The Transport equation.

Let's now collect all the pieces of information that we have seen up to now, and merge them in the most general diffusion equation for a species $N(E, \vec{r}, t)$.

$$
\underset{(1)}{\frac{\partial N}{\partial t}} + \underset{(2)}{\left[- \vec{\nabla} \cdot \left(k(E, \vec{r})\, \vec{\nabla} \right) + \vec{\nabla} \cdot \vec{V}_c(r) \right] N} +
$$

$$
+ \underset{(3)}{\frac{\partial}{\partial E} \left[b(E)\, N - c(E) \frac{\partial N}{\partial E} \right]} +
$$

$$
+ \underset{(4)}{\left(\Gamma_{inel} + \Gamma_{rad} \right) N} =
$$

$$
\underset{(5)}{Q(E, \vec{r})} + \sum_{A_i > A_j} \Gamma^{i \to j} N
$$

1. The spectrum of CRs can in principle be time dependent, if sources, diffusion or any of the other terms are. Usually it is assumed steady state, and the term $\partial N / \partial t$ is set to zero.

2. This is the SPATIAL TRANSPORT term. It is due to diffusion on the inhomogeneities of the magnetic field, and to the convective wind. Diffusion acts randomly on the charged particle, which is spread out in any direction (if D is isotropic of course). Convection pushes particles vertically out of the galactic plane.

3. Energy losses and gains. Losses are more relevant for $e^{\pm}$ than for nuclei. Gain is due to reacceleration, Fermi 2^{nd} order process.

4. It describes catastrophic, NUCLEAR losses of the nucleus through an inelastic scattering off the ISM, or the β decay.

5. These are the SOURCES of CRs. The first term is for the acceleration of particles in some specific source.

$\mathcal{J}_\Gamma$ describes PRIMARY CRs, injected in the ISM by:

- supernova remnants (SNR) $\Rightarrow$ e^-; p, C, (N), O, Fe and all the species in the ISM close to the SNR;
- pulsars, for $e^+ e^-$;
- eventually Dark Matter.

$\mathcal{J}_\Gamma$ also includes the production of SECONDARY CRs through the spallation of heavier species on the ISM.
A species j can be produced by the spallation of a number of species i with atomic number $A_i > A_j$.

$$N_1 + (p, He)_{ISM} \longrightarrow N_2 + X$$

SOURCES I : SNR

SNRs originate from the violent explosion in the final stage of a star with $M > 8 M_\odot$. In the shocked plasma around these sources, particles can be accelerated through diffusive shock acceleration, as we have seen. The typical total energy output of the SNR ejecta is $E_{SNR} \sim 10^{51}$ erg. We expect 1-3 SNRs every century in the MW.

So $L_{kin} \simeq 10^{51}$ erg;

explosion rate $= \dfrac{3}{\text{century}}$

10% efficiency for conversion into relativistic particles

$$\Rightarrow P_{CR} \gtrsim 10^{51} \text{ erg} \cdot \frac{3}{100 \cdot 3 \cdot 10^{4} s} \cdot 0.1 \cong$$

$$\cong 10^{50} \cdot 10^{-9} \frac{\text{erg}}{s} = 10^{41} \text{ erg}/s$$

which is the power needed for the observed CR energy density.

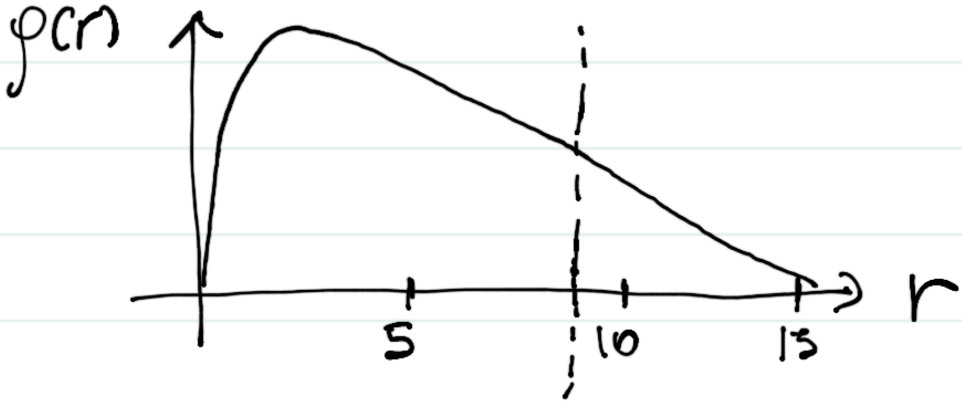
The energy spectrum of a particle injected by a SNR is a power law, as we have seen:

