Exercices

[Tight-coupling approximation]

The first two moments of the photon and baryon Boltzmann equations read

$$\dot{\Theta}_0 = -\frac{k}{3}\Theta_1 + \dot{\Phi} \tag{2.55}$$

$$\dot{\Theta}_1 = k\left(\Theta_0 + \Psi - \frac{2}{5}\Theta_2\right) - \dot{\tau}_c\big(\Theta_1 - V_b\big) \tag{2.56}$$

$$\dot{\delta}_b = -kV_b + 3\dot{\Phi} \tag{2.57}$$

$$\dot{V}_b = -\mathcal{H}V_b + k\Psi + \dot{\tau}_c\frac{\big(\Theta_1 - V_b\big)}{R} \; . \tag{2.58}$$

While V_b represents the baryon bulk velocity, Θ_1 is formally *not* a bulk velocity because it does not exist for photons (and, generally, for collisionless particles that can propagate in all directions). We should interpret it as the bulk velocity of the *photon temperature perturbation*.

This system of coupled ODEs simplifies in the so-called *tight-coupling limit*. Let $k \sim L^{-1}$ be the wavenumber of the fluctuations. There are two important characteristic timescales:

$$k^{-1} \quad = \quad \text{travel time across the perturbation} \tag{2.59}$$

$$\dot{\tau}_c^{-1} \quad = \quad \text{time between scattering events} \tag{2.60}$$

Tight coupling between the photons and baryons occurs when

$$\frac{\dot{\tau}_c^{-1}}{k^{-1}} = \frac{k}{\dot{\tau}_c} \ll 1 \; . \tag{2.61}$$

In regime, photons experience so many scattering as they travel across a perturbation that they remain strongly coupled to the baryons.

(a) since the baryon bulk velocity V_b varies on a timescale much longer than $\dot{\tau}_c^{-1}$, show that this implies

$$\Theta_1 \simeq V_b \quad \Leftrightarrow \quad \Theta_2 \simeq 0 \; . \tag{2.62}$$

The photon temperature quadrupole, or anisotropic stress, can thus be neglected, which closes the Boltzmann hierarchy. We will hereafter assume the tight-coupling limit, so that we can ignore all multipoles with $\ell \geqslant 2$.

[Acoustic Oscillations]

One generally expands the Boltzmann hierarchy in powers of $k/\dot{\tau}_c$ (the inverse of the optical depth through a wavelength k) and $\omega/\dot{\tau}_c$ (the inverse of the optical depth through a period of oscillation ω). We will remain at first order in $\dot{\tau}_c^{-1}$, which leads to a driven harmonic oscillator equation describing acoustic waves in the photon-baryon fuild. At second order in $\dot{\tau}_c^{-1}$, acoustic oscillations of the monopole and dipole are damped owing to the imperfect coupling between photons and baryons. Photon diffusion creates heat conduction through $\Theta_1 - V_b$ and shear viscosity through Θ_2.

(b) Extract the term $\dot{\tau}_c(\Theta_1 - V_b)$ from Eq.(2.58) and substitute into Eq.(2.56). Show that, after some manipulations, one obtains

$$\big(1+R\big)\ddot{\Theta}_0 + \mathcal{H}R\dot{\Theta}_0 + \frac{k^2}{3}\Theta_0 = -\frac{k^2}{3}\big(1+R\big)\Psi + \mathcal{H}R\dot{\Phi} + \big(1+R\big)\ddot{\Phi}\ . \tag{2.63}$$

Use the fact that $R = 3\bar{\rho}_b/4\bar{\rho}_\gamma \propto a$, i.e. $\dot{R} = \mathcal{H}R$ to reexpress this relation as

$$\frac{d}{d\eta}\Big[\big(1+R\big)\dot{\Theta}_0\Big] + \frac{k^2}{3}\Theta_0 = -\frac{k^2}{3}\big(1+R\big)\Psi + \frac{d}{d\eta}\Big[\big(1+R\big)\dot{\Phi}\Big]\ . \tag{2.64}$$

This is the equation of an oscillator with a time-varying mass $m_{\rm eff} = 1 + R$. The homogeneous equation can be solved by employing the fact that variations over a single period of the oscillation are small.

(c) We have thus far not used Einstein equations. One can show that, in the absence of anisotropic stress (that is, $\pi_\gamma = \pi_\nu = ... = 0$), Einstein equations imply that the two potentials are equal: $\Psi = \Phi$. Neglect the time-dependence of R and show that

$$\frac{d^2}{d\eta^2}\big(\Theta_0 + \Psi\big) + k^2c_s^2\big(\Theta_0 + \Psi\big) = -k^2c_s^2 R\Psi + 2\ddot{\Psi}\ , \tag{2.65}$$

where

$$c_s^2 \equiv \frac{\dot{P}}{\dot{\rho}} \approx \frac{1}{3(1+R)} \tag{2.66}$$

is the adiabatic sound speed of the photon-baryon fluid. c_s defines a characteristic comoving length scale $r_s(\eta)$ known as the *sound horizon*,

$$r_s(\eta) = c_s\int\frac{dt}{a(t)} \equiv c_s\eta\ , \tag{2.67}$$

which represents the comoving distance traveled by a sound perturbation. Calculate r_s when the CMB decouples from the plasma at $\eta_{\rm dec}$ assuming $z_{\rm dec} = 10^3$ and an EdS universe.

