Emergent topological structures in supergravity

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JHEP 1603 (2016) 169, J. Bae, C.I. S.J Rey, D. Rosa
JHEP 1805 (2018) 112, C. I. and D. Rosa,
C. I., V. Pedemonte and D. Rosa, forthcoming

The topological sectors of supergravity

- Today I am going to describe two different topological structures which sit inside supergravity.
- The first one is universal: it exists in any dimensions and in any supergravity.
- The second structure exists for a certain class of supergravity theories, which includes $N = (2, 2)$ and $N = (4, 4)$ in $d = 2$ and $N = 2$ in $d = 4$. For lack of a better name, I am going to refer to this class as “twistable” supergravities.

Two emergent structures

- The first structure consists of topological gravity coupled to topological Yang-Mills.
- The second structure consists of several topological scalar multiplets sitting in a global duality group, coupled to topological gravity.
- Both structures are emergent in the sense that (part of) the topological fields are composites of the microscopic supergravity fields.

Applications

- These structures might be relevant for an “effective” low energy description of topological sector of quantized supergravity. This is still far ahead.
- A more mundane application, which I am going to sketch in this talk, will be to localization of supersymmetric matter theories.

Topological integrability conditions

- It will turn out that, on the space of classical supersymmetric vacua, the two topological structures are related by certain polynomial relations, which are equivalent to the integrability conditions of the equations for generalized covariantly constant spinors.
- This fact provides a powerful tool to analyse the space of classical supersymmetric vacua of supergravity.

Topological gravity

- The fields of topological gravity include the space-time metric $g_{\mu\nu}$ and the topological gravitino $\psi_{\mu\nu}$.
- “Morally" these fields should be related by the simple BRST transformation rules

$$s_0\, g_{\mu\nu} = \psi_{\mu\nu} \qquad s_0\, \psi_{\mu\nu} = 0$$

BRST vs De Rham cohomology

- Obviously $s_0^2 = 0$.
- s_0 has the geometric interpretation of De Rham differential on the space Met of the space-times metrics, $\psi_{\mu\nu}$ being the differential on this space:

$$s_0\, Z(g_{\mu\nu}, \psi_{\mu\nu}) = \int \psi^{\mu\nu}(x)\, \frac{\delta Z(g, \psi)}{\delta\, g^{\mu\nu}(x)}$$

The quotient with respect to diffeomorphisms

- Met however is a topologically trivial space, and the BRST cohomology of s_0 is empty:

$$s_0\, Z(g_{\mu\nu}, \psi_{\mu\nu}) = 0 \Rightarrow Z(g_{\mu\nu}, \psi_{\mu\nu}) = s_0\, S(g_{\mu\nu}, \psi_{\mu\nu})$$

- The interesting space is $\mathrm{Met}/\mathrm{Diff}$, where Diff is the group of space-time diffeomorphisms.
- Topological gravity should compute the form cohomology of such quotient space.

Equivariant cohomology

- The machinery to compute the cohomology of a quotient space $\mathcal{M} = X/G$ defined by the action of a group G is G-equivariant cohomology.
- The basic trick of equivariant form cohomology is to consider not just standard forms on $X \ni y^i$, but equivariant forms

$$Z(y^i, dy^i) \rightarrow Z(y^i, dy^i, \gamma^a)$$

where γ^a are commuting variables with values in the Lie algebra of G.

The equivariant differential

- One then defines an extension D_γ of the De Rham differential d on equivariant forms

$$D_\gamma = dy^i \frac{\partial}{\partial y^i} - \gamma^a \, i_{V^a}$$

where $V^{a\,i}(y)$ are the vector fields describing the action of G on X and

$$i_V = V^i \frac{\partial}{\partial dy^i}$$

is the contraction of a form with the vector V^i.

The equivariant algebra

- The equivariant differential does not square to zero:

$$D_\gamma^2 = \mathcal{L}_{\gamma^a V^a}$$

where $\mathcal{L}_V$ is the Lie derivative along the vector $V = \gamma^a V^a$.

- However on the space of G-invariant equivariant forms, this is as good.
- The main theorem is that the cohomology of D_γ on the space of G-invariant equivariant forms is the same as the standard de Rham cohomology on the quotient X/G, when the latter is smooth. Otherwise it provides its proper definition.

