QUANTUM STATISTICS IN NETWORKS GEOMETRY WITH FRACTIONAL FLAVOR

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SIMPLICIAL COMPLEXES

A simplicial complex is a generalized network structure that allows the description of many-body interactions between a set of nodes.

Simplicial complexes are not only formed by nodes and links like networks, but they are instead also formed by triangles, tetrahedra and so on.

Simplicial complexes are natural structures to model higher order interactions in real data.

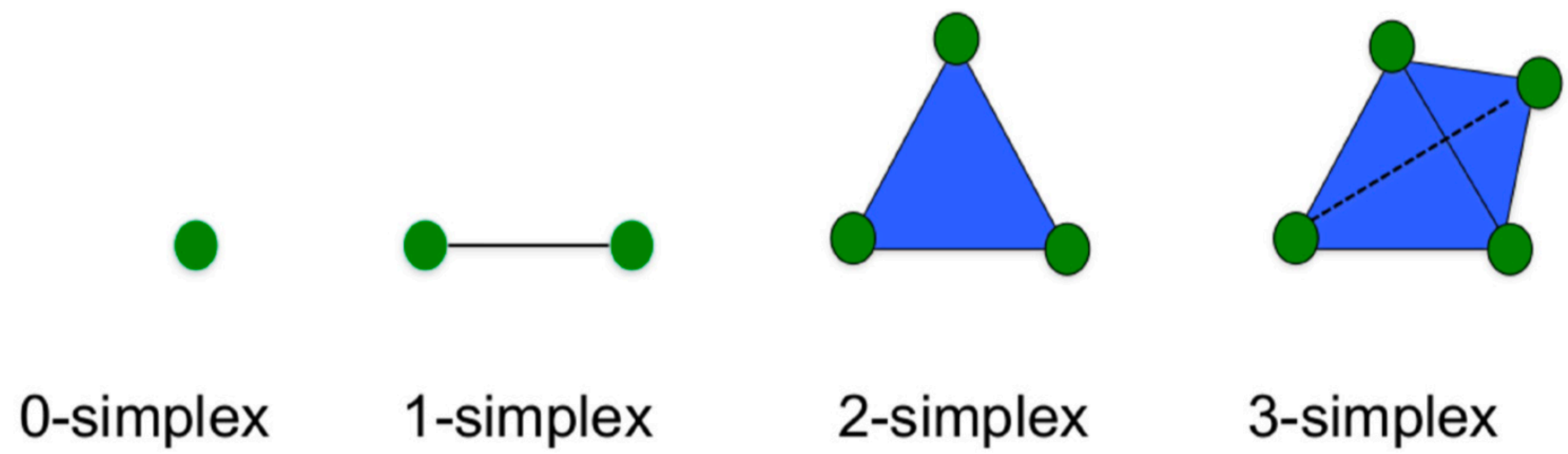


Figure 1. Examples of δ-dimensional simplices with $\delta = 0, 1, 2, 3$.

