Zeta regularization in the scalar Casimir effect

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based on joint works with
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Vacuum Fluctuations at Nanoscale and Gravitation: Theory and Experiments
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Background and basic ideas

Zeta Regularization (ZR) gives meaning via analytic continuation to ill-defined or divergent expressions appearing in mathematics and physics.

- A textbook example ($\zeta =$ Riemann zeta function) :

$$\sum_{n=1}^{+\infty} n \text{ ``}=\text{''} \left[\sum_{n=1}^{+\infty} \frac{1}{n^s}\right]_{s=-1} \text{ ``}=\text{''} \Big[\zeta(s)\Big]_{s=-1} \overset{a.c.}{=} -\frac{1}{12} \,.$$

- First in number theory [Ramanujan, Hardy, Littlewood (1916)], then to study pseudo-differential operators [Minakshisundaram, Pleijel ('45), Seeley ('67)] and geometric invariants (analytical/topological indexes) [Ray, Singer ('71)] .

- Later in QFT, to define renormalized Vacuum Expectation Values (VEVs) [Dowker, Critchley ('75); Hawking ('77); Zimerman *et al.* ('77), Wald ('79)] [Albeverio, Actor, Cognola, Elizalde, Kirsten, Moretti, Vanzo, Zerbini, ... ('85 - today)] .
  - In recent works with L. Pizzocchero (2011-today), we focus on:
    i. developing a general framework for ZR in QFT, using canonical quantization and results on integral kernels of pseudo-diff. operators;
    ii. analysing some exactly solvable cases, related to the Casimir effect.

The reference framework: scalar field theory

- Consider $(d+1)$-dim. Minkowski spacetime with inertial coordinates
$$(x^\mu)_{\mu=0,1,\dots,d} = (t, \mathbf{x}) \in \mathbb{R}\times\mathbb{R}^d \qquad (c=1,\ \hbar=1)\,;$$
the metric is $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, \dots, 1)$.

- Refer to the **classical field theory** described by the action functional
$$\mathcal{S}[\phi] \,=\, -\frac{1}{2}\int_{\mathbb{R}\times\Omega} dt\, d\mathbf{x} \left(\partial^\mu\phi\,\partial_\mu\phi + V(\mathbf{x})\,\phi^2\right)$$
  - $\Omega \subset \mathbb{R}^d$ is a space domain, with boundary $\partial\Omega$;
  - $\phi = \phi(t, \mathbf{x})$ is a real scalar field on Ω, fulfilling boundary cond. on $\partial\Omega$;
  - $V = V(\mathbf{x})$ is an external static potential ($\sim$ position-dependent mass).

- The (Euler-Lagrange) field eq. and the (improved) stress-energy tensor are
$$\left(\partial_{tt} - \Delta + V\right)\phi = 0 \quad \text{plus b.c. for } \phi|_{\partial\Omega} \qquad \left(\Delta := \textstyle\sum_{i=1}^d \partial_{ii}\right),$$
$$T_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi - \tfrac{1}{2}\left(\partial^\lambda\phi\,\partial_\lambda\phi + V\phi^2\right) - \xi\left(\partial_{\mu\nu} - \eta_{\mu\nu}\,\Box\right)\phi^2 \quad (\xi\in\mathbb{R}).$$
  - Term $\propto\xi$ added by [Callan,Coleman,Jackiw ('70)] to improve quantum features of $T_{\mu\nu}$, is a remnant of coupling to gravity ($\xi = 0$ minimal; $\xi = \frac{d-1}{4d}$ conformal).

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○ Working hypothesis: b.c. on $\partial\Omega$ and V are such that

$\mathcal{A} := -\Delta + V$ on $\mathrm{Dom}\mathcal{A} \subset L^2(\Omega)$ is self-adjoint and non-negative

$\left(L^2(\Omega) = \text{single particle Hilbert space with inner prod. } \langle f|g\rangle = \int_\Omega \overline{f} g\right)$.

