Dual formulations of gauge models with static quarks at finite baryon density

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I. Duals of lattice models and sign problem

- Dual representations based on the plaquette formulation. Dual variables are introduced as variables conjugate to local Bianchi identities. The dual model is non-local due to the presence of connectors

- Dual representations based on 1) the character expansion of the Boltzmann weight and 2) the integration over link variables using Clebsch-Gordan expansion

$$Z = \sum_{r_p, r_l} \prod_p C_{r_p}(\beta_{\mu\nu}) \prod_x (6j \text{ links}) \prod_{\text{C}} (\text{6j cubes})$$

- Recent approaches: n-link action, abelian colour cycles, Weingarten functions, $\cdots$

How useful is dual approach

Model	Dual positive form	Algorithm
$Z(N)$ and XY spin models, $\mu \neq 0$	Yes	Yes
$O(N)$ linear and non-linear sigma models, $\mu \neq 0$	Yes	Yes
$O(3)$ non-linear sigma model, $\theta \neq 0$	No	No
Principal chiral models, $\mu \neq 0$	Yes	Yes
* Polyakov loop spin models, $\mu \neq 0$	Yes	Yes
Pure abelian LGT	Yes	Yes
Pure non-abelian LGT	No	No
Pure LGT with θ-term, $2d$	Yes	Yes
Pure LGT with θ-term, $4d$	No	No
* Abelian LGT, $\mu \neq 0$, static quarks	Yes	Yes
Abelian LGT, $\mu \neq 0$, $2d$	Yes, if $m = 0$	Yes
Abelian LGT, $\mu \neq 0$, $d > 2$	No	No
Lattice QCD, $\mu \neq 0, \beta = 0$	Yes (partially)	Yes
Full lattice QCD, $\mu \neq 0$	No	No

II. Polyakov loop models and their duals

Let $U_l \equiv U_\mu(x) \in Z(N), U(N), SU(N)$. LGT in $(d+1)$-dimensions on anisotropic periodic lattice $\Lambda = N_t \times L^d$ with N_f quark flavours

$$Z_\Lambda = \int \prod_l dU_l \ \exp \sum_{x;\mu<\nu} \beta_{\mu\nu} \, \mathrm{Re}\, \chi_\mathrm{f}(\mathrm{U_p}) \ \prod_{f=1}^{N_f} \ \mathrm{Det}\, \mathrm{Q}(\mathrm{m_f}, \mu_\mathrm{f})$$

$$U_p \ = \ U_\mu(x) U_\nu(x+\mu) U_\mu^\dagger(x+\nu) U_\nu^\dagger(x) \ .$$

$\chi_f(U)$ is the character of the fundamental representation. Staggered or Wilson fermion matrix (colour indices omitted and $\xi = a_t/a_s$)

$$Q(m_f, \mu_f) = V_{tt'}(x)\delta_{xx'} + \xi \sum_{n=-d}^{d} \Gamma_n \, U_n(x,t) \delta_{x+n,x'} \delta_{tt'} \ ,$$

$$V_{tt'}(x) = 2a_t m_f \delta_{tt'} + e^{a_t \mu_f} U_0(x,t) \delta_{t,t'-1} - e^{-a_t \mu_f} U_0^\dagger(x,t') \delta_{t,t'+1}$$

Strategy

- Integration over fermion fields.

- Integration over spatial gauge fields (usually requires some approximation for fermion determinant and/or Wilson gauge action). Resulting theory is an effective d-dimensional spin model of interacting Polyakov loops in the external complex (if $\mu \neq 0$) field.

- Construction of a dual representation for the effective spin model.

- If the dual Boltzmann weight is positive one uses the conventional MC to study the model.