FACES OF SIMPLICIAL COMPLEXES

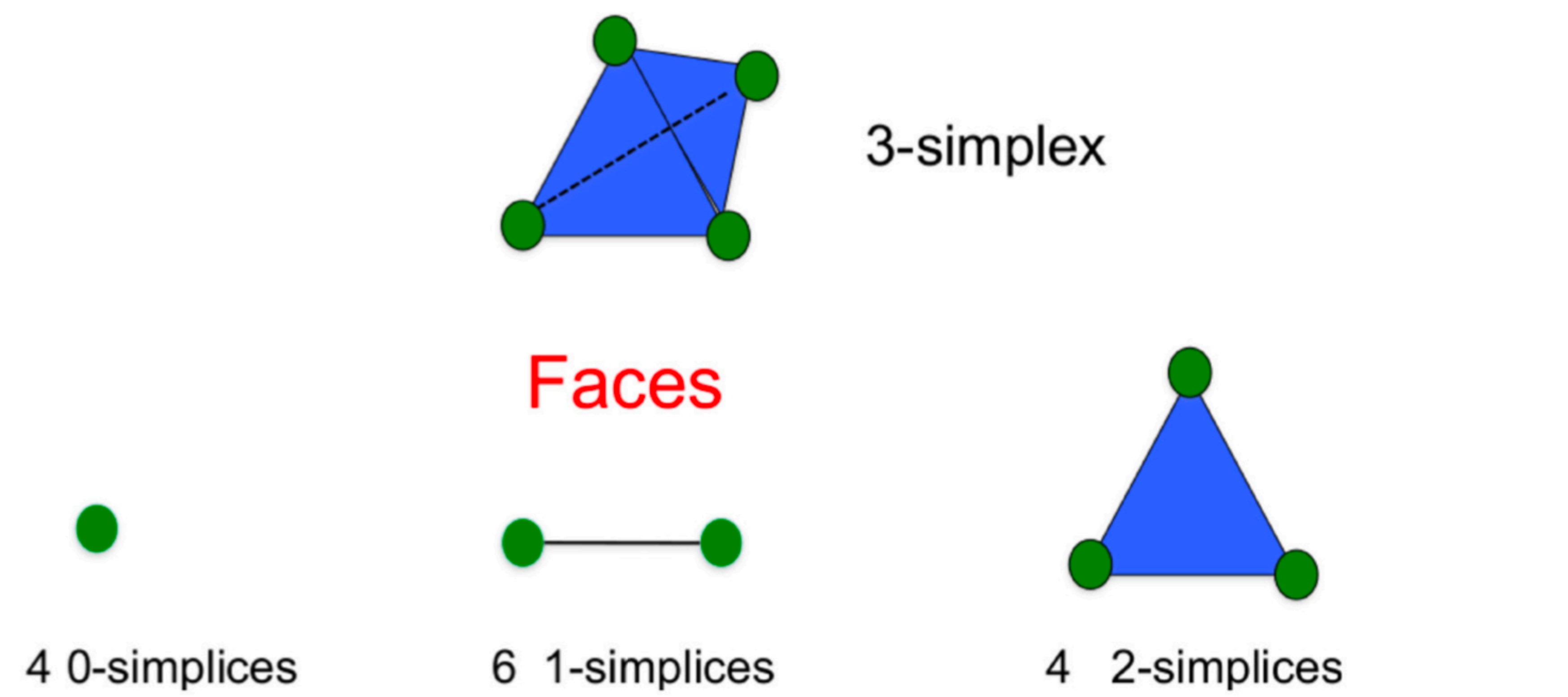


Figure 2. The faces of a $\delta = 3$ dimensional simplex (a tetrahedron) are shown and their relative number is indicated correspondingly.

To each (d−1)-dimensional face α of a *d*-dimensional simplicial complex one can associate an ***incidence number*** $\boldsymbol{n_\alpha}$ given by the number of incident *d*-dimensional simplices minus one, i.e.

$$n_\alpha = k_{d,d-1}(\alpha) - 1.$$

ENERGY AND FITNESS OF THE SIMPLICES

To each simplex $\alpha \in K$ one can associate an energy ε_α that does not change in time.

The energy of a face describes its intrinsic and heterogeneous properties and has an important effect on the simplicial complex evolution.

The energy ε_i of node *i* is drawn randomly from a given distribution $g(\varepsilon)$. To every face α one can associate an *energy* ε_α given by the sum of the energy of the nodes that belong to the face α,

$$\epsilon_\alpha = \sum_{i \in \alpha} \epsilon_i .$$

Thus the energy of a link is given by the sum of energies of the two nodes that belong to it, the energy of a triangular face is given by the sum of the energy of the three nodes belonging to it and so on.

ENERGY AND FITNESS OF THE SIMPLICES

For each simplex $\alpha \in K$ one can associate a ***fitness*** $\boldsymbol{\eta_\alpha}$ given by

$$\eta_\alpha = \mathrm{e}^{-\beta\epsilon_\alpha}$$

where β is an external parameter of the model called *inverse temperature*.