$\Rightarrow \exists\ (F_k)_{k\in\mathcal{K}}$ orthonormal basis of $L^2(\Omega)$ of (generalized) eigenfunctions with eigenvalues $\omega_k^2 \geq 0$, labelled by a measure space $(\mathcal{K}, dk)$:

$$\mathcal{A}\, F_k = \omega_k^2\, F_k\,, \qquad \langle F_h | F_k\rangle = \delta(h,k)$$

$$\left(\begin{array}{c}\mathcal{K} \text{ discrete set: } dk = \text{counting meas., } \delta(h,k) = \text{Kronecker delta, } \int_{\mathcal{K}} dk = \sum_{k\in\mathcal{K}} \\ \mathcal{K} \text{ continuous set: } dk = \text{Lebesgue meas., } \delta(h,k) = \text{Dirac delta}\end{array}\right).$$

• **Canonical quantization** : $(\phi, \partial_t\phi) \to (\hat{\phi}, \hat{\pi})$ self-adj. operators on the bosonic Fock space $\mathfrak{F} = \bigoplus_{n\in\mathbb{N}} L^2(\Omega)^{\otimes_s n}$, fulfilling equal-time CCR

$$[\hat{\phi}(t,\mathbf{x}), \hat{\phi}(t,\mathbf{y})] = [\hat{\pi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})] = 0\,, \quad [\hat{\phi}(t,\mathbf{x}), \hat{\pi}(t,\mathbf{y})] = i\,\delta(\mathbf{x}-\mathbf{y})\,.$$

$\Rightarrow$ Field expansion via annihilation/creation operators on $\mathfrak{F}$:

$$\hat{\phi}(t,\mathbf{x}) = \int_{\mathcal{K}} \frac{dk}{\sqrt{2\omega_k}} \left[e^{-i\omega_k t} F_k(\mathbf{x})\, \hat{a}_k + e^{i\omega_k t}\, \overline{F_k}(\mathbf{x})\, \hat{a}_k^\dagger\right];$$

$$[\hat{a}_h, \hat{a}_k] = [\hat{a}_h^\dagger, \hat{a}_k^\dagger] = 0\,, \qquad [\hat{a}_h, \hat{a}_k^\dagger] = \delta(h,k)\,;$$

$\hat{a}_k|\mathbf{v}\rangle = 0, \quad |\mathbf{v}\rangle \in \mathfrak{F}$ the vacuum state (NB: ***not*** Minkowski vacuum).

Zeta regularization and renormalization

- VEVs like $\langle \mathbf{v}|\hat{\phi}^2(t,\mathbf{x})|\mathbf{v}\rangle = \langle \mathbf{v}|\hat{\phi}(t,\mathbf{x})\hat{\phi}(t,\mathbf{y})|\mathbf{v}\rangle\big|_{\mathbf{y}=\mathbf{x}} = \frac{1}{2}\int_{\mathcal{K}} \frac{dk}{\omega_k}\, F_k(\mathbf{x})\,\overline{F_k}(\mathbf{x})$ are plagued by **UV divergences**, corresp. to $\mathbf{y}\to\mathbf{x}$ or $\omega_k\to\infty$ (*IR divergences* also appear if $\omega_k\to 0$ and *must be treated separately*).

- Define $\mathcal{A}^{-s}$ for $s\in\mathbb{C}$ via functional calc. for self-adj. oper. ($\mathcal{A}=-\Delta+V$).
  - $\mathcal{A}^{-s}$ is a pseudo-diff. oper. with distrib. integral kernel ($\mathcal{A}\,F_k=\omega_k^2 F_k$)

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \langle\delta_{\mathbf{x}}|\mathcal{A}^{-s}\delta_{\mathbf{y}}\rangle = \int_{\mathcal{K}} \frac{dk}{\omega_k^{2s}}\, F_k(\mathbf{x})\,\overline{F_k}(\mathbf{y})\;;$$

$$(\mathcal{A}^{-s}f)(\mathbf{x}) = \int_\Omega d\mathbf{y}\;\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})\,f(\mathbf{y}) \quad \text{for } f:\Omega\to\mathbb{R} \text{ regular enough}.$$

  - $\mathcal{A}^{-s}$ regularizing for $\Re s>0$ large. Especially, for $j\in\mathbb{N}$ and $\Re s>\frac{j+d}{2}$, $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})\in C^j(\Omega\times\Omega)$ and $s\mapsto\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$ is analytic for fixed $\mathbf{x},\mathbf{y}$.

