An introduction to gravitational waves

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Outline of lectures (1/2)

- The world's shortest introduction to General Relativity
- The linearized Einstein equations and the degrees of freedom of General Relativity
- Gravitational waves in linearized gravity and the quadrupole formula
- Gravitational waves in the geometric optics regime and their stress energy tensor
- A detector's response to gravitational waves

Outline of lectures (2/2)

- GW detectors and their sources
- (A little about) matched filtering and parameter estimation
- Source modelling:

  - Numerical relativity in a nutshell: 3+1 form of the Einstein equations

  - Analytic approximations: The Post-Newtonian expansion, the self-force formalism, the effective one-body model
- Fundamental physics, astrophysics and cosmology with gravitational-wave detectors: a few examples

References

- Einstein equations: any GR textbook (Misner, Thorne & Wheeler, Wald, Carroll, …)
- Basics of gravitational waves:

  - Flanagan, E. E. & Hughes, S. A. 2005, New Journal of Physics, 7, 204 (arXiv:gr-qc/0501041)

  - Rezzolla, L. 2003, ICTP Lecture Series, Vol. 3 (arXiv:gr-qc/0302025)

  - Thorne, K., ”Gravitational Waves and Experimental Tests of General Relativity” www.pma.caltech.edu/Courses/ph136/yr2004/0426.1.K.pdf

  - Maggiore, M., ”Gravitational waves. Vol. 1: Theory and experiments”
- 3+1 formulation of Einstein equations and numerical relativity: Gourgoulhon, E., gr-qc/0703035
- LISA: Pau Amaro-Seoane et al, arXiv:1201.3621
- More specialized references for some slides

General Relativity: a description of gravity

- Newtonian mechanics (v<< c and weak gravitational fields M/r << c^2): gravity is a force

  - Gravitational potentials satisfies Poisson's equation (aka Newton's law of gravitation): $\nabla^2 \varphi = 4\pi G \rho$

  - Motion described by 3 laws of Newtonian mechanics and namely $\vec{F} = m\vec{a}$

- Special relativity generalizes Newtonian mechanics (but not Newton's law of gravitation) to v ~ c by requiring that speed of light be the same and finite in all inertial reference systems (cf Michelson-Morley experiment!) →

  Minkoswki metric $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2$

- General relativity generalizes Newton's law of gravitation to v ~ c and strong gravitational fields, but gravity is not a force any more!

General Relativity in a nutshell (1/5)

- Gravity is not a force, but geometrical effect encoded in 4D metric
  $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$
- Metric measures "distance" between events $x_1^\mu = (ct, x, y, z)$ and $x_2^\mu = (ct, x, y, z)$, is symmetric, has signature Lorentz signature (-,+,+,+)
- Particles move along lines that minimize distance (geodesics)

$$u^\mu = \frac{dx^\mu}{d\lambda} \qquad a^\mu = u^\nu \nabla_\nu u^\mu = 0 \qquad \nabla_\nu u^\mu = \partial_\nu u^\mu + \Gamma^\mu_{\nu\alpha} u^\alpha u^\nu = 0$$

$$g_{\mu\nu} u^\mu u^\nu = -1 \quad \text{(particles with mass)}$$

$$\Gamma^\mu_{\nu\alpha} = \frac{1}{2} g^{\mu\sigma} (\partial_\nu g_{\alpha\sigma} + \partial_\alpha g_{\nu\sigma} - \partial_\sigma g_{\alpha\nu}) \qquad g_{\mu\nu} u^\mu u^\nu = 0 \quad \text{(light rays)}$$

- General covariance: equations of motion take same form in any coordinate system (because defined in terms of spacetime geometry)

  In locally flat coordinates near moving particle (ie free-falling frame), $g_{\mu\nu} = \eta_{\mu\nu} + O(x)^2$ ⇒ non-gravitational law of physics reduce to special relativity, and gravitational forces disappear (cf free-falling spacecraft in Newtonian gravity)

General Relativity in a nutshell (2/5)

- Geodesic motion generalizes Newtonian/special relativistic mechanics, but how do we choose the metric, ie how do we generalize Poisson's equation?
- Requirements for generalization