If $\boldsymbol{\beta = 0}$ we have that $\eta_\alpha = 1$ for every simplex $\alpha \in K$, therefore every simplex has the same fitness independently of their differences in energy.

On the contrary, when **β is large**, small differences in energy lead to large differences in the fitness of different simplices.

NGF MODEL

The **Network Geometry with Flavor** (NGF) is a non-equilibrium model of growing simplicial complexes with fitness that has been proposed to study emergent network geometry in Bianconi and Rahmede, Phys Rev. E **93**, 032315 (2016)

The ***flavor s*** of the NGF is a parameter that can change the topological nature of the simplicial complex and its evolution.

For ***s* = −1** the NGF is a manifold;

for ***s* = 0** the network grows by uniform attachment of *d*-dimensional simplices on (d−1)-dimensional faces;

for ***s* = 1** the network evolves according to a generalized preferential attachment rule of *d*-dimensional simplices on (d−1)-dimensional faces.

EVOLUTION OF THE NETWORK GEOMETRY WITH FLAVOR

The stochastic evolution of NGF is determined by the ***flavor s*** and by the **fitness** of the simplices of the simplicial complex.

The evolution of the NGF obeys a **simple iterative algorithm:**

At time $\boldsymbol{t=1}$ the simplicial complex is formed by a single d-dimensional simplex.

At time $\boldsymbol{t>1}$ one glues a d-dimensional simplex to a (d−1)-face α chosen with probability