- Fix $\kappa>0$ (a mass parameter) and define the smeared **ZR-field** for $u\in\mathbb{C}$

$$\hat{\phi}^u(t,\mathbf{x}) := \big((\mathcal{A}/\kappa^2)^{-u/4}\hat{\phi}\big)(t,\mathbf{x}) = \kappa^{u/2}\int_\Omega d\mathbf{y}\;\mathcal{A}^{-u/4}(\mathbf{x},\mathbf{y})\,\hat{\phi}(t,\mathbf{y})$$

$$= \kappa^{u/2}\int_{\mathcal{K}} \frac{dk}{\sqrt{2}\,\omega_k^{(1+u)/2}}\left[e^{-i\omega_k t}F_k(\mathbf{x})\,\hat{a}_k + e^{i\omega_k t}\,\overline{F_k}(\mathbf{x})\,\hat{a}_k^\dagger\right].$$

- Quantized ZR-observables are obtained from corresp. classical expressions with the replacements $\phi \to \hat{\phi}^u$ and $A \cdot B \to \hat{A} \circ \hat{B} := \frac{1}{2}\left(\hat{A}\hat{B} + \hat{B}\hat{A}\right)$.
  $\hookrightarrow$ **Casimir physics**: evaluate VEVs without normal ordering.

- A relevant observable is the (on-shell) **ZR-stress-energy tensor**

$$\hat{T}^u_{\mu\nu} := (1-2\xi)\,\partial_\mu\hat{\phi}^u \circ \partial_\nu\hat{\phi}^u + \left(\frac{1}{2} - 2\xi\right)\eta_{\mu\nu}\left(\partial^\lambda\hat{\phi}^u\partial_\lambda\hat{\phi}^u + V\,(\hat{\phi}^u)^2\right) - 2\xi\,\hat{\phi}^u \circ \partial_{\mu\nu}\hat{\phi}^u .$$

The VEVs $\langle \mathbf{v}|\,\hat{T}^u_{\mu\nu}|\mathbf{v}\rangle$ can be expressed in terms of (derivatives* of) the integral kernels $\mathcal{A}^{-\frac{u\pm 1}{2}}(\mathbf{x},\mathbf{y})$, evaluated along the diagonal $\mathbf{y} = \mathbf{x}^\star$; e.g.,

$$\langle \mathbf{v}|\hat{T}^u_{00}(\mathbf{x})|\mathbf{v}\rangle = \kappa^u \left[\left(\frac{1}{4} + \xi\right)\mathcal{A}^{-\frac{u-1}{2}}(\mathbf{x},\mathbf{y}) + \left(\frac{1}{4} - \xi\right)\left(\partial^{x^\ell}\partial_{y^\ell} + V(\mathbf{x})\right)\mathcal{A}^{-\frac{u+1}{2}}(\mathbf{x},\mathbf{y})\right]_{\mathbf{y}=\mathbf{x}} .$$

  - $\langle \mathbf{v}|\,\hat{T}^u_{\mu\nu}(\mathbf{x})|\mathbf{v}\rangle$ is t-independent (as expected for static config.).

$\star$ $\Re u > j+d+1 \Rightarrow \langle \mathbf{v}|\,\hat{T}^u_{\mu\nu}(\mathbf{x})|\mathbf{v}\rangle \in C^j(\Omega)$ (hence, locally bounded);
$\Rightarrow u \mapsto \langle \mathbf{v}|\,\hat{T}^u_{\mu\nu}(\mathbf{x})|\mathbf{v}\rangle$ analytic for fixed $\mathbf{x} \in \Omega$,
with meromorphic continuation (possible simple poles).

- Define **renormalized VEVs** by analytic continuation at $u = 0$ $\left(\hat{\phi}^u \xrightarrow{u\to 0} \hat{\phi}\right)$.

◦ The **Casimir stress-energy tensor** is

$$\langle \mathbf{v}|\, \hat{T}_{\mu\nu}(\mathbf{x})|\mathbf{v}\rangle_{ren} \;:=\; \mathrm{RP}\Big|_{u=0} \langle \mathbf{v}|\, \hat{T}^u_{\mu\nu}(\mathbf{x})|\mathbf{v}\rangle \;,$$

$$\Big(\langle \mathbf{v}|\, \hat{T}^u_{\mu\nu}(\mathbf{x})|\mathbf{v}\rangle = \text{anal. cont.}\,; \;\; T(u) = \textstyle\sum_{\ell\in\mathbb{Z}} T_\ell\, u^\ell \;\Rightarrow\; \mathrm{RP}\Big|_{u=0} T(u) = T_0\Big).$$

- *No pole at $u = 0$* $\Rightarrow$ finite result *independent of* κ, *without subtraction of any divergent expression*.
- *Pole at $u = 0$* $\Rightarrow$ *minimal subtraction scheme* [Wald ('79), Visser *et al.* ('88)] $\leftrightarrow$ addition of local counter-terms in the Lagrangian; $\Rightarrow$ *explicit dependence on* κ.