Equivariant Topological Gravity

- In the context of topological gravity, this means that we have to add to the fields $g_{\mu\nu}(x)$ and $\psi_{\mu\nu}(x)$ also the commuting fields $\gamma^{\mu}(x)$.
- The BRST equivariant transformations are accordingly

$$
\begin{aligned}
S\, g_{\mu\nu} &= \psi_{\mu\nu} \\
S\, \psi_{\mu\nu} &= -\mathcal{L}_{\gamma}\, g_{\mu\nu} \\
S\, \gamma^{\mu} &= 0
\end{aligned}
$$

- $\psi_{\mu\nu}$ has ghost number +1 and γ^{μ} ghost number +2.

Topological gravity BRST cohomology

- The equivariant topological gravity BRST operator squares to the Lie derivative along $\gamma^\mu(x)$:

$$S^2 = \mathcal{L}_\gamma$$

- Observables of topological gravity are therefore BRST classes of S on the space of reparametrization invariant (local) functionals.
- Their expectation values compute intersection numbers of $\mathrm{Met}/\mathrm{Diff}$.

Topological Yang-Mills

- Much of the same story repeats for topological Yang-Mills :

$$\begin{aligned} S\, A^I &= \lambda^I \\ S\, \lambda^I &= -\delta_\phi\, A^I = -D\,\phi^I \\ S\, \phi^I &= 0 \end{aligned}$$

- $A^I = A^I_\mu dx^\mu$ is the connection and λ is the topological gaugino.
- $\phi^I(x)$ is commuting field of ghost number +2, with values in the YM Lie algebra.
- The BRST cohomology of S computes form cohomology on moduli space of gauge connections $\mathcal{A}/\mathcal{G}$.

Topological Yang-Mills coupled to Topological Gravity

- One can also consider the coupling of topological YM to topological gravity

$$
\begin{aligned}
S\,A &= \lambda \\
S\,\lambda &= -D\,\phi + i_\gamma(F) \\
S\,\phi &= i_\gamma(\lambda)
\end{aligned}
$$

Topological YM coupled to topological gravity

- This theory computes the De Rham cohomology on the quotient $(\mathrm{Met}(M) \times \mathcal{A}(M))/(\mathrm{Diff} \rtimes \mathcal{G})$ of metrics and connections on a manifold M modulo to combined action of Diffeos and gauge transformations.
- This theory should provide a field theoretical way to study the metric dependence of Donaldson invariants, wall-crossing phenomena, quantum topological anomalies etc. As far as I know, it has not been explored yet.

The BRST formulation of supergravity

- I will start by revisiting the BRST formulation of supergravity, for the purpose of setting the notation.
- This formulation requires introducing:
  - anti-commuting ghosts for bosonic symmetries;
  - commuting ghosts for fermionic symmetries;
  - a nilpotent operator s acting on ghost and other matter fields.

Ghosts and superghosts

- (Poincaré) Supergravity bosonic symmetries include:
  - Diffeos with ghost ξ^μ
  - YM symmetries like local Lorentz and local R-symmetries with ghost c, living in the total = Lorentz+YM Lie algebra
- Fermionic symmetries:
  - Local supersymmetries with ghosts ζ^i, with $i = 1, \ldots N$, which are Majorana spinors.

The BRST transformations of the supersymmetric ghost

- The action of the BRST s on the supersymmetric ghost is

$$s\,\zeta^i = i_\gamma(\psi^i) + \text{diffeos} + \text{gauge}$$

where i_γ is the contraction of a form by the composite vector ghost bilinear γ^μ

$$\gamma^\mu \equiv -\frac{1}{2}\,\bar{\zeta}_i\,\Gamma^A\,\zeta^i\,e^\mu_A$$

and $\psi^i = \psi^i_\mu\,dx^\mu$ are the gravitinos.

The BRST transformations of the metric

- The BRST transformation of the vierbein is also universal

$$s\, e^A = \bar{\zeta}_i\, \Gamma^A \psi^i + \text{diffeos} + \text{gauge}$$

The BRST algebra

- One can show that the BRST algebra of any supergravity theory takes the form

$$S^2 = \mathcal{L}_\gamma + \delta_{i_\gamma(A)+\phi}$$

- S is obtained from s by subtracting the transformations associated to the bosonic gauge symmetries

$$S = s + \delta_c + \mathcal{L}_\xi$$

The γ^μ and ϕ ghost bilinears

We see that the BRST algebra is fully characterized by two bilinears of the commuting ghosts ζ^i

$$\boxed{\gamma^\mu \quad : \text{a commuting vector fields}}$$

$$\boxed{\phi \quad : \text{scalars in the total gauge Lie algebra}}$$

The γ^μ and ϕ ghost bilinears

- The vector bilinear γ^μ has an universal expression

$$\gamma^\mu \equiv -\frac{1}{2}\,\bar{\zeta}_i\,\Gamma^A\,\zeta^i\,e^\mu_A$$

- The scalar ghost bilinear

$$\phi = \phi^{AB}\,\frac{1}{2}\,\sigma_{AB} + \phi^I\,T^I$$

valued in the total = Lorentz +R-symmetry gauge Lie algebra is model dependent: it characterises the specific supergravity one is considering.

