Neglected modular bootstrap equations and 3d Quantum Gravity

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Challenges in Computational Theoretical Physics

Quantizing 3*d* pure gravity?

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- ✓ $\Lambda < 0$ can be formulated in an asymptotically anti de Sitter spacetime ➪ *AdS/CFT* correspondence

- ❋ According to holographic duality, solving pure quantum gravity means finding the 2*d* CFT on the boundary of the asymptotically AdS spacetime in the large *c* (=central charge) limit, with

$$c = \frac{3\ell}{2G}, \qquad (\Lambda = -\frac{1}{\ell^2})$$

- ❋ Euclidean AdS has a toroidal boundary, ➪ *modular invariance*
- ✗ The degrees of freedom of pure gravity correspond, on the boundary, to multitraces of the stress-tensor belonging to the Virasoro module of the identity ➪ not modular invariant
- ➪ Modular invariance of the boundary theory requires additional degrees of freedom on the bulk
- ❋ What is the meaning of modular invariance on the gravity side?

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- There is a one-to-one correspondence between the BTZ black holes and the primary states of the CFT on the boundary
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- Proving that a primary with $\Delta < \Delta_{BTZ}$ is necessary for a consistent CFT would be sufficient to argue that pure quantum gravity does not exist

❋ The modular invariance of the partition function of a CFT defined on a torus constrains the spectrum. It reads

$$Z(\tau,\bar\tau) = Z\left(\frac{a\tau+b}{c\tau+d},\frac{a\bar\tau+b}{c\bar\tau+d}\right),\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \equiv PSL(2,\mathbb{Z}),\ \tau\in H_+$$

it becomes particularly constraining on the special points of H_+ invariant under the action of some non-trivial subgroup of G

❋ G admits three of them: the cusp at $\tau = i\,\infty$ is stabilized by $T:\ \tau\to\tau+1$, the $\mathbb{Z}_2$ elliptic point at $\tau = i$ stabilized by $S:\ \tau\to -1/\tau$, the elliptic point at $\tau = e^{2i\pi/3}$ stabilized by $ST:\ \tau\to\frac{-1}{\tau+1}$, the generator of a $\mathbb{Z}_3$ subgroup of G.

➪ Consistency conditions at $\tau = i\,\infty$ imply integer spins $j = h-\bar h$ and a constraint on the the twist gap $\Delta - j$ at large j

❋ smooth derivatives at $\tau = i$ demand an infinite set of eq.s, dubbed modular bootstrap (Hellerman 2009):

$$(\tau\partial_\tau)^m\,(\bar\tau\partial_{\bar\tau})^n\,Z(\tau,\bar\tau)|_{\tau=-\bar\tau=i} = 0 \text{ for } m+n \text{ odd}$$

- ➪ Helllerman analytic upper bound for the scaling dimensions Δ of the most relevant primary (very far from the threshold Δ_{BTZ})

$$\Delta_0 \leq \frac{c}{6} + 0.4737 \qquad (c > 1)$$

- ❋ In spite of many analytical as well as numerical efforts on these modular bootstrap equations, small improvements have been found. Best results to date:
- ❋ analytic bound $\Delta_0 \leq c/8.503 \quad (c \to \infty)$ (T. Hartman, D. Mazàc and L. Rastelli, 2019)
- ❋ numerical bound extrapolating large-c data $\Delta_0 \leq c/9.08 \quad (c \to \infty)$ (N. Afkhami-Jeddi, T. Hartman and A. Tajdini, 2019)

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⇨ $$\partial_\tau^n Z|_{\tau=\rho} - \sum_{m=1}^n \rho^{m+n} \frac{n!}{m!} \binom{n-1}{m-1} \partial_\tau^m Z|_{\tau=\rho} = 0$$

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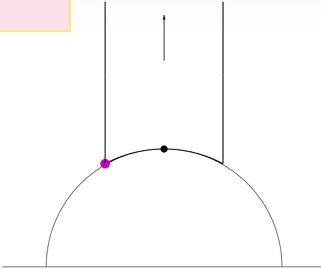
- Set $\tau = -\frac{1}{2} + i\,\frac{\beta}{2\pi}$, $\bar\tau = -\frac{1}{2} - i\,\frac{\bar\beta}{2\pi}$

$Z = \sum_{h,\bar h} q^{h-\frac{c}{24}} \bar q^{\bar h - \frac{c}{24}}$, $(\Delta = h + \bar h,\ j = h - \bar h)$

$q \equiv e^{2i\pi\tau} = -e^{-\beta},\ \ \bar q \equiv e^{-2i\pi\bar\tau} = -e^{-\bar\beta}$