◦ The **Casimir pressure** at $\mathbf{x}\in\partial\Omega$ is ($n(\mathbf{x})$ = outer normal)

$$p_i^{ren}(\mathbf{x}) := \mathrm{RP}\Big|_{u=0} \langle \mathbf{v}|\, \hat{T}^u_{ij}(\mathbf{x})|\mathbf{v}\rangle\, n^j(\mathbf{x}) \qquad (i,j = 1,...,d)\,.$$

◦ The **Casimir total energy** is $\mathcal{E}^{ren} := \mathrm{RP}\Big|_{u=0} \int_\Omega \langle \mathbf{v}|\, \hat{T}^u_{00}|\mathbf{v}\rangle = E^{ren} + B^{ren}$:

$$E^{ren} := \mathrm{RP}\Big|_{u=0} \frac{\kappa^u}{2}\, \mathrm{Tr}\, \mathcal{A}^{\frac{1-u}{2}} = \mathrm{RP}\Big|_{u=0} \frac{\kappa^u}{2} \sum_{k\in\mathcal{K}} {\omega_k}^{1-u} \quad (\textit{bulk term})\,,$$

$$B^{ren} := \mathrm{RP}\Big|_{u=0} \kappa^u \left(\frac{1}{4} - \xi\right) \int_{\partial\Omega} d\sigma(\mathbf{x})\; \partial_{n_\mathbf{y}} \mathcal{A}^{-\frac{u+1}{2}}(\mathbf{x},\mathbf{y})\Big|_{\mathbf{y}=\mathbf{x}} \quad (\textit{boundary term})\,.$$

Analytic continuation of $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$

- $e^{-\mathfrak{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) = \langle \delta_{\mathbf{x}} | e^{-\mathfrak{t}\mathcal{A}} | \delta_{\mathbf{y}} \rangle$ is the **heat kernel** (smooth for $\mathfrak{t} > 0$);
  $e^{-\mathfrak{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y}) = \langle \delta_{\mathbf{x}} | e^{-\mathfrak{t}\sqrt{\mathcal{A}}} | \delta_{\mathbf{y}} \rangle$ is the **wave kernel** (smooth for $\mathfrak{t}>0$).

- For $\Re s > d/2$, there hold the **Mellin identities** (even on the diag. $\mathbf{x}=\mathbf{y}$):

$$\begin{aligned}\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) &= \frac{1}{\Gamma(s)} \int_0^\infty d\mathfrak{t}\ \mathfrak{t}^{s-1}\, e^{-\mathfrak{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) \\ &= \frac{1}{\Gamma(2s)} \int_0^\infty d\mathfrak{t}\ \mathfrak{t}^{2s-1}\, e^{-\mathfrak{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y})\end{aligned}$$

$\hookrightarrow$ Starting point to construct the analytic continuation of $\mathcal{A}^{-s}(\mathbf{x},\mathbf{y})$.

1. Heat asymptotics: assume $e^{-\mathfrak{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) = \frac{1}{\mathfrak{t}^{d/2}}\Big(\sum_{n=0}^{N-1} a_n(\mathbf{x},\mathbf{y})\,\mathfrak{t}^n + O(\mathfrak{t}^N)\Big)$ for $\mathfrak{t}\to 0$ (HDMS expansion); then for $\Re s > \frac{d}{2} - N$

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{1}{\Gamma(s)}\left(\sum_{n=0}^{N-1} \frac{a_n(\mathbf{x},\mathbf{y})}{s+n-\frac{d}{2}} + \int_0^1 d\mathfrak{t}\ \mathfrak{t}^{s-\frac{d}{2}-1} O(\mathfrak{t}^N) + \int_1^\infty d\mathfrak{t}\ \mathfrak{t}^{s-1} e^{-\mathfrak{t}\mathcal{A}}(\mathbf{x},\mathbf{y})\right)$$

(possible simple poles at $s = \frac{d}{2} - n$, for $n = 0, \dots, N-1$).

