A numerical solution method for noise calculations

R. Kovács, T. Fülöp, M. Szücs, P. Ván

Department of Energy Engineering, BME
Department of Theoretical Physics, Wigner RCP,
and
Montavid Thermodynamic Research Group
Budapest, Hungary

May 2, 2019

Noise

- Disturbances in the environment.
- Various sources: wind, seismic events, etc.
- Underground specificity: behavior of rocks.
- Rheology of rocks $\Rightarrow$ newtonian noise?
- Space and time evolution of these effects?

Wave propagation in solids

Elastic: Hamiltonian structure,
Hooke + small deformations + momentum balance.

$$\rho\partial_t \mathbf{v} + \nabla\cdot\boldsymbol{\sigma} = 0, \tag{1}$$

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\boldsymbol{\varepsilon}^{sph}, \tag{2}$$

$$\boldsymbol{\varepsilon} = (\nabla\mathbf{u})_{\mathrm{sym}},$$

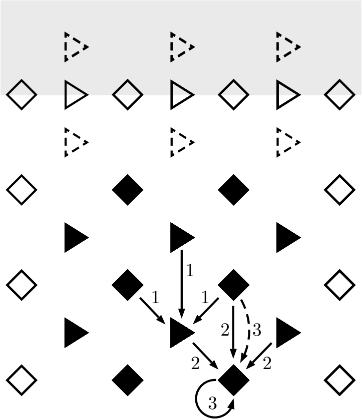
$$\partial_t\mathbf{u} = \mathbf{v}, \Rightarrow \partial_t\boldsymbol{\varepsilon} = (\nabla\mathbf{v})_{\mathrm{sym}} \tag{3}$$

Rheological: modified constitutive relation between σ and ε, e.g. Poynting-Thomson (in 1D):

$$\sigma + \tau\partial_t\sigma = E\varepsilon + E_1\partial_t\varepsilon \tag{4}$$

1D setting: elastic waves in a rod

Initial and boundary conditions? $\Rightarrow$ Staggered discretization.



$$\rho \partial_t v = -E \partial_x \varepsilon,$$
$$\partial_t \varepsilon = \partial_x v$$

$$\varepsilon_j^{n+1} = \varepsilon_j^n + \frac{\Delta t}{\Delta x} \left(v_{j+1}^n - v_j^n \right),$$
$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x} \frac{E}{\rho} \left(\varepsilon_j^{n+1} - \varepsilon_{j-1}^{n+1} \right).$$

Symplectic in time

- Staggered discretization: for spatial derivatives due to the BC's.
- What about the time integration?
  $\rightarrow$ Hamiltonian system: conservative. $\rightarrow$ **SYMPLECTIC**

Semi-implicit Euler: simplest method, first order.

$$\varepsilon_j^{n+1} = \varepsilon_j^n + \frac{\Delta t}{\Delta x}\left(v_{j+1}^n - v_j^n\right),$$
$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}\frac{E}{\rho}\left(\varepsilon_j^{n+1} - \varepsilon_{j-1}^{n+1}\right).$$

Others: Störmer-Verlet (second order), Symplectic-RK, etc.

Stability analysis: Neumann + Jury

There is an explicit part: stability is a key question.

$$\varepsilon_j^n = \xi^n e^{ikj\Delta x}, \tag{5}$$

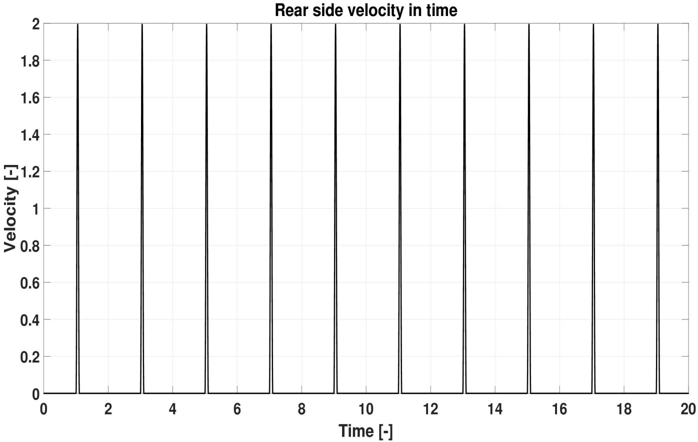
and the same for v_j^n. **Condition:** $|\xi| \leq 1$.