Finite-temperature effective models

1. $\beta_s = 0$, $\beta_t < 1$, large quark masses

$$S = \beta \sum_{x,n} \mathrm{ReTr} W(x) \mathrm{Tr} W^{\dagger}(x + e_n) + \sum_x \left(h_r \mathrm{Tr} W(x) + h_i \mathrm{Tr} W^{\dagger}(x) \right)$$

$\beta = \beta(\beta_t)$; $h_r = h_r(m_q, \mu)$ and $h_i = h_i(m_q, \mu)$ are functions of quark mass m_f and baryon chemical potential μ_f

2. $\beta_s = 0$, arbitrary β_t, static quarks

$$Z = \int \prod_x dW(x) \prod_{x,n} \left[\sum_\lambda D_\lambda^{N_t}(\beta_t) \chi_\lambda(W(x)) \chi_\lambda(W^{\dagger}(x + e_n)) \right] \times \prod_{f=1}^{N_f} \prod_x \mathrm{Det}\, \mathrm{V}_{\mathrm{tt}'}(\mathrm{x}).$$

$D_\lambda(\beta)$ - coefficients of the character expansion.

3. Non-local Polyakov loop model by J. Greensite, R. Hollwieser

$$S = \beta \sum_{x,y} \mathrm{Re}\, \mathrm{Tr} W(x)\, K(x - y)\, \mathrm{Tr} W^{\dagger}(y) + S_q.$$

Dual formulation of the model 1

$$Z \ = \ \int \prod_x dW(x) \ \exp\left[S[\{W\}]\right] \ .$$

(for $SU(3)$ see: C. Gattringer, Nucl.Phys. B850 2011)

- Taylor expansion of the Boltzmann weight
- Group integration: $Q_N(s,\bar{s}) =$

  $$\int dW \ (\mathrm{Tr}W)^s \ (\mathrm{Tr}W^*)^{\bar{s}} = \begin{cases} \delta_{s,\bar{s}} \sum_\lambda d^2(\lambda) \ , \ U(N) \\ \sum_k \delta_{\bar{s}-s,kN} \sum_\lambda \ d(\lambda) d(\lambda+k) \ , \ SU(N) \end{cases}$$

  $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0)$ is a partition of s, *i.e.* $\sum_{i=1}^N \lambda_i = s$.
  $d(\lambda)$ - dimension of the permutation group S_s

- Solution of constraints if possible

The dual form of $SU(N)$ spin model (flux representation)

$$Z = \prod_l \left[\sum_{p(l)=-\infty}^{\infty} \sum_{q(l)=0}^{\infty} \left(\frac{\beta}{2}\right)^{|p(l)|+2q(l)} \frac{1}{(q(l)+|p(l)|)!q(l)!} \right]$$

$$\prod_x \left[\sum_{k(x)=-\infty}^{\infty} \sum_{t(x)=0}^{\infty} \frac{(h_r h_i)^{t(x)+\frac{1}{2}|m(x)|}}{t(x)!\,(t(x)+|m(x)|)!} \left(\frac{h_r}{h_i}\right)^{\frac{1}{2}m(x)} Q_N(s(x)) \right] ,$$

$$s(x) = \sum_{i=1}^{2d} \left(q(l_i) + \frac{1}{2}\,|p(l_i)| \right) + t(x) + \frac{1}{2} \sum_{n=1}^{d} (p_n(x) - p_n(x-e_n)) + \frac{1}{2}|m(x)| + \frac{1}{2}m(x) + \eta(x) ,$$

$$m(x) = \sum_{n=1}^{d} (p_n(x-n) - p_n(x)) - k(x)N + \bar{\eta}(x) - \eta(x) .$$

The complex action problem is solved for all $U(N)$ and $SU(N)$ models in any dimension if the product $h_r h_i$ and the ratio h_r/h_i is non-negative.

Dual formulation of the model 2

- Sum over representations

$$\sum_{\lambda} F(\lambda) = \sum_{r=0}^{\infty} \sum_{\lambda \vdash r} F(\lambda) \; , \; \sum_{i=1}^{N} \lambda_i = r$$

- Static quark determinant (staggered fermions)

$$\mathrm{Det}\, \mathrm{V}_{\mathrm{tt}'}(\mathrm{x}) \sim \mathrm{Det}_{\mathrm{C}} \left(1 + \kappa_{+}^{\mathrm{f}} \prod_{\mathrm{t}=1}^{\mathrm{N_t}} \mathrm{U}_0(\mathrm{x},\mathrm{t}) \right) \left(1 + \kappa_{-}^{\mathrm{f}} \prod_{\mathrm{t}=1}^{\mathrm{N_t}} \mathrm{U}_0^{\dagger}(\mathrm{x},\mathrm{t}) \right)$$

