Vacuum Fluctuations and Traversable Wormholes

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Introduction

- A wormhole can be represented by two asymptotically flat regions joined by a bridge
- One very simple and at the same time fundamental example of wormhole is represented by the Schwarzschild solution of the Einstein's field equations.
- One of the prerogatives of a wormhole is its ability to connect two distant points in space-time. In this amazing perspective, it is immediate to recognize the possibility of traveling crossing wormholes as a short-cut in space and time.
- A Schwarzschild wormhole does not possess this property.

Traversable wormholes

The traversable wormhole metric

M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395 (1988).

$$ds^2 = -\exp\left(-2\phi(r)\right)dt^2 + \frac{dr^2}{1-\frac{b(r)}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

Condition

b(r) is the shape function

f(r) is the redshift function

$$r \in \left[r_0, +\infty\right)$$

$$b_\pm(r_0) = r_0$$
$$b_\pm(r) < r$$

Proper radial distance

$$l(r) = \pm\int_{r_0}^{r} \frac{dr'}{\sqrt{1-b_\pm(r')/r'}}$$

$$\lim_{r\to\infty} b_\pm(r) = b_\pm \quad \text{Appropriate asymptotic}$$
$$\lim_{r\to\infty} \phi_\pm(r) = \phi_\pm \quad \text{limits}$$

Einstein Field Equations

Orthonormal frame

$$b'(r) = 8\pi G\rho c^2 r^2$$

$$\phi'(r) = \frac{b + 8\pi G p_r r^3}{2r^2\left(1 - b(r)/r\right)} \qquad \tau(r) = -p_r$$

$$p'_r(r) = \frac{2}{r}\left(p_t(r) - p_r(r)\right) - \left(\rho(r) + p_r(r)\right)\phi'(r)$$

Exotic Energy

$\rho(r) + p_r(r) < 0 \quad r \in [r_0, r_0 + \varepsilon]$ $\Longleftrightarrow$ $b'(r) < b(r)/r \quad r \in [r_0, r_0 + \varepsilon]$

Flare-Out Condition

Candidate $\Longrightarrow$ **Casimir Energy**

Traversable Wormholes
Example

$$ds^2 = -dt^2 + dl^2 + (r_0^2 + l^2)(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

M. S. Morris and K. S. Thorne, Am. J. Phys. 56, 395 (1988).
H. G. Ellis, J. Math. Phys. 14, 104 (1973). K.A. Bronnikov, Acta Phys. Pol. B 4, 251 (1973).

The new coordinate *l* covers the range $-\infty < l < +\infty$. The constant time hypersurface Σ is an Einstein-Rosen bridge with wormhole topology $S^2 \times R^1$. The Einstein-Rosen bridge defines a bifurcation surface dividing Σ in two parts denoted by Σ_+ and Σ_-.

In Schwarzschild coordinates becomes

$$ds^2 = -dt^2 + \frac{dr^2}{1 - r_0^2/r^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$$b(r_0) = r_0$$ Throat Condition

$$\text{Minimum at the throat} \Rightarrow \frac{d^2 r}{dl^2} > 0 \Leftrightarrow \frac{b(r)}{r} > b'(r)$$

Features of the EB Wormhole

$M(r)=\frac{r_0^2}{2Gr}\rightarrow 0 \quad when\ r\rightarrow\infty$ Zero Mass Wormhole

But $M_P(r)=\pm\int_{r_0}^{r}\frac{4\pi\rho(r')r'^2}{\sqrt{1-b(r')/r'}}dr'$
$\rightarrow\mp\frac{\pi r_0}{4G} \quad when\ r\rightarrow\infty$