$$p(\xi) = \xi^2 - \xi \left(2 + \frac{\Delta t^2}{\Delta x^2}\frac{E}{\rho}(2\cos(k\Delta x) - 2)\right) + 1. \tag{6}$$

- $p(\xi = 1) \geq 0 \rightarrow p(\xi = 1) = 4\frac{\Delta t^2}{\Delta x^2}\frac{E}{\rho} > 0.$
- $p(\xi = -1) \geq 0 \rightarrow \frac{\Delta x}{\Delta t} \geq v_0,\ v_0 = \sqrt{\frac{E}{\rho}}.$
- $|a_2| \geq |a_0|$, trivially fulfilled.

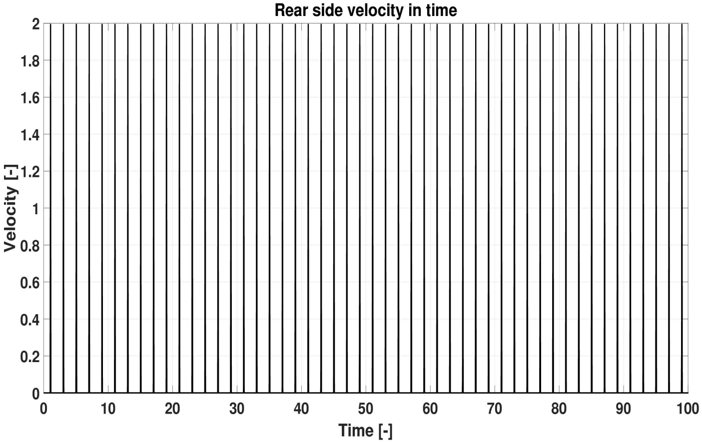
Solution I.

Preserved amplitude and width = preserved energy.



Solution II.

On a large interval...



Rheology in 1D: Kelvin body

Damped wave like relaxation.
Eq. (7) is NOT the time evolution of ε!

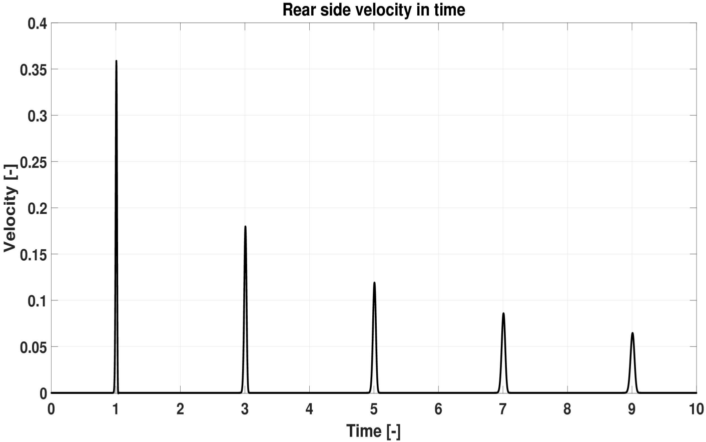
$$\begin{aligned} \rho \partial_t v &= -E \partial_x \varepsilon - E_1 \partial_{xx} v, \\ \sigma &= E\varepsilon + E_1 \partial_t \varepsilon, \\ \partial_t \varepsilon &= \partial_x v \end{aligned} \tag{7}$$

$$\varepsilon_j^{n+1} = \varepsilon_j^n + \frac{\Delta t}{\Delta x}\left(v_{j+1}^n - v_j^n\right),$$

$$v_j^{n+1} = v_j^n + \frac{\Delta t}{\Delta x}\frac{E}{\rho}\left(\varepsilon_j^{n+1} - \varepsilon_{j-1}^{n+1}\right) + \frac{\Delta t}{\Delta x^2}\frac{E_1}{\rho}\left(v_{j+1}^n - 2v_j^n + v_{j-1}^n\right)$$

$+$ Stability analysis in the same way...

