= \; 1 \cdot \delta_{jk} \, , \quad \{Q_j, Q_k\} \; = \; 0 \, , \quad \{P_j, P_k\} \; = \; 0 \\ \{Q_j, 1\} \; &= \; \{P_j, 1\} \; = \; 0 \end{aligned}$$

Canonical transformations connect different representations of w_N.

Quantization

Quantization

The passage from classical to quantum mechanics is realized by taking unitary representations of the algebra of the observables on an Hilbert space $\mathcal{H}$[3]:

$$\pi(f) \;=\; -i\hat{O}_f$$

where $\hat{O}_f$ is a self-adjoint operator. The Lie algebra homomorphism property reads

$$\pi\left(\{f\,,\,g\}\right) \;=\; \left[\pi(f)\,,\,\pi(g)\right]$$

Explicitly:

$$-i\,\hat{O}_{\{f\,,\,g\}} \;=\; -\left[\hat{O}_f\,,\,\hat{O}_g\right]$$

[3] P. Woit, *Quantum Theory, Groups and Representations* (Springer, 2017)

Schrödinger representation of w_1

The Schrödinger representation of w_1 is defined as the pair $\left(\Gamma'_S, L^2(\mathbb{R})\right)$

$$\begin{aligned} \Gamma_S(q)\,\psi(q) &= -i\,\hat{q}\,\psi(q) \;=\; -i\,q\,\psi(q) \\ \Gamma_S(p)\,\psi(q) &= -i\,\hat{p}\,\psi(q) \;=\; -\frac{\mathrm{d}}{\mathrm{d}q}\,\psi(q) \\ \Gamma_S(1)\,\psi(q) &= -i\,\hat{\mathbb{1}}\,\psi(q) \;=\; -i\,\psi(q) \end{aligned}$$

with $\psi(q) \in L^2(\mathbb{R})$.

We have

$$[\hat{q}\,,\,\hat{p}] \;=\; i\,\hat{\mathbb{1}}\,, \quad \left[\hat{q}\,,\,\hat{\mathbb{1}}\right] \;=\; \left[\hat{p}\,,\,\hat{\mathbb{1}}\right] \;=\; 0\,,$$

which are the Heisenberg CCR.

Schrödinger representation of W_1

Weyl-Heisenberg group W_1 (Schrödinger representation) elements are:

$$\left(\begin{pmatrix} x \\ y \end{pmatrix}, y\right) \equiv \exp\left(-iz\hat{\mathbb{1}}\right) \exp\left[-i\left(x\hat{q} + y\hat{p}\right)\right]$$

The group analog of the Heisenberg CCR is:

$$\exp\left(-ix\hat{q}\right) \exp\left(-iy\hat{p}\right) = \exp\left(-xy\right) \exp\left(-iy\hat{p}\right) \exp\left(-ix\hat{q}\right)$$

This is the Weyl form of CCR.

Stone–von Neumann theorem

- The Schrödinger representation of W_1 is irreducible.
- Given a basis e_1, e_2, e_3 of w_1:

Every irreducible representation of W_1, on an Hilbert space $\mathcal{H}$, so that

$$\pi(e_3) \; = \; -i \, \hat{\mathbb{1}}$$

is unitarily equivalent to the Schrödinger representation.

This is the Stone–von Neumann theorem. It can be generalized to W_N for a finite N.

Canonical transformations in quantum field theory

Inequivalent representations: fermion fields (1)

Let us consider a massless Dirac field

$$i\gamma^\mu \partial_\mu \hat{\psi} \;=\; 0$$

In a finite volume V:

$$\hat{\psi}(x) \;=\; \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left(\tilde{u}^r_{\mathbf{k}} \hat{\tilde{\alpha}}^r_{\mathbf{k}}(t) + \tilde{v}^r_{-\mathbf{k},j} \hat{\tilde{\beta}}^{r\dagger}_{-\mathbf{k}}(t) \right) e^{i\mathbf{k}\cdot\mathbf{x}}$$

where $\hat{\tilde{\alpha}}^r_{\mathbf{k}}(t) \;\equiv\; \hat{\tilde{\alpha}}^r_{\mathbf{k}} e^{-i\tilde{\omega}_{\mathbf{k}} t}$, $\hat{\tilde{\beta}}^r_{\mathbf{k}}(t) \;\equiv\; \hat{\tilde{\beta}}^r_{\mathbf{k}} e^{-i\tilde{\omega}_{\mathbf{k}} t}$. Let us perform a Bogoliubov transformation:

$$\hat{\tilde{\alpha}}^r_{\mathbf{k}}(t) \;=\; \hat{B}^{-1}_m(t)\, \hat{\alpha}^r_{\mathbf{k}}(t)\, \hat{B}_m(t) = \hat{\alpha}^r_{\mathbf{k}}(t) \cos\Theta_{\mathbf{k}} - \epsilon^r\, \hat{\beta}^{r\dagger}_{-\mathbf{k}}(t)\, \sin\Theta_{\mathbf{k}}$$

$$\hat{\tilde{\beta}}^{r\dagger}_{-\mathbf{k}}(t) = \hat{B}^{-1}_m(t)\, \hat{\beta}^{r\dagger}_{-\mathbf{k}}(t)\, \hat{B}_m(t) \;=\; \epsilon^r\, \hat{\alpha}^r_{\mathbf{k}}(t) \sin\Theta_{\mathbf{k}} + \hat{\beta}^{r\dagger}_{-\mathbf{k}}(t) \cos\Theta_{\mathbf{k}}$$