$$\Pi_{d,d-1}(\alpha) = \frac{\eta_\alpha(1+sn_\alpha)}{Z^{[s]}}$$

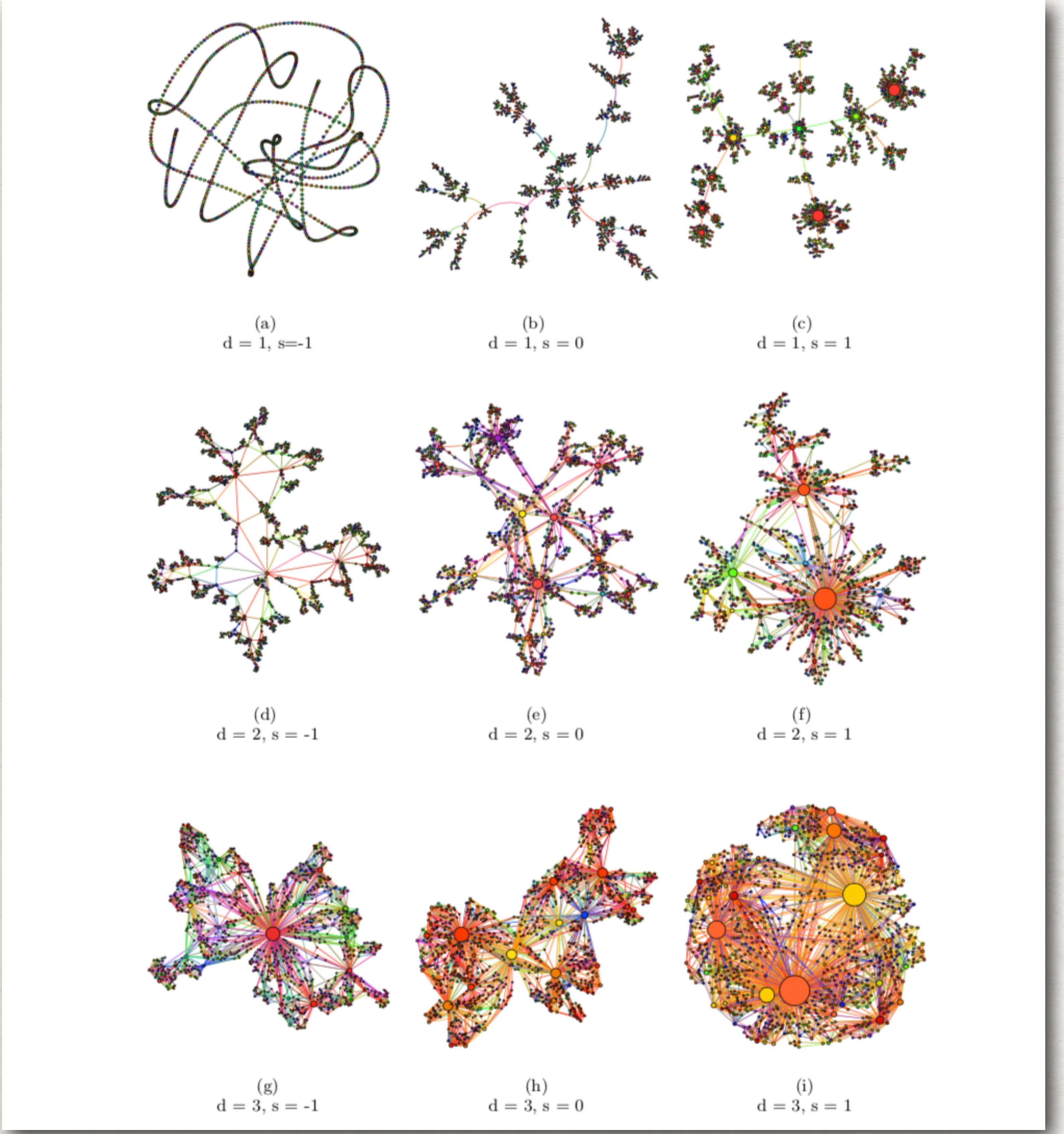
and

$$\eta_\alpha = e^{-\beta\epsilon_\alpha}$$

where $Z^{[s]}$ is called the ***partition function*** of the NGF and is given by

$$Z^{[s]} = \sum_{\alpha' \in S_{d,d-1}} \eta_{\alpha'}(1+sn_{\alpha'}).$$

SIMPLICIAL COMPLEXES WITH THE NGF MODEL



Examples of simplicial complexes obtained with the NGF model with flavor $s = -1, 0, 1$ and dimension $d = 1,2,3$.

The NGFs have $N = 10^3$ nodes, $\beta = 0.1$ and a uniform distribution of the energy of the nodes $g(\varepsilon) = 1/10$ for $0 < \varepsilon < 10$.

The color of the nodes indicates their energy, in order of increasing energy we have red, orange, yellow, green, blue, violet;

the size of the nodes is proportional to their degree.

POSSIBLE VALUES OF THE FLAVOR AND THEIR TOPOLOGICAL IMPLICATIONS

The NGF describes a growing simplicial complex that depends on the value of the flavor s.

The attachment probability **$\Pi_{d,d-1}(\alpha)$** for **$\beta = 0$** and the integer flavors **$s \in \{-1, 0, 1\}$** is

$$\Pi_{d,d-1}(\alpha) \propto (1 + sn_\alpha) = \begin{cases} 1 - n_\alpha & \text{for} \quad s = -1, \\ 1 & \text{for} \quad s = 0, \\ k_{d,d-1}(\alpha) & \text{for} \quad s = 1. \end{cases}$$

The NGF with integer flavor reduces to several known models for different values of the parameters s,d and β.

For **$d = 1, s = 1, \beta = 0$** the NGF reduces to the Barabási– Albert model

for **$d = 1, s = 1, \beta > 0$** the NGF reduces to the Bianconi–Barabási model

for **$d = 2, s = 0, \beta = 0$** the NGF reduces to the model proposed in Dorogovtsev et al 2001 Phys. Rev. E **63** 062101

for **$d = 3, s = -1, \beta = 0$** the NGF reduces to a random Apollonian network proposed by Andrade et al 2005 Phys. Rev. Lett. **94** 018702

NGF MODEL WITH INTEGER FLAVOUR

Short summary of the results for the NGF model with integer flavor

Table 1. Distribution of generalized degrees of faces of dimension δ in a d-dimensional NGF of flavor s at $\beta = 0$. For $d \geqslant 2\delta + 2 - s$ the power-law distributions are scale-free, i.e. the second moment of the distribution diverges.

