The Fisher matrix approach

At the 1.5PN order in the frequency domain, the waveform of a two-body system can be described by the so called TaylorF2 approximant, $\tilde{h}(f) = \mathcal{A}f^{-7/6}e^{i\psi(f)}$, where the phase is given a sum of the PN terms:

$$\psi(f) = 2\pi f\tau_c - \phi_c - \frac{\pi}{4} + \frac{3}{128}(\mathcal{M}\pi f)^{-5/3}\sum_{i=0}^{3}(M\pi f)^{i/3}\alpha_i^{\rm PN} \ , \tag{1}$$

$$\alpha_0 = 1 \ , \ \alpha_1 = 0 \ , \ \alpha_2 = \frac{3715}{765} + \frac{55\nu}{9} \ , \ \alpha_3 = -16\pi, \tag{2}$$

being (τ_c, ϕ_c) the time and phase at the coalescence, $M = m_1 + m_2$ the total mass, $\nu = m_1 m_2/M^2$ the symmetric mass ratio, $\mathcal{M} = M\nu^{3/5}$ the chirp mass. The amplitude of the signal $\mathcal{A}$ reads

$$\mathcal{A} = \mathcal{C}\frac{\mathcal{M}^{5/6}}{\pi^{2/3}d}\sqrt{\frac{5}{24}} \tag{3}$$

where d is the source distance, and $\mathcal{C}$ is a geometrical factor equal to 1 for binaries optimally oriented with respect to the detector, and 2/5 if we consider an average over the angles which define the sky position. The gravitational wave template (1) is then fully specified by a set of 5 parameters $\theta = (\mathcal{A}, \mathcal{M}, \nu, \tau_c, \phi_c)$.

1. Given two prototype systems, specified by the true parameters:

$$d^{(1)} = 450{\rm Mpc} \ , \ \mathcal{M}^{(1)} = 30M_\odot \ , \ \nu^{(1)} = 0.25 \ , \ \tau_c^{(1)} = \phi_c^{(1)} = 0 \ , \tag{4}$$

$$d^{(2)} = 40{\rm Mpc} \ , \ \mathcal{M}^{(2)} = 1.2M_\odot \ , \ \nu^{(2)} = 0.24 \ , \ \tau_c^{(2)} = \phi_c^{(2)} = 0 \ , \tag{5}$$

compute the number of cycles spent by the binaries and that and the signal-to-noise ratio (SNR):

$$\mathcal{N}_{\rm GW} = \int_{f_{\rm min}}^{f_{\rm max}} \frac{f}{df/dt}df \quad , \quad \rho = \left[4\Re\int_{f_{\rm min}}^{f_{\rm max}} \frac{\tilde{h}(f)^\star\tilde{h}(f)}{S_n(f)}df\right]^{1/2} \ , \tag{6}$$

where $df/dt = \frac{96}{5\mathcal{M}^2\pi}(\pi\mathcal{M}f)^{11/3}$, and $S_n(f)$ is the noise spectral density of the specific detector. For the latter, use the LIGO-O1 and the design sensitivity curves provided as data files with the exercises. Moreover, set $f_{\rm min} = 10$Hz, and $f_{\rm max} = f_{\rm ISCO}$, where $f_{\rm ISCO}$ is the (Newtonian) frequency at the Innermost Stable Circular Orbit in the Schwarzschild spacetime, i.e. $r_{\rm ISCO} = 6M$.

2. Compute the 5×5 Fisher and covariance matrices for the binary parameters defined before

$$\Gamma_{ab} = 4\Re\int_{f_{\rm min}}^{f_{\rm max}} \frac{\tilde{h}^\star_{,a}(f)\tilde{h}_{,b}(f)}{S_n(f)}df \quad , \quad \Sigma_{ab} = (\Gamma^{-1})_{ab} \ , \tag{7}$$

and estimate the relative errors on the mass parameters $(\mathcal{M}, \nu)$.

3. How does the amplitude correlate with the other parameters?

4. At the leading order in the spin, the template (1) acquires a 1.5 PN correction, such that $\alpha_3 = -16\pi + \frac{113}{3}\chi$, where χ is an *effective* spin. Repeat the analysis and compute the uncertainties for a 6×6 Fisher which takes into account χ as extra parameter. Assume for the two binaries, $\chi^{(1)} = 0.5$ and $\chi^{(2)} = 0$. How do the statistical errors change?

[Results for Advanced LIGO and a 5×5 Fisher matrix:

$$\mathcal{N}_{\rm GW}^{(1)} \simeq 73 \ , \ \rho^{(1)} \simeq 33 \ , \ \sigma_{\mathcal{M}}^{(1)}/\mathcal{M} \simeq 0.8\%, \ \sigma_{\nu}^{(1)}/\nu \simeq 8\%]$$