Inequivalent representations: fermion fields (2)

$$\hat{B}_m(t) \;=\; \exp\left[\sum_{\mathbf{k},r} \epsilon^r\,\Theta_{\mathbf{k}}\,\left(\hat{\alpha}^r_{\mathbf{k}}(t)\hat{\beta}^r_{-\mathbf{k}}(t) - \hat{\beta}^{r\dagger}_{-\mathbf{k}}(t)\hat{\alpha}^{r\dagger}_{\mathbf{k}}(t)\right)\right]$$

where $\Theta_{\mathbf{k}} = 1/2\cot^{-1}(|\mathbf{k}|/m)$ and $\hat{\alpha}^r_{\mathbf{k}}(t) \;\equiv\; \hat{\alpha}^r_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t}$, $\hat{\beta}^r_{\mathbf{k}}(t) \;\equiv\; \hat{\beta}^r_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t}$.

Defining

$$u^r_{\mathbf{k}} \;=\; \tilde{u}^r_{\mathbf{k}}\cos\Theta_{\mathbf{k}} + \epsilon^r\,v^r_{-\mathbf{k}}\,\sin\Theta_{\mathbf{k}} \tag{1}$$

$$v^r_{-\mathbf{k}} \;=\; \tilde{v}^r_{-\mathbf{k}}\cos\Theta_{\mathbf{k}} - \epsilon^r\,u^r_{\mathbf{k}}\,\sin\Theta_{\mathbf{k}} \tag{2}$$

Inequivalent representations: fermion fields (3)

The field can be expanded as:

$$\hat{\psi}(x) \; = \; \frac{1}{\sqrt{V}} \sum_{\mathbf{k},r} \left(u^r_{\mathbf{k}} \hat{\alpha}^r_{\mathbf{k}}(t) + v^r_{-\mathbf{k},j} \hat{\beta}^{r\dagger}_{-\mathbf{k}}(t) \right) e^{i\mathbf{k}\cdot\mathbf{x}}$$

This satisfies[4]:

$$(i\gamma^\mu \partial_\mu - m)\, \hat{\psi} \; = \; 0$$

[4] V.A. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories*, (World Scientific, London, 1993)

Inequivalent representations: fermion fields (4)

The respective vacua $|0\rangle$ and $|\tilde{0}\rangle$

$$\hat{\alpha}^r_{\mathbf{k}}\,|0\rangle \;=\; \hat{\beta}^r_{\mathbf{k}}\,|0\rangle \;=\; 0 \qquad \hat{\tilde{\alpha}}^r_{\mathbf{k}}\,|\tilde{0}\rangle \;=\; \hat{\tilde{\beta}}^r_{\mathbf{k}}\,|\tilde{0}\rangle \;=\; 0$$

are related as

$$|0\rangle \;=\; \hat{B}_m(0)\,|\tilde{0}\rangle$$

Removing the regulator:

$$\lim_{V\to\infty}\langle 0|\tilde{0}\rangle \;=\; 0$$

Orthogonal Fock spaces: Inequivalent representations of anticommutation relations.

Neutrino mixing

- Two-flavor neutrino mixing Lagrangian density:

$$\hat{\mathcal{L}} \;=\; \sum_{\sigma=e,\mu} \hat{\bar{\nu}}_\sigma(x)\,(i\gamma^\mu\partial_\mu - m_\sigma)\,\hat{\nu}(x) \;-\; m_{e\mu}\left(\hat{\bar{\nu}}_e(x)\hat{\nu}_\mu(x) + \hat{\bar{\nu}}_\mu(x)\hat{\nu}_e(x)\right)$$

- Mixing transformations define fields with definite mass ν_1, ν_2:

$$\begin{aligned}\hat{\nu}_e(x) &= \cos\theta\;\hat{\nu}_1(x) + \sin\theta\;\hat{\nu}_2(x)\\ \hat{\nu}_\mu(x) &= -\sin\theta\;\hat{\nu}_1(x) + \cos\theta\;\hat{\nu}_2(x)\end{aligned}$$

with

$$\tan 2\theta \;=\; \frac{2m_{e\mu}}{m_e - m_\mu}$$

Mixing generator

- At finite volume, mixing relations are rewritten as

$$
\begin{aligned}
\hat{\nu}_e^\alpha(x) &= \hat{G}_\theta^{-1}(t)\hat{\nu}_1^\alpha(x)\,\hat{G}_\theta(t) \\
\hat{\nu}_\mu^\alpha(x) &= \hat{G}_\theta^{-1}(t)\,\hat{\nu}_2^\alpha(x)\hat{G}_\theta(t)
\end{aligned}
$$

Mixing generator:

$$\hat{G}_\theta(t) = \exp\left[\theta \int d^3\mathbf{x}\,\left(\hat{\nu}_1^\dagger(x)\hat{\nu}_2(x) - \hat{\nu}_2^\dagger(x)\hat{\nu}_1(x)\right)\right]$$