(d) Eq.(2.65) is a driven harmonic oscillator equation for the effective temperature $\Theta_0 + \Psi$, in which the gravitational blueshift due to the infall onto a potential well is exactly compensated by Ψ. The frequency $\omega = kc_S$ increasing with decreasing comoving scale k^{-1}. Ignoring the time variation of c_s, R and, especially, $\ddot{\Psi}$, show that the solution to Eq.(2.65) is of the form

$$\Theta_0 + \Psi = -R\Psi + A\cos\big(kc_s\eta\big) + B\sin\big(kc_s\eta\big)\ . \tag{2.68}$$

$\Theta_0 + \Psi$ oscillates around $-R\Psi$ and not zero owing to the baryons, which drag the photons into the potential wells (an effect known as *baryon drag*).

(e) To fix the initial conditions which determine A and B, we take the limit $\eta \to 0$ (or, equivalently, $c_s\eta \equiv r_s \to 0$), in which case (not demonstrated here)

$$\Theta_0 + \Psi \approx \frac{1}{3}\Psi \qquad \text{(adiabatic ICs)} \tag{2.69}$$

$$\Theta_0 + \Psi \approx 2\Psi \qquad \text{(isocurvature ICs)} \tag{2.70}$$

Assuming adiabatic perturbations (generated by inflation for instance) and zero initial "velocity" $\dot{\Theta}_0$, show that the solution is

$$\big(\Theta_0+\Psi\big)(\eta)=\frac{1}{3}\big(1+R\big)\Psi\cos\big(kc_s\eta\big)-R\Psi\;. \tag{2.71}$$

The baryon drag increases the amplitude of the cosine term. Overall, it accounts for the alternate height of the acoustic peaks (compression peaks occuring at $kc_s\eta=\pi$, 3π, ..., are enhanced relative to the rarefaction peaks at $kc_s\eta=2\pi$, 4π, ...) and their enhancement with $R\propto\Omega_b h^2$.

[Integral solution to CMB anisotropies]

Rather than decomposing the Boltzmann equation Eq.(2.37),

$$\frac{d\Theta}{d\eta}=-\hat{n}^i\partial_i\Psi+\dot{\Phi}+\dot{\tau}_c\Big[-\Theta+\frac{1}{4}\delta_\gamma+\hat{\mathbf{n}}\cdot\mathbf{v}_b\Big]\;,\qquad \frac{d}{d\eta}\equiv\partial_\eta+\hat{n}^i\partial_i \tag{2.72}$$

in Legendre polynomials and solve for the multipoles $\Theta_\ell(\eta,\mathbf{k})$, it can be formally integrated to yield the CMB temperature anisotropy $\Theta(\eta_0,\hat{\mathbf{n}})$ as seen by an observer at time η_0:

$$\Theta(\eta_0,\hat{\mathbf{n}})=\int_0^{\eta_0}d\eta\,e^{-\tau(\eta)}\frac{d\Theta}{d\eta}\qquad\text{where}\qquad \tau(\eta)=\int_\eta^{\eta_0}d\eta'\,\dot{\tau}_c(\eta')\;. \tag{2.73}$$

Here, $\tau(\eta)$ is the (average) optical depth along the line of sight.

(f) To see this, show that

$$\begin{aligned}\Theta(\eta_0,\hat{\mathbf{n}})&=\int_0^{\eta_0}d\eta\,e^{-\tau}\left(\frac{d\Theta}{d\eta}+\dot{\tau}_c\Theta\right)\\ &=\int_0^{\eta_0}d\eta\,e^{-\tau}\left[-\hat{n}^i\partial_i\Psi+\dot{\Phi}+\dot{\tau}_c\Big(-\Theta+\frac{1}{4}\delta_\gamma+\hat{\mathbf{n}}\cdot\mathbf{v}_b\Big)\right]\;.\end{aligned} \tag{2.74}$$

Next, demonstrate that

$$\int_0^{\eta_0}d\eta\,e^{-\tau}\big(-\hat{n}^i\partial_i\Psi\big)=-e^{-\tau}\Psi\Big|_0^{\eta_0}+\int_0^{\eta_0}d\eta\,e^{-\tau}\Big(\dot{\tau}_c\Psi+\dot{\Psi}\Big) \tag{2.75}$$

Argue that the first term in the right-hand side can be neglected, and substitute this result into the expression of $\Theta(\eta_0,\hat{\mathbf{n}})$ to obtain

$$\Theta(\eta_0,\hat{\mathbf{n}})=\int_0^{\eta_0}d\eta\,e^{-\tau}\dot{\tau}_c\left(\frac{1}{4}\delta_\gamma+\Psi+\hat{\mathbf{n}}\cdot\mathbf{v}_b\right)+\int_0^{\eta_0}d\eta\,e^{-\tau}\big(\dot{\Psi}+\dot{\Phi}\big)\;. \tag{2.76}$$

Argue that the visibility function

$$g(\eta)\equiv\dot{\tau}_c e^{-\tau}=\frac{d\tau}{d\eta}e^{-\tau} \tag{2.77}$$

is sharply peaked around the decoupling or last scattering epoch $\eta_{\rm dec}$ to approximate $\Theta(\eta_0,\hat{\mathbf{n}})$ as

$$\Theta(\eta_0,\hat{\mathbf{n}})\approx\left(\frac{1}{4}\delta_\gamma+\Psi+\hat{\mathbf{n}}\cdot\mathbf{v}_b\right)(\eta_{\rm dec},\hat{\mathbf{n}})+\int_0^{\eta_0}d\eta\,\big(\dot{\Psi}+\dot{\Phi}\big) \tag{2.78}$$

The first encode the contributions of the intrinsic photon density perturbation ($\frac{1}{4}\delta_\gamma$), Sachs-Wolfe effect from the gravitational potential (Ψ) and Doppler effect from the photon-baryon relative motion ($\hat{\mathbf{n}}\cdot\mathbf{v}_b$). The second term is the integrated Sachs-Wolfe (ISW) effect, which vanishes for time-independent potentials.