$\kappa_{\pm} = \exp(-N_t \sinh^{-1} a_t m \pm \beta\mu)$. By Cauchy identity

$$\prod_{f=1}^{N_f} \mathrm{Det}\, \mathrm{V}_{\mathrm{tt}'}(\mathrm{x}) = \sum_{\mathrm{m},\mathrm{n}} \chi_{\mathrm{m}}(\mathrm{W}(\mathrm{x}))\, \chi_{\mathrm{n}}(\mathrm{W}^*(\mathrm{x}))\, \chi_{\mathrm{m}'}(\kappa_{+}^{\mathrm{f}})\, \chi_{\mathrm{n}'}(\kappa_{-}^{\mathrm{f}})$$

For one flavour it gives

$$\sum_{m,m'=0}^{N} \kappa_{+}^{m} \kappa_{-}^{m'} \chi_{1^m}(W(x))\, \chi_{1^{m'}}(W^*(x))$$

- Group integration

$$\int dW \prod_{n=1}^{d} \chi_{\lambda(x,n)}(W)\chi_{\lambda(x-e_n,n)}(W^*)\ \chi_{1^{m(x)}}(W)\ \chi_{1^{m'(x)}}(W^*)$$

can be done by expanding integrand into the Littlewood-Richardson (LR) coefficients

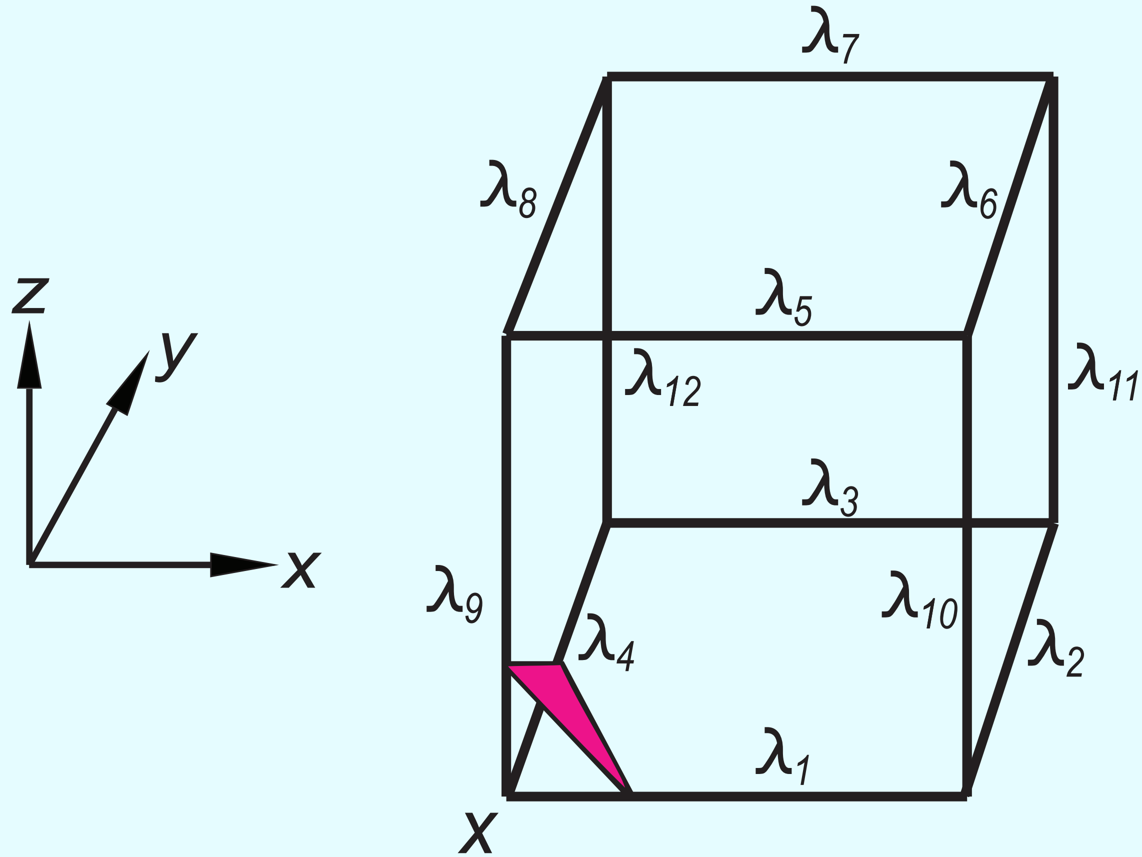
$$\chi_{\lambda_1}\,\chi_{\lambda_2} = \sum_{\lambda} C^{\lambda}_{\lambda_1\,\lambda_2}\,\chi_\lambda$$