Decomposition of the mixing generator (1)

- The mixing generator can be decomposed as[5]:

$$\hat{G}_\theta \;=\; \hat{B}(\Theta_1, \Theta_2) \;\; \hat{R}(\theta) \;\; \hat{B}^{-1}(\Theta_1, \Theta_2)$$

where $\hat{B}(\Theta_1, \Theta) \equiv \hat{B}_1(\Theta_1)\, \hat{B}_2(\Theta_2)$,

$$\hat{R}(\theta) \equiv \exp\left\{\theta \sum_{\mathbf{k},r} \left[\left(\hat{\alpha}^{r\dagger}_{\mathbf{k},1}\hat{\alpha}^{r}_{\mathbf{k},2} + \hat{\beta}^{r\dagger}_{-\mathbf{k},1}\hat{\beta}^{r}_{-\mathbf{k},2}\right) e^{i\psi_{\mathbf{k}}} - h.c.\right]\right\}$$

$$\hat{B}_i(\Theta_i) \equiv \exp\left\{\sum_{\mathbf{k},r} \Theta_{\mathbf{k},i}\; \epsilon^r \left[\hat{\alpha}^{r}_{\mathbf{k},i}\hat{\beta}^{r}_{-\mathbf{k},i} e^{-i\phi_{\mathbf{k},i}} - \hat{\beta}^{r\dagger}_{-\mathbf{k},i}\hat{\alpha}^{r\dagger}_{\mathbf{k},i} e^{i\phi_{\mathbf{k},i}}\right]\right\}, \quad i = 1, 2$$

and $\Theta_{\mathbf{k},i} = 1/2\, \cot^{-1}(|\mathbf{k}|/m_i)$, $\quad \psi_{\mathbf{k}} = (\omega_{\mathbf{k},1} - \omega_{\mathbf{k},2})t$, $\quad \phi_{\mathbf{k},i} = 2\omega_{\mathbf{k},i}t$.

[5] M.Blasone, M.V.Gargiulo and G.Vitiello, Phys. Lett.B **761**, 104 (2016)

Decomposition of the mixing generator (2)

$\hat{B}_i(\Theta_{\mathbf{k},i})$, $i = 1, 2$ are Bogoliubov transformations which induces a mass shift and $\hat{R}(\theta)$ is a rotation.

Their action on the mass vacuum is:

$$
\begin{aligned}
|\widetilde{0}\rangle_{1,2} &\equiv \hat{B}^{-1}(\Theta_1, \Theta_2)|0\rangle_{1,2} \\
&= \prod_{\mathbf{k},r} \left[\cos\Theta_{\mathbf{k},i} + \epsilon^r \sin\Theta_{\mathbf{k},i} \alpha^{r\dagger}_{\mathbf{k},i} \beta^{r\dagger}_{-\mathbf{k},i} \right] |0\rangle_{1,2} \\
\hat{R}^{-1}(\theta)|0\rangle_{1,2} &= |0\rangle_{1,2}
\end{aligned}
$$

- A rotation of fields is not a rotation at the level of creation and annihilation operators!

Flavor Vacuum

- The flavor vacuum is defined by[6]:

$$|0\rangle_{e,\mu} \equiv \hat{G}_\theta^{-1}(0)\; |0\rangle_{1,2}$$

In the infinite volume limit:

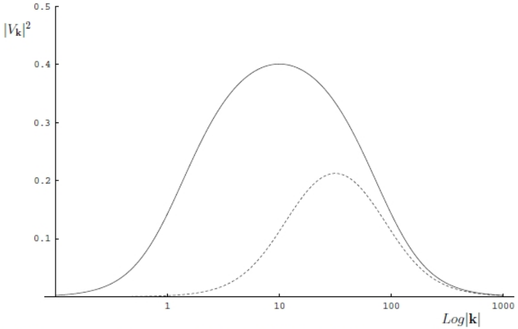
$$\lim_{V\to\infty}\ {}_{1,2}\langle 0|0\rangle_{e,\mu} = \lim_{V\to\infty}\ e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \ln\left(1-\sin^2\theta\,|V_\mathbf{k}|^2\right)^2} = 0$$

where

$$|V_\mathbf{k}|^2 \ \ \equiv\ \ \sum_{r,s} |\ v^{r\dagger}_{-\mathbf{k},1} u^s_{\mathbf{k},2}\ |^2 \neq 0 \quad for \quad m_1 \neq m_2$$

[6]M.Blasone and G.Vitiello, Ann. Phys. **244**, 283 (1995)

Vacuum condensate



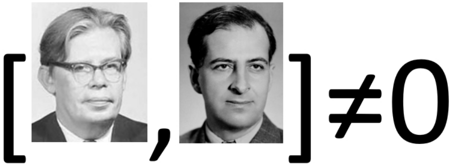
Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- Condensation density: ${}_{e,\mu}\langle 0|\alpha^{r\dagger}_{\mathbf{k},i}\alpha_{\mathbf{k},i}|0\rangle_{e,\mu} = \sin^2\theta\,|V_{\mathbf{k}}|^2$, with $i = 1, 2$. Same result for antiparticles.

- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

Bogoliubov vs Pontecorvo

- $\left[B(m_1, m_2)\ ,\ R^{-1}(\theta)\right] \neq 0$: Bogoliubov and Pontecorvo do not commute!!



