SPECTRAL GEOMETRY OF SUPERSYMMETRIC GAUGE THEORIES

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Fundamental interactions and Geometry

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ABSTRACT: MAKE USE OF GEOMETRIC ENGINEERING OF GAUGE THEORIES IN SUPERSTRING/M-THEORY TO GET NEW INSIGHTS IN THEIR NON-PERTURBATIVE FORMULATION. WE'LL FOCUS ON SEIBERG-WITTEN THEORY IN D=4 AND D=5.

Geometry and Physics

spectral geometry of integrable systems

Non perturbative definition of QFTs

Geometric engineering of QFTs in String theory

QFT $\longleftrightarrow$ String

Perturbative expansions in QFTs are asympthotic series

zero radius of convergence

(DYSON '52)

$$\int D\Phi\, e^{-S_0 - g S_I} = \int D\Phi\, e^{-S_0} \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} (S_I)^n =$$

$$= \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} \int D\Phi\, e^{-S_0} (S_I)^n$$

WRONG

perturbation theory works because the first few terms are representative up to small corrections

LOOK FOR CASES WHEN ONE CAN EXACTLY COMPUTE THE PATH INTEGRAL TO LEARN NON-PERTURBATIVE PHYSICS.

$\longrightarrow$ SUSY QFTs

$\longrightarrow$ 2D QFTs

$\longrightarrow$ TQFTs

Are connected by a rich web of

DUALITIES

Supersymmetry is a beautiful organizing principle for quantum theories

→ postpones to a broken IR phase the difficult

→ allows exact evaluation of its vacua structure

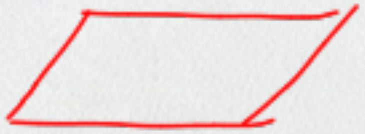
Generally intricate non-linear problems in QFT reveal to be integrable at SUSY points in the space of couplings (to all orders)

Geometry is behind exact integrability in the form of spectral data

Riemann surface + auxiliary gauge theory

LAX CONNECTION/HITCHIN SYSTEM

Geometric engineering of QFT $\longleftrightarrow$ string



QFT

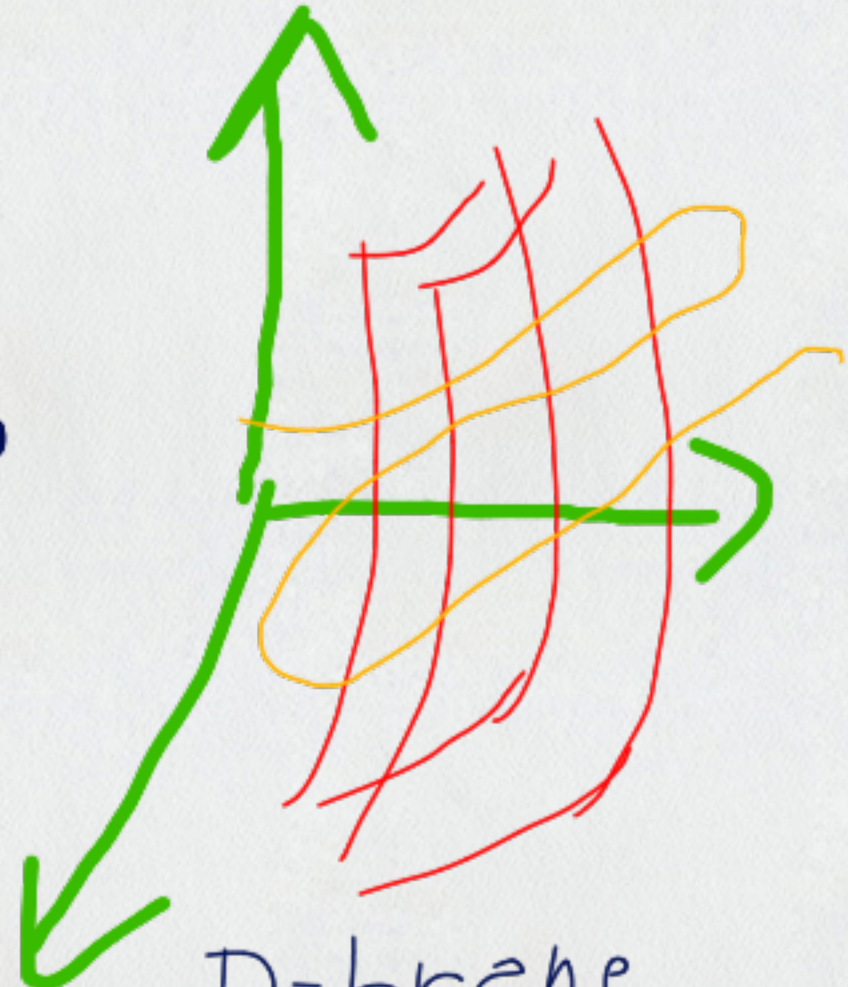
space time manifold

$\times$

spectral curve

$\Big\}$ QFT Moduli

$\longleftrightarrow$



D-brane system

EXTRA DIMENSIONS ARE NEEDED TO ENCODE MODULI SPACE GEOMETRY

What do you gain from Geom. Eng. ?