LR coefficients are positive integers (multiplicities) given by

$$C^{\lambda}_{\lambda_1\,\lambda_2} = \int dW \chi_{\lambda_1}(W)\ \chi_{\lambda_2}(W)\chi_\lambda(W^*)$$

For $SU(N=2,3)$ exact formulae are known.

- Special coupling of characters as shown below for $d=3$



Multiplication of characters in the group integrals can be done such that all summations over representations are closed inside even cubes, $d = 3$

Partition function with positive weight

$$
\begin{aligned}
Z \; &= \sum_{m(x)=0}^{N} \; \sum_{m'(x)=0}^{N} \; \sum_{r_l=0}^{\infty} \; \sum_{\rho(x)} \; \prod_{x} \; \sum_{k(x)=-\infty}^{\infty} \; \delta(G(x) - k(x)N) \\
&\times \; \prod_{x} \kappa_{+}^{m(x)} \, \kappa_{-}^{m'(x)} \; \prod_{p,even} \; B_p
\end{aligned}
$$

$$
\begin{aligned}
B_p \; &= \sum_{\lambda_1 \vdash r_1} D_{\lambda_1}^{N_t}(\beta) \ldots \sum_{\lambda_4 \vdash r_4} D_{\lambda_4}^{N_t}(\beta) \sum_{\mu_1} \ldots \sum_{\mu_4} C_{\lambda_1 \lambda_4}^{\mu_1} C_{\lambda_1' \lambda_2}^{\mu_2} C_{\lambda_2' \lambda_3'}^{\mu_3} C_{\lambda_3 \lambda_4'}^{\mu_4} \\
&\times \; C_{\mu_1 1^{m(x)}}^{\rho(x)} C_{\mu_2 1^{m'(x+e_1)}}^{\rho(x+e_1)} C_{\mu_3 1^{m'(x+e_1+e_2)}}^{\rho(x+e_1+e_2)} C_{\mu_4 1^{m(x+e_2)}}^{\rho(x+e_2)}
\end{aligned}
$$

$$
G(x) = \sum_{n} (r_n(x) - r_n(x - e_n)) + m(x) - m'(x)
$$

For $SU(N)$ the dependence on chemical potential appears as $e^{\beta\mu N \sum_x k(x)}$. For $U(N)$ the dependence drops out. The dual Boltzmann weight is positive (in any dimension).

III. Abelian models with static quarks

Can we avoid approximations used so far: 1) $\beta_s = 0$ and 2) static quarks? For abelian models we do not need $\beta_s = 0$ and can use the full Wilson or any other local action

$$\sum_{r(p)=-\infty}^{\infty} D_{r(p)} \, e^{ir(p)\phi(p)}$$

The underlying reason for that is we know exact and positive dual form of any abelian $U(1)$ and $Z(N)$ pure gauge theory in any dimension. Adding static fermions does not violate positivity of the dual Boltzmann weight even at finite density. Fermion determinant for abelian models

$$\sum_{m(x)=0}^{1} \sum_{m'(x)=0}^{1} \prod_x \kappa_+^{m(x)} \, \kappa_-^{m'(x)} \, W^{m(x)-m'(x)}(x)$$

