The phase diagram of QCD: some reasons for *optimism*

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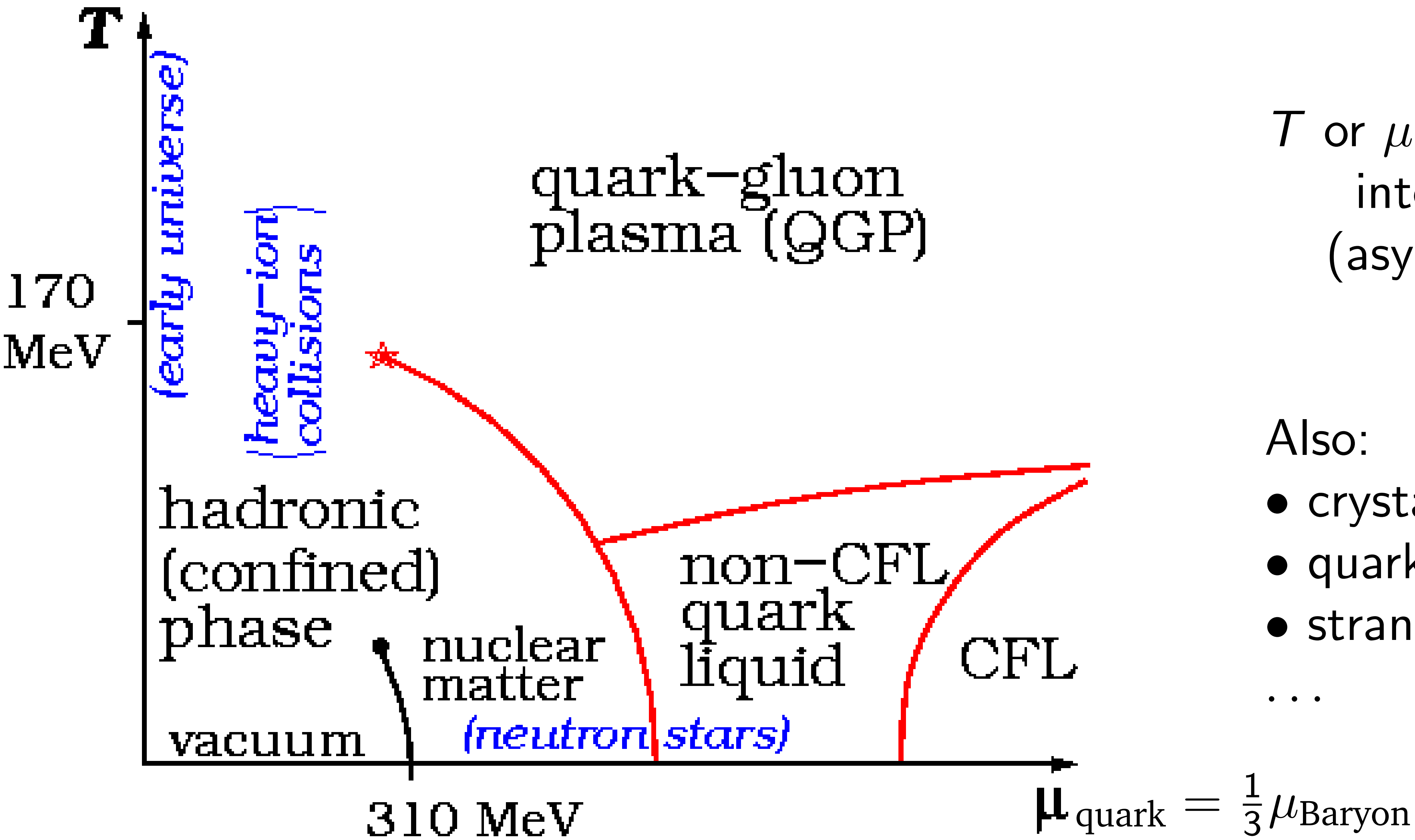
Bari, SM&FT, dec.12, 2019