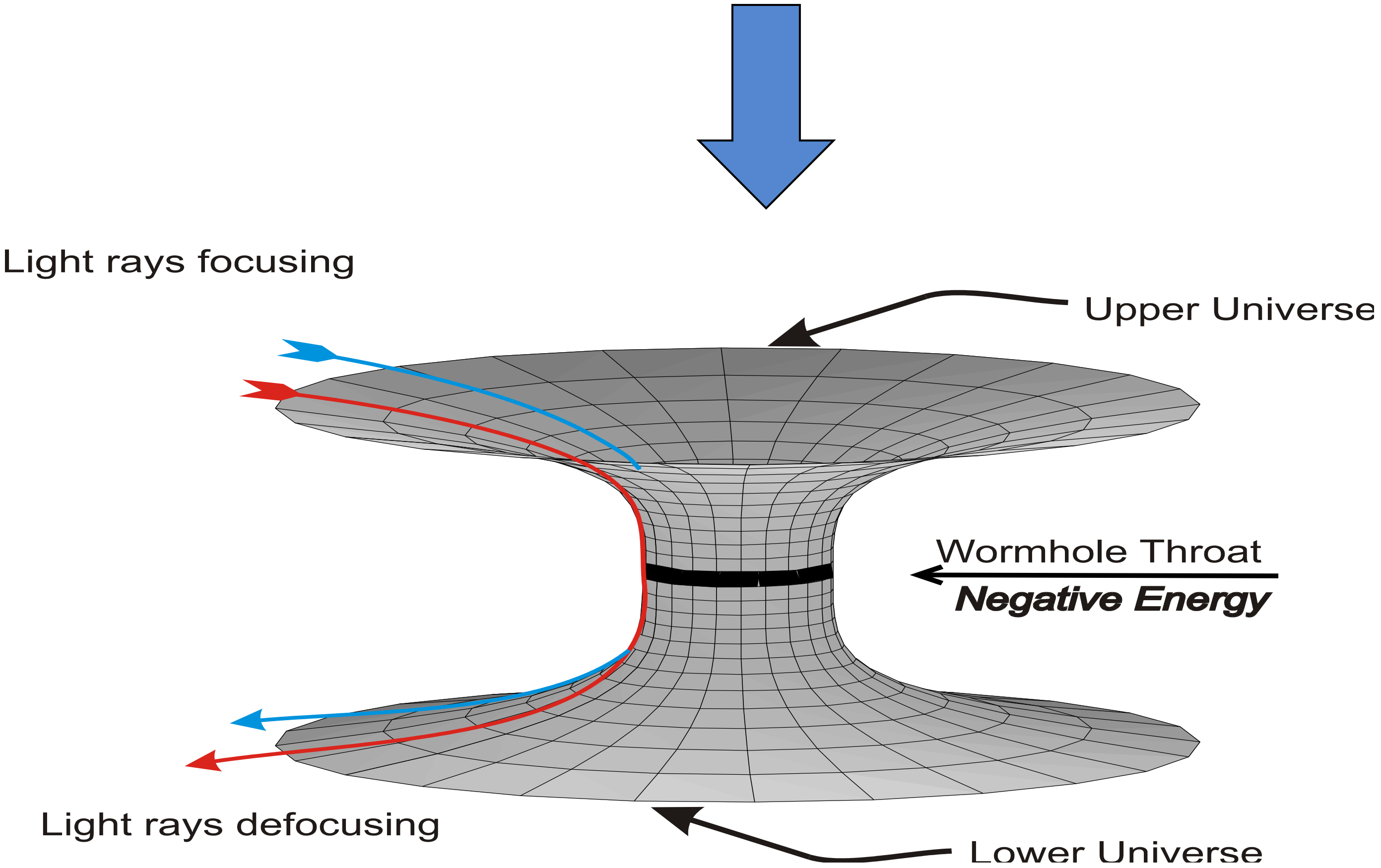
$E_G(r)=M(r)-M_P(r)=\pm\frac{\pi r_0}{4G}$ Total Energy

$l(r)=\pm\sqrt{r^2-r_0^2}$ Proper length

$p_r(r)=-\frac{b(r)}{8\pi Gr^3}=-\frac{r_0^2}{8\pi Gr^4}\Longrightarrow-\frac{1}{8\pi Gr_0^2}$ On the throat

$p_t(r)=-\frac{b'(r)r-b(r)}{16\pi Gr^3}=\frac{r_0^2}{8\pi Gr^4}\Longrightarrow\frac{1}{8\pi Gr_0^2}$ On the thro