-> Manifest duality invariance

-> Control on strong coupling phases

-> Synthetic language for phase transitions

-> New ways to compute QFT effects

-> Extension of the space of QFTs to non-Lagrangean ones

EXAMPLE : SEIBERG - WITTEN THEORY

Adjoint QCD with 2 extra massless scalars and constrained couplings (8 super ch.)

in susy path integral a huge boson/fermion canceletion reduces the ∞-dim integrel to finitely meny veriebles

$\longrightarrow$ EQUIVARIANT LOCALIZATION

this is a refined version of well known

NO RENORMALIZATION THEOREMS

Equivariant Localization

Many new results

quantitative and qualitative

new observables new dualities

New connections Mathematics/Pysics

$$\int D\phi\, e^{-S(\phi)}\, \mathcal{O}(\phi) = \sum_{\substack{P \\ \text{BPS}}} [1\text{-loop}]_P\, \mathcal{O}(\phi_P)$$

EXPLICIT EXPRESSION IN SPECIAL FUNCTIONS

(W-W Revival ?)

- WITTEN INDEX $\mathrm{Tr}(-1)^F e^{-\beta H} = Z\,(T^D)$
  - → orderparameter x susy breaking
  - → computes $(\#b - \#F)_{vac.}$
- Supersymmetric Quantum mechanics
- $\mathcal{N}=2$ $D=4$ gauge theories (SW)
  - → solution in the Ω-background via explicit instanton counting

$$Z_{Nek} = e^{-\frac{1}{\epsilon_1 \epsilon_2}(\mathcal{F}_{SW} + \mathcal{O}(\epsilon))}$$ [NEKRASOV]

  - → Correspondence with CFT_2 [AGT]
  - → Non-perturbative definition via Fredholm determinant (lift to 5D)

    Quantum mirror curve [GHM]

- Cascade of results in $D=2,3$ Eg: $D=2$ MIRROR SY
  $D=3$ QUIVER/CS

- $Z_{gauge}(X)$ where $X = S^D, T^D, AdS$, toric $\uparrow \mathbb{CP}^N$
  - -> exact dependence on fugacities
  - -> new scale parameters / gravitational bkgr
  - -> new bounds on RG flows (a-, c-fun)
  - -> new exact tests/results in AdS/CFT

RG-Dynamics

- > SDuality (strong-weak; e.m. duality) match of partition functions and observables ('tHooft/wilson loops)
- > Strongly coupled IR fixed pts of RGE IR dualities (Seiberg, Mirror, ...) exact matching of spectra in the dual phases

NEW CONNECTIONS

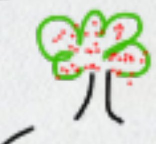
AGT $\quad \mathcal{Z}^{\mathcal{N}=2}_{\text{quiver}(\Sigma)}(X_4) = \mathcal{Z}^{D=2}_{T(X)}(\Sigma_2)$

$\Sigma_{0,4} \to N_f = 2N_c$
$T(S^4) = \text{Toda}$ CFT W_{N_c}

3D/3D Corr. $\quad \mathcal{Z}^{\mathcal{N}=1}_{\text{quiver}}(S^3) = \mathcal{Z}_{CS}(M_3)$

Application to knot th.

PROOF OF MIRROR SYMM $\quad \mathcal{Z}^{(2,2)}_{GL\sigma M} = \mathcal{Z}^{LG}_{\text{mirror}}$

EXACT PROBE OF C.Y. AT  LEVEL IN STRING

Bethe/gauge $\quad \mathcal{Z}^{\mathcal{N}=2}_{\text{gauge}}(\mathbb{R}^2_\Omega) = \tau$ int. syst

Sources $\longleftrightarrow$ Whithem times

ALL THIS NATURALLY ENCODED IN

geometric engineering

gauge $\longleftrightarrow$ string $\Big\|$ HARD TO SEE OTHERWISE

Let's focus on S.W. theory $SU(2) = G$

$D=4$ $\quad A_\mu + \phi^{adj}_{cplx} + \lambda^{adj}_2$ [8 supercharges]

$V(\phi) = |[\phi, \bar{\phi}]|^2 \longrightarrow$ IR Coulomb branch

$SU(2) \longrightarrow U(1)$

Moduli space $\quad u = \mathrm{tr}\,\phi^2$ order parameter

$[u \propto a^2]$

CLASSICALLY $\quad a = \oint x \frac{dw}{w} \quad ; \quad w = x^2 - u$

QUANTISTIC

$$a = \oint_\alpha x \frac{dw}{w}$$

$$\frac{\partial \mathcal{F}}{\partial a} = \oint_\beta x \frac{dw}{w}$$

$$\boxed{w + \frac{\Lambda^4}{w} = x^2 - u}$$

$\mathcal{F}$: prepotential

SEIBERG-WITTEN CURVE

$$\mathcal{L}_{\text{eff}}^{U(1)} = \operatorname{Im}\left[\int d^4\theta \, \frac{\partial \mathcal{F}}{\partial z}(A)\,\bar{A} + \tfrac{1}{2}\int d^2\theta\, W \frac{\partial^2 \mathcal{F}}{\partial^2 z} W\right]$$

kahler pot. $\qquad \tau = \mathcal{F}''$ gauge funct.

$$\mathcal{F} = \frac{1}{4} z^2 \ln\left(\frac{z^2}{\Lambda^2}\right) + \sum_{k=0}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{z}\right)^{4k}$$