CERN

ETH zürich

The wonderland phase diagram of QCD from Wikipedia



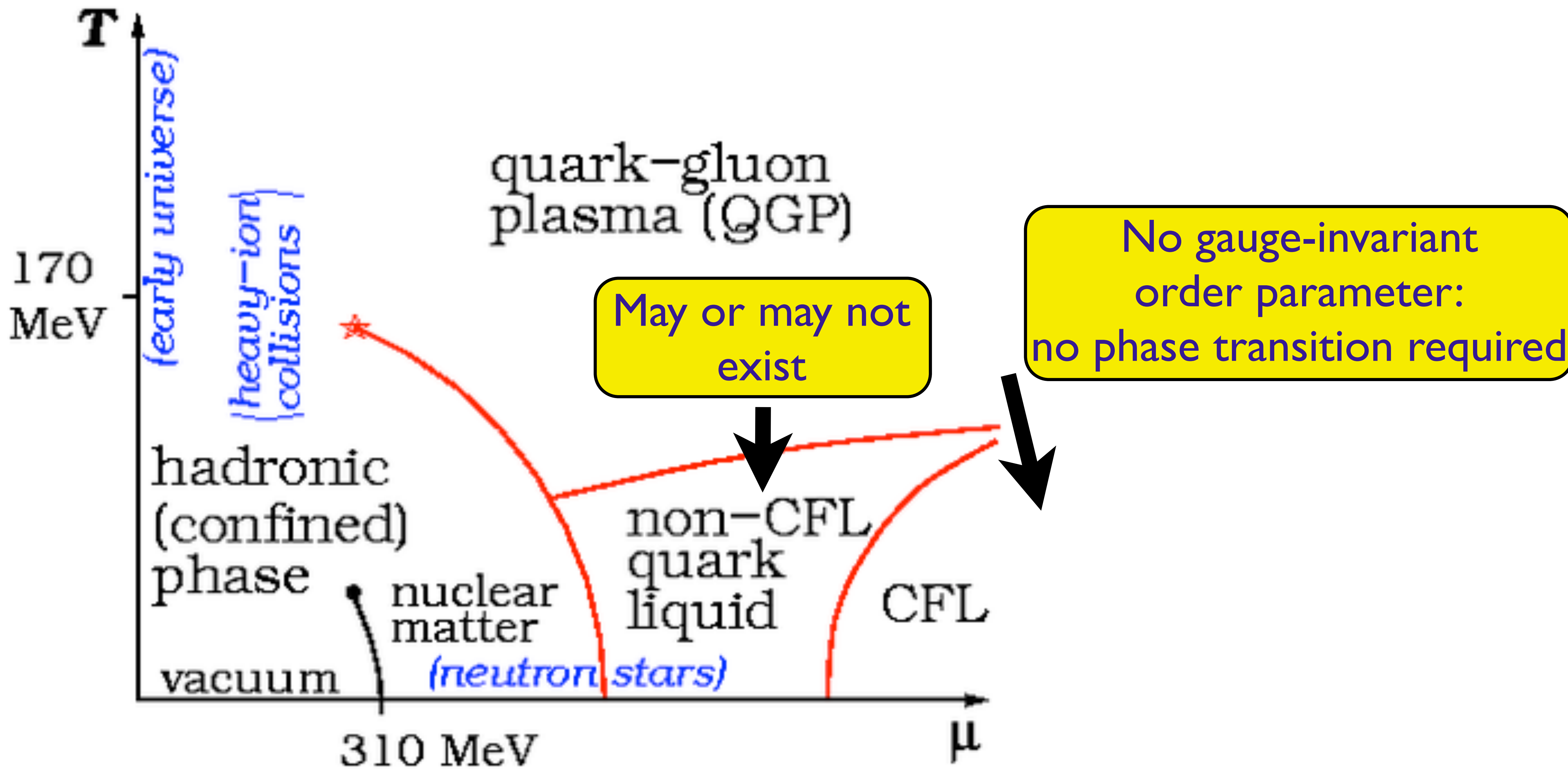
T or $\mu \to \infty$:
interaction weak
(asymptotic freedom)

Also:

- crystal phase(s)
- quarkyonic phase
- strangelets

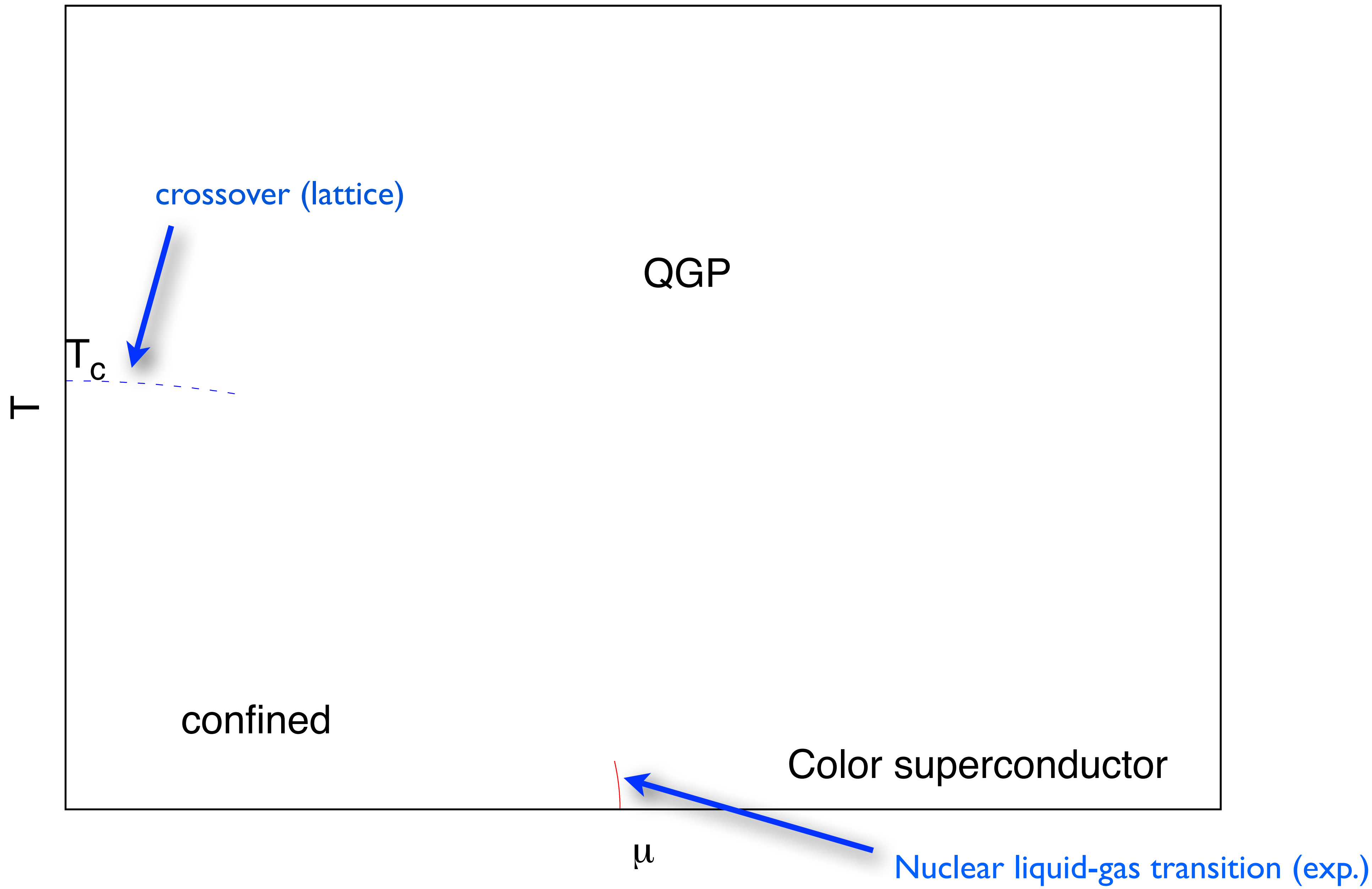
...

Caveat: everything in red is a conjecture


T
170
MeV
(early universe)
(heavy-ion collisions)
quark-gluon plasma (QGP)
May or may not exist
No gauge-invariant order parameter: no phase transition required
hadronic (confined) phase
nuclear matter
non-CFL quark liquid
CFL
vacuum
(neutron stars)
310 MeV
μ

Finite μ: what is known?