  1) Must reduce to Poisson equation for v<<c and weak fields

  2) General covariance: equation for the gravitational field must be the same in all coordinate systems (must be defined in terms of 4D tensors)

  3) Gravity described by metric alone (eg no gravitational scalars)

  4) Possion equation is linear and second order in the derivatives of φ : look for simplest equation that is linear in 2nd derivatives of metric and satisfies first 3 conditions

→ Einstein equations

General Relativity in a nutshell (3/5)

The Einstein equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4}$

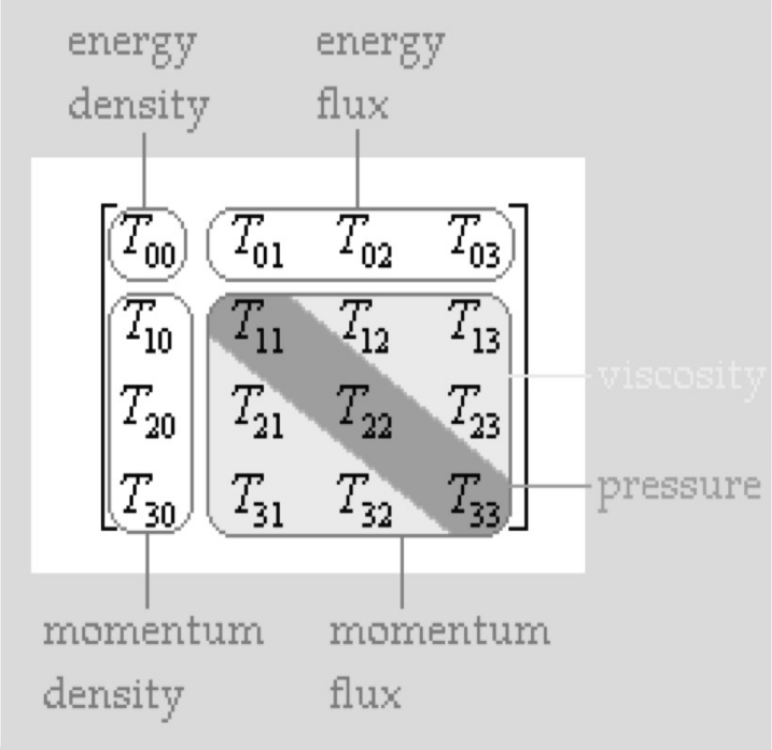
$$R^{\alpha}_{\beta\gamma\delta} \equiv \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\nu}_{\beta\delta}\Gamma^{\alpha}_{\nu\gamma} - \Gamma^{\nu}_{\beta\gamma}\Gamma^{\alpha}_{\nu\delta}$$ (Riemann tensor)

$$R_{\alpha\beta} \equiv R^{\gamma}_{\alpha\gamma\beta}$$ (Ricci tensor)

$$R = g^{\alpha\beta} R_{\alpha\beta}$$ (Ricci scalar)

- Stress-energy tensor $T^{\mu\nu}$ describes matter content of spacetime,

  eg for perfect fluid $T^{\mu\nu} = (\rho + p) u^{\mu} u^{\nu} + p g^{\mu\nu}$



General Relativity in a nutshell (4/5)

Bianchi identity $\nabla_\nu G^{\mu\nu} = 0$ **+** $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^4}$

$\Rightarrow$ $\nabla_\nu T^{\mu\nu} = 0$

- 4 independent components: conservation of energy and linear momentum
- For a perfect fluid, energy conservation and Euler equation

$$u^\mu \partial_\mu \rho = -(p+\rho) \nabla_\mu u^\mu \qquad a^\mu = -\frac{(g^{\mu\nu} + u^\mu u^\nu) \partial_\nu p}{p+\rho}$$

- For dust (p=0) we get the geodesic equation. Same if we use stress energy tensor for a single particle

Equations of motion of matter follow from Einstein equations

General Relativity in a nutshell (5/5)

The stress energy tensor of a point particle

$$S_{\text{mat}} = -mc \int ds.$$

$$S_{\text{mat}} = \int d^4x \sqrt{-g} \left(-\frac{mc}{\sqrt{-g}} \sqrt{-g_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}} \delta^3(x^i - x^i_P) \right)$$