1-loop $\qquad$ instantons

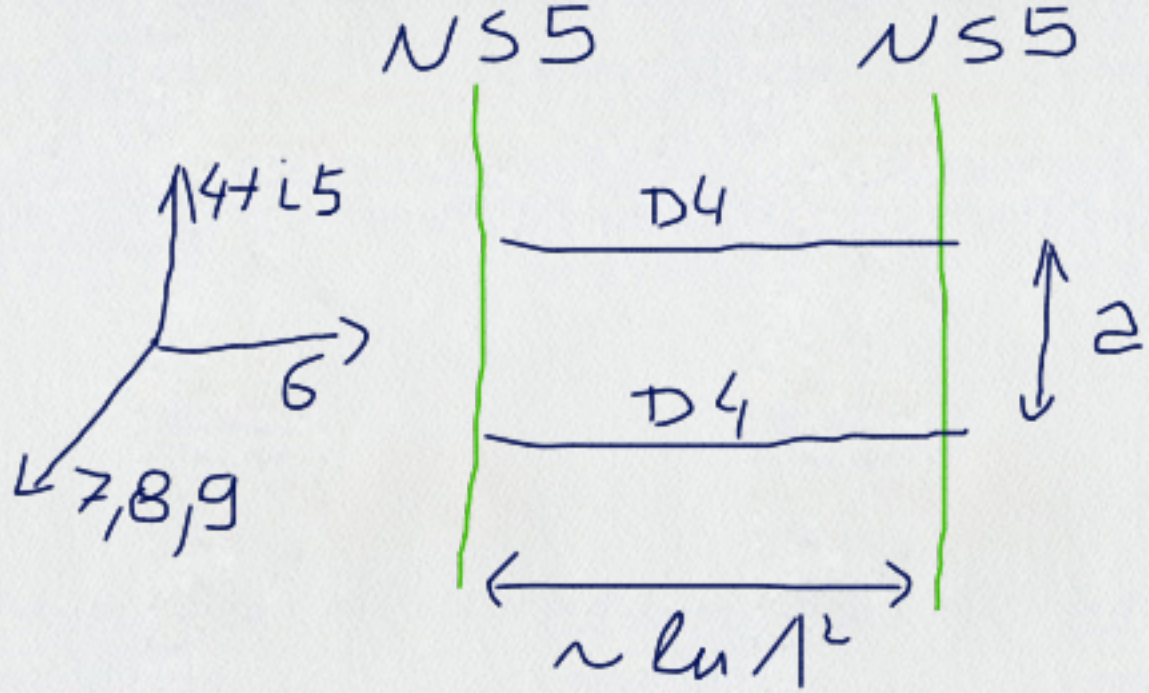
SW solution $\Rightarrow$ τ = period of Σ

$\Rightarrow$ e.m. duality = mapping class group of Σ_{SW}

SU(2) gauge th solved in IR by complexified grav. pendulum

Brane construction

$\otimes = 0,1,2,3$

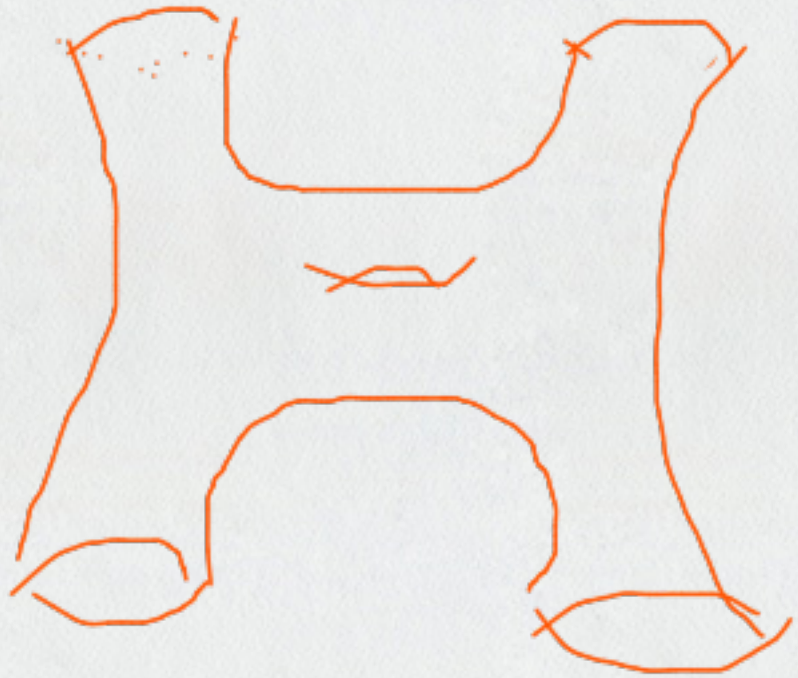


LEET of stretched open strings are vector bosons

M-theory lift

$\underbrace{6 \times \bigcirc\ S^1_M}_{\downarrow \atop w \in \mathbb{C}^*}$

IR - a single M5 brane



Materialize Σ_{SW} by M-theory geometry $\Big] \; w - (x^2 - u) + \dfrac{\Lambda^4}{w} = 0$

The SW solution can be computed by explicit evaluation of the susy path-integral $\equiv$ crucially one has to consider the gauge theory in a gravitational back-ground to tame the integral over ADHM Moduli space (instantons)

Ω-background $\Rightarrow$ attractive potential (cplex rotations $\varepsilon_1, \varepsilon_2$) × instantons -> origin $\mathbb{C}^2$

$$Z_{NEK}(\mathbb{C}^2_{\varepsilon_1,\varepsilon_2}) = \sum_m q^m \int_{\mathcal{M}_m} [d\mu]_{eq}$$

$$= e^{-\frac{1}{\varepsilon_1\varepsilon_2}\left(\mathcal{F}_{SW} + \mathcal{O}(\varepsilon\text{'s})\right)}$$

[Nekrasov]