really

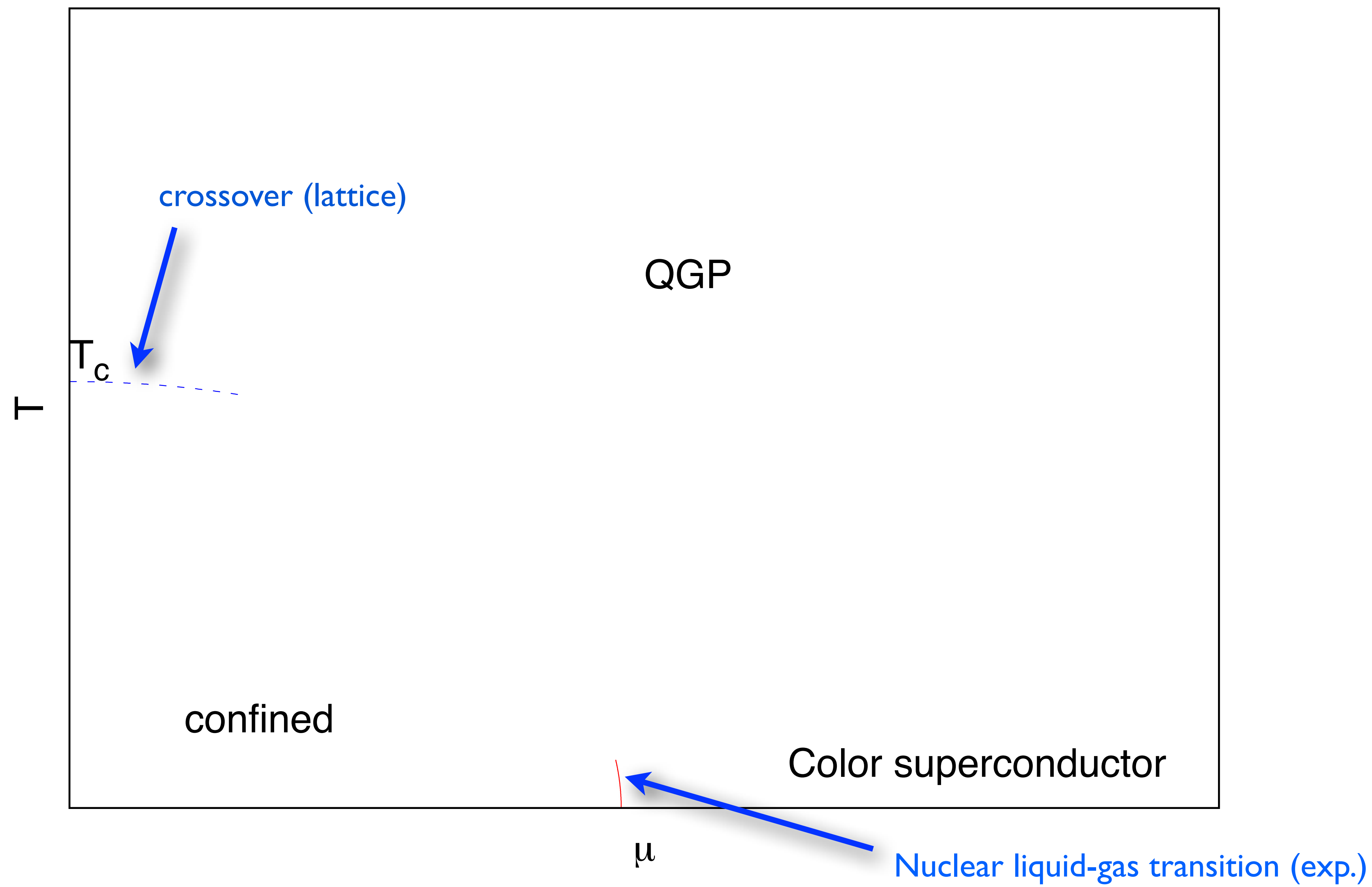


Minimal, possible phase diagram

Finite μ: what is known?

Lattice: Sign problem *as soon as* $\mu \neq 0$

Complex weights
No importance sampling



Minimal, possible phase diagram

Small-μ approach: Taylor expansion

Karsch et al., 2002

- Expansion parameter $\mu/T \lesssim 1$

$$P(T,\mu) - P(T,0) = \sum_{k=1} c_{2k}(T) \left(\frac{\mu}{T}\right)^{2k}$$

$$c_{2k} = \langle \mathrm{Tr} \; (\text{degree } 2k \text{ polynomial in } \not{D}^{-1}, \frac{\partial \not{D}}{\partial \mu}) \; \rangle_{\mu=0}$$

Standard $\mu = 0$ simulation & *noise vectors* to estimate Trace

- Combinatorial complexity in k $\rightarrow$ c_8 not quite yet

$c_4 : 2002$
$c_6 : 2005$

high orders???