Flavor	$s = -1$	$s = 0$	$s = 1$
$\delta = d - 1$	Bimodal	Exponential	Power-law
$\delta = d - 2$	Exponential	Power-law	Power-law
$\delta \leqslant d - 3$	Power-law	Power-law	Power-law

for $\beta > 0$

Table 2. The average $\langle k_{d,\delta} - 1 | \epsilon \rangle$ of the generalized degrees $k_{d,\delta} - 1$ of δ-faces with energy ϵ in a d-dimensional NGF of flavor s follows either the Fermi–Dirac, the Boltzmann or the Bose–Einstein statistics depending on the values of the dimensions d and δ.

Flavor	$s = -1$	$s = 0$	$s = 1$
$\delta = d - 1$	Fermi–Dirac	Boltzmann	Bose–Einstein
$\delta = d - 2$	Boltzmann	Bose–Einstein	Bose–Einstein
$\delta \leqslant d - 3$	Bose–Einstein	Bose–Einstein	Bose–Einstein

Bianconi and Rahmede Phys Rev. E **93**, 032315 (2016)

NGF MODEL WITH FRACTIONAL FLAVOR

For negative values of the flavor s the requirement of observing a well defined attachment probability $\Pi_{d,d-1}(\alpha) \in [0, 1]$ implies some restriction on the possible values of s.

In particular if $s < 0$, s should be of the form

$$s = -\frac{1}{m},$$

with $m \in N$.

For such values of the flavor s the incidence number of any $(d-1)$-dimensional face α can only take $m + 1$ values, i.e. $n_\alpha \in \{0,1,2,...,m\}$.

Therefore as long as $\boldsymbol{m > 1}$, NGF with $\boldsymbol{s = -1/m}$ are not anymore manifolds, but they have a generalized degree of the $(d-1)$-dimensional faces bounded by $m + 1$, i.e. $k_{d,d-1}(\alpha) \in \{1, 2, \ldots, m + 1\}$.

This case, that we will call **NGF with fractional flavor**, is therefore expected to display statistical properties that are not equivalent to the ones observed for any of the integer flavors $s \in \{-1, 0, 1\}$.

NGF MODEL WITH FRACTIONAL FLAVOR

Results related to Network Geometry with Fractional Flavor with $s = -1/m$ and $m > 1$.

In principle, since for these networks the incidence number of the (d−1)-faces is allowed to take only values $n_\alpha \in \{0,1,2...,m\}$ we might expect to find that (d−1)-faces are described by fractional statistics.

Contrary to this naive expectation, we show that also in this case (d−1)-dimensional faces are described by the Fermi–Dirac and Bose-Einstein statistics.

The main difference with the NGF with integer flavor $s = -1$ is that we do not find any face described by the Boltzmann statistics.

This result sheds light on the effect that dimensionality and flavor have on the emergence of quantum statistics in NGFs.

RESULTS FOR THE CASE $\beta = 0$

Table 3. Distribution of generalized degrees of faces of dimension δ in a d-dimensional NGF of flavor s at $\beta = 0$. Only for $d - 2\delta \geqslant 2 + \frac{3}{m}$ the power-law distributions are scale-free, i.e. the second moment of the distribution diverges.

Flavor	$s = -1/m$
$\delta = d - 1$	Bounded $k \leqslant m + 1$
$\delta \leqslant d - 2$	Power-law

MASTER EQUATION RESULTS

For the case $\delta = d - 1$ the recursive master equation for the generalized degree distribution is

$$P_{d,d-1}(k) = \frac{m+2-k}{(md-1)} P_{d,d-1}(k-1)(1-\delta_{k,1}) - \frac{m+1-k}{(md-1)} P_{d,d-1}(k) + \delta_{k,1}$$

The latter has the following solution

$$P_{d,d-1}(k) = \frac{md-1}{m(d+1)-1} \frac{\Gamma(m+1)}{\Gamma(md+m-1)} \frac{\Gamma(md+m-k)}{\Gamma(m-k+2)}$$

for $1 \leqslant k \leqslant m$

Therefore the **generalized degree distribution is** indeed **bounded** for δ = d – 1 dimensional faces