Traversable wormholes



Casimir Effect

Hendrik Casimir 1909-2000

H.B.G.Casimir and D.Polder,
Phys. Rev., 73, 360, 1948

(ZPE) responsible for the Casimir effect. This was predicted by Casimir [1] and confirmed experimentally in the Philips laboratories†. This is induced when the presence of electrical conductors distorts the zero-point energy of the quantum electrodynamics vacuum. Two parallel conducting surfaces, in a vacuum environment, attract one another by a very weak force that varies inversely as the fourth power of the distance between them. This kind of energy is a purely quantum effect; no real particles are involved, only virtual ones. The difference between the stress–energy computed in the presence and in the absence of the plates with the same boundary conditions gives

$$\Delta\langle T^{\mu\nu}\rangle = \langle T^{\mu\nu}\rangle^{p}_{\rm vac} - \langle T^{\mu\nu}\rangle_{\rm vac} = \frac{\pi^2}{720a^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}. \tag{1}$$

It is evident that separately, each contribution coming from the summation over all possible resonance frequencies of the cavities is divergent and devoid of physical meaning but the *difference* between them in the two situations (with and without the plates) is well defined. Note that the energy density

$$\rho = E/V = \Delta\langle T^{00}\rangle = -\frac{\pi^2}{720a^4} \tag{2}$$

Only wavelength less than d

Any wavelength is possible

Take seriously the Result

$$G_{\mu\nu}=\frac{8\pi G}{c^4}T_{\mu\nu} \quad \Longrightarrow \quad G_{\mu\nu}=\frac{8\pi G}{c^4}\langle T_{\mu\nu}\rangle^{Ren}$$

See also

M.S. Morris, K.S. Thorne, U. Yurtsever (Caltech). 1988. 4 pp.
Published in Phys.Rev.Lett. 61 (1988) 1446-1449

AND

M. Visser, Lorentzian Wormholes: From Einstein to Hawking
(American Institute of Physics, New York), 1995.

Take seriously the Result

$$\rho(a)=-\frac{\hbar c\pi^2}{720a^4} \qquad p_r(a)=-\frac{3\hbar c\pi^2}{720a^4} \qquad p_t(a)=\frac{\hbar c\pi^2}{720a^4}$$

The Casimir tensor is traceless and divergenceless

$$b'(r)=\frac{8\pi G}{c^4}\rho(a)r^2$$

$$b(r)-b(r_0)=\frac{8\pi G}{c^4}\int_{r_0}^{r}\left(-\frac{c\hbar\pi^2}{720a^4}\right)r'^2\,dr'$$

$$b(r)=r_0-\frac{\pi^3 G\hbar}{270c^3a^4}\left(r^3-r_0^3\right)$$

This is not a TW because there is no A. Flatness. It is Asymptotically de Sitter

It can be transformed into a TW with the junction condition method matching the solution with the Schwarzschild metric at some point r=c

Take seriously the Result

$Zero\ Tidal\ Forces\ \phi(r)=0$ $\qquad a \longrightarrow r$

$$\rho(r)=-\frac{\hbar c\pi^2}{720 r^4} \qquad p_r(r)=-\frac{3\hbar c\pi^2}{720 r^4} \qquad p_t(r)=\frac{\hbar c\pi^2}{720 r^4}$$

If we impose the EoS $\qquad p_r(r)=\omega\rho(r) \qquad \omega=3$

$$\Rightarrow ds^2=-dt^2+\frac{dr^2}{1-\left(\frac{r_0}{r}\right)^{4/3}}+r^2 d\Omega^2$$

No Connection with the original Casimir Stress-Energy tensor

$$T_{\mu\nu}=\frac{1}{\kappa r^2}\left(\frac{r_0}{r}\right)^{\frac{\omega+1}{\omega}}\left(-\frac{1}{\omega},-1,\frac{\omega+1}{\omega},\frac{\omega+1}{\omega},\right)$$

Take seriously the Result

$Strategy \Longrightarrow Impose\ an\ EoS$

$$b'(r)=\frac{8\pi G}{c^4}\rho(r)r^2 \qquad \rho(r)=-\frac{c\hbar\pi^2}{720r^4}$$