As a result, flavor vacuum gets a non-trivial term:

$$|0\rangle_{e,\mu} \equiv G_\theta^{-1}|0\rangle_{1,2} = |0\rangle_{1,2} + \left[B(m_1, m_2)\ ,\ R^{-1}(\theta)\right]\ |\widetilde{0}\rangle_{1,2}$$

Flavor Vacuum and Condensate Structure

The flavor vacuum is characterized by a condensate structure:

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k}} \prod_{r} \Big[\left(1 - \sin^2\theta \, |V_{\mathbf{k}}|^2\right) - \epsilon^r \sin\theta \, \cos\theta \, |V_{\mathbf{k}}| \left(\hat{\alpha}^{r\dagger}_{\mathbf{k},1} \hat{\beta}^{r\dagger}_{-\mathbf{k},2} + \hat{\alpha}^{r\dagger}_{\mathbf{k},2} \hat{\beta}^{r\dagger}_{-\mathbf{k},1} \right)$$

$$+ \epsilon^r \sin^2\theta \, |V_{\mathbf{k}}||U_{\mathbf{k}}| \left(\hat{\alpha}^{r\dagger}_{\mathbf{k},1} \hat{\beta}^{r\dagger}_{-\mathbf{k},1} - \hat{\alpha}^{r\dagger}_{\mathbf{k},2} \hat{\beta}^{r\dagger}_{-\mathbf{k},2} \right) + \sin^2\theta \, |V_{\mathbf{k}}|^2 \hat{\alpha}^{r\dagger}_{\mathbf{k},1} \hat{\beta}^{r\dagger}_{-\mathbf{k},2} \hat{\alpha}^{r\dagger}_{\mathbf{k},2} \hat{\beta}^{r\dagger}_{-\mathbf{k},1} \Big] |0\rangle_{1,2}$$

- SU(2) (Perelomov) coherent state.
- This vacuum structure can be dynamically generated in an effective model within a string inspired framework[7]. The same it can be generally proved by means of algebraic reasonings[8]

---

[7] N.E. Mavromatos, S. Sarkar and W. Tarantino, Phys. Rev. D **80**, 084046 (2009)
[8] M.Blasone, P. Jizba, N.E. Mavromatos and L.S., arXiv:1807.07616 [hep-th] (2018)

Flavor-Energy uncertainty (1)

The flavor-charges

$$\hat{Q}_{\nu_\sigma}(t) \ = \ \int \mathrm{d}^3\mathbf{x}\, \hat{\nu}_\sigma^\dagger(x)\ \hat{\nu}_\sigma(x)\,, \qquad \sigma \ = \ e, \mu$$

are not conserved, i.e. $\left[\hat{Q}_{\nu_\sigma}(t)\,,\, \hat{H}\right] \neq 0$. It follows a flavor-energy uncertainty relation[9]:

$$\langle \Delta \hat{H}\rangle\, \langle \Delta \hat{Q}(t)\rangle \ \geq \ \frac{1}{2}\left|\frac{\mathrm{d}\langle \hat{Q}(t)\rangle}{\mathrm{d}t}\right|$$

It follows

$$\Delta E\, T \geq \mathcal{Q}_{\sigma\to\rho}(T)\,, \quad \sigma \neq \rho$$

[9] M. Blasone, P. Jizba and L.S., arXiv:1810.01648v2 [hep-ph] (2018)

Flavor-Energy uncertainty (2)

$$\mathcal{Q}_{e\to\mu}(t) \;=\; \sin^2(2\theta)\left[|U_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{\mathbf{k},1}-\omega_{\mathbf{k},2}}{2}t\right)+|V_{\mathbf{k}}|^2\sin^2\left(\frac{\omega_{\mathbf{k},1}+\omega_{\mathbf{k},2}}{2}t\right)\right]$$

is the exact QFT oscillation formula[10] with $|U_{\mathbf{k}}|^2 = 1-|V_{\mathbf{k}}|^2$. When $m_i/|\mathbf{k}| \to 0$:

$$\Delta E \;\geq\; \frac{2\sin^2(2\theta)}{L_{osc}}$$

where $L_{osc} = 4\pi|\mathbf{k}|/\delta m^2$. Corrections beyond ultra-relativistic regime:

$$\Delta E \;\geq\; \frac{2\,\sin^2 2\theta}{L_{osc}}\left[1-\varepsilon(\mathbf{k})\,\cos^2\left(\frac{|\mathbf{k}|L_{osc}}{2}\right)\right]$$

with $\varepsilon(\mathbf{k}) \equiv (m_1-m_2)^2/(4|\mathbf{k}|^2)$.