One can add BPS observables in the path-integral

Nekrasov's result is obtained by EXACT
semi-classical expansion of the path-integral

$\to F^{+}=0 \Rightarrow$ ADHM moduli space $\qquad [B_1, B_2] + IJ = 0 \,/\, GL(k,\mathbb{C})$

$B_i \in Mat(k\times k, \mathbb{C})$; $I, J^{+} \in Mat(k\times 2, \mathbb{C})$

$\to$ Equivariant configurations $\to$ instantons packed at origin (f.pt $\mathbb{C}^2_{\varepsilon_1,\varepsilon_2}$)

$$Z_{Full} = Z_{cl}\, Z_{1\text{-loop}}\, Z_{inst}$$

$$Z_{cl} = e^{-a^2/\varepsilon_1\varepsilon_2} \;;\; Z_{1\text{-loop}} = \Gamma_2^{\prime}(a; \varepsilon_1, \varepsilon_2)$$

$\hookrightarrow \Gamma_2^{-1} \sim \prod (a + m\varepsilon_1 + m\varepsilon)$

$$Z_{inst} = \sum_{Y_1, Y_2} q^{|Y_1|+|Y_2|} \prod_{i,j=1}^{2} \prod_{s \in Y_i} \left[E(s)\left(E(s)-\varepsilon_+\right)\right]^{-1}$$