Solution

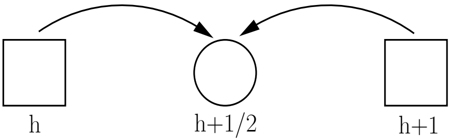

Rear side velocity in time
Velocity [-]
Time [-]
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9
10

Rheology in 1D: Poynting-Thomson model

Eq. (8) is NOT the time evolution of ε!

$$\begin{aligned} \rho\partial_t v &= -\partial_x \sigma, \\ \sigma + \tau\partial_t\sigma &= E\varepsilon + E_1\partial_t\varepsilon, \\ \partial_t\varepsilon &= \partial_x v \end{aligned} \tag{8}$$

Derivatives on a lattice: (h could be n or j)



$\Rightarrow$ σ and ε are at the wrong place to calculate $\partial_t\sigma$

$\Rightarrow$ convex combination of the terms n and $n+1$

Rheology in 1D: Poynting-Thomson model

Discrete time evolution equations:

$$
\begin{aligned}
\varepsilon_j^{n+1} &= \varepsilon_j^n + \frac{\Delta t}{\Delta x}\left(v_{j+1}^n - v_j^n\right), \\
\left(1 + (1-\alpha)\frac{\Delta t}{\tau}\right)\sigma_j^{n+1} &= \sigma_j^n\left(1 - \frac{\alpha \Delta t}{\tau}\right) + E\alpha\frac{\Delta t}{\tau}\varepsilon_j^n + \\
&\quad + E(1-\alpha)\frac{\Delta t}{\tau}\varepsilon_j^{n+1} + \frac{E_1}{\tau}\left(\varepsilon_j^{n+1} - \varepsilon_j^n\right), \\
v_j^{n+1} &= v_j^n - \frac{\Delta t}{\Delta x \rho}(\sigma_{j+1}^{n+1} - \sigma_j^{n+1}).
\end{aligned}
$$

Stability analysis

Goes a bit inconvenient for higher order systems...

$$p(\xi) = a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0$$

with

$$a_0 = \frac{\Delta t\alpha}{\tau} - 1,$$

$$a_1 = 3 + \frac{\Delta t}{\tau}(1-\alpha) + 2\big(\cos(k\Delta x) - 1\big)\left(\frac{\Delta t^2 E_1}{\Delta x^2\rho\tau} - \frac{\Delta t^3 E\alpha}{\Delta x^2\rho\tau} - \frac{2\Delta t\alpha}{\tau}\right)$$

$$a_2 = -2\big(\cos(k\Delta x) - 1\big)\left(\frac{\Delta t^2 E_1}{\Delta x^2\rho\tau} + \frac{\Delta t^3 E(1-\alpha)}{\Delta x^2\rho\tau}\right) + \frac{\Delta t}{\tau}(3\alpha - 2) - 3,$$

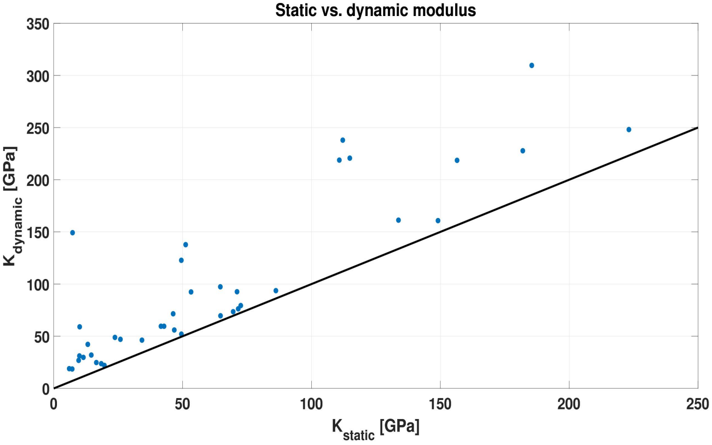
$$a_3 = 1 + \frac{\Delta t}{\tau}(1-\alpha),$$

$\alpha = 1/2$ is the best...