[10] M. Blasone, P.A. Henning and G. Vitiello, Phys. Lett. B **451**, 140 (1999)

GGF for mixed representation Green's functions

Scalar field Mixing

Let us consider the Lagrange density

$$\hat{\mathcal{L}}(x) \;=\; \partial_\mu \hat{\boldsymbol{\varphi}}_f^\dagger(x) \partial_\mu \hat{\boldsymbol{\varphi}}_f(x) \;-\; \hat{\boldsymbol{\varphi}}_f^\dagger(x)\, M\, \hat{\boldsymbol{\varphi}}_f(x)$$

where

$$\hat{\boldsymbol{\varphi}}_f(x) \;=\; \begin{bmatrix} \hat{\varphi}_A(x) \\ \hat{\varphi}_B(x) \end{bmatrix}, \qquad M = \begin{bmatrix} m_A^2 & m_{A\,B}^2 \\ m_{B\,A}^2 & m_B^2 \end{bmatrix}$$

We proceed as in the fermion case[11]. Mixing transformation:

$$\hat{\boldsymbol{\varphi}}_f(x) \;=\; \hat{G}_\theta^{-1}(t)\, \hat{\boldsymbol{\varphi}}_m(x)\, \hat{G}_\theta(t)$$

[11] M. Blasone, A. Capolupo, O. Romei and G. Vitiello, Phys. Rev. D **63**, 125015 (2001)

Flavor vacuum

$$\hat{G}_\theta(t) \;=\; \exp\left\{\theta\left[\hat{S}_+(t) - \hat{S}_-(t)\right]\right\}$$

with $\tan 2\theta \;=\; 2\, m_{AB}^2/\left(m_B^2 - m_A^2\right)$, and

$$\hat{S}_+(t) \;=\; \hat{S}_-^\dagger(t) \;=\; -i\int \mathrm{d}^3\mathbf{x}\;\left[\hat{\pi}_1(x)\hat{\varphi}_2(x) \;-\; \hat{\varphi}_1^\dagger(x)\hat{\pi}_2^\dagger(x)\right]$$

Flavor vacuum is defined as for fermions

$$|0\rangle_{A,B} \;=\; \hat{G}_\theta^{-1}(0)|0\rangle_{1,2}$$

Green's functions on the flavor vacuum

Let us consider the Green's functions:

$$\mathcal{G}_{\rho\sigma}(x) \; = \; {}_{A,B}\langle 0|T[\varphi_\rho(x)\varphi^\dagger_\sigma(0)]|0\rangle_{A,B} \, , \qquad \rho,\sigma \; = \; A, B$$

We would find a generating functional of these Green's functions:

$$\mathcal{G}_{\rho\sigma}(x) \; = \; \left. \frac{\delta^2 \mathcal{Z}[J]}{\delta J_\rho(x) \delta J_\sigma(0)} \right|_{J=0}$$

We start studying the problem in QM[12]

[12] M. Blasone, P. Jizba and L.S., Phys. Rev. A **96**, 052107 (2017)

Canonical transformations in QM

Let us consider a canonical transformation:

$$\hat{q}(t;\alpha) \equiv \hat{G}^\dagger_\alpha(t)\, \hat{q}(t)\, \hat{G}_\alpha(t)$$

$$\hat{p}(t;\alpha) \equiv \hat{G}^\dagger_\alpha(t)\, \hat{p}(t)\, \hat{G}_\alpha(t)$$

where

$$\hat{G}_\alpha(t) = \exp\left[i\alpha \hat{K}(\hat{q},\hat{p})\right]$$

We define the new “vacua”:

$$|0(\alpha,t)\rangle \equiv \hat{G}^\dagger_\alpha(t)|0\rangle$$

Green's functions

Our problem is to find a generating functional for all these sets of Green's functions:

$$
\begin{aligned}
i\mathcal{G}_0(t'-t) \;\equiv\; i\mathcal{G}_{00}(t'-t) &= \langle 0|T\left[\hat{q}(t')\hat{q}(t)\right]|0\rangle \\
i\mathcal{G}_{0\alpha}(t'-t) &= \langle 0|T\left[\hat{q}(t';\alpha)\hat{q}(t;\alpha)\right]|0\rangle \\
i\mathcal{G}_{\beta 0}(t'-t) &= \langle 0(\beta,t)|T\left[\hat{q}(t')\hat{q}(t)\right]|0(\beta,t)\rangle \\
i\mathcal{G}_{\beta\beta}(t'-t) &= \langle 0(\beta,t)|T\left[\hat{q}(t';\beta)\hat{q}(t;\beta)\right]|0(\beta,t)\rangle
\end{aligned}
$$

Generating functional

Defining

$$\mathcal{Z}_{0\alpha}[J_q] \;=\; \lim_{\substack{t_f \to +\infty \\ t_i \to -\infty}} \frac{\int \mathcal{D}\mu(\alpha)\, e^{iS[p,q;\alpha]+i\int_{-\infty}^{+\infty} J_q(t;\alpha)q(t;\alpha)}}{\int \mathcal{D}\mu(\alpha)\, e^{iS[p,q;\alpha]}}$$

where $q(t_f;\alpha) = q_f(\alpha)$, $q(t_i;\alpha) = q_i(\alpha)$ and

$$S(p,q;\alpha) = \int_{t_i}^{t_f} \mathrm{d}t \left[p(t;\alpha)\dot{q}(t;\alpha) - H(q(\alpha),p(\alpha))\right]$$

We get

$$\langle 0|T\left[\hat{q}(t_2;\alpha)\hat{q}(t_1;\alpha)\right]|0\rangle = (-i)^n \frac{\delta^n}{\delta J_q(t_2;\alpha)\delta J_q(t_1;\alpha)} \mathcal{Z}_{0\,\alpha}[J_q]\bigg|_{J_q=0}$$