⇨ Z real, even spin contributions positive, odd spin contributions negative

New modular bootstrap equations:

$$\partial_\beta Z|_{\beta=\bar\beta=\beta_c} = 0\,, \quad \left(\partial_\beta^4 + \frac{2\sqrt{3}}{\pi}\partial_\beta^3\right) Z|_{\beta=\bar\beta=\beta_c} \;=\; 0\,,$$

$$\partial_\beta^2 Z|_{\beta=\bar\beta=\beta_c} = 0\,, \quad \left(\partial_\beta^5 - \frac{10}{\pi^2}\partial_\beta^3\right) Z|_{\beta=\bar\beta=\beta_c} \;=\; 0\,,$$

$$\left(\partial_\beta^7 + \frac{525}{\pi^4}\partial_\beta^3 + \frac{7\sqrt{3}}{\pi}\partial_\beta^6\right) Z|_{\beta=\bar\beta=\beta_c} \;=\; 0\,,$$

$$\left(\partial_\beta^8 - \frac{1470\sqrt{3}}{\pi^5}\partial_\beta^3 - \frac{98}{\pi^2}\partial_\beta^6\right) Z|_{\beta=\bar\beta=\beta_c} \;=\; 0\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \quad ,$$

$$\beta_c = \sqrt{3}\,\pi$$

Z can be expanded in Virasoro characters. If $c > 1$ and the theory is unitary, the modules of the Virasoro algebra are the identity degenerate module $\chi_0(q)$ and a family of non-degenerate modules $\chi_A(q)$

$$\chi_0(q) = \frac{q^{-\frac{c-1}{24}}}{\eta(\tau)}(1-q)\,, \quad \chi_A(q) = \frac{q^{h_A - \frac{c-1}{24}}}{\eta(\tau)} \quad (q = -e^{-\beta}\,, \bar{q} = -e^{-\bar{\beta}})$$

➪ $Z = \chi_0(q)\chi_0(\bar{q}) + \sum_A N_A\,\chi_A(q)\,\chi_A(\bar{q}) \quad (N_A \in \mathbb{N})$

❄ $\sqrt{\tau - \bar{\tau}}\,\eta(\tau)\eta(\bar{\tau}) = Z_b^{-1}$ is modular invariant

➪ $Z/Z_b = Z_{vac} + \sum_A N_A Z_A$ modular invariant

$$\begin{aligned} Z_{vac} &= \sqrt{\beta + \bar{\beta}}\; e^{\beta\frac{c-1}{24}} e^{\bar{\beta}\frac{c-1}{24}}\,(1+e^{\beta})(1+e^{\bar{\beta}})\,, \\ Z_A &= \sqrt{\beta + \bar{\beta}}\; e^{\beta\frac{c-1}{24}} e^{\bar{\beta}\frac{c-1}{24}}\; e^{-\beta h_A} e^{-\bar{\beta}\bar{h}_A} \end{aligned}$$

❄ Applying the new bootstrap equations ➪ $\Delta_A = h_A + \bar{h}_A$ always appear in the combination $\Delta_A - \Delta_+$ with

$$\Delta_+ = \frac{c-1}{12} + \frac{1}{2\sqrt{3}\,\pi}$$

resulting sum rules:

$$\sum_A (-1)^{j_A} N_A \, e^{-\beta_c \Delta_A} (\Delta_A - \Delta_+) = v^2 \Delta_+ - uv \,,$$

$$\sum_A (-1)^{j_A} N_A \, e^{-\beta_c \Delta_A} \, j_A^2 = -u \,,$$

$$\sum_A (-1)^{j_A} N_A \, e^{-\beta_c \Delta_A} \left((\Delta_A - \Delta_+)^2 - \frac{1}{6\pi^2} \right) = -v^2 \Delta_+^2 + 2uv\Delta_+ - u(1+u) + \tfrac{v^2}{6\pi^2} \,,$$

$u = 2\, e^{-\beta_c}$, $v = 1 + e^{-\beta_c}$, $\beta_c = \sqrt{3}\, \pi$

large-c limit

$$\sum_A (-1)^{j_A} N_A \, e^{-\beta_c \Delta_A} \left(\frac{\Delta_A - \Delta_+}{\Delta_+} \right)^n = v^2 (-1)^{n+1} + O(\tfrac{1}{c})$$

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- *Theorem*: There is at least a primary of odd spin with

$$\Delta < \Delta_+ \equiv \frac{c-1}{12} + \frac{1}{2\sqrt{3}\,\pi}$$

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- This theorem can be proved by *reductio ad absurdum*, i.e. negating the theorem implies no solution of the above equations (see FG arXiv:1908.00029 for a complete proof)

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Hint: Assume that the sum of the first two terms has the same sign of the total sum, i.e. (a: even spin term, b: odd spin term, $k > 0$ multiplicity ratio)

1. $a > k\, b$
2. $a^2 < k\, b^2$
3. $a^3 > k\, b^3$

if $b > 0$ ⇨

❶×❸= $a^4 > k^2\, b^4$

while

❷×❷= $a^4 < k^2\, b^4$

Conclusions

1. There is an infinite set of modular bootstrap equations so far neglected
2. They generate an upper bound Δ_0 for the first non-trivial primary which is remarkably close to the BTZ black hole threshold
3. Even a modest improvement of the upper bound could push Δ_0 down Δ_{BTZ} implying that pure Einstein gravity in AdS_3 do not exist as a quantum theory
4. Further study of the new set of equations would give important information on the spectrum of primaries in a general $2d$ CFT