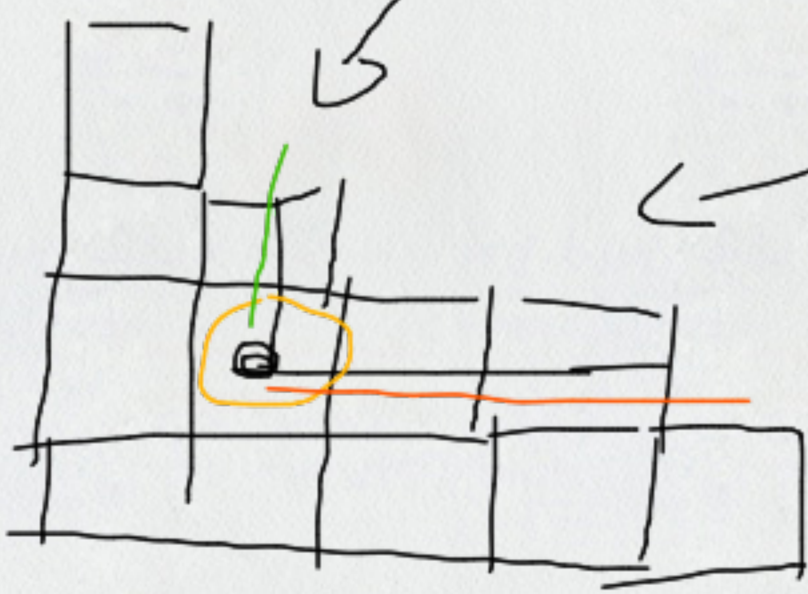
Young diagrams

$$E(s) = a - a_{Y_i}(s)\,\varepsilon_1 - \ell_{Y_j}(s)\,\varepsilon_2$$

arm

leg

$|Y| = \#$boxes



s-box

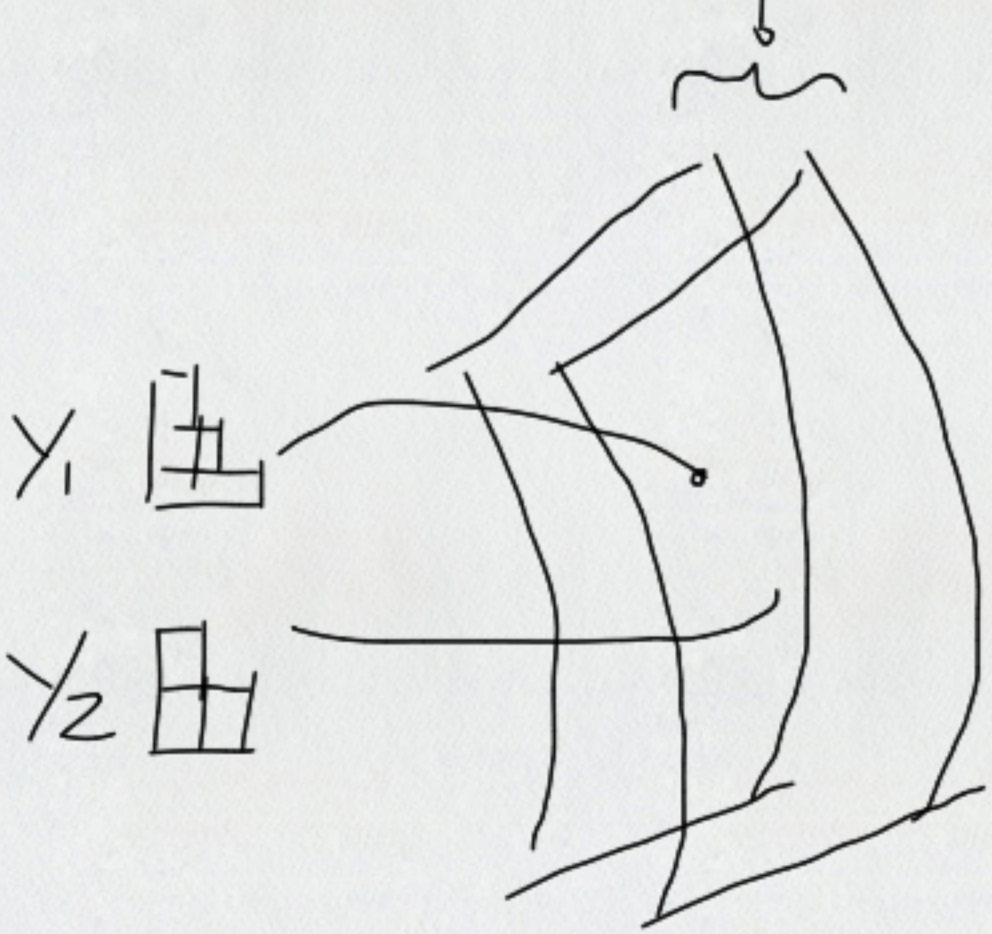
$$q = \Lambda^4$$

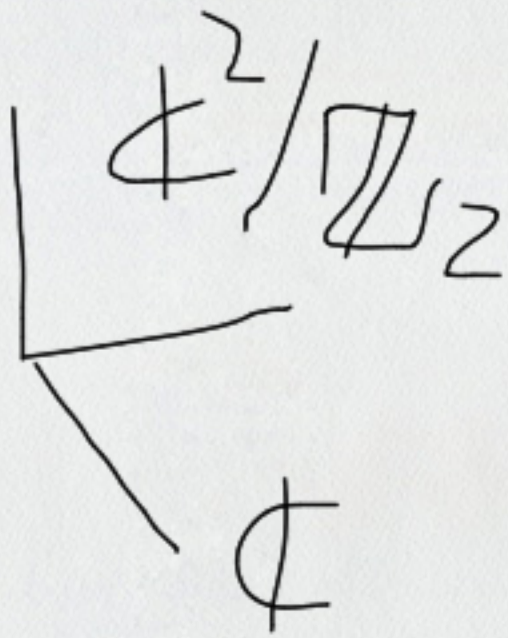
D-brane realization

D(-1) / D3 System on the tip of $\mathbb{C}^2/\mathbb{Z}_2$

k = inst #

$N_c = 2$

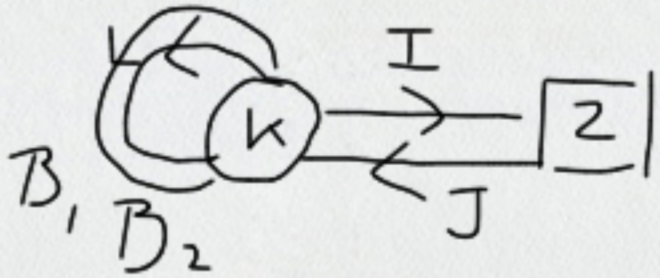




open strings

-> B_i D(-1) D(-1)

-> I, J D(-1) D3



$0 = [B_1, B_2] + IJ^t$

related to open strings potential

Y_i -> representation of S_{k_i} [k_i : D(-1)] there has to be much more behind this formula...

$\to$ $\mathcal{M}_{ADHM} \sim \bigcup_k \mathcal{M}_k$ clustering

$\to$ topological charge is additive $\mathcal{M}_k \times \mathcal{M}_{k'} \hookrightarrow \mathcal{M}_{k+k'}$

$\to$ $\mathcal{M}_{ADHM}$ is an Hopf-space

$\to$ $\exists$ graded coproduct in $H_T^\bullet(\mathcal{M}_{ADHM})$ $\Leftarrow$ (BPS observables)

$\to$ Indeed there's an infinite dimensional Lie algebra action behind

$\mathcal{U}(Vir)$ $\leftarrow$ Vertex Algebra

$\hookrightarrow$ Virasoro Algebra $\times$ $SU(2)$

[Nakajima]

QUANTUM INTEGRABLE SYSTEM @ WORK

? suspect relation with CFT_2 ?

AGT correspondence

Indeed $\mathcal{N}=2$ $D=4$ gauge theories are linked to CFT_2

Let's keep the pure SYM $SU(2)$

$$\overbrace{Z^{SYM}_{\mathcal{N}=2}(S^4) = \langle \phi(\infty) | \phi(0) \rangle_{\text{Liouville}}}$$

irregular states created by scaled OPE of primaries

$$\int Z^{SYM}(\mathbb{R}^4_+)\, Z^{SYM}(\mathbb{R}^4_-)\, da$$

$$Z_c^+ Z_1^+ Z_{inst}^+ \qquad Z_c^- Z_1^- Z_{inst}^-$$

$$\int da\, C^{(3pt)} B^{CB}_{vir}$$

$$[\text{inst}] = B^{CB}_{vir}$$

$$[1\text{loop}] = C^{(3pt)}$$

Extends to higher rank
any matter repr
quiver gauge theories
method to generate non-Lagrangean theories

AGT Correspondence

→ Map 1-1 $[\text{BPS Operators}]_{\text{gauge th}} \longleftrightarrow [\text{Obs.}]_{\text{CFT}}$

→ Extends to other M_4, CFT (M_4)

→ Maps EM-Duality $\longleftrightarrow$ modular invariance

HARD PART $\quad Z_{inst} \longleftrightarrow$ Vir conformal blocks

HARD PART

$\approx\rangle$ Geometric description in M-theory

M-theory 5branes naturally engineer $\mathcal{N}=4, D=4$ SYM

$$N_c \text{ M5 s on } T^2 \times \mathbb{R}^{1,3} \hookrightarrow T^2 \times \mathbb{R}^{1,3} \times \mathbb{R}^5$$