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_{\text{mat}}}{\delta g_{\mu\nu}} \qquad \delta S_{\text{mat}}(g_{\mu\nu}) = \int d^4x \frac{\delta S_{\text{mat}}}{\delta g_{\mu\nu}(x)} \delta g_{\mu\nu}(x),$$

$$T^{\mu\nu} = mc \frac{u^\mu u^\nu}{u^0 \sqrt{-g}} \delta^3(x^i - x^i_P(\tau))$$

The degrees of freedom of GR

- 4D metric has 10 independent components vs 1 potential of Newtonian theory. What are the other degrees of freedom?
- Let's consider linear perturbations over Minkoskwi background metric, ie $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$, with $|h_{\mu\nu}|\ll 1$ and $|T_{\mu\nu}|\ll 1$ (from now on, *G=c=1*)
- If $T_{\mu\nu}, h_{\mu\nu}\to 0$ as $r\to\infty$, most general decomposition is

$$h_{tt} = 2\phi\ ,$$
$$h_{ti} = \beta_i + \partial_i\gamma\ ,$$
$$h_{ij} = h_{ij}^{\rm TT} + \frac{1}{3}H\delta_{ij} + \partial_{(i}\varepsilon_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\lambda\ ,$$

$$\partial_i\beta_i = 0$$
$$\partial_i\varepsilon_i = 0$$
$$\partial_i h_{ij}^{\rm TT} = 0$$
$$\delta^{ij}h_{ij}^{\rm TT} = 0$$

$$T_{tt} = \rho\ ,$$
$$T_{ti} = S_i + \partial_i S\ ,$$
$$T_{ij} = P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma,$$

$$\partial_i S_i = 0\ ,$$
$$\partial_i\sigma_i = 0\ ,$$
$$\partial_i\sigma_{ij} = 0\ ,$$
$$\delta^{ij}\sigma_{ij} = 0,$$

Gauge transformations

- Physics does not depend on choice of coordinates, ie we are free to use any coordinate system
- Metric and stress energy transform as

$$\tilde{g}_{\mu\nu}(\tilde{x}) = g_{\alpha\beta}(x(\tilde{x}))\frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}}(\tilde{x})\frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}}(\tilde{x}) \qquad \tilde{T}_{\mu\nu}(\tilde{x}) = T_{\alpha\beta}(x(\tilde{x}))\frac{\partial x^{\alpha}}{\partial \tilde{x}^{\mu}}(\tilde{x})\frac{\partial x^{\beta}}{\partial \tilde{x}^{\nu}}(\tilde{x})$$

- For a ”small” coordinate change $\tilde{x}^{\mu} = x^{\mu} + \xi^{\mu}, \quad |\xi^{\mu}| \ll 1$

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} \qquad \tilde{T}_{\mu\nu} = T_{\mu\nu} - \xi^{\alpha}\partial_{\alpha}T_{\mu\nu} - \partial_{\mu}\xi^{\alpha}T_{\alpha\nu} - \partial_{\nu}\xi^{\alpha}T_{\mu\alpha}$$

- Decomposing $(\xi_t, \xi_i) \equiv (A, B_i + \partial_i C)$, the metric transforms as

$$\begin{aligned} \phi &\rightarrow \phi - \dot{A}\,, \\ \beta_i &\rightarrow \beta_i - \dot{B}_i\,, \\ \gamma &\rightarrow \gamma - A - \dot{C}\,, \\ H &\rightarrow H - 2\nabla^2 C\,, \end{aligned} \qquad \begin{aligned} \lambda &\rightarrow \lambda - 2C\,, \\ \varepsilon_i &\rightarrow \varepsilon_i - 2B_i\,, \\ h_{ij}^{\mathrm{TT}} &\rightarrow h_{ij}^{\mathrm{TT}}\,. \end{aligned}$$