- Progress: μ on the lattice
  - Linear: $U_4 \rightarrow (1 + a\mu)U_4$, UV divergence

1983 • Hasenfratz & Karsch: $U_4 \rightarrow \exp(a\mu)U_4$, cures UV divergence

2011 • Gavai & Sharma: linear + subtract UV divergence by hand ??

Taylor expansion: nitty-gritty

$$
\begin{aligned}
\frac{\partial^6 \ln\det M}{\partial\mu^6} \quad = \quad & \mathrm{tr}\left(M^{-1}\frac{\partial^6 M}{\partial\mu^6}\right) - 6\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^5 M}{\partial\mu^5}\right) \\
& -15\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\right) - 10\mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial\mu^3}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
& +30\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^4 M}{\partial\mu^4}\right) + 60\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
& +60\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) + 30\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
& -120\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^3 M}{\partial\mu^3}\right) \\
& -180\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
& -90\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
& +360\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial^2 M}{\partial\mu^2}\right) \\
& -120\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}M^{-1}\frac{\partial M}{\partial\mu}\right) .
\end{aligned}
$$

Now estimate all Traces by sandwiching between noise vectors... GPUs

Taylor expansion: nitty-gritty

$$\frac{\partial^6 \ln \det M}{\partial \mu^6} = \mathrm{tr}\left(M^{-1}\frac{\partial^6 M}{\partial \mu^6}\right) - 6\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^5 M}{\partial \mu^5}\right)$$

$$-15\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^4 M}{\partial \mu^4}\right) - 10\mathrm{tr}\left(M^{-1}\frac{\partial^3 M}{\partial \mu^3}M^{-1}\frac{\partial^3 M}{\partial \mu^3}\right)$$

$$+30\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^4 M}{\partial \mu^4}\right) + 60\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^3 M}{\partial \mu^3}\right)$$

$$+60\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^3 M}{\partial \mu^3}\right) + 30\mathrm{tr}\left(M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}\right)$$

$$-120\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^3 M}{\partial \mu^3}\right)$$

$$-180\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial^2 M}{\partial \mu^2}\right)$$

$$-90\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial^2 M}{\partial \mu^2}\right)$$

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$$-120\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}M^{-1}\frac{\partial M}{\partial \mu}\right).$$

Only term surviving with linear μ

$$\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}\right)^k$$

Fewer traces $\rightarrow$ less work and more precise estimates

Even better: much faster method for $\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}\right)^k$

- $\frac{\partial M}{\partial \mu}$ only hops in time direction $\rightarrow$ easy to invert
- Diagonalize $A \equiv \frac{\partial M}{\partial \mu}^{-1} M \;\rightarrow$ eigenvalues $|\lambda_1| < |\lambda_2| < \ldots$
- $\mathrm{tr}\left(M^{-1}\frac{\partial M}{\partial \mu}\right)^k = \sum_j \lambda_j^{-k}$; truncate the sum (ARPACK)

cheaper & more accurate than noise vectors

Get all susceptibilities at no extra cost!

1812.00869, PdF & Jaeger; Schmidt et al.

Convergence radius of Taylor expansion

$$P(T,\mu) - P(T,0) = \sum_{k=1} c_{2k}(T) \left(\frac{\mu}{T}\right)^{2k}$$

- Susceptibility diverges at phase transition $\to$ pole $\hat{\mu} \in \mathcal{C}$
- For k "large enough", $c_{2k} \propto 1/\hat{\mu}^{2k} \quad \to \quad \hat{\mu}^2 = \lim_{k\to\infty} \frac{c_{2k}}{c_{2k+2}}$
- Complex equation: estimate modulus first

  naive: $|\hat{\mu}| \sim \left|\frac{1}{c_{2k}}\right|^{\frac{1}{2k}}$; improved (Mercer-Roberts) $\left|\frac{c_{2k+2}c_{2k-2} - c_{2k}^2}{c_{2k+4}c_{2k} - c_{2k+2}^2}\right|^{\frac{1}{4}}$
- Then estimate phase, eg. $\cos 2\theta \sim \frac{(2k+2)c_{2k+2}|\hat{\mu}|^2 + (2k-2)c_{2k-2}}{2kc_{2k}|\hat{\mu}|}$

1904.01974, Giordano & Pasztor

With phase, can separate real singularity (*phase transition*) and complex conjugate pair (*crossover*)

Last problem: "gauge noise"

- Taylor coeff $c_{2k}(T)$ is averaged over gauge ensemble

  Config-to-config fluctuations rise rapidly with k

  Exponential statistics? (and no free lunch?)