MASTER EQUATION RESULTS

For the case $\delta \leqslant d-2$ the recursive master equation can be written as

$$P_{d,\delta}(k) = \frac{[m(d-\delta-1)-1]k+m+1}{md-1} P_{d,\delta}(k-1)(1-\delta_{k,1}) - \frac{[m(d-\delta-1)-1]k+m+1}{md-1} P_{d,\delta}(k) + \delta_{k,1}.$$

The latter has the following solution

$$P_{d,\delta}(k) = \frac{md-1}{m(2d-\delta)-1} \frac{\Gamma\left(2+\frac{m(d+1)}{m(d-\delta-1)-1}\right)}{\Gamma\left(1+\frac{m+1}{m(d-\delta-1)-1}\right)} \times \frac{\Gamma\left(k+\frac{m+1}{m(d-\delta-1)-1}\right)}{\Gamma\left(k+1+\frac{m(d+1)}{m(d-\delta-1)-1}\right)}.$$

which for k>>1 decays as a **power law**

$$P_{d,\delta}(k) \simeq k^{-\gamma_{d,\delta}}$$

with

$$\gamma_{d\cdot\delta} = 1 + \frac{md-1}{m(d-\delta-1)-1}.$$

where $\gamma \leqslant 3$

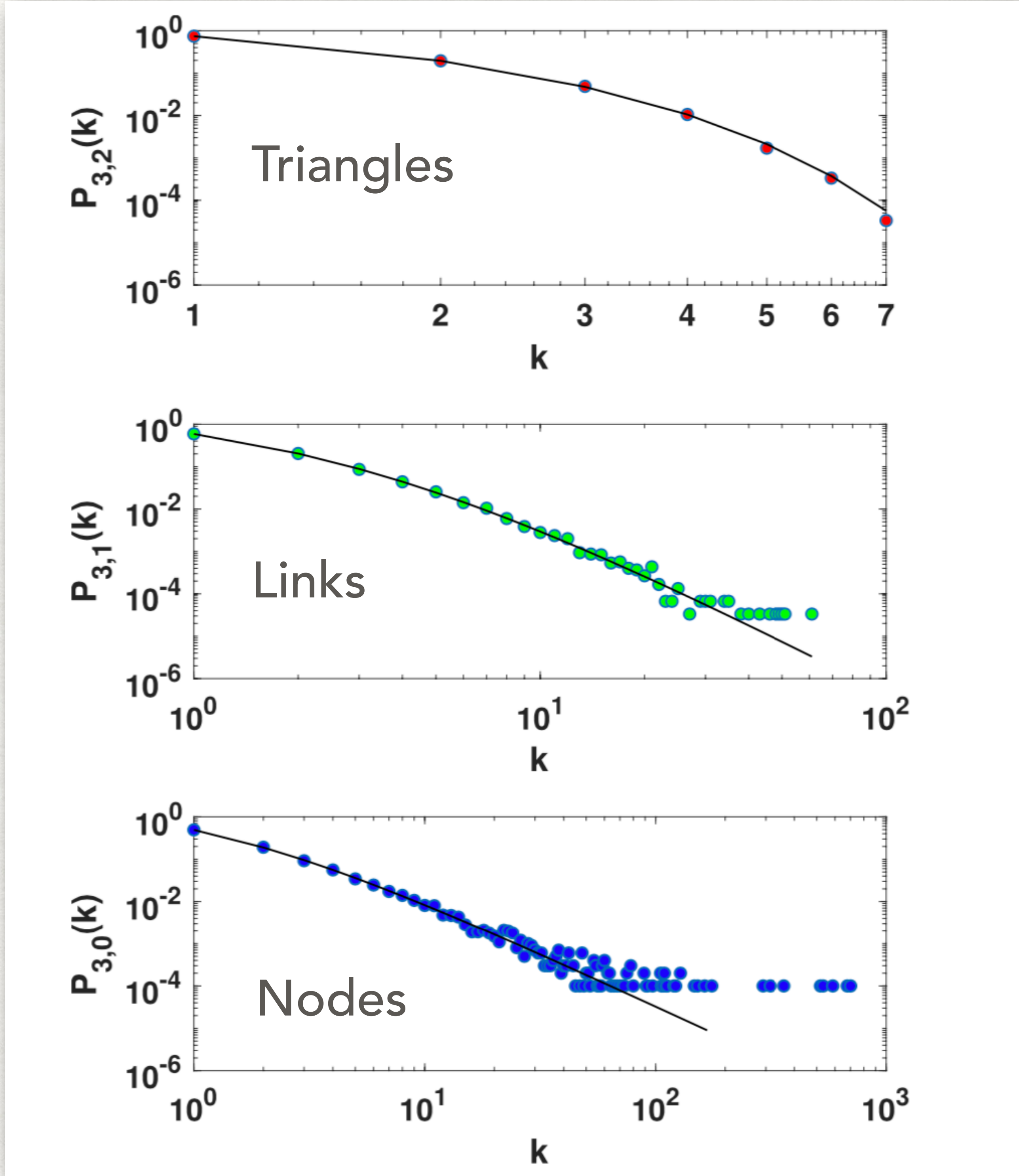
if and only if $d-2\delta \geqslant 2+\frac{3}{m}$