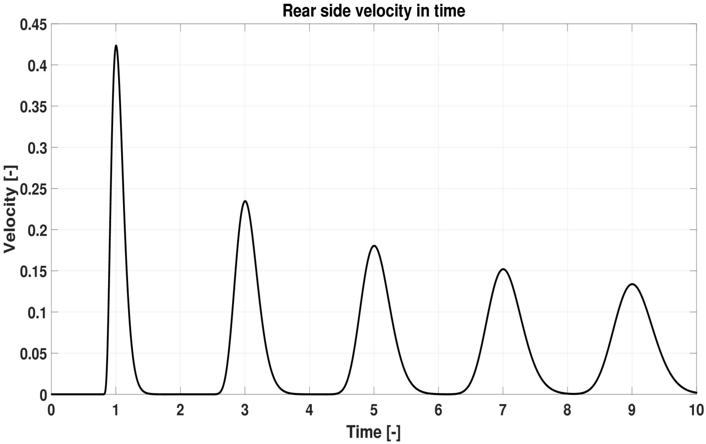
- $p(\xi = 1) = \frac{4\Delta t^3 E}{\Delta x^2 \rho \tau} > 0$, ✓
- $p(\xi = -1) < 0$, $\rightarrow \frac{E_1}{\rho \tau} < \frac{\Delta x^2}{\Delta t^2}$ when $\alpha = 1/2$.
  Presence of rheological time scale! ✓
- $|a_0| \leq |a_3| \rightarrow \alpha < 1/2 + \tau/\Delta t$. ✓
- $|b_0| > |b_2|$ where $b_0 = \begin{vmatrix} a_0 & a_3 \\ a_3 & a_0 \end{vmatrix}$ and $b_2 = \begin{vmatrix} a_0 & a_1 \\ a_3 & a_2 \end{vmatrix}$.
  $\rightarrow b_0 = \frac{\Delta t}{\tau^2}\big(\Delta t(2\alpha - 1) - 2\tau\big)$ and
  $b_2 = \frac{\Delta t}{\Delta x^2 \rho \tau^2}\left(\Delta t \Delta x^2 (2\alpha - 1)\rho - 2\Delta x^2 \rho \tau + 4\Delta t^2 (E_1 - E\tau)\right)$.
  $\rightarrow \alpha = 1/2 \rightarrow \frac{\Delta x^2}{\Delta t^2} > \frac{E_1}{\rho \tau} - \frac{E}{\rho}$ ✓
  Moreover: $E_1/E > \tau$ thermodynamic requirement!

Static vs. dynamic parameters

Depends on the material...but still remarkably differs.



Solution


Rear side velocity in time
Velocity [-]
Time [-]
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
1
2
3
4
5
6
7
8
9
10

Newtonian noise calculations in 1D

Based on the local density variations of the conducting medium.

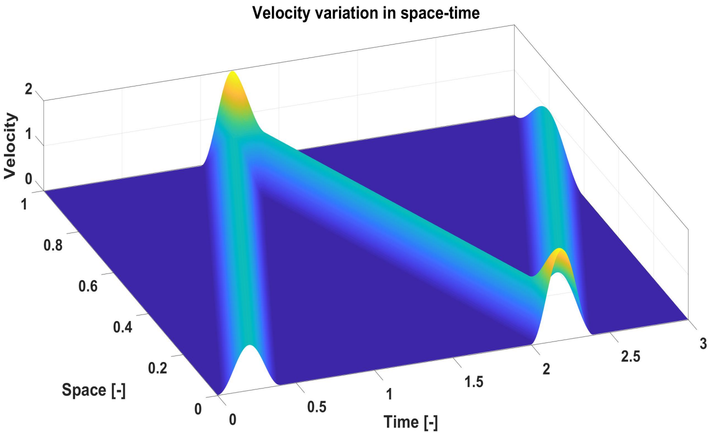
$$\begin{aligned}
\partial_t \rho &= -\rho \partial_x v = -\rho \partial_t \varepsilon, \\
\rho \partial_t v &= \partial_x P, \\
\partial_t \varepsilon &= \partial_x v, \\
\rho \partial_t \phi &= L_1(\partial_{xx}\phi - 4\pi G \rho).
\end{aligned}$$

P: total pressure $= -E\varepsilon + (\partial_x \phi)^2/(4\pi G)$.
Boundary for $\partial_x \phi$! Let it be a new variable γ and...

$$\rho \partial_t \gamma = L_1(\partial_{xx}\gamma - 4\pi G \partial_x \rho). \tag{9}$$