- No: $c_{2k}, c_{2k\pm 2}$ strongly correlated $\rightarrow$ measure $\frac{\langle c_{2k} \rangle}{\langle c_{2k\pm 2} \rangle}$

1904.01974, Giordano & Pasztor

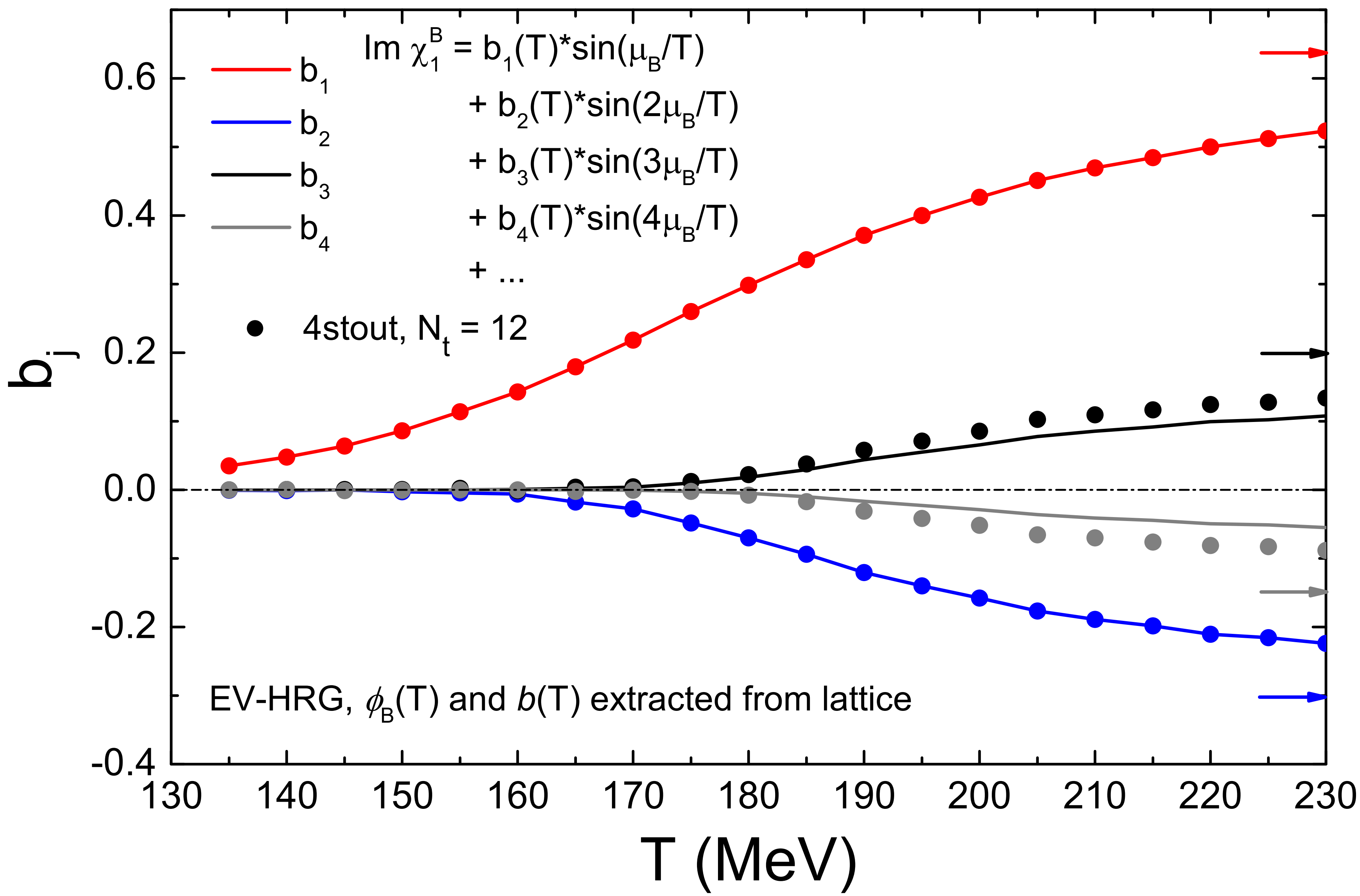
Looking at the elephant from different angles