$$Q(E) = Q_0 \, (E/E_0)^{-\gamma} \qquad [Q_0] = \text{GeV}^{-1}$$

We also need a space distribution for SNRs, which is taken from (radio) catalog:

$$\rho(r)_{\text{SNR, Green}} \propto \left(\frac{r}{r_\odot}\right)^{\alpha_1} \exp\left[-\alpha_2 \frac{(r-r_\odot)}{r_\odot}\right]$$

$$\alpha_1 = 1.09 \;, \quad \alpha_2 = 3.87$$



$r_\odot = 8$ kpc

SOURCES II : PWN

Pulsars are fast rotating, strongly magnetized neutron stars produced in the collapse of a SNR. The rapidly spinning neutron star dissipates its energy by a wind of $e^\pm$, confined by the SNR ejecta, and forming a PULSAR WIND NEBULA (PWN).

Because of the fast rotating B, an electric field is induced, that extracts e^- from the star's surface. These e^- lose E through synchrotron emission. Given the high B of the star ($10^{11} \div 10^{13}$ G), the photons can form $e^\pm$ pairs. They in turn produce other photons, till a multiplicity of $10^4 \div 10^5$.

The SN blast wave, while propagating in the ISM, bounds the ejecta. A termination shock is produced, which accelerated $e^\pm$. The pairs are <u>confined</u> in the B, and loose E untie they are finally released in the ISM.

The lost E is $\sim 10^{49}$ erg in 10 kyr. The $e^\pm$ are probably released in the ISM after the PWN exits the SNR.

The energetics of $e^\pm$ from PWN can be derived from the pulsar spin-down power, which is converted into $e^\pm$ with a given efficiency η.

The energy spectrum of $e^{\pm}$ produce by PWN is shaped by:

$$Q(E) = Q_0 \, (E/E_0)^{-\gamma} \exp\left(-E/E_c\right).$$

$$[Q_0] = \mathrm{GeV}^{-1}$$

$$E_c \sim \mathcal{O}(\mathrm{TeV})$$

The TOTAL ENERGY emitted in e^- (SNR) or $e^{\pm}$ (PWN) is:

$$E_{TOT} = \int_{E_{min}}^{\infty} E \, Q(E) \, dE$$

For PWN: $E_{TOT} = \eta W_0$,
W_0 being the total spin-down energy.

SOURCES III : SPALLATIONS

The primary CR nuclei can be the sources of secondary CRs through spallation reactions on the ISM.
The source term is: $(\Gamma_{inel} = \sigma v n_{ISM})$

$$N_{sec}^{i\to j}(E,\vec{r},t) = \sum_{p \ni CR} \sum_{k=ISM} \int_{E_{th}}^{\infty} \frac{d\sigma^{pk}}{dE}(E_i \to E/n_k(\vec{r})\, v_p' \cdot N_i(E')\, dE'$$

ISM = 90% H, 10% He

We therefore need a plethora of cross sections $\sigma^{k\to j}$ for the PRODUCTION of nucleus j by a heavier nucleus k. These σ^{kj} are often poorly known. They can only be determined by high-energy accelerator experiments.