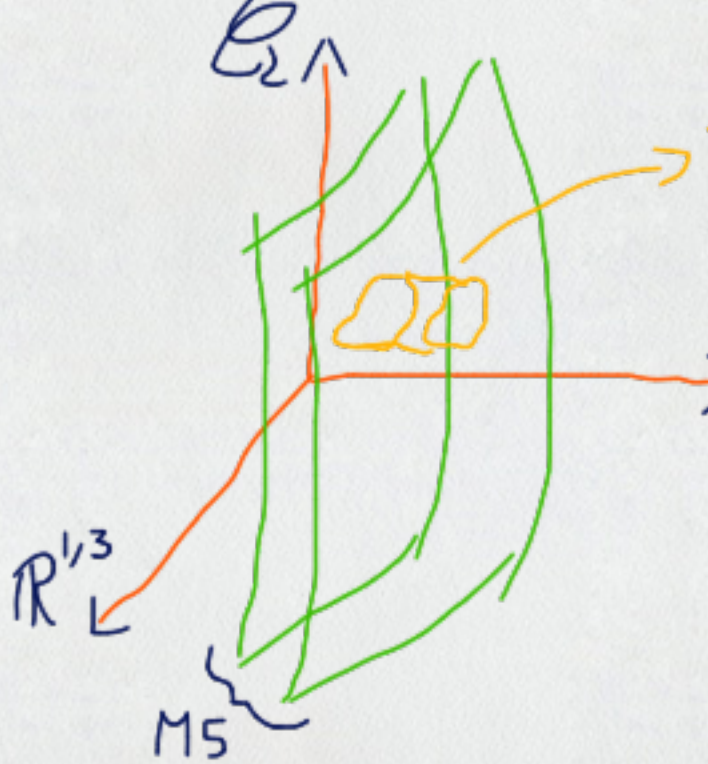
$\hookrightarrow \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2}$

To break further susy $\mathcal{N}=4 \to \mathcal{N}=2$, replace $T^2 \longrightarrow \mathcal{C}_2$

cotangent bundle (mom comp. k3)

$$N_c \text{ M5 on } \mathcal{C}_2 \times \mathbb{R}^{1,3} \hookrightarrow T^*\mathcal{C}_2 \times \mathbb{R}^{1,3} \times \mathbb{R}^3$$

$\otimes = \mathbb{R}^3$



$\to$ M2 branes stretched between M5-branes

$\hookrightarrow$ M-Theory version of OPEN strings stretched between D-branes

$\partial(M2) = $ strings $\in$ M5 w.v.

$\hookrightarrow$ particle w.l. in $\mathcal{C}_2$
particle w.l. in $\mathbb{R}^{1,3}$

TWO VIEWPOINTS OF THE SAME THEORY !

COMPARE THE TWO VIEWPOINTS:

→ $\mathbb{R}^{1,3}$: $SU(N_c)$ SYM $\mathcal{N}=2$ BPS particles

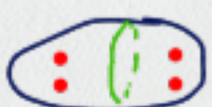
→ $\mathcal{C}_2$: BPS config described by the dimensional reduction of $F^+=0$ (ASD condition) on $T^*\mathcal{C}_2$

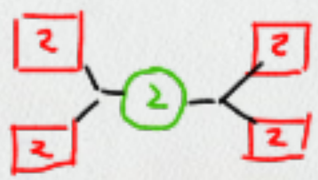
$\Rightarrow \left(F_{\mathcal{C}} + [\phi, \bar{\phi}] = 0, \ \bar{\partial}_A \phi = 0\right)$: $N_c = 2 \approx$ Liouville

n.b. the picture with S^4 is given by M-theory on $T^*\mathcal{C}_2 \times ASD(S^4)$

$\Rightarrow$ $\mathcal{C}_2$ defines a quiver gauge theory by pants decomp'n

→ each complex modulus corresponds to a gauge cplg

e.g. 
Liouville

= 

$N_F = 4$ $SU(2)$

primary conf. weights ↔ mass of matter mult.s

pure $SU(2)$: collision v.s $m_f \to \infty$

- Topological strings dual of S.W. theories in 5D
- M-theory geometric engineering via M2-branes
- Consider M-theory on $CY_6 \times S^1 \times \mathbb{C}^2$

N.B. Decouple the gravitational sector via LARGE VOLUME limit of CY_6 $\Rightarrow$ Non compact

Count BPS M2 on $\Sigma_2 \times$ w.l.

Topological Strings on local surface $\longleftrightarrow$ BPS sector of gauge theory w/8 sch. on $S^1 \times \mathbb{C}^2$

$\Rightarrow$ The topological string dual is intrinsecally

NON-PERTURBATIVE

because instanton particles arise as a particuler S^1-winding sector of the 5D theory

$\Rightarrow$ topological string is the theory which counts holomorphic maps $\Sigma_2 \hookrightarrow CY_6$ (A-model)

$$F_{[\Sigma]} = \sum_{w} e^{-w \cdot T} n_{[\Sigma]}$$

w ← hol. maps

$[\Sigma]$: homotopy type is the # (handles) ← genus

T : set of volume parameters of non contractible 2-cycles in CY_6

$$n_g = \left[\frac{\text{multiplicities}}{\#(\text{discrete AUT})}\right]_g \in \mathbb{Q}$$

called Gromov-Witten invariants

the total free energy of the topological string, summed over the genus is an asymptotic series

[SHANKER]

$$F_{A\,model} = \sum_{g \geq 0} g_s^{2g-2} F_g \quad \text{diverges because} \quad F_g \sim (g-1)!$$

Therfore, if we want to use the TS duality to give a non-pertubative definition of the D=5 gauge theory, better to solve the problem for the TS itself.