State of the art: *imaginary mu*

1708.02852

Vovchenko et al

$$\frac{\rho_B}{T^3} = \frac{\partial(p/T^4)}{\partial(\mu_B/T)} = \sum_{k=1}^{\infty} \mathbf{b_k(T)} \sinh\left(\frac{k\mu_b}{T}\right)$$

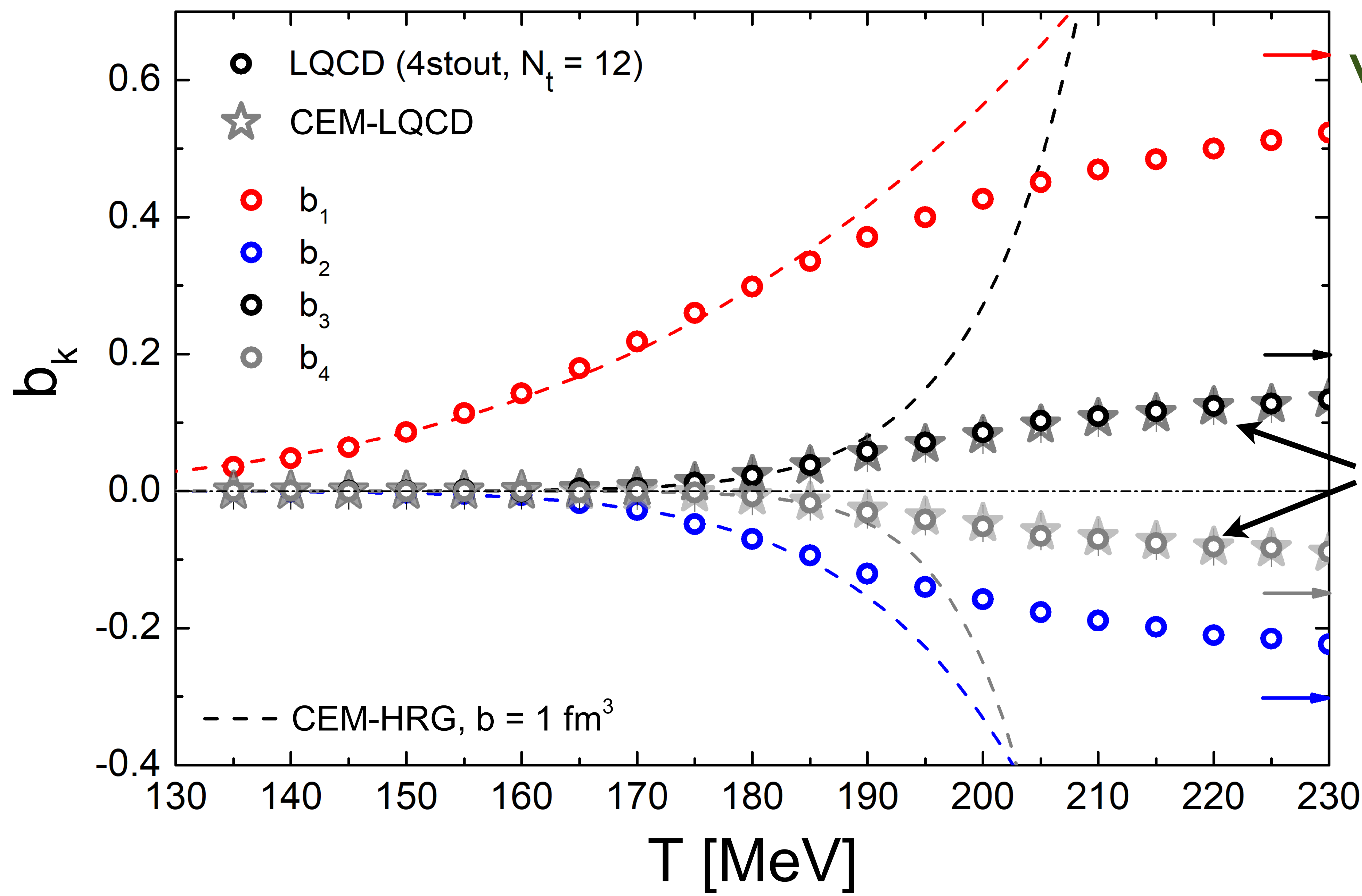


- Hadron Resonance Gas (non-interacting): $b_2 = b_3 = \cdots = 0$
- Here, $b_2 < 0 \Longrightarrow$ baryons *repel above* T_c -- input for model