The BRST transformations of ghost bilinears

- The basic observation is that ghost bilinears γ^μ and ϕ have remarkable and universal BRST transformation properties:

$$S\,\gamma^\mu = 0$$
$$S\,\phi = i_\gamma(\lambda)$$

Topological gravity inside supergravity

- The BRST transformation rule of γ^μ is the one of the superghost of topological gravity.
- Indeed one also finds

$$S\, g_{\mu\nu} = \bar{\zeta}_i\, \Gamma_{(\mu}\, \psi^i_{\nu)} \equiv \psi_{\mu\nu} \equiv \text{Topological gravitino}$$
$$S\, \psi_{\mu\nu} = \mathcal{L}_\gamma\, g_{\mu\nu}$$

which are precisely the BRST transformations of topological gravity

Topological YM inside supergravity

- The BRST transformation rules of ϕ are precisely those of the superghost of topological YM coupled to topological gravity.
- One also finds the rest of the topological BRST rules, after introducing the composite topological gaugino:

$$S\,A = \lambda \equiv \text{Topological gaugino}$$
$$S\,\lambda = -D\,\phi + i_\gamma(F)$$
$$S\,\phi = i_\gamma(\lambda)$$

Topological YM coupled to topological gravity

- Summarizing, there exists a universal subsector of composites of supergravity fields transforming under BRST precisely as the fields of topological YM coupled to topological gravity.
- This topological structure is emergent, and, therefore it is not obvious, yet, what is its fate at quantum level.
- Later I will discuss its relevance to localization, for which supergravity is a classical background.

The second topological structure of supergravity

- More topological multiplets emerge whenever gauge invariant scalar bilinears F_a of the commuting ghosts ζ^i — not depending on other bosonic fields — exist.
- For lack of a better name, I will refer to supergravities with this property as “twistable”.

Gauge invariant scalar ghost bilinears

- $d = 2\ N = 2$

$$F_1 = \bar{\zeta}\,\zeta \qquad F_2 = \bar{\zeta}\,\Gamma_3\,\zeta$$

- $d = 2\ N = 4$

$$F_1 = \bar{\zeta}_i\,\zeta^i \quad F_2 = \bar{\zeta}_i\,\Gamma_3\,\zeta^i \quad F_3 + i\,F_4 = \bar{\zeta}^c_i\,\epsilon^{ij}\,\zeta_j$$

- $d = 4\ N = 2$

$$F = \epsilon^{\alpha\beta}\,\epsilon_{ij}\,\zeta^i_\alpha\,\zeta^j_\beta \qquad \overline{F} = \epsilon_{\dot\alpha\dot\beta}\,\epsilon^{ij}\,\bar{\zeta}^{\dot\alpha}_i\,\bar{\zeta}^{\dot\beta}_j$$

The BRST transformations of the F_a

- Let us set, schematically, $F_a = \bar{\zeta}\, X_a\, \zeta$.
- Then, since,

$$S\,\zeta^i = i_\gamma(\psi^i)$$

one obtains

$$S\,F_a = i_\gamma(\chi_a^{(1)})$$

where $\chi_a^{(1)}$ is a fermionic one-form of ghost number 1

$$\chi_a^{(1)} = \bar{\zeta}\, X_a\, \psi + \bar{\psi}\, X_a\, \zeta$$

The BRST multiplet of gauge invariant ghost bilinears

- BRST descent equations ensue from the supergravity BRST algebra

$$
\begin{aligned}
S\,F_a &= i_\gamma(\chi_a^{(1)}) \\
S\,\chi_a^{(1)} &= -d\,F_a + i_\gamma(N_a^{(2)}) \\
S\,(N_a^{(2)}) &= -d\,\chi_a^{(1)}
\end{aligned}
$$

where $N_a^{(2)}$ is a bosonic two-form of ghost number 0.