The Poisson gauge

- Defined $\partial_i h^{ti} = \partial_i h^{ij} = 0$ $\longrightarrow$ $\gamma = \lambda = \epsilon_i = 0$

$$
\begin{aligned}
h_{tt} &= 2\phi \,, \\
h_{ti} &= \beta_i + \cancel{\partial_i \gamma} \,, \\
h_{ij} &= h_{ij}^{\mathrm{TT}} + \frac{1}{3} H \delta_{ij} + \cancel{\partial_{(i}\epsilon_{j)}} + \cancel{\left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \lambda} \,,
\end{aligned}
$$

- Equivalent to using gauge invariant combinations

$$
\begin{aligned}
\Phi &\equiv -\phi + \dot{\gamma} - \frac{1}{2}\ddot{\lambda} \,, \\
\Theta &\equiv \frac{1}{3}\left(H - \nabla^2 \lambda \right) \,, \\
\Xi_i &\equiv \beta_i - \frac{1}{2}\dot{\epsilon}_i \,;
\end{aligned}
$$

and h_{ij}^{TT}

(already gauge-invariant)

The linearized Einstein equations

$$G_{tt} = -\nabla^2\Theta \ ,$$
$$G_{ti} = -\frac{1}{2}\nabla^2\Xi_i - \partial_i\dot{\Theta} \ ,$$
$$G_{ij} = -\frac{1}{2}\Box h^{\rm TT}_{ij} - \partial_{(i}\dot{\Xi}_{j)} - \frac{1}{2}\partial_i\partial_j\left(2\Phi+\Theta\right) + \delta_{ij}\left[\frac{1}{2}\nabla^2\left(2\Phi+\Theta\right) - \ddot{\Theta}\right] \ .$$

$$T_{tt} = \rho \ ,$$
$$T_{ti} = S_i + \partial_i S \ ,$$
$$T_{ij} = P\delta_{ij} + \sigma_{ij} + \partial_{(i}\sigma_{j)} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma,$$

$$\nabla^2 S = \dot{\rho} \ ,$$
$$\nabla^2\sigma = -\frac{3}{2}P + \frac{3}{2}\dot{S} \ , \qquad \left(\text{from } \partial_\mu T^{\mu\nu} = 0\right)$$
$$\nabla^2\sigma_i = 2\dot{S}_i \ .$$

$$\nabla^2\Theta = -8\pi\rho \ ,$$
$$\nabla^2\Phi = 4\pi\left(\rho + 3P - 3\dot{S}\right) \ ,$$
$$\nabla^2\Xi_i = -16\pi S_i \ ,$$
$$\Box h^{\rm TT}_{ij} = -16\pi\sigma_{ij} \ .$$

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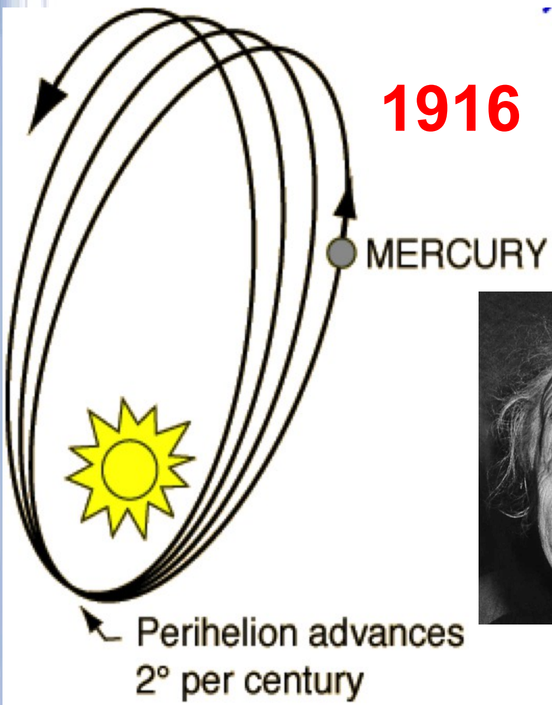
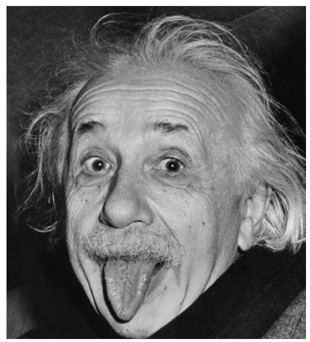
$$\nabla^2\Xi_i = -16\pi S_i \,,$$ → h_{ti}, appears at 1PN order, ie suppressed by $(v/c)^2$

$$\Box h_{ij}^{\rm TT} = -16\pi\sigma_{ij} \,.$$ → TT part of h_{ij}, appears at 2PN (conservative part) and 2.5PN order (dissipative part)