N.B. the inverse problem was used by Gopakumar and Vafa who proposed a definition of the TS partition fcn as a supersymmmetric index counting BPS particles in 5D

$$F_{GV} = \sum_{\substack{g \geq 0 \\ w, \beta}} \underbrace{N_{g,\beta}}_{\text{multiplicity of BPS multiplets = G.V. inv}} \frac{1}{w} \underbrace{\left(2\sin\left[\frac{w g_s}{2}\right]\right)^{2g-2}}_{\text{Determinant of particle multiplets on the M-theory circle}} e^{-w T_\beta}$$

] ← type of BPS particles are 2-cycles in CY_6

w: winding

β: type of BPS Part.

For small g_s
$F_{GV} \sim F_{A\,model}$

[N.B. $g_s \sim \hbar$] ??

THERE EXISTS A CONJECTURAL NON-PERTURBATIVE DEFINITION OF TOPOLOGICAL STRINGS WHICH APPLIES IN SOME CASES

IF THE CY_6 IS A TORIC LOCAL SURFACE

$CY_6 = S \to K_S$

S is toric if acted on by $\mathbb{C}^2_*$

examples: $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$

THEN ONE CAN ASSOCIATE A MIRROR $\widetilde{CY}_6$ geometry $wz = \mathcal{O}(p,q) + K$

where $\mathcal{O}(p,q) = \sum_{(\nu,\mu) \in \text{polygon}(S)} e^{-p\nu - q\mu} \cdot k_{\nu,\mu}$

↳ moduli of S are hidden here!

the curve $\mathcal{O}(p,q) + K = 0$ is called mirror curve

THE MIRROR CURVE IS THE 5D VERSION OF THE SW CURVE OF THE DUAL GAUGE THEORY IN D=5

(LATER AN EXAMPLE)

The non perturbative definition of the TS partition function is given in terms of the QUANTUM MIRROR CURVE

$\hat{\mathcal{O}}(\hat{p},\hat{q}) \leftarrow$ q.m. operator $[\hat{q},\hat{p}] = i\hbar$

[Grassi, Hatsuda, Mariño]

CHALLENGE: PROOF !!

[GHM] Conjecture: $\Xi^{TS} = \det\left(1 + \kappa\, \hat{\mathcal{O}}^{-1}\right)$

simplified to one Kahler modulus $[\kappa]$!

the operator $1+\kappa\,\hat{\mathcal{O}}^{-1}$ is a Fredholm (indeed trace class) $\Rightarrow$ the det is an entire function in κ

Grand Potential

$$\Xi^{TS} = e^{J_X}\,\Theta_X \qquad J_X = F_{GV} + \text{(non pert)}$$

gen. Θ-fnct $\quad [\kappa = e^{\mu}]$

$$\Theta_X = \sum_{n\in\mathbb{Z}} e^{J_X(\mu+2\pi i n) - J(\mu)}$$

or, inverting,

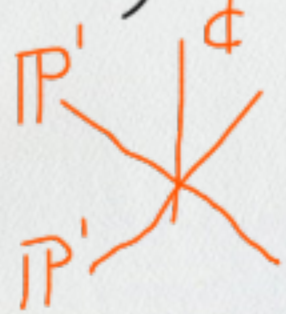
$$J_X + \ln\Theta_X = \log \Xi^{TS} = \sum_{n>0} \frac{(-\kappa)^n}{n} Z_n\,, \quad Z_n = \mathrm{tr}\,\mathcal{O}^{-n}$$

spectral traces

IF SO, THEN THE SPECTRAL DETERMINANT HAS TO BE RELATED TO THE NEKRASOV 5D PARTITION FUNCTION OF THE DUAL GAUGE THEORY.

EXAMPLE:

TOP. STRING REALIZATION OF S.W. theory (5D version)



The Amodel dual is the local $\mathbb{P}^1 \times \mathbb{P}^1$

the mirror curve is $e^p + e^{-p} + m\, e^q + e^{-q} + \kappa = 0$

N.B. for small p, $z \sim e^q$

$$p^2 - u = z + \frac{\Lambda^4}{z}$$ [up to normaliz.]

↳ 4D SW curve pure $SU(2)$

IT CAN BE EXACTLY QUANTIZED => TESTABLE CONJECTURE

For $\mathbb{P}^1 \times \mathbb{P}^1$

$$\hat{O} = e^{\hat{p}} + e^{-\hat{p}} + m e^{\hat{q}} + e^{-\hat{q}}$$

Its inverse is an integral trace class operator

$$\hat{O}^{-1}(x,y) = \frac{e^{-V(x)-V(y)}}{ch\left(\frac{x-y}{2}\right)}$$

For a given $V(x)$ [complicated $\ln \Phi_b$ + q dilog

The spectral determinant

$$\det(1 + \kappa\, \hat{O}^{-1})$$

can be exactly computed

It results in a grand canonical partition function

for a matrix model $Z_N = \int d^N x \prod_{i<j} tg(x_i - x_j) \cdot e^{-\sum_i V(x_i)}$

Cauchy det ↗

1^{st} check : In the 4D limit $V(x) \sim ch(x)$

$\Rightarrow O(2)$ matrix model

[Zamolodchikov] $\Xi_{O(2)} = \tau_{P\mathrm{III}_3}$ Painlevé / m.m.