RESULTS:COMPARISON WITH SIMULATIONS

So the theoretical predictions show that the generalized degree distribution is **bounded** for $\delta = d - 1$

and **power-law** for $\delta \leqslant d - 2$

Predictions compare well with numerical simulations



Generalized degree distribution $P_{d,\delta}(k)$ of nodes ($\delta = 0$), links ($\delta = 1$) and triangles ($\delta = 2$) of a NGF with $N = 5000$ nodes, flavor $s = -1/6$, dimension $d = 3$ and inverse temperature $\beta = 0$.

RESULTS NETWORK GEOMETRY WITH FRACTIONAL FLAVOR FOR $\beta > 0$

For the generalized degree minus one averaged over faces of energy **ε**, we get the exact results

$$\langle k_{d,d-1} - 1|\epsilon\rangle = \frac{m}{e^{\beta(\epsilon_\alpha - \mu_{d,d-1})} + 1} = m n_F(\epsilon). \qquad \delta = d-1,$$

$$\langle k_{d,\delta} - 1|\epsilon\rangle = A_\delta \frac{1}{e^{\beta(\epsilon_\alpha - \mu_{d,\delta})} - 1} = A_\delta n_B(\epsilon), \qquad \delta \leqslant d-2.$$

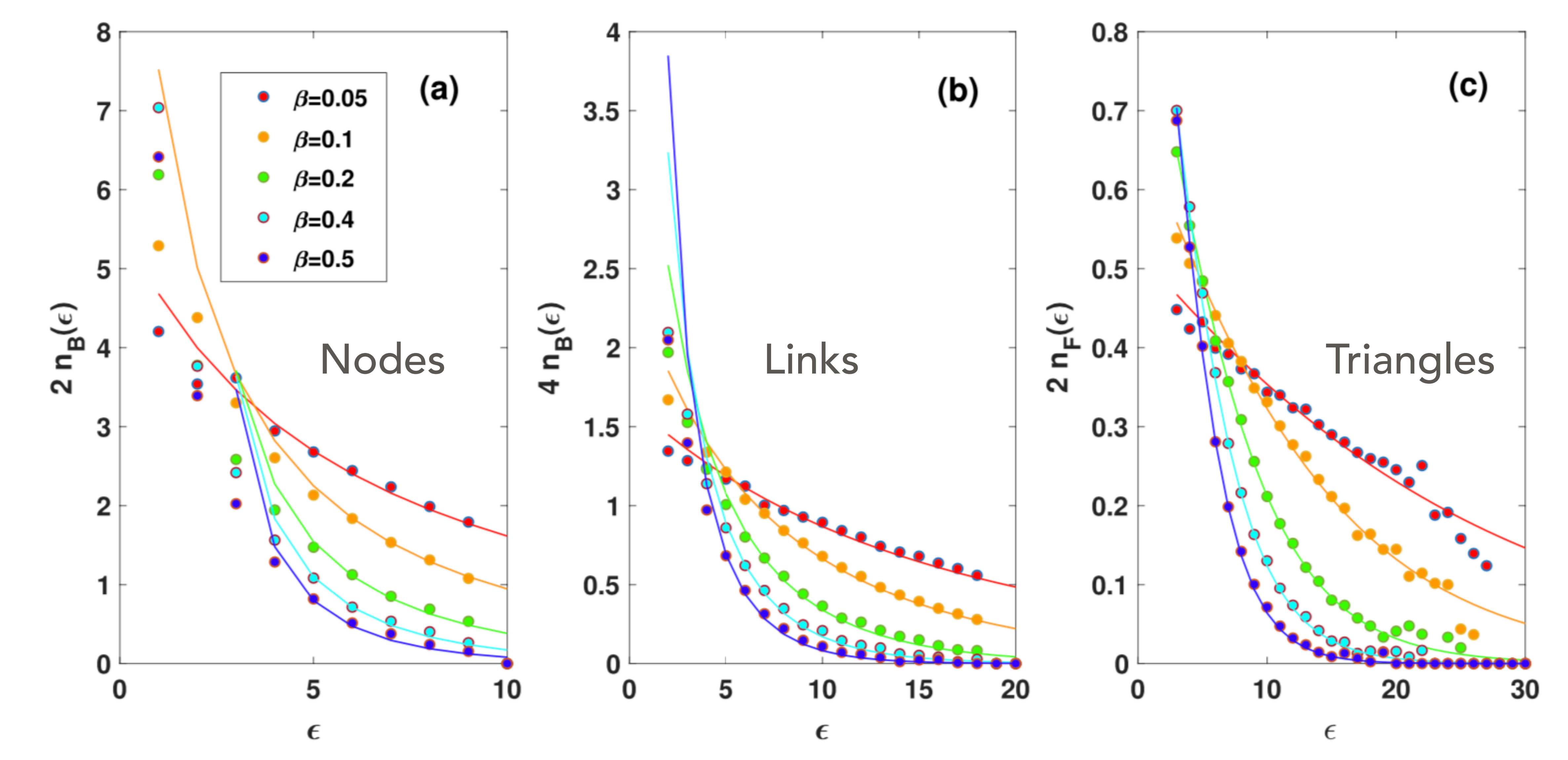
$$A_\delta = 1 + a = \frac{m(d-\delta)}{m(d-\delta-1)-1} \qquad e^{\beta\mu_{d,d-1}} = \lim_{t\to\infty} \frac{t}{mZ^{[s]}},$$

Thus the generalized degree minus one averaged over faces of energy **ε**

is proportional to the **Fermi–Dirac** distribution for $\delta = d-1,$

is proportional to the **Bose–Einstein** distribution for faces of dimension $\delta \leqslant d-2.$

RESULTS NETWORK GEOMETRY WITH FRACTIONAL FLAVOR FOR $\beta > 0$



Average generalized degree minus one, over faces of energy ε and dimension δ = 0 (panel (a)) δ = 1 (panel (b)) or δ = 2 (panel (c)) for d = 3 dimensional NGF with fractional flavor s = −1/2 are plotted for different values of β (symbols) and compared to the theoretical expectations (Fermi–Dirac and Bose–Einstein statistics). The simulations are performed for NGF with N = 3000 nodes. The data are averaged over 50 NGF realizations.

CONCLUSIONS

For the **NGF model with fractional flavor s = −1/m** we observe different statistics as a function of the dimensionality of the faces, but **the only two types of statistics emerging are the Fermi–Dirac and Bose–Einstein statistics** as long as the NGF evolution reaches a steady state.

The proposed NGF model with fractional flavor can be used to model real simplicial complexes in which nodes have some intrinsic features that can be associated with their fitness.

The NGF with fractional flavor can be used as well in artificial models to test dynamical processes defined on simplicial complexes, such as percolation, synchronization or social contagion.

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