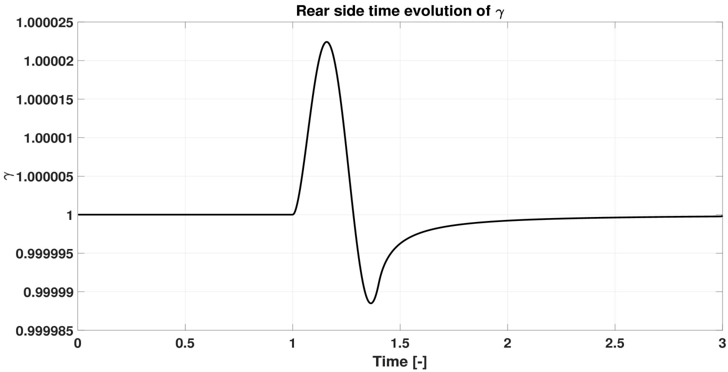
...and applying the same discretization as previously.

Solution I.



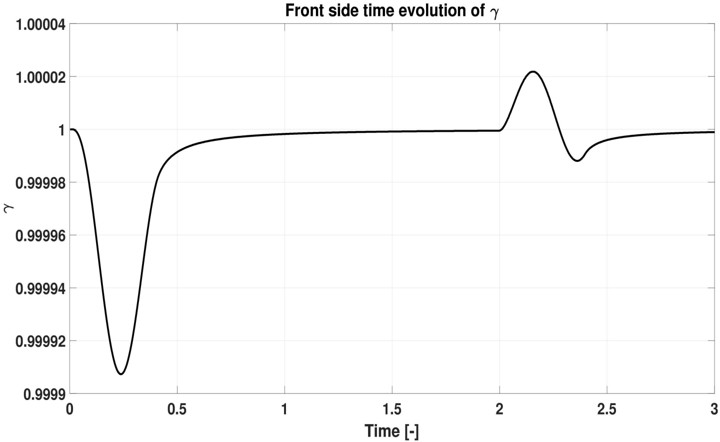
Velocity variation in space-time
Velocity
2
1
0
1
0.8
0.6
0.4
0.2
0
Space [-]
0
0.5
1
1.5
2
2.5
3
Time [-]

Solution II.



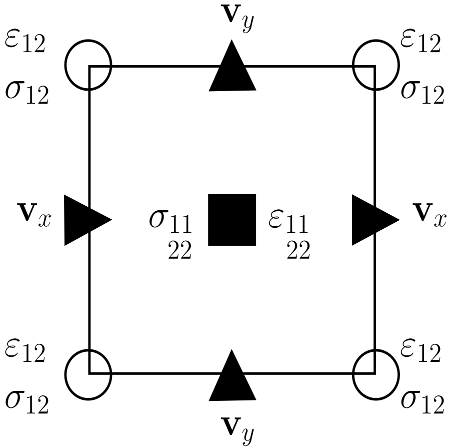
with γ being dimensionless by the factor $\tilde{\gamma} = \sqrt{2\pi G v_0^2 \rho_0} \approx 10^0$, with $v_0^2 = E/\rho_0$, $E = 10^{11}$ Pa, $\rho_0 = 8000$ kg/m^3.

Solution III.



2D simulations

Spatial discretization in 2D



$$
\begin{aligned}
\rho \partial_t \mathbf{v} + \nabla \cdot \boldsymbol{\sigma} &= 0, \\
\boldsymbol{\sigma} &= 2\mu \boldsymbol{\varepsilon} + \lambda \boldsymbol{\varepsilon}^{sph}, \\
\boldsymbol{\varepsilon} &= (\nabla \mathbf{u})_{\mathrm{sym}}, \\
\partial_t \mathbf{u} = \mathbf{v}, \Rightarrow \partial_t \boldsymbol{\varepsilon} &= (\nabla \mathbf{v})_{\mathrm{sym}}
\end{aligned}
$$

+ Symplectic in time...
+ Square "tunnel"...
See the video!

Summary

- Effective treatment of boundary conditions.
- Role of symplectic integration in time is essential.
- Further developments are on the horizion: 2D rheology, cylindrical and spherical problems, etc.
- Role of thermodynamics: structure of PDEs, stability, numerical method.

Thank you for your kind attention!