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2. Integration by parts: assume $e^{-\mathfrak{t}\mathcal{A}}(\mathbf{x},\mathbf{y}) = \frac{1}{\mathfrak{t}^{d/2}}\,H(\mathfrak{t};\mathbf{x},\mathbf{y})$ for some H of class $C^N([0,\infty))$ w.r.t. $\mathfrak{t}$ (for fixed $\mathbf{x},\mathbf{y}$); then for $\Re s > \frac{d}{2} - N$

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{(-1)^N}{(s-\frac{d}{2})\ldots(s-\frac{d}{2}+N-1)\,\Gamma(s)}\int_0^\infty d\mathfrak{t}\;\mathfrak{t}^{s-\frac{d}{2}+N-1}\;\partial_{\mathfrak{t}}^N H(\mathfrak{t};\mathbf{x},\mathbf{y})$$

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3. Hankel representation: assume $e^{-\mathfrak{t}\sqrt{\mathcal{A}}}(\mathbf{x},\mathbf{y}) = \frac{1}{\mathfrak{t}^d}\,H(\mathfrak{t};\mathbf{x},\mathbf{y})$ for some H admitting analytic extension to $\mathcal{U}\subset\mathbb{C}$ with $[0,\infty)\subset\mathcal{U}$ (for fixed $\mathbf{x},\mathbf{y}$); then for all $s\in\mathbb{C}$ ($\mathfrak{H}=$ Hankel contour)

$$\mathcal{A}^{-s}(\mathbf{x},\mathbf{y}) = \frac{e^{-2\pi i}\,\Gamma(1-2s)}{2\pi i}\int_{\mathfrak{H}} d\mathfrak{t}\,\mathfrak{t}^{2s-d-1}\,H(\mathfrak{t};\mathbf{x},\mathbf{y})$$

   (possible simple poles at $s=\frac{d-n}{2}$, for $n=0,...,d-1$) .
   $\triangledown$ For $s=-\ell/2$, $\ell\in\mathbb{N}$ $\Rightarrow$ *easy explicit computations* (residue thm.) .

Parallel plates [Prog.Teor.Phys. **126**(3) ('11), World Scientific ('17)]

- *Case study*: massless scalar field ($V = 0$) on $\Omega = (0, a) \times \mathbb{R}^2$ ($a > 0$), with Dirichlet b.c. on $\partial\Omega$

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_\parallel\,;$$

$$\mathcal{A}_1 = -\partial_{11} \text{ on } (0,a) \text{ with Dirichlet b.c.}\,, \qquad \mathcal{A}_\parallel = -\partial_{22} - \partial_{33} \text{ on } \mathbb{R}^2\,.$$

- Reduced $1D$ problem on $(0, a)$: the wave kernel is (via eigenfun. expan.)

$$e^{-\mathfrak{t}\sqrt{\mathcal{A}_1}}(x^1, y^1) = \frac{1}{2a}\left[\frac{\cos\left(\frac{\pi}{a}(x^1 - y^1)\right) - e^{-\frac{\pi}{a}\mathfrak{t}}}{\cosh\left(\frac{\pi}{a}\,\mathfrak{t}\right) - \cos\left(\frac{\pi}{a}(x^1 - y^1)\right)} - \left(y^1 \to -y^1\right)\right]$$

$\Rightarrow \mathcal{A}^{-s}(x^1, y^1)$ via Hankel repres. + residue thm.; e.g.,

$$\mathcal{A}^{1/2}(x^1, x^1) = -\,\mathrm{Res}\left(\mathfrak{t}^{-2}\, e^{-\mathfrak{t}\sqrt{\mathcal{A}_1}}(x^1, x^1)\,;\, 0\right) = \frac{\pi}{12a^2}\,\frac{3 - \sin^2\left(\frac{\pi}{a}\,x^1\right)}{\sin^2\left(\frac{\pi}{a}\,x^1\right)}\,.$$