Partition function is expressed in terms of fermion numbers $m(x), m'(x)$ and plaquette occupation numbers (their number can be reduced by solving constraints on all plaquettes but p_t in one fixed time slice)

$$Z = \sum_{m(x)=0}^{1} \sum_{m'(x)=0}^{1} \prod_x \kappa_+^{m(x)} \, \kappa_-^{m'(x)} \sum_{r(p)} \prod_{p_s} D_{r(p_s)}(\beta_s) \prod_{p_t} D_{r(p_t)}(\beta_t)$$

$$\prod_{l_s} \delta\left(\sum_{p \in l_s} r(p)\right) \prod_{l_t} \delta\left(\sum_{p \in l_t} r(p) + m(x) - m'(x)\right)$$

- The final form depends on the model, $Z(N)$ or $U(1)$, and on the space dimension (solution of constraints). Dual weight is positive.

- For $Z(N)$ the dependence on chemical potential appears in the form $e^{\beta \mu N \sum_x k(x)}$. For $U(1)$ the dependence drops out (only for one flavour).

- Can be trivially extended to any number of flavours and to Wilson fermions.

IV. Applications of the dual formulation

Large-N solution

$$Z_G(\beta) \;=\; \int \, dU \;\; \mathrm{exp}\left[\frac{\beta}{2}\,\mathrm{Tr}U\right]$$

Large N ’t Hooft limit:

$$Z_{U(N)}(\beta) \;=\; 1 \;, \;\text{for}\;\;\text{all}\; N$$

$$F(\beta) = \lim_{N\to\infty} \frac{1}{N^2} \ln Z_{SU(N)}(\beta) \;=\begin{cases} 0 \;, \;\beta \le \beta_c = 2/e \;, \\ A(\beta - 2/e)^3 \;, \;\beta \gtrapprox 2/e \end{cases}$$

Third order phase transition occurs at $\beta_c = 2/e$. Using dual representation it is strightforward to extend this result to a model of interacting Polyakov loops.

Polyakov loop correlations

Motivation:

- computation of screening masses at finite density
- search for oscillatory behaviour

$$\langle W(x)W^*(y)\rangle \; \sim \; e^{-m_r R} \cos m_i R$$

- Theoretical arguments advanced by M. Ogilvie et.al (2014,2015).
- $Z(3)$ spin model in external complex field studied by Ph. de Forcrand et.al (2016)