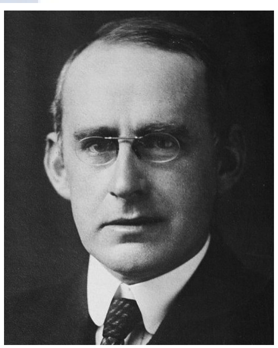
1PN effects observed for a century!

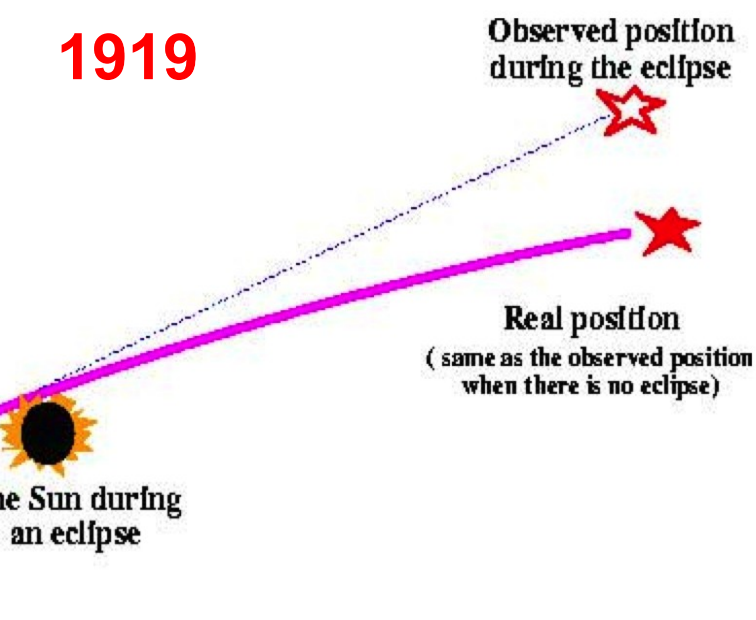
1916





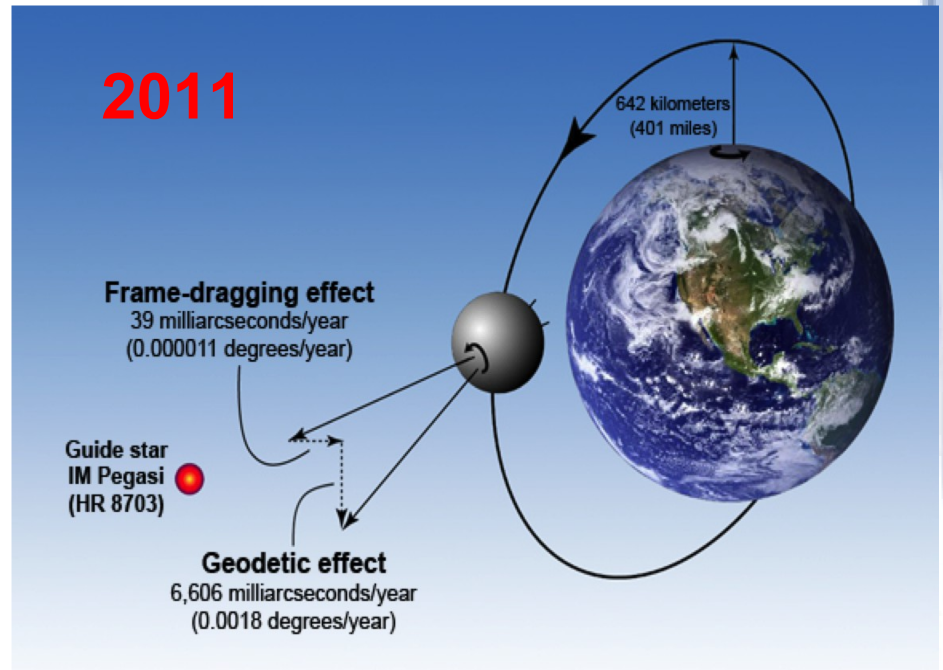
1919





Credit: Jose Wudka