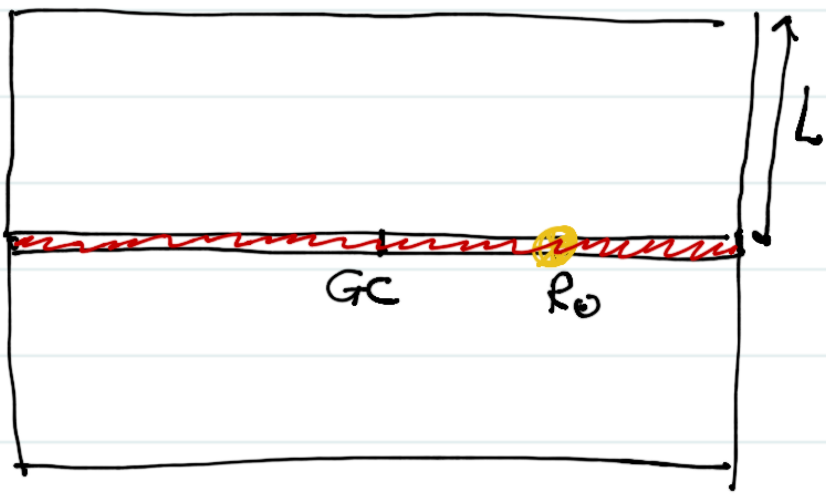
⟹ See figure for Li and Be production channels

⟹ See figure with some cross section data

⟹ See figure with TABLE prim / sec / mixed nuclei

The Geometry of The Galaxy

The Transport equation needs a geometry of The system – The Milky Way, specifically – in which To be solved.
The simplest geometry for The Milky Way (MW) is cylindric. It is made of a disc, where The sources (SNR, PWN, ISM) are located, and a diffusive halo, where B is present and particles diffuse.



R disc $= 20$ kpc
$2h$ (disc) $= 200$ pc
$R_\odot \cong 8 \div 8.5$ kpc
$L =$ size diffusive halo – $3 \div 10$ kpc

In The simplest case, The disc is isotropic (The ISM). Indeed, we know it has spiral arms, but in a first approximation, and for charged and stable particles it works well.
Looking at The disc and The halo, They contain:

	DISC	HALO
SNR, PWN	✓	—
ISM (destr.; source)	✓	—
B (diffusion)	✓	✓
V_c (convection)	✓	✓
V_A (reacceleration)	✓	(✓)
Coulomb, ioniz. losses	✓	—
synchrotron losses	✓	✓ ($e^\pm$)
IC losses	✓	✓ ($e^\pm$)
Dark matter	✓	✓

More complicated geometries are assumed when dealing with γ-ray emission, with local sources ($e^+ e^-$), radioactive nuclei, ...

We only concentrate here on the simplest description of MW. That means that there is an azimuthal symmetry and we can solve the transport equation at the position (r, z).

If the energy losses take place in the disc, then the transp. eq. can be solved first in space and then in energy. The two parts are separated.

Given the cylindrical geometry, we can develop $N(r, z)$ (let's omit E for a while) on the basis of the

Bessel functions.

$$N(r,z) = \sum_{i=1}^{\infty} N_i(z)\, J_0(x_i \rho)$$

$\rho = r/R$, $R \equiv R_{disc}$

J_0 = Bessel function of 0-th order

$$N_i(z) = \frac{2}{J_1^2(x_i)} \int_0^1 \rho\, N(\rho R, z)\, J_0(x_i \rho)\, d\rho$$

x_i = i-th zero of J_0.

The source in the disc write as:

$$q(r,z) = q_{disc}(r)\, \delta(z)$$

which is valid for primary in SNR/PWN and spallations.