[BGT] $$\sum_{n \in \mathbb{Z}} Z_{NEK}(a + n\hbar) = \tau_{P\mathrm{III}_3}$$ Painlevé / gauge

N.B. $P\mathrm{III}_3$ = radial shG in the gauge coupling

SAME INITIAL CONDITIONS

$\Rightarrow$ Can be extended to $SU(N_c)$ SW theory [BGT]

WHAT ARE PAINLEVÈ EQTNs DOING HERE ??

Painlevé/gauge correspondence

[BLMST]

$$\begin{array}{ccccccccc} N_F=4 & \longrightarrow & N_F=3 & \longrightarrow & N_F=2 & \longrightarrow & N_F=1 & \longrightarrow & N_F=0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & AD_3 & & AD_2 & & AD_3 & & \end{array}$$

$$\begin{array}{ccccccccc} P_{VI} & \longrightarrow & P_V & \longrightarrow & P_{III_1} & \longrightarrow & P_{III_2} & \longrightarrow & P_{III_3} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & P_{IV} & & P_{II} & & P_I & & \end{array}$$

$$\sum_{n\in\mathbb{Z}} Z_{NEK}^{(a+n\hbar)} = \tau_{P_T}$$

$\llcorner$ 4D SU(2) gauge th. T

$\searrow$ Corresponding P. equation

there is a geometric theory beyond Painlevé equations

Remind: multi M5 engineering on $T^*\mathcal{C}_2$ associates BPS configs to the solution of

INTEGRABLE

Hitchin System $\longrightarrow$ $\bar{\partial}_A \Phi = 0$ on $\mathcal{C}_2$

A — connection encoding gauge theory data [gauge gr. + couplings, matter + masses]

A — Painlevé equations arise as isomonodromic deformation eqs of the Hitchin connection A

EX:

Pure $SU(2)$

$\mathcal{C}_2 = \mathbb{P}^1$ 2 irregular singularities in A

$\Rightarrow P_{III_3}$

Check 2) 5D pure SU(2)

Consider again the spectral determinant for the local $\mathbb{P}^1 \times \mathbb{P}^1$

$$\Xi = \det\left(1 - \kappa\, \hat{\rho}^{-1}\right)$$

it can be shown to

→ satisfies a difference quadratic eq.

qP_{III_3}

$$\Xi \cdot \Xi = 0$$

descrete version of P_{III_3}

↓

same b.c

$$\rightarrow \Xi = \sum_{m \in \mathbb{Z}} Z^{5D}_{Nek}(\alpha + m\xi)$$

Geometry behind qP_{III_3} [Bershtein et al]

-> Put the gauge theory on $T^*\mathbb{P}^1$ spacetime

-> its path integral factorizes in two factors (N and S poles)

-> specify the flux of the gauge field on $\mathbb{P}^1$

=> super selection rule by fermionic zero modes

-> insert a surface operator $\mathcal{S}$ VIOLATING the SS-rule

-> $0 = \langle \mathcal{S} \rangle = Z^{5D}_{North} \cdot Z^{5D}_{South}$

NB: the sum over fluxes makes $Z^{5D}_{N/S} = \sum_n Z^{5D}_{Nek}(a + n\hbar)$

this is the qP_{III_3} eq.

It is called blow up eq

$\mathbb{C}^2/\mathbb{Z}_2 \to T^*\mathbb{P}^1$

Check 3 (in progress [BdMT])

Back to D=4

→ $\mathcal{N}=2^*$ theory SU(2) SYM coupled to an adj chiral

with mass m $\begin{cases} \to 0 \Rightarrow \mathcal{N}=4 \text{ SYM} \\ \to \infty \Rightarrow \mathcal{N}=2 \text{ pure} \end{cases}$

→ SW curve $p^2 + m^2 \wp(z) + u = 0$ Calogero Moser

↳ there is a torus around ⇐ Dbrane realisation

→ Relevant Painlevé ⇒ P_{VI} at special values $\overbrace{e_2}$

→ The τ-function solving the isomonodroic equations on $T^2\setminus\{pt\}$

can be obtained by a CFT_2 on T^2 1-pt function

$$\tau_{VI} = \langle P \rangle_{T^2} = \sum_n C_n B_{VIR}(m, a+n\hbar) = \sum_n Z^{\mathcal{N}=2^*}_{gauge}(m, a+n\hbar)$$

$\hbar = \epsilon_1 = -\epsilon_2$

still missing the full TS dual

GENERAL PICTURE

X = local toric surface $tot(K_S)$

$$\boxed{TS_X}$$

Sakai's class'n of P_{eq} $\rightarrow$ $\boxed{P_X}$

geometric engineering $\rightarrow$ $\boxed{\text{gauge dual th}}$

quantum mirror curve $\rightarrow$ $\boxed{\hat{\mathcal{O}}_X}$

$$\tau_{P_X} = Z_{NO}^{gauge} = \det(1 - \varkappa\, \hat{\mathcal{O}}_X^{-1})$$

To be explored with care!

Conclusions

→ There are QFTs whose partition function/particular corr. fn can be defined non perturbatively as certain solutions of differential/difference equations

→ these are susy QFTs in $D=4,5$

→ their embeddings in string theory describe the above equations in a geometrical set-up which naturally explains the appearing of integrable systems in the game

→ different embeddings (dual to each others) describe different aspects of the QFT and different phases

Open Questions

- → Explore more theories in full detail
- → Role of integrable systems in QFTs
- → Lower Susy or no Susy : D=3 QFTs

THANK YOU