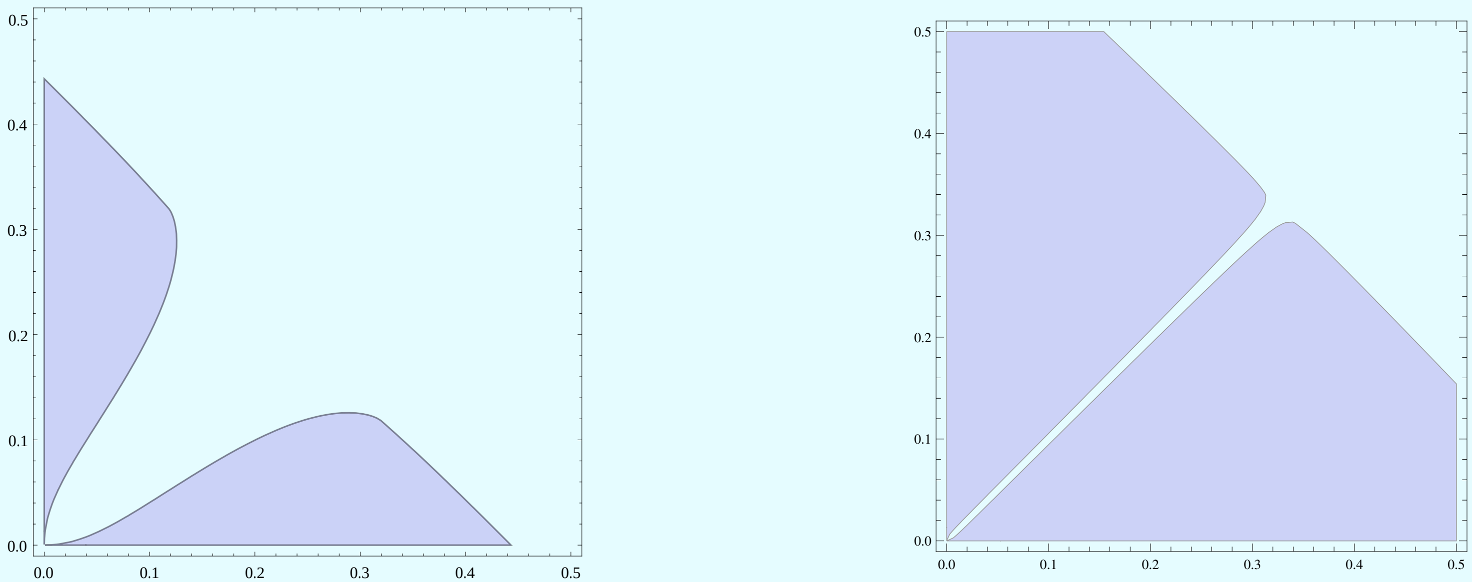
Preliminary results

- Transfer matrix applied to the dual of one-dimensional Polyakov loop model exhibits existence of oscillating phase

- Large-N solution also confirms such behaviour

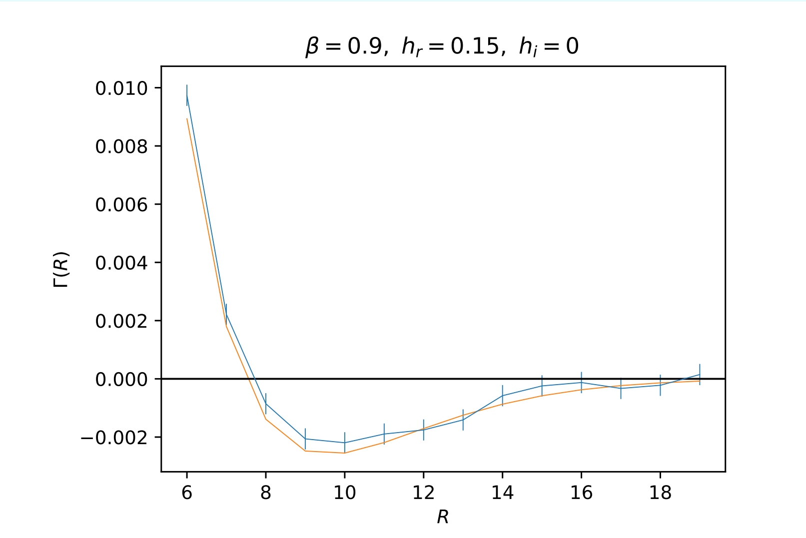
- Monte-Carlo simulations of dual version of $3d$ $SU(3)$ PL model

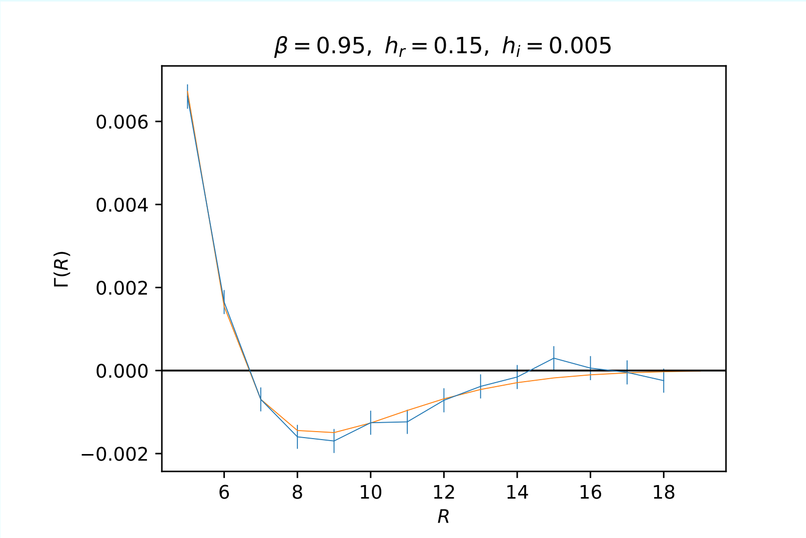
$$S = \beta \sum_{x,n} \mathrm{ReTr} W(x) \mathrm{Tr} W^{\dagger}(x+e_n) + \sum_x \ln \mathrm{Det}\, \mathrm{V}_{tt'}(x; h, \mu)$$