Here, excluded volume

Models: excluded volume $\longrightarrow$ cluster expansion

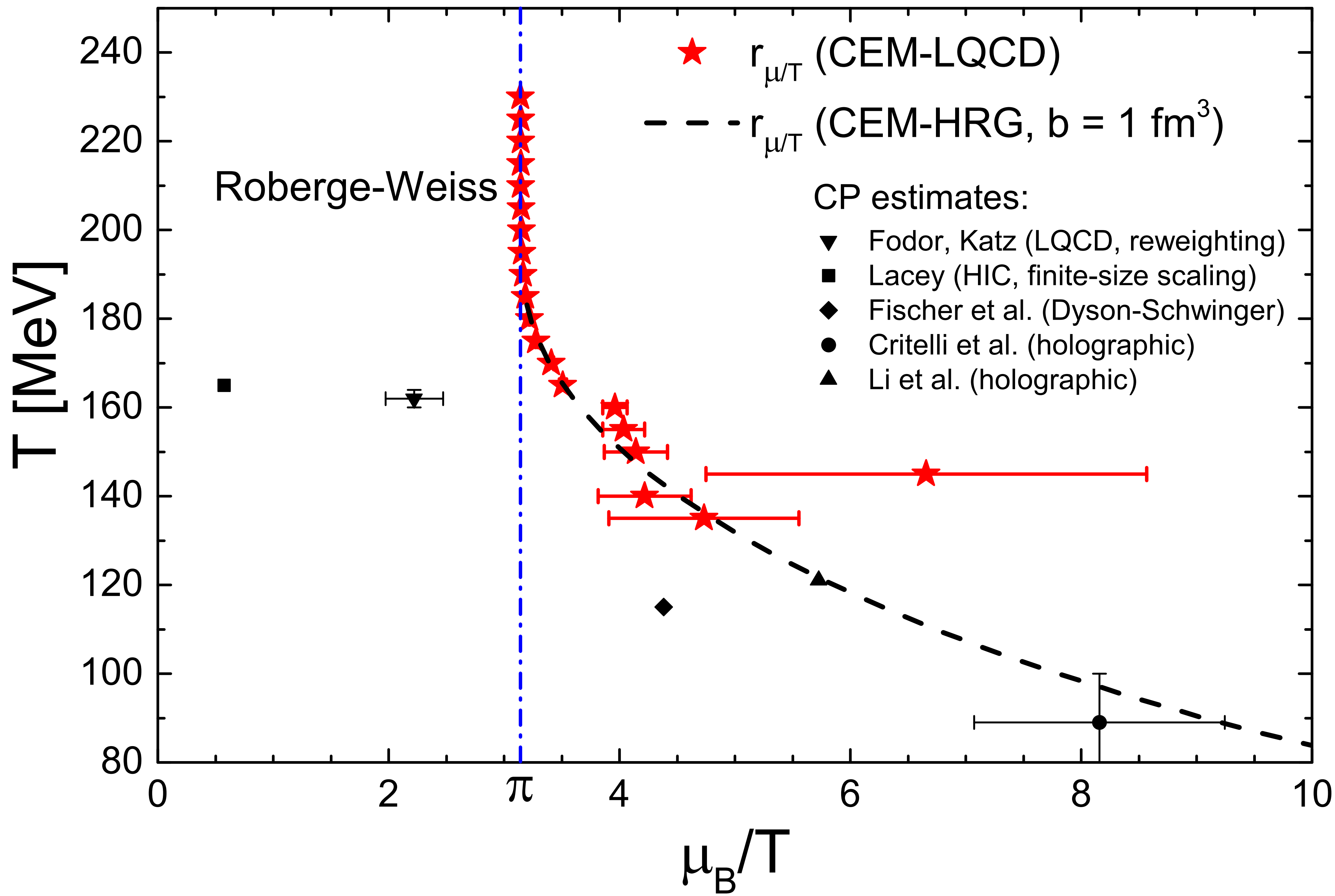
1711.01261
Vovchenko et al



stars & circles on top of each other

- At each T, b_1 & b_2 set to lattice data $\Rightarrow$ *all* higher coeffs fixed
- Analytically solvable
- Analytic ansatz (no phase transition) describes *all* MC data

Radius of convergence



- Nearest *complex* singularity: $\approx$ real (low T), imaginary (high T)
- Incompatible with Critical Point estimates (black symbols)