The solution for $N_i(z)$ turns out to be:

$$N(r,z) = \exp\left(\frac{V_c z}{2k}\right) \cdot \sum_{i=1}^{\infty} \frac{Q_i}{A_i} \frac{\sinh[S_i(L-z)/2]}{\sinh[S_i L/2]} J_0\left(x_i \frac{r}{R}\right)$$

$$A_i = 2h\,\Gamma_{inel} + V_c + k\, S_i \coth\left(\frac{S_i L}{2}\right)$$

$$S_i = \sqrt{\frac{4x_i^2}{R^2} + \frac{V_c^2}{k^2} + 4\frac{\Gamma_{rad}}{k}}$$

$k \equiv K(E) \equiv D(E)$ diffusion coefficient

$$Q_i = \begin{cases} \to q_0\, Q(E)\, q_{i,disc} & \text{primary} \\ \to \sum_{k}^{m_k > m_j} \mu^{kj} N_i^k(0) & \text{secondary} \end{cases}$$

$q_{i,disc}$ is the Bessel expansion for $q(r)$
j is the nucleus considered in the equation (j usually omitted)

In case of primary nuclei and diffusion dominated transport;

$$N(r,z) \simeq \sum_i \frac{Q_i}{A_i} J_0(\varkappa_i \rho) \propto \frac{Q(E)}{K(E)}$$

If $Q(E) \propto E^{-\alpha}$ and $K(E) = \beta K_0 R^{\delta} \propto E^{\delta}$:

$$N_{prim}(r,z,E) \propto E^{-\alpha-\delta}$$

The equation is a 2^{nd} order differential equation in space, in energy, and eventually $\partial/\partial t$. There is a set of coupled equations from the heaviest nucleus to the j nucleus.

$$^{56}Fe \ldots \to {}^{16}O \to \ldots {}^{12}C \to \ldots {}^{10}B \to \ldots \to {}^{4}He \to H$$

Each species (i.e. isotope) can be produced by SNR, or by spallation of heavier ones. The solution is iterative for nuclei.

Note about secondaries

$$Q_N^{sec}(r,E) = \sum_{m_p > m_s} n_{ISM}\, v\, \sigma^{ps} N^p(r,E)$$

There are secondary particles such as:

$\bar{p}, \bar{D}, e^+, \gamma, \nu$

which are produced after spallation + hadronization (+) decays. These particles are described by:

$$Q_{\bar{p},\bar{D},e^+,\gamma,\nu}^{sec}(r,E) = \sum_{p=CR} \sum_{k=ISM} \int n_k \frac{d\sigma}{dE}(E',E)\, N^p(E')\, dE'$$

which is a convolution of the cross section on the CR fluxes.
projection

The solution we have seen is SEMI-ANALYTICAL: space eq. is solved analytically, while for the E-dep. eq. one has to use numerical solvers. $\rightarrow$ Usine

Other techiques to solve the transport equation are:

1) NUMERICAL with numerical routines for space and energy.
   It is easy to include new ingredients.
   Easy to trace γ-rays.
   Huge computational time.
   Difficult to get physical insights.
   $\rightarrow$ Galprop - Dragon - Picard

2) MONTE CARLO, following each particle.
   Stochastic differential eqs.
   Very slow. Statistical (Theoretical) errors.

The solution to the transport eq. requires to fix some parameters.

- Diffusion : $K(E) = K_0 \beta R^{\delta}$ $\qquad R = pc/Ze$
  K_0, δ free parameters
- acceleration : $Q = Q_0 E^{-\alpha}$
  α free parameter, Q_0 fixed to source abundances
- convection : V_c
- reacceleration : V_A
- geometry : L

$K_0, \delta, \alpha, V_c, V_A, L; \ldots$ free parameters

Free parameters of the diffusive model

They are L (kpc), K_0 (kpc^2/Myr), δ, (α), V_c (km/s), V_A (km/s) at least.

They are <u>degenerate</u> in most cases, and one can only make educated guess. However, at high energies ($\gtrsim 10$ GeV/n), V_c and V_A can be $\sim$ neglected.

The slope δ can be derived by the flux ratio:

Boron / Carbon $\equiv$ B/C

$$\frac{B}{C} = \frac{\text{sec}}{\text{prim}} \propto \frac{Q_{sec}}{Q_{prim}} \propto \frac{1}{Q_{prim}} \cdot \frac{Q_{prim}}{K(E)} \propto \frac{1}{K(E)} \propto E^{-\delta}$$

$\Longrightarrow$ figure B/C

MORE on how to fix the free parameters