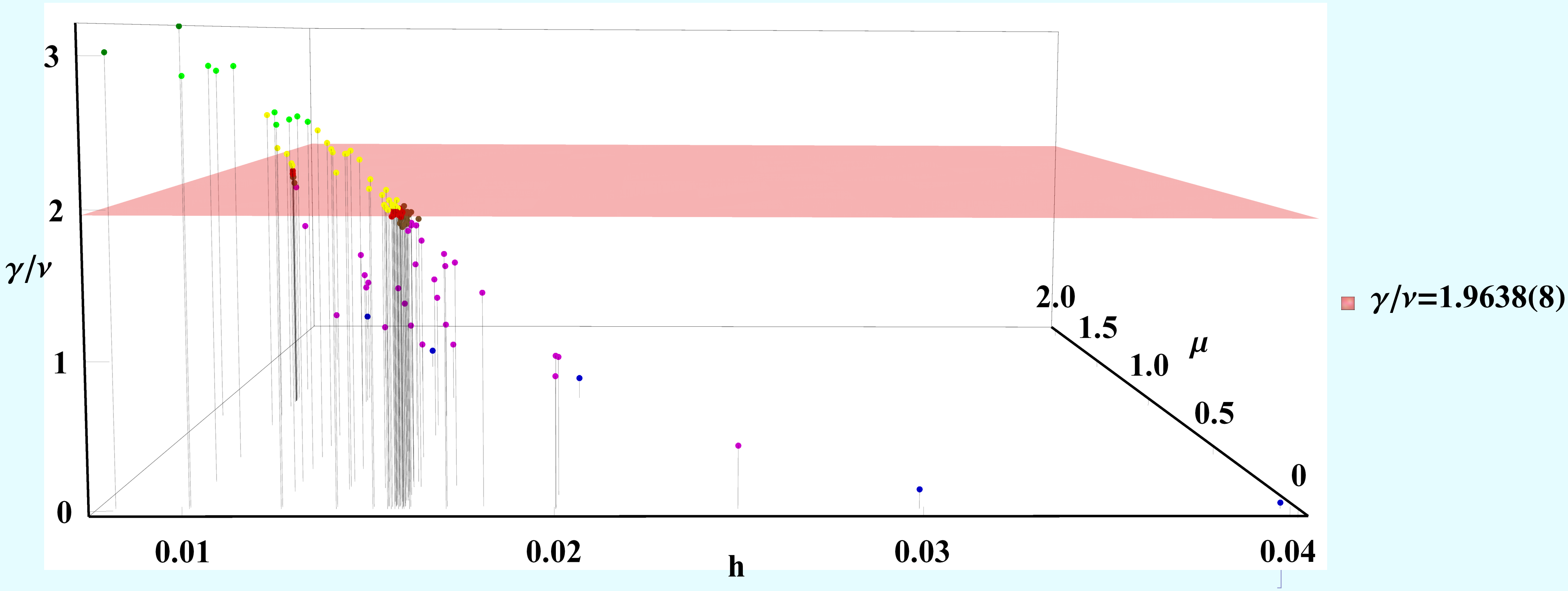


Regions of oscillating phase for $SU(3)$, $\beta = 0.9$ (left), $\beta = 1.2$ (right).
$h_r = h(m)\exp(\mu), h_i = h(m)\exp(-\mu)$.