Superfields

- In short, when the supergravity is “twistable”, topological scalar multiplets coupled to topolgical gravty exist

$$\mathbb{H}_a = F_a + \chi_a^{(1)} + N_a^{(2)}$$
$$(S + d - i_\gamma)\,\mathbb{H}_a = 0$$

  with a which labels the invariant ghost bilinears.
- We will see that on the space of supersymmetric configurations this scalar topological structure is related to the topological gravity structure.

Application to Localization

- Localization is a long-known property of both supersymmetric (SQFT) and topological (TQFT) theories, by virtue of which semi-classical approximation becomes, in certain cases, exact. [Witten '88, Pestun '07,...]

Localization and supergravity

- Conserved currents of the SQFT that one would like to probe couple to gauge fields which must sit in supergravity multiplets.
- Therefore to identify localizable backgrounds of SQFT one couples supersymmetric matter field theories to classical supergravity: setting the supersymmetry variations of the fermionic supergravity fields — both gravitinos and gauginos — to zero, one obtains equations for the local supersymmetry spinorial parameters.

Generalized Killing Spinor equations

- These differential equations, that are often named generalized Killing spinor (GKS) equations, admit non-trivial solutions only for special configurations of the bosonic fields of the supergravity multiplet.
- The relevant supergravity and the particular GSK equations depend on the global symmetries of the specific SQFT one is interested in.

GKS equations for $N = 2$ $d = 2$

The $N = (2,2)$ $d = 2$ GKS equations write

$$S\,\psi_\mu = (\partial_\mu + \frac{i}{2}\omega_\mu - i\mathcal{A}_\mu)\,\zeta - \frac{i}{2}N_1\,\Gamma_\mu\zeta - \frac{i}{2}N_2\,\Gamma_\mu\Gamma_3\zeta = 0$$

where

- $\mathcal{A}_\mu$ is the $U(1)_R$ gauge field
- N_1 and N_2 are scalars duals of the graviphoton backgrounds.

GKS equations for $N = 4$ $d = 2$

The $N = (4, 4)$ $d = 2$ GKS equations write

$$
\begin{aligned}
S\,\psi_\mu = \partial_\mu\,\zeta^i + i\,\mathcal{A}^I_\mu\,(\tau^I)^i{}_j\,\zeta^j + \frac{1}{2}\,i\,\omega_\mu\,\Gamma_3\,\zeta^i + \\
+2\,i\,\big[N_1\,\Gamma_\mu\,\zeta^i + N_0\,\Gamma_3\,\Gamma_\mu\,\zeta^i + \\
-(N_2 + i\,N_3)\,\Gamma_\mu\,\Gamma_3\,\epsilon^{ij}\,\zeta^c_j\big] = 0
\end{aligned}
$$

where:

- $\mathcal{A}^I_\mu$, $I = 1, 2, 3$ are the $SU(2)_R$ gauge fields;
- N_a, $a = 0, 1, 2, 3$ are scalars backgrounds.

The old topologically twisted solution

- It has been known for a long time that the $N=(2,2)$ $d=2$ GKS equations admit, for generic space-time topologies, the topologically twisted solution

$$\mathcal{A}_\mu = \frac{1}{2}\omega_\mu \qquad N_1 = N_2 = 0$$

Modern genus zero solutions

- More solutions of the $d = 2$ GKS eqs were found for spheric world-sheet topology [Benini& Cremonesi: '12, Doroud et al.: '12, Closset&Cremonesi: '14, Closset, Cremonesi, Park: '15].
- GKS equations have also been studied in higher dimensions and a host of new solutions have been found as well [Hama et al. '12, Klare et al. '12...]

GKS equations and topological structures

- There is no general strategy to construct solutions of GKS equations.
- One application of the topological structures of supergravity is to provide a systematic way to find and classify solutions of GKS equations.
- I am going to sketch how to obtain the general solutions $d = 2$ $N = (4,4)$ GKS equations (which include as a specific case the $d = 2$ $N = (2,2)$ GKS equations).

Supersymmetric supergravity backgrounds

- Supersymmetric bosonic backgrounds are obtained by setting to zero the supergravity BRST variations of all the fermionic supergravity fields. We will refer to the set of such backgrounds as the localization locus.