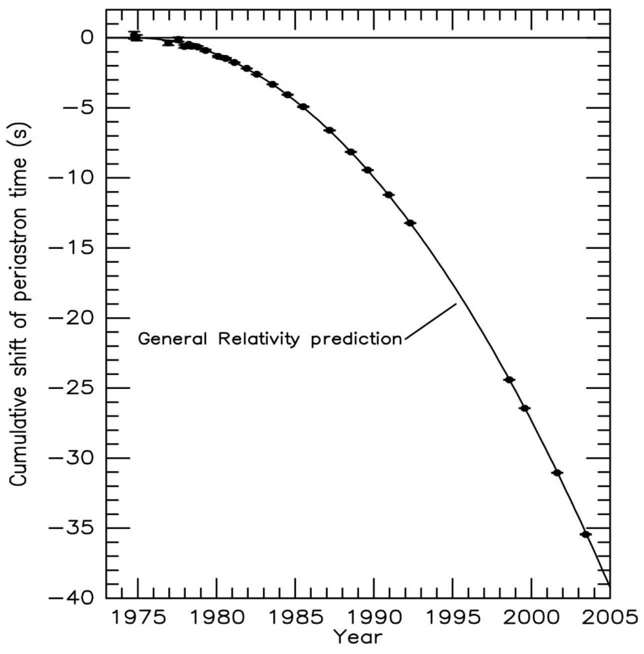
2011



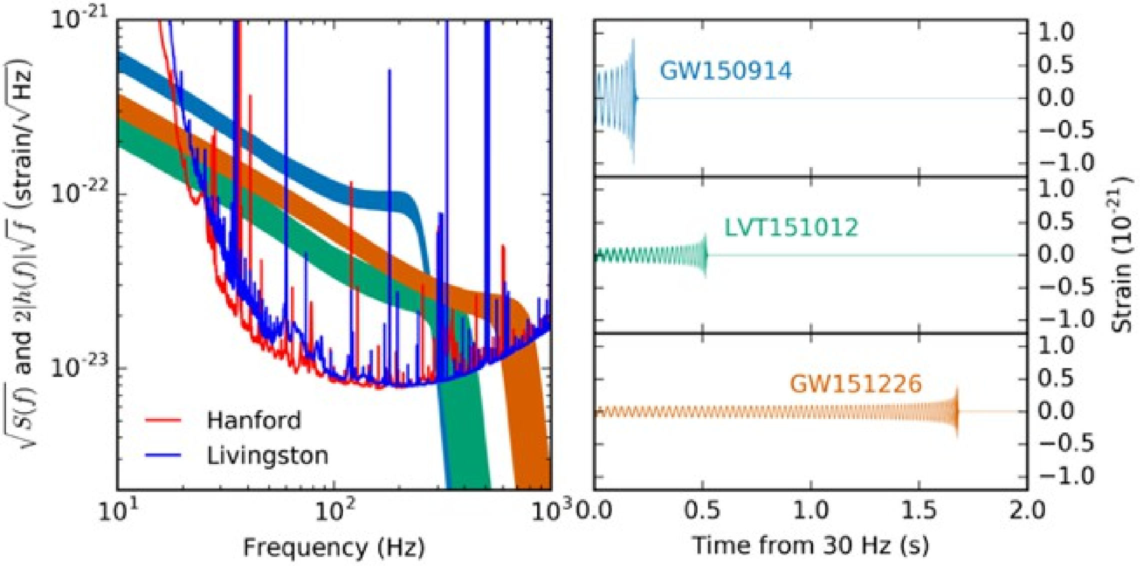
	Measured	Predicted
Geodetic precession (mas)	6602 ± 18	6606
Frame-dragging (mas)	37.2 ± 7.2	39.2

How about hTT?

Gravitational waves!