Generalized generating functional

By definition

$$\langle 0(\beta, t_+)|T[\hat{q}(t_2;\alpha)\hat{q}(t_1;\alpha)]|0(\beta, t_-)\rangle$$

$$= \langle 0|\hat{G}_\beta(t_+)T[\hat{q}(t_2;\alpha)\hat{q}(t_1;\alpha)]\hat{G}^\dagger_\beta(t_-)|0\rangle$$

If t_+ is greater than t_1 and t_2, and t_- is the smallest:

$$\langle 0(\beta, t_+)|T[\hat{q}(t_2;\alpha)\hat{q}(t_1;\alpha)]|0(\beta, t_-)\rangle$$

$$= \lim_{\substack{t_f\to+\infty \\ t_i\to-\infty}} \frac{\langle q_f(\alpha), t_f|T[\hat{G}_\beta(t_+)\hat{q}(t_2;\alpha)\hat{q}(t_1;\alpha)\hat{G}^\dagger_\beta(t_-)]|q_i(\alpha), t_i\rangle}{\langle q_f(\alpha), t_f|q_i(\alpha), t_i\rangle}$$

qp–ordering

We introduce the qp–ordering as follows

$$\mathcal{O}^q\left[e^{iK(\hat{p}(t;\alpha),\hat{q}(t;\alpha))}\right] \equiv \sum_{k,l=0}^{\infty} K_{kl}\ \hat{q}^k(t;\alpha)\hat{p}^l(t;\alpha)$$

The latter orders the operator in such a way that all $\hat{q}(\tau)$'s are on the left and $\hat{p}(\tau)$'s are on the right.

The "classical" qp–ordering is

$$\mathcal{O}^q_{cl}\left[e^{iK(\hat{p}(t;\alpha),\hat{q}(t;\alpha))}\right] = \frac{\langle q(\alpha),t|\mathcal{O}^q\left[e^{iK(\hat{p}(t;\alpha),\hat{q}(t;\alpha))}\right]|p(\alpha),t\rangle}{\langle q(\alpha),t|p(\alpha),t\rangle}$$

Schwinger CTP (1)

Thanks to Feynman-Matthews-Salam formula, we get:

$$\langle 0(\beta, t_+)|T[\hat{q}(t_n;\alpha)\ldots\hat{q}(t_1;\alpha)]|0(\beta,t_-)\rangle$$

$$= \lim_{\substack{t_f \to +\infty \\ t_i \to -\infty}} \frac{\int \mathcal{D}\mu(\alpha)\, \mathcal{O}^q_{cl}\left[G_\beta(t_+)\right]\, \mathcal{O}^q_{cl}\left[G_{-\beta}(t_-)\right]\, q(t_2;\alpha) q(t_1;\alpha)\;\; e^{iS(p,q;\alpha)}}{\int \mathcal{D}\mu(\alpha)\;\; e^{iS(p,q;\alpha)}}$$

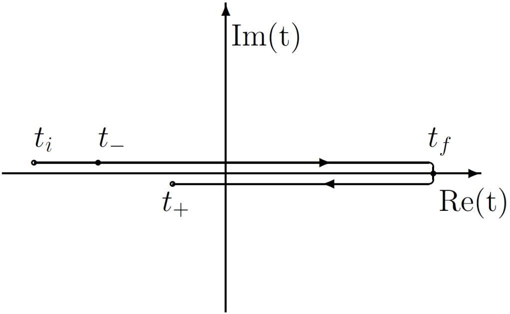
Taking the limit $t_+ \to t_-$:

$$\langle 0(\beta, t)|T[\hat{q}(t_n;\alpha)\ldots\hat{q}(t_1;\alpha)]|0(\beta,t)\rangle$$

$$= \lim_{\substack{t_f \to +\infty \\ t_i \to -\infty}} \frac{\int_{\mathcal{C}} \mathcal{D}\mu(\alpha)\, \mathcal{O}^q_{cl}\left[G_\beta(t_+)\right] \mathcal{O}^q_{cl}\left[G_{-\beta}(t_-)\right]\, q(t_n;\alpha)\ldots q(t_1;\alpha)\;\; e^{iS(p,q;\alpha)}}{\int \mathcal{D}\mu(\alpha)\;\; e^{iS(p,q;\alpha)}}$$

Schwinger CTP(2)

The Schwinger contour $\mathcal{C}$ is shown in figure:



GGF: definition

We define a generalized generating functional of Green's functions:

$$\mathcal{Z}^{+\,-}_{\beta\,\alpha}[J_q] \;=\; e^{if(\beta)K\left(\frac{\delta}{\delta J_p(t_+;\alpha)},\frac{\delta}{\delta J_q(t_+;\alpha)}\right)} \times e^{if(-\beta)K\left(\frac{\delta}{\delta J_p(t_-;\alpha)},\frac{\delta}{\delta J_q(t_-;\alpha)}\right)}\mathcal{Z}_{0\,\alpha}[J_q,J_p]\Big|_{J_p=0}$$

where

$$\mathcal{Z}_{0\,\alpha}[J_q,J_p] \;=\; \lim_{\substack{t_f\to+\infty\\ t_i\to-\infty}} \frac{\int\mathcal{D}\mu(\alpha)\, e^{iS(p,q;\alpha)+i\int_{t_i}^{t_f}\,\mathrm{d}t[J_q(t)q(t;\alpha)+J_q(t)p(t;\alpha)]}}{\int\mathcal{D}\mu(\alpha)\;\; e^{iS(p,q;\alpha)}}$$