The topological localization equations

- On the localization locus also the BRST variation of the composite topological fermions must vanish as well

$$S\,\psi_{\mu\nu} = 0 \Leftrightarrow \text{The topological gravitino eq.}$$
$$S\,\lambda = 0 \Leftrightarrow \text{The topological gaugino eq.}$$
$$S\,\chi_a = 0 \Leftrightarrow \text{The topological scalar eq.}$$

The topological gravitino equations

- The first equation

$$S\,\psi_{\mu\nu} = D_\mu\,\gamma_\nu + D_\nu\,\gamma_\mu = 0$$

  states that the vector bilinear γ^μ is an isometry of the space-time metric $g_{\mu\nu}$
- This is a well-known result which was obtained quite early in the GKS literature.

The topological gaugino equations

- The topological gaugino equation $S\lambda = 0$

$$D\phi - i_\gamma(F) = 0$$

appears to be a novel equation which has not been yet explored in either supergravity or topological field theory literature.

The topological gaugino equations

- In the context of supergravity, the topological gaugino equation splits into equations valued in the Lorentz local algebra and in the R-symmetry YM symmetry algebra.

$$D\,\phi_{Lorentz} - i_\gamma(R^{(2)}) = 0$$
$$D\,\phi_{gauge} - i_\gamma(\mathcal{F}^{(2)}_{gauge}) = 0$$

- When either one of these algebras is non-abelian, these equations are non-linear.

The equivariant Chern classes

- To extract the gauge invariant content of the topological gaugino equation, define the generalized field strength

$$\mathbb{F} = F^{(2)} + \gamma\,\phi$$

which is an equivariant form with respect to the $U(1)$ isometry, with γ being the associated commuting parameter.

- The generalized Chern classes

$$\mathrm{Tr}\,\mathbb{F}^n = \mathrm{Tr}(F+\phi)^n = \mathrm{Tr}\,F^n + n\,\gamma\,\mathrm{Tr}\,F^{n-1}\,\phi + \cdots + \gamma^n\,\mathrm{Tr}\,\phi^n$$

are gauge invariant $U(1)$-equivariant forms.

De Rham γ-equivariant cohomology

- The topological gaugino equation implies that the generalized Chern classes $\mathrm{Tr}\,\mathbb{F}^n$ are equivariant extensions of the ordinary Chern classes:

$$(d - i_\gamma)\,\mathrm{Tr}\,\mathbb{F}^n = 0$$

- The differential

$$\mathcal{D}_\gamma \equiv d - i_\gamma \qquad \mathcal{D}_\gamma^2 = -\mathcal{L}_\gamma$$

is the coboundary operator defining the de Rham cohomology on space-time forms, equivariant with respect to the γ^μ isometry.

Integral invariants of the localization locus

- The ordinary Chern classes are integer-valued.
- In the examples we computed so far also their $U(1)$-equivariant extensions $\mathrm{Tr}\,\mathbb{F}^n$ are also integer.
- $U(1)$-equivariant Chern classes provide therefore a topological classification of the space of supersymmetric backgrounds of supergravity.

- It should be stressed that the topological gravitino and gaugino equations do not, in general, completely characterize the localization locus.
- Additional, independent, equations are obtained by setting to zero the variations of other, independent, gauge invariant composite fermions.
- When the ghost bilinears F_a exists one gets precisely these extra topological equations.

The scalar topological equations

- For each F_a, the $S\,\chi_a^{(1)} = 0$ equation leads to the topological scalar equation

$$d\,F_a - i_\gamma(N_a^{(2)}) = 0$$

- We see that $\mathbb{H}_a \equiv \gamma\,F_a + N_a^{(2)}$ are also a equivariant closed form, with respect to the γ-isometry.

Cohomological invariance

- Given a solution $\mathbb{H}_a$ of the scalar topological equations, cohomologically equivalent solutions are associated to every $\mathcal{L}_\gamma$ invariant 1-form $\omega_a^{(1)}$

$$
\begin{aligned}
&\mathbb{H}'_a = \mathbb{H}_a + \mathcal{D}_\gamma\, \omega_a^{(1)} \Leftrightarrow \\
\Leftrightarrow \qquad & F'_a = F_a + i_\gamma(\omega_a^{(1)}) \quad N_a^{(2)\,\prime} = N_a^{(2)} + d\,\omega_a^{(1)} \\
& \mathcal{L}_\gamma\, \omega_a^{(1)} = 0
\end{aligned}
$$

- Since the $\mathcal{D}_\gamma$ cohomological symmetry is inherited by the original supergravity BRST symmetry, it is natural to conjecture that localizing backgrounds corresponding to cohomologically equivalent solutions give rise to the same partition function.