Energy Density

$$b(r)=b(r_0)+\frac{8\pi G}{c^4}\int_{r_0}^{r}\left(-\frac{c\hbar\pi^2}{720r'^4}\right)r'^2\,dr'=r_0-\frac{\pi^3 G\hbar}{90c^3}\left(\frac{1}{r_0}-\frac{1}{r}\right)$$

$$r_1^2=\frac{\pi^3}{90}\frac{G\hbar}{c^3}=\frac{\pi^3}{90}l_p^2$$

$$b(r)=r_0-r_1^2\left(\frac{1}{r_0}-\frac{1}{r}\right)$$

$$p_r(r)=\omega\rho(r) \qquad EoS$$

With the second EFE, we can compute $\phi(r)$

Depending on the value of ω we can have BH, TW or a singularity

Take seriously the Result

When $\omega = r_0^2 / r_1^2$ $\Longrightarrow$ Traversable Wormhole

$$\phi(r)=\frac{1}{2}(\omega-1)\ln\left(\frac{r(\omega+1)}{\omega r+r_0}\right) \xrightarrow{\omega=3} \phi(r)=\ln\left(\frac{4r}{3r+r_0}\right)$$

Planckian

$$\Longrightarrow \quad b(r)=\left(1-\frac{1}{\omega}\right)r_0+\frac{r_0^2}{\omega r} \xrightarrow{\omega=3} b(r)=\frac{2}{3}r_0+\frac{r_0^2}{3r}$$

$$SET \quad T_{\mu\nu}=\left(\frac{r_0^2}{3\kappa r^4}\right)\left[diag(-1,-3,1,1)+\left(\frac{6r}{3r+r_0}\right)diag(0,0,1,1)\right]$$

Take seriously the Result

When $\omega = \frac{r_0^2}{r_1^2} \Longrightarrow$ Traversable Wormhole

$$\phi(r) = \frac{1}{2}(\omega - 1)\ln\left(\frac{r(\omega+1)}{\omega r + r_0}\right) \xrightarrow{\omega=1} \phi(r) = 0$$

Sub-Planckian

$$\Longrightarrow b(r) = \left(1 - \frac{1}{\omega}\right) r_0 + \frac{r_0^2}{\omega r} \xrightarrow{\omega=1} b(r) = \frac{r_0^2}{r}$$

EB Wormhole

$$SET \quad T_{\mu\nu} = \left(\frac{r_0^2}{\kappa r^4}\right)\left[diag(-1,-3,1,1) + 2\, diag(0,1,0,0)\right] = \left(\frac{r_0^2}{\kappa r^4}\right)\left[diag(-1,-1,1,1)\right]$$

Take seriously the Result

When $\omega = r_0^2 / r_1^2$ $\Rightarrow$ Traversable Wormhole

$\phi(r)=\frac{1}{2}(\omega-1)\ln\left(\frac{r(\omega+1)}{\omega r+r_0}\right) \xrightarrow{\omega=1} \phi(r)=0$ Sub-Planckian

$\Rightarrow$ $b(r)=\left(1-\frac{1}{\omega}\right)r_0 + \frac{r_0^2}{\omega r} \xrightarrow{\omega=1} b(r)=\frac{r_0^2}{r}$ EB Wormhole

SET $T_{\mu\nu}=\frac{r_0^2}{2\kappa r^4}\left[diag(-1,-3,1,1)+diag(-1,1,1,1)\right]=\frac{r_0^2}{\kappa r^4}\left[diag(-1,-1,1,1)\right]$

Canonical Decomposition

QWEC Equation of State

P. Martin-Moruno and M. Visser, JHEP 1309 (2013) 050; arXiv:1306.2076 [gr-qc].
M. Bouhmadi-Lopez, F. S. N. Lobo and P. Martin-Moruno, JCAP 1411 (2014) 007 [arXiv:1407.7758 [gr-qc]]

$p_r(r)+\rho(r)=-f(r)$ $\qquad$ $f(r)$ *is an energy density*

$$\frac{b'(r)}{r}+\left[2\left(1-\frac{b(r)}{r}\right)\phi'(r)-\frac{b(r)}{r^2}\right]=-(8\pi G)rf(r)$$

Introduce $u(r)=1-\frac{b(r)}{r}\Rightarrow b(r)=r\left[1-(8\pi G)\exp(2\phi(r))\int_{r_0}^{r}\exp(-2\phi(r'))f(r')r'^2\,dr'\right]$

Example: *Assume the Casimir Profile* $f(r)=-\frac{4c\hbar\pi^2}{720r^4}$

Impose the ZTF

$$\Rightarrow b(r)=r\left[1+\frac{G\pi^3}{45}\left(\frac{1}{r^2}-\frac{r^2}{r_0^2}\right)\right]$$