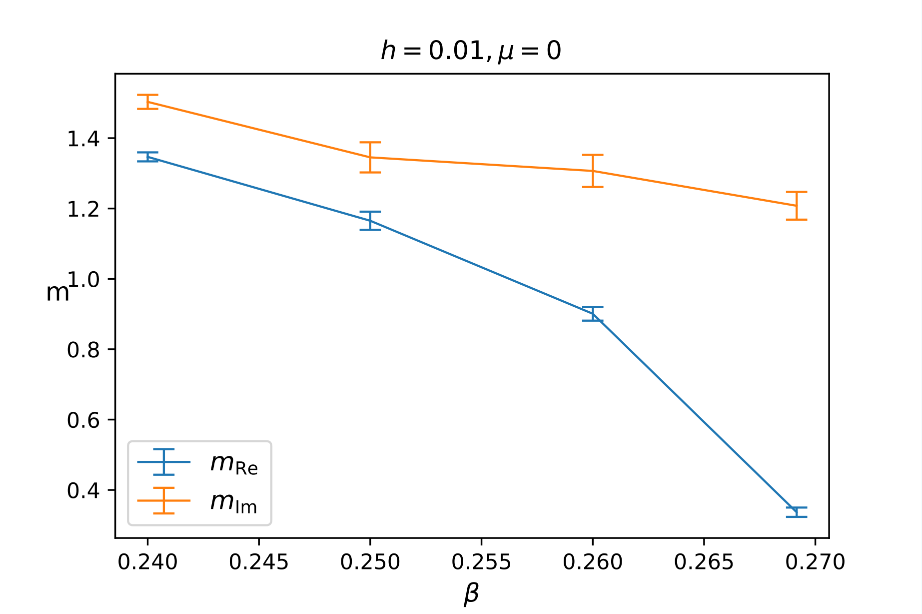
Polyakov loop correlation function in one-dimensional $SU(3)$ model ($2d$ QCD with static quarks).



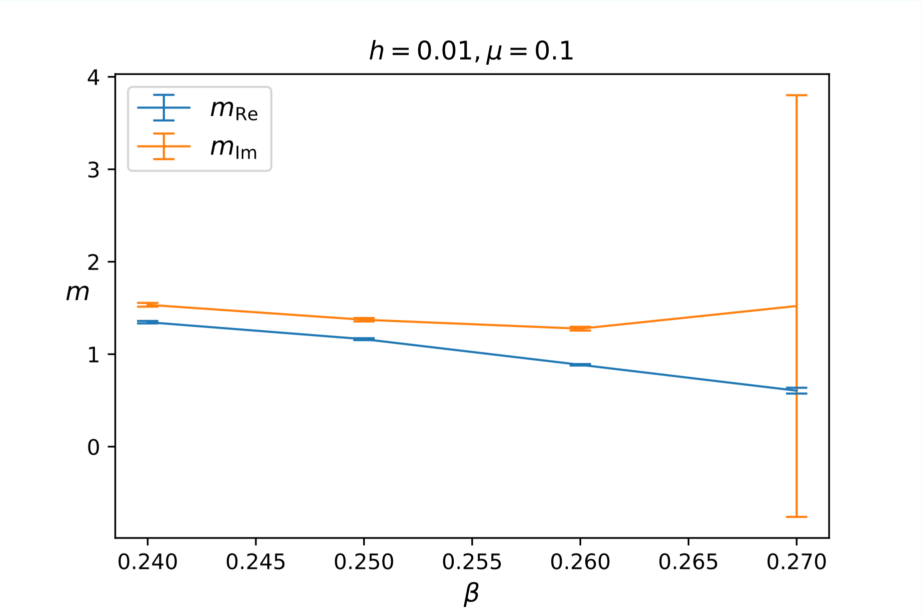




Phase diagram of $3d$ $SU(3)$ Polyakov loop model from simulations on $L = 10, 16, 24, 32$ lattices. Green points correspond to 1st order phase transition, the red ones - 2nd order, magenta and blue points represent crossover. The value $\gamma/\nu = 1.9638$ corresponds to 2nd order phase transition in three-dimensional Ising model.



Typical behaviour of masses. Blue (orange) line describe correlation of real (imaginary) parts of Polyakov loops.



Masses are calculated in the vicinity of the 2nd order phase transition. No oscillatory behaviour was found.

V. Summary

- Dual formulation is constructed for all $U(N)$ and $SU(N)$ Polyakov loop models
- Dual Boltzmann weight is positive in the presence of baryonic chemical potential for all N
- Dual formulation of abelian models with full gauge action and static quarks
- Exact solution is given in the large-N limit
- Numerical simulations: liquid phase at finite-density (oscillatory behaviour of the Polyakov loop correlators)
- Corrections to the static quark contribution: in progress