Mixed representation Green's functions

Mixed representation Green's functions can be thus obtained as

$$i\mathcal{G}_{\beta 0}(t'-t) = \lim_{t_+ \to t_- = t} (-i)^2 \frac{\delta^2}{\delta J_q(t')\delta J_q(t)} \mathcal{Z}^{+-}_{\beta 0}[J_q]\Big|_{J_q=0}$$

$$i\mathcal{G}_{\beta\beta}(t'-t) = \lim_{t_+ \to t_- = t} (-i)^2 \frac{\delta^2}{\delta J_q(t';\beta)\delta J_q(t;\beta)} \mathcal{Z}^{+-}_{\beta\beta}[J_q]\Big|_{J_q=0}$$

GGF: perturbative approach

If $f(\beta) = o(\beta)$, we can write a perturbative expansion in β. The leading-order reads:

$$\begin{aligned} \mathcal{Z}^{+-}_{\beta\alpha}[J_q] &\approx& \mathcal{Z}_{0\alpha}[J_q] \;+\; i\beta \left[K\left(\frac{\delta}{\delta J_p(t_+;\alpha)}, \frac{\delta}{\delta J_q(t_+;\alpha)} \right) \right. \\ &-& \left. \left. K\left(\frac{\delta}{\delta J_p(t_-;\alpha)}, \frac{\delta}{\delta J_q(t_-;\alpha)} \right) \right] \mathcal{Z}_{0\alpha}[J_q, J_p] \right|_{J_p=0} \end{aligned}$$

Vacuum-to-vacuum amplitude

Generalizing our previous definition:

$$\begin{aligned}\mathcal{Z}^{+-}_{\gamma\beta\alpha}[J_q] &= e^{if(\gamma)K\left(\frac{\delta}{\delta J_p(t_+;\alpha)},\frac{\delta}{\delta J_q(t_+;\alpha)}\right)} \\ &\times \; e^{if(-\beta)K\left(\frac{\delta}{\delta J_p(t_-;\alpha)},\frac{\delta}{\delta J_q(t_-;\alpha)}\right)}\mathcal{Z}_{0\alpha}[J_q, J_p]\Big|_{J_p=0}\end{aligned}$$

we find that the vacuum-to-vacuum transition amplitude:

$$\langle 0|0(\beta, t)\rangle = \lim_{t_-\to t}\mathcal{Z}^{+-}_{0\beta 0}[0]$$

Translations

Let us start by considering translations

$$\hat{q}(t;\alpha) \;=\; \hat{q}(t) \;+\; \alpha\,, \qquad\qquad \hat{p}(t;\alpha) \;=\; \hat{p}(t)$$

The generator of this transformation is

$$\hat{G}_\alpha(t) \;=\; \exp[-i\alpha\hat{p}(t)]\,.$$

We now consider the Hamiltonian

$$\begin{aligned}
&\hat{H}\,(\hat{p}(t;\alpha), \hat{q}(t;\alpha)) \\
&\qquad = \;\frac{\hat{p}^2(t;\alpha)}{2} \;+\; \frac{\hat{q}^2(t;\alpha)}{2} \;-\; \alpha\,\hat{q}(t;\alpha) \;+\; \frac{\alpha^2}{2}
\end{aligned}$$

Translations: vacuum to vacuum transition amplitude

We can calculate the generalized GGF:

$$\begin{aligned} \mathcal{Z}^{+-}_{0\beta 0}[J_q] &= e^{-\frac{i}{2}\int_{-\infty}^{+\infty}\mathrm{d}\tau \int_{-\infty}^{+\infty}\mathrm{d}\tau' J_q(\tau)\mathcal{G}_0(\tau-\tau')J_q(\tau')} \\ &\times \quad e^{i\beta\int_{-\infty}^{+\infty}\mathrm{d}\tau J_q(\tau) - \frac{\beta^2}{4} - i\beta\int_{-\infty}^{+\infty}\mathrm{d}\tau J_q(\tau)\partial_{t_-}\mathcal{G}_0(\tau-t_-)} \end{aligned}$$

We can evaluate the vacuum-vacuum transition amplitude

$$\langle 0|0(\beta,t)\rangle = \lim_{t_-\to t}\mathcal{Z}^{+-}_{0\beta 0}[0] = \exp\left[-\frac{\beta^2}{4}\right]$$

Translations: Mixed representation Green's functions

We can calculate mixed representation Green's functions:

$$i\mathcal{G}_{\beta\,0}(t'-t) \;=\; i\mathcal{G}_0(t'-t) \;+\; \beta^2\cos(t-t')$$

$$i\mathcal{G}_{\beta\,\beta}(t'-t) \;=\; i\mathcal{G}_0(t'-t)$$

where

$$i\mathcal{G}_0(t'-t) = \frac{1}{2}\left[\theta(t'-t)e^{-i(t'-t)} + \theta(t-t')e^{i(t'-t)}\right]$$

is the harmonic oscillator Feynman propagator.