- Casimir stress-energy tensor, pressure and total energy per unit surface:

$$\langle \mathbf{v}|\,\hat{T}_{00}(x^1)\,|\mathbf{v}\rangle_{ren} = -\frac{\pi^2}{1440a^4} + \left(\xi - \frac{1}{6}\right)\frac{\pi^2}{8a^4}\,\frac{3 - 2\sin^2(\frac{\pi}{a}\,x^1)}{\sin^4(\frac{\pi}{a}\,x^1)} \xrightarrow{x^1 \to 0, a} \infty\,;$$

$$p_1^{ren}(0) = \frac{\pi^2}{480a^4}\,, \quad p_1^{ren}(a) = -\frac{\pi^2}{480a^4}\,; \qquad E^{ren} = -\frac{\pi^2}{1440a^3}\,, \quad B^{ren} = 0\,.$$

Rectangular box [Int.J.Mod.Phys.A **31**(04&05) (2016)]

- *Case study*: massless scalar field ($V = 0$) on $\Omega = (0, a_1) \times (0, a_2)$ ($a_1, a_2 > 0$), with Dirichlet b.c. on $\partial\Omega$

$$\mathcal{A} = -\Delta\,.$$

  - Previous analyses: mostly on global aspects, only partly on local aspects [Lukosz ('71), Mamaev, Trunov ('79), Ambjørn, Wolfram ('83), Actor ('95), Elizalde *et al.* ('95), Svaiter *et al.* ('01), Bordag *et al.* ('09), Fulling *et al.* ('09), ...] .

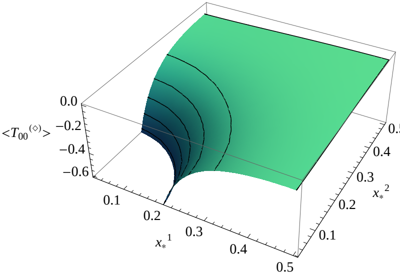
- Computational methods :
  - heat kernel $e^{-t\mathcal{A}}(\mathbf{x}, \mathbf{y})$: explicit series expansions for small and large $\mathfrak{t}$;
  - $\mathcal{A}^{-s}(\mathbf{x}, \mathbf{y})$ via Mellin identity (with the split $\int_0^\infty = \int_0^1 + \int_1^\infty$) ;
  - uniform estimates $\Rightarrow$ *exchange summation/integration orders* $\Rightarrow$ series repres. of $\mathcal{A}^{-s}(\mathbf{x}, \mathbf{y})$ giving the analytic continuation to $\mathbb{C}$.

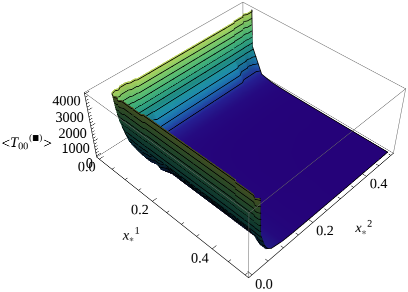
- Main advances :
  - unified analysis of Casimir observables: stress-energy tensor, pressure, total energy and total force ;
  - series repres. with exponential convergence and explicit error estimates .

- **Numerical evaluation** of $\langle \mathbf{v} | \hat{T}_{\mu\nu}(\mathbf{x}) | \mathbf{v} \rangle_{ren}$:
  - $\langle \mathbf{v} | \hat{T}_{\mu\nu} | \mathbf{v} \rangle_{ren} = \langle T^{(\Diamond)}_{\mu\nu} \rangle + \left(\xi - \frac{1}{8}\right) \langle T^{(\blacksquare)}_{\mu\nu} \rangle$ ($\Diamond$ = conf., $\blacksquare$ = non-conf.);
  - rescaled variables: $x^i_\star := x^i / a_i$ $(i = 1, 2)$.

○ An example: $a_1 = a_2 = 1$, $x^1_\star, x^2_\star \in (0, 1/2)$ (by symmetry) ,
series truncation at order 7 $\Rightarrow$ $|\text{errors}| \leqslant 3 \cdot 10^{-11}$.

$\langle T^{(\Diamond)}_{00} \rangle$ $\langle T^{(\blacksquare)}_{00} \rangle$





▼ Divergences near the boundaries $\Rightarrow$ $\int_\Omega d\mathbf{x} \, \langle \mathbf{v} | \hat{T}_{00}(\mathbf{x}) | \mathbf{v} \rangle_{ren} = \infty$.