The equivariant Fierz identity

- The ghost bilinears are not independent, but satisfy the Fierz identity

$$\eta^{ab} F_a F_b = \gamma^2$$

- This extends to an identity of equivariantly closed forms

$$\eta^{ab} \mathbb{H}_a \mathbb{H}_b = \gamma^2 + \star\phi_{Lorentz} \qquad \phi_{Lorentz} = \sqrt{g}\, \epsilon_{\mu\nu} D^{\mu} \gamma^{\nu}$$

- This together with other equivariant Fierz identities allow to express the equivariant curvatures forms $\mathbb{R}$ and $\mathbb{F}^I$ in terms of the scalar $\mathbb{H}_a$ ones.

The dual of γ-equivariant polyforms

- To obtain these relations one introduces a derivation L which maps equivariantly closed forms to equivariantly closed forms:

$$L(\mathbb{H}_a) \equiv \gamma\, N_a + \star\Delta_\gamma\, F_a$$

where

$$\Delta_\gamma\, F_a \equiv d^\dagger \frac{1}{\gamma^2} \star d\, F_a$$

$$\mathcal{D}_\gamma L(\mathbb{H}_a) = 0$$

Curvature backgrounds in terms of the scalar closed polyforms

- $N = 2 \; d = 2$

$$\mathbb{R} = \eta^{ab}\, \mathbb{H}_a\, L(\mathbb{H}_b)$$
$$\mathbb{F} = \epsilon^{ab}\, \mathbb{H}_a\, L\,(\mathbb{H}_b)$$

- $N = 4 \; d = 2$

$$\mathbb{R} = \eta^{ab}\, \mathbb{H}_a\, L(\mathbb{H}_b)$$
$$\mathbb{D}^{ab} = \epsilon^{abcd}\, \mathbb{H}_c\, L\,(\mathbb{H}_d)$$
$$\mathrm{Tr}\, \mathbb{F}^2 = \mathbb{D}^{ab}\, \mathbb{D}_{ab}$$

The integrability of GKS equations

- These relations between the equivariant curvatureforms and the $\mathbb{H}_a$ are the topological counterpart of the integrability equations of the GKS equations.
- These relations are manifestly invariant under the global duality symmetry which acts on the $\mathbb{H}_a$ in the vector representation. Thus this duality group acts on the space of the supersymmetric backgrounds.

A host of new localizing backgrounds

- In $N = 4$ $d = 2$ one obtains a huge amount of new localizable backgrounds, which include all the previously known $N = 2$ $d = 2$ backgrounds and many both with more and with less supersymmetry.
- It would be interesting to compute matter partition function as functions of the data $\{g_{\mu\nu}, \gamma^{\mu}, F_a\}$ which determine the supersymmetric backgrounds, to verify, among other things, its cohomological properties.

Conclusions

- Supergravity contains an emergent universal topological subsector, described by topological gravity coupled to topological YM.
- Certain “twistable” extended supergravities contain a second topological structure which consists of scalar topological multiplets $\mathbb{H}_a = F_a + \chi_a^{(1)} + N_a^{(2)}$ coupled to topological gravity.
- The two structures are related, “on shell”, by certain topological “integrability” conditions that we worked out explicitly in $d = 2$ and $N = 2$ and $N = 4$

Open problems and outlook

- Explore the new host of localizing backgrounds of $N = (4, 4)$ in $d = 2$ that we found.
- Find the relation between the two structures for $N = 2$ $d = 4$ supergravity. This might lead to the solution of the long standing problem of the classification of localizing backgrounds for this theory.
- Find the fate of the topological emergent structures of supergravity at quantum level.

An effective topological sigma model on the space of supersymmetric vacua?

- We have seen that the classical supersymmetric vacua of $d=2$ $N=(4,4)$ can be parametrized by "on shell" topological multiplets $\mathbb{H}_a$ and topological gravity backgrounds $\{g_{\mu\nu},\gamma^\mu\}$, with

$$\eta^{ab}\,\mathbb{H}_a\,\mathbb{H}_b=\gamma^2+\star\phi_{Lorentz} \qquad \mathcal{D}_\gamma\mathbb{H}_a=0$$

- Could one construct a non-linear topological sigma models with coordinates $\mathbb{H}_a$ coupled to topological gravity which describes, in an effective way, the quantum fluctuations of supergravity?