Mass vacuum partition function

The generating functional of Green's functions on the mass vacuum can be derived:

$$\mathcal{Z}[\boldsymbol{J}_\varphi] \;=\; e^{-i\int \mathrm{d}^4x \int \mathrm{d}^4y\, \boldsymbol{J}^\dagger_\varphi(x)\, \Delta^f_F(x-y)\, \boldsymbol{J}_\varphi(y)}$$

where the flavor propagator is:

$$\Delta^f_F(x-y) = \begin{bmatrix} \Delta^1_F(x-y)\cos^2\theta + \Delta^2_F(x-y)\sin^2\theta & (\Delta^2_F(x-y) - \Delta^1_F(x-y))\cos\theta\sin\theta \\ (\Delta^2_F(x-y) - \Delta^1_F(x-y))\cos\theta\sin\theta & \Delta^2_F(x-y)\cos^2\theta + \Delta^1_F(x-y)\sin^2\theta \end{bmatrix}.$$

Here Δ^1_F and Δ^2_F are scalar propagator of fields with definite masses.

Flavor generating functional: a perturbative approach

For $\theta \ll 1$ we get[13]:

$$\mathcal{Z}_f[\boldsymbol{J}_\varphi] \approx \mathcal{Z}[\boldsymbol{J}_\varphi] - i\theta\Big[\left(S_+(\tau_+) - S_-(\tau_+) + S_-(\tau_-) - S_+(\tau_-)\right)\mathcal{Z}[\boldsymbol{J}_\varphi, \boldsymbol{J}_\pi]\Big]_{\mathbf{J}_\pi=0}$$

where

$$S_+(t) = S_-^*(t) = -i\int \mathrm{d}^3\mathbf{x}\, \left[\frac{\delta}{\delta J_{\pi\, A}(t,\mathbf{x})}\frac{\delta}{\delta J^*_{\phi\, B}(t,\mathbf{x})} - \frac{\delta}{\delta J_{\phi\, A}(t,\mathbf{x})}\frac{\delta}{\delta J^*_{\pi\, B}(t,\mathbf{x})}\right]$$

[13] M. Blasone, P. Jizba and L. Smaldone, J. Phys. Conf. Ser. **880**, 012051 (2017)

Generating functional on the flavor vacuum

The result is:

$$
\begin{aligned}
&\mathcal{Z}_f[J_\varphi] \approx \ \mathcal{Z}[J_\varphi]\,\{1- \\
&\theta \int \mathrm{d}^3\mathbf{x}_\pm \ \int \mathrm{d}^4x \int \mathrm{d}^4y \left[-J^*_{\varphi,\alpha}(x)\partial_{\tau_+}\Delta^{\alpha A}(x-x_+)\Delta^{B\beta}(x_+-y)J_{\varphi,\beta}(y)\right. \\
&+\ J^*_{\varphi,\alpha}(x)\Delta^{\alpha A}(x-x_+)\partial_{\tau+}\Delta^{B\beta}(x_+-y)J_{\varphi,\beta}(y) \\
&+J_{\varphi,\alpha}(x)\partial_{\tau_-}\Delta^{\alpha A}(x-x_-)\Delta^{B\beta}(x_--y)J^*_{\varphi,\beta}(y) \\
&-\ J_{\varphi,\alpha}(x)\Delta^{\alpha A}(x-x_-)\partial_{\tau-}\Delta^{B\beta}(x_--y)J^*_{\varphi,\beta}(y) \\
&-J_{\varphi,\beta}(x)\Delta^{B\beta}(x-x_+)\partial_{\tau_+}\Delta^{\alpha A}(x_+-y)J^*_{\varphi,\alpha}(y) \\
&+\ J_{\varphi,\beta}(x)\partial_{\tau_+}\Delta^{\beta B}(x-x_+)\Delta^{A\alpha}(x_+-y)J^*_{\varphi,\alpha}(y) \\
&+J^*_{\varphi,\beta}(x)\Delta^{B\beta}(x-x_-)\partial_{\tau_-}\Delta^{\alpha A}(x_--y)J_{\varphi,\alpha}(y) \\
&\left.-\ J^*_{\varphi,\beta}(x)\partial_{\tau_+}\Delta^{\beta B}(x-x_-)\Delta^{A\alpha}(x_--y)J_{\varphi,\alpha}(y)\right]\} \ .
\end{aligned}
$$

where

$$x_+ \equiv (\tau_+, \mathbf{x}) \qquad x_- \equiv (\tau_-, \mathbf{x})$$

Conclusions and perspectives

Conclusions and perspectives

- QFT is characterized by unitarily inequivalent representations of canonical (anti)commutation relations.
- The study of flavor mixing in QFT reveals an exotic vacuum structure. The problem of evaluating Green's functions on the flavor vacuum led us to a generalization of the standard generating functional of Green's functions. This opens new opportunities in the study of flavor oscillations.
- These objects can be related to coherent state path integrals[14]. The relation with Perelomov coherent state path integrals has to be established.

[14] M. Blasone, P. Jizba, L. S., J. Phys. Conf. Ser. **965**, 012008 (2018)

Thank you for the attention!