Indirect detection: GWs carry energy away from binary, which shrinks (ie period decreases)



Direct detection by LIGO (2015)

The generation of GWs

$$\Box h_{ij}^{\mathrm{TT}} = -16\pi\sigma_{ij}$$

$$G(t,\boldsymbol{x}) = -\frac{1}{4\pi|\boldsymbol{x}|}\delta(t-|\boldsymbol{x}|), \qquad \Box G(t,\boldsymbol{x}) = \delta(t)\delta^{(3)}(\boldsymbol{x}),$$

$$h_{ij}^{\mathrm{TT}}(t,x^i) = 4\int \frac{\sigma_{ij}(t-|\boldsymbol{x}-\boldsymbol{x}'|,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x'$$

$$\sigma_{ij} = P_i^k P_j^l T_{kl} - P_{ij}P^{kl}T_{kl}/2$$

$$P_{ij} = \delta_{ij} - \nabla^{-2}\partial_i\partial_j$$

The generation of GWs

$$h_{ij}^{\mathrm{TT}} = -16\pi\Box^{-1}\sigma_{ij} = -16\pi\Box^{-1}\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)T_{kl}$$

$$= -16\pi\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)\Box^{-1}T_{kl}$$

$$= 4\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)\int \frac{T_{ij}\left(t - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}'\right)}{|\boldsymbol{x} - \boldsymbol{x}'|}\mathrm{d}^3x'$$

$$\approx \frac{4}{r}\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)\int T_{ij}(t - r, \boldsymbol{x}')\mathrm{d}^3x'$$

The generation of GWs

From stress-energy tensor conservation:

$$\partial_t^2 \left(T^{tt} x^i x^j\right) = \partial_k \partial_l \left(T^{kl} x^i x^j\right) - 2\partial_k \left(T^{ik} x^j + T^{kj} x^i\right) + 2T^{ij}$$

$$\frac{4}{r} \int d^3x' \, T_{ij} = \frac{4}{r} \int d^3x' \left[\frac{1}{2}\partial_t^2 \left(T^{tt} x'^i x'^j\right) + \partial_k \left(T^{ik} x'^j + T^{kj} x'^i\right) \; -\frac{1}{2}\partial_k \partial_l \left(T^{kl} x'^i x'^j\right)\right]$$

$$= \frac{2}{r} \int d^3x' \, \partial_t^2 \left(T^{tt} x'^i x'^j\right) \; = \frac{2}{r} \frac{\partial^2}{\partial t^2} \int d^3x' \, \rho \, x'^i x'^j \; = \frac{2}{r} \frac{d^2 I_{ij}(t-r)}{dt^2}$$

$$I_{ij}(t) = \int d^3x' \, \rho(t, \mathbf{x}') \, x'^i x'^j$$

The quadrupole formula, finally!

$$h_{ij}^{\mathrm{TT}} \approx \frac{4}{r}\left(P_i^k P_j^l - \frac{1}{2}P_{ij}P^{kl}\right)\int T_{ij}(t-r,\boldsymbol{x}')\mathrm{d}^3x'$$

$$= \frac{2}{r}\frac{d^2\mathcal{I}_{kl}(t-r)}{dt^2}\left[P_{ik}(\mathbf{n})P_{jl}(\mathbf{n}) - \frac{1}{2}P_{kl}(\mathbf{n})P_{ij}(\mathbf{n})\right]\frac{G}{c^4}$$

$$P_{ij} = \delta_{ij} - n_i n_j \qquad \mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij} I,$$

Quadrupole tensor

small number!

$$I_{ij}(t) = \int d^3x'\,\rho(t,\mathbf{x}')\,x'^i x'^j \qquad I = I_{ii}\ .$$

Not a rigorous procedure

- We have still started from linerized theory over Minkowski
- This implies that stress energy tensor is conserved wrt to Minkowski metric ...
- … and is used to go from ”Green formula” to ”quadrupole formula”
- This is inconsistent as binary system in GW-dominated regimes does NOT move on Minkowski geodesics (i.e. straight lines)
- Exercise: compute GWs from Green formula for a system of two unequal masses on Keplerian orbits one around the other and verify that the GW amplitudes differ by a factor 2 (assume propagation along z axis)
- Which one is correct? Quadrupole or Green?
- One would expect Green, but actually the quadrupole formula is the correct one