When $r\to\infty$ $\quad b(r)\simeq r\left[1-\frac{G\pi^3}{45r_0^2}\right]$

Global monopole $\Rightarrow$ *Excess of the Solid Angle* $\qquad$ *Rescale* $r^2=45r^2/G\pi^3$

Asymptotic limit $\Rightarrow ds^2=-dt^2+dr^2+\frac{G\pi^3}{45r_0^2}r^2\,d\Omega^2$

QWEC Equation of State

TW returns if $\frac{G\pi^3}{45 r_0^2} = 1 \Rightarrow ds^2 = -dt^2 + \frac{dr^2}{\left[1 - \frac{r_0^2}{r^2}\right]} + r^2 d\Omega^2$

Abandon the ZTF $\quad \phi(r) = \ln\left(\frac{4r}{3r + r_0}\right)$

$$b(r) = r\left(1 - \frac{\hbar G \pi^3}{30 r_0^2 c^3}\right) + \frac{\hbar G \pi^3}{45 r_0 c^3} + \frac{\hbar G \pi^3}{90 r c^3}$$

When $r \to \infty \quad b(r) \simeq r\left[1 - \frac{\hbar G \pi^3}{30 r_0^2 c^3}\right]$

Global monopole $\Rightarrow$ *Excess of the Solid Angle*

TW returns if $\frac{\hbar G \pi^3}{30 r_0^2 c^3} = 1$ $\Rightarrow ds^2 = -\left(\frac{4r}{3r + r_0}\right)^2 dt^2 + \frac{dr^2}{\left[1 - \frac{2r_0}{3r} - \frac{r_0^2}{3r^2}\right]} + r^2 d\Omega^2$

Different Point of View

$b(r)=\frac{r_0^2}{r}$ *EB Wormhole* $\Longleftrightarrow$ $b(r)=\lim_{\mu\to 0}\frac{r_0^2}{r}e^{-\mu(r-r_0)}$

Yukawa Wormhole

Motivations

- Nuclear Physics
- Yukawa constraints on the recent measurement of the Casimir force

(Bordag,Gillies and Mostepanenko Phys.Rev.D56:6-10,1997 arXiv:hep-th/9705101)

- Van der Waals forces described in a Yukawa form

(*Milonni The Quantum Vacuum: An Introduction to Quantum Electrodynamics*)

- *Black Holes in Modified Gravity (MOG)*

(*J.W. Moffat, arXiv:1412.5424 [gr-qc]*)

- *Modified Theory of Gravity*
- *Many other contexts!!!*

Different Point of View

$b(r)=\frac{r_0^2}{r}e^{-\mu(r-r_0)}$

Yukawa Wormhole

$M(r)=\frac{r_0^2 e^{-\mu(r-r_0)}}{2Gr}\rightarrow 0$ $when\ r\rightarrow\infty$

Zero Mass Wormhole

$\rho(r)=-\frac{r_0^2}{8\pi G r^4}e^{-\mu(r-r_0)}(1+\mu r)\longrightarrow-\frac{1+\mu r_0}{8\pi G r_0^2}$ $p_r(r)=-\frac{r_0^2}{8\pi G r^4}e^{-}$

On the throat

But $\mp\frac{\pi r_0}{4G}\lesseqgtr M^P(r)\lesseqgtr 0\quad when\ r\rightarrow\infty$

$\pm\frac{\pi r_0}{4G}\gtreqless E_G(r)\gtreqless 0$ Total Energy

Out of the throat $b(r)$ and $b'(r)\rightarrow 0$ for $\mu\rightarrow\infty$

Conclusions and Perspectives

- Casimir energy is the only source of exotic matter that can be generated in laboratory.
- Traversable wormholes can be sustained by Casimir Energy.
- The Wormhole is traversable in principle but not in practice.
- The QWEC condition supports the Casimir wormhole.
- For appropriate choices of the parameters we have global monopoles carried by TW. For other choices of the same parameters we describe black holes, traversable wormholes or singularities.
- Yukawa Wormholes generalize the Casimir wormhole.
- At this stage the TW is completely useless, we need an amplification mechanism.

Thank You for Your Attention