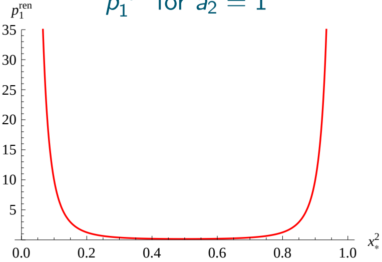
- Pressure on the side $\{x^1 = 0\}$

$$p_1^{ren}(x^2) = -\mathrm{RP}\Big|_{u=0} \langle \mathbf{v} | \hat{T}^u_{11}(0, x^2) | \mathbf{v} \rangle = -\lim_{x_1 \to 0} \langle \mathbf{v} | \hat{T}_{11}(x^1, x^2) | \mathbf{v} \rangle \,.$$

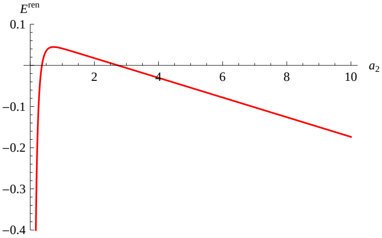
- Total energy: analogous series expansions for E^{ren} $\big(B^{ren} = 0\big)$.

- An example: $a_1 = 1$

p_1^{ren} for $a_2 = 1$



$E^{ren}(a_2)$



- **Remarks**:
  - $p_1^{ren} = \frac{1}{32\pi(x^2)^3} + O\big((x^2)^2\big)$ for $x^2 \to 0$ $\Rightarrow$ divergences near the corners;
  - $E^{ren}(a_2)$ finite for all $a_2 > 0$ $\Rightarrow$ $E^{ren} \neq \int_\Omega d\mathbf{x} \, \langle \mathbf{v} | \hat{T}_{00}(\mathbf{x}) | \mathbf{v} \rangle_{ren}$.

Harmonic potential [Int.J.Mod.Phys.A **30**(35) (2015)]

- *Case study*: massless scalar field on $\Omega = \mathbb{R}^3$, with isotropic harmonic potential $V(\mathbf{x}) := \lambda^4 |\mathbf{x}|^2$ $(\lambda > 0)$

$$\mathcal{A} = -\Delta + \lambda^4 |\mathbf{x}|^2$$

  - Previous analysis: total energy determined by [Actor, Bender ('95)] .

◦ Computational methods :

- heat kernel $e^{-t\mathcal{A}}(\mathbf{x}, \mathbf{y}) =$ Mehler kernel (explicit expression) ;
- pass to the rescaled spherical coordinate $r := \lambda|\mathbf{x}| \in (0, \infty)$;
- for $\Re u > 4$, the Mellin identity gives

$$\langle \mathbf{v}|\, \hat{T}^u_{\mu\nu}(r)|\mathbf{v}\rangle = \frac{\lambda^4}{\Gamma(\frac{u+1}{2})} \left(\frac{\kappa}{\lambda}\right)^u \int_0^\infty d\tau\; \tau^{-3+\frac{u}{2}}\, H^{(u)}_{\mu\nu}(\tau\,;r)\;,$$

$$H^{(u)}_{\mu\nu}(\tau; r) \text{ smooth on } [0,\infty) \text{ w.r.t. } \tau\;;$$

- 3-fold integration by parts ⇒ analytic continuation for $\Re u > -2$

$$\langle \mathbf{v}|\, \hat{T}^u_{\mu\nu}(r)|\mathbf{v}\rangle = -\frac{\lambda^4}{\Gamma(\frac{u+1}{2})(\frac{u}{2}-2)(\frac{u}{2}-1)\frac{u}{2}} \left(\frac{\kappa}{\lambda}\right)^u \int_0^\infty d\tau\; \tau^{\frac{u}{2}}\, \partial^3_\tau H^{(u)}_{\mu\nu}(\tau\,;r)\,.$$

○ The renormalized stress-energy VEV is ($\Diamond$ = conf., $\blacksquare$ = non-conf.)

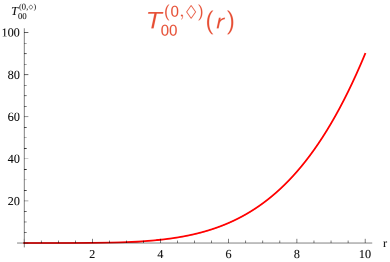
$$\langle \mathbf{v} | \hat{T}_{\mu\nu}(r) | \mathbf{v} \rangle_{ren} =$$

$$\lambda^4 \left[\left(T^{(0,\Diamond)}_{\mu\nu}(r) + M_{\kappa\lambda}\, T^{(1,\Diamond)}_{\mu\nu}(r) \right) + \left(\xi - \frac{1}{6} \right) \left(T^{(0,\blacksquare)}_{\mu\nu}(r) + M_{\kappa\lambda}\, T^{(1,\blacksquare)}_{\mu\nu}(r) \right) \right],$$

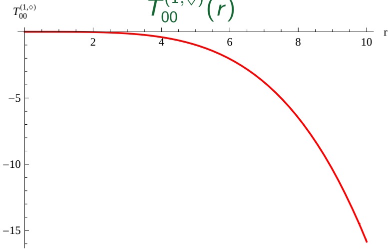
$$T^{(a,\bullet)}_{\mu\nu}(r) := \int_0^{+\infty} d\tau\, e^{-r^2 \tanh \tau} \left[\begin{array}{c} \text{polinomial in } r^2 \text{ with} \\ \text{coefficients depending on } \tau \end{array} \right], \quad M_{\kappa\lambda} := \gamma_{EM} + 2 \ln\left(\frac{2\kappa}{\lambda}\right).$$

• **Numerical evaluation** of the above integral representations:

$T^{(0,\Diamond)}_{00}(r)$



$T^{(1,\Diamond)}_{00}(r)$



○ Asymptotics for small/large $r = \lambda|\mathbf{x}|$: exact express. + remainder estimates.

○ The total energy : $E^{ren} = -(0.0079 \pm 10^{-4})k,\ B^{ren} = 0$.

$\Rightarrow\ E^{ren} \neq \int_\Omega d\mathbf{x}\, \langle \mathbf{v} | \hat{T}_{00}(\mathbf{x}) | \mathbf{v} \rangle_{ren} = \infty$.

- The renormalized stress-energy VEV is $(\Diamond = \text{conf.}, \ \blacksquare = \text{non-conf.})$

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$$\lambda^4 \left[\left(T^{(0,\Diamond)}_{\mu\nu}(r) + M_{\kappa\lambda}\, T^{(1,\Diamond)}_{\mu\nu}(r) \right) + \left(\xi - \tfrac{1}{6} \right) \left(T^{(0,\blacksquare)}_{\mu\nu}(r) + M_{\kappa\lambda}\, T^{(1,\blacksquare)}_{\mu\nu}(r) \right) \right],$$

$$T^{(a,\bullet)}_{\mu\nu}(r) := \int_0^{+\infty} d\tau \, e^{-r^2 \tanh \tau} \left[\begin{array}{c} \text{polinomial in } r^2 \text{ with} \\ \text{coefficients depending on } \tau \end{array} \right], \qquad M_{\kappa\lambda} := \gamma_{EM} + 2 \ln\left(\frac{2\kappa}{\lambda} \right).$$

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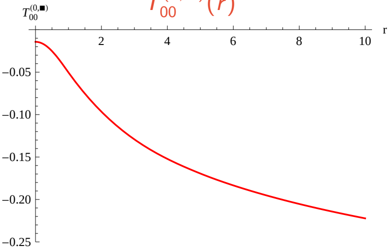
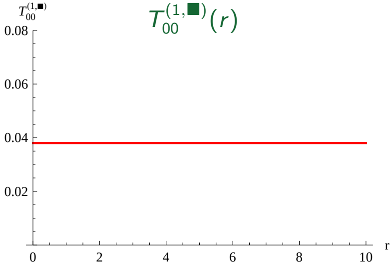


$T^{(1,\blacksquare)}_{00}(r)$



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Summary and outlook

- **Summary**
  - general ZR approach: unified analysis of local and global observables ;
  - computational effectiveness in some models .

- **Generalizations and further developments**
  - Curved (ultrastatic) background spacetimes
    $\hookrightarrow$ semi-classical Einstein eq.s : $G_{\mu\nu} = \kappa_g \langle \mathbf{v} | \hat{T}_{\mu\nu} | \mathbf{v} \rangle$.
  - Analysis of boundary divergences/energy anomalies :
    - delta-like potentials (semi-transparent boundaries) ;
    - semi-classical boundaries (position = r.v.) [Ford, Svaiter ('98)] .
  - Higher spin fields : EM and Dirac .
  - Equivalence with other renorm. schemes [Hack, Moretti ('12)] .

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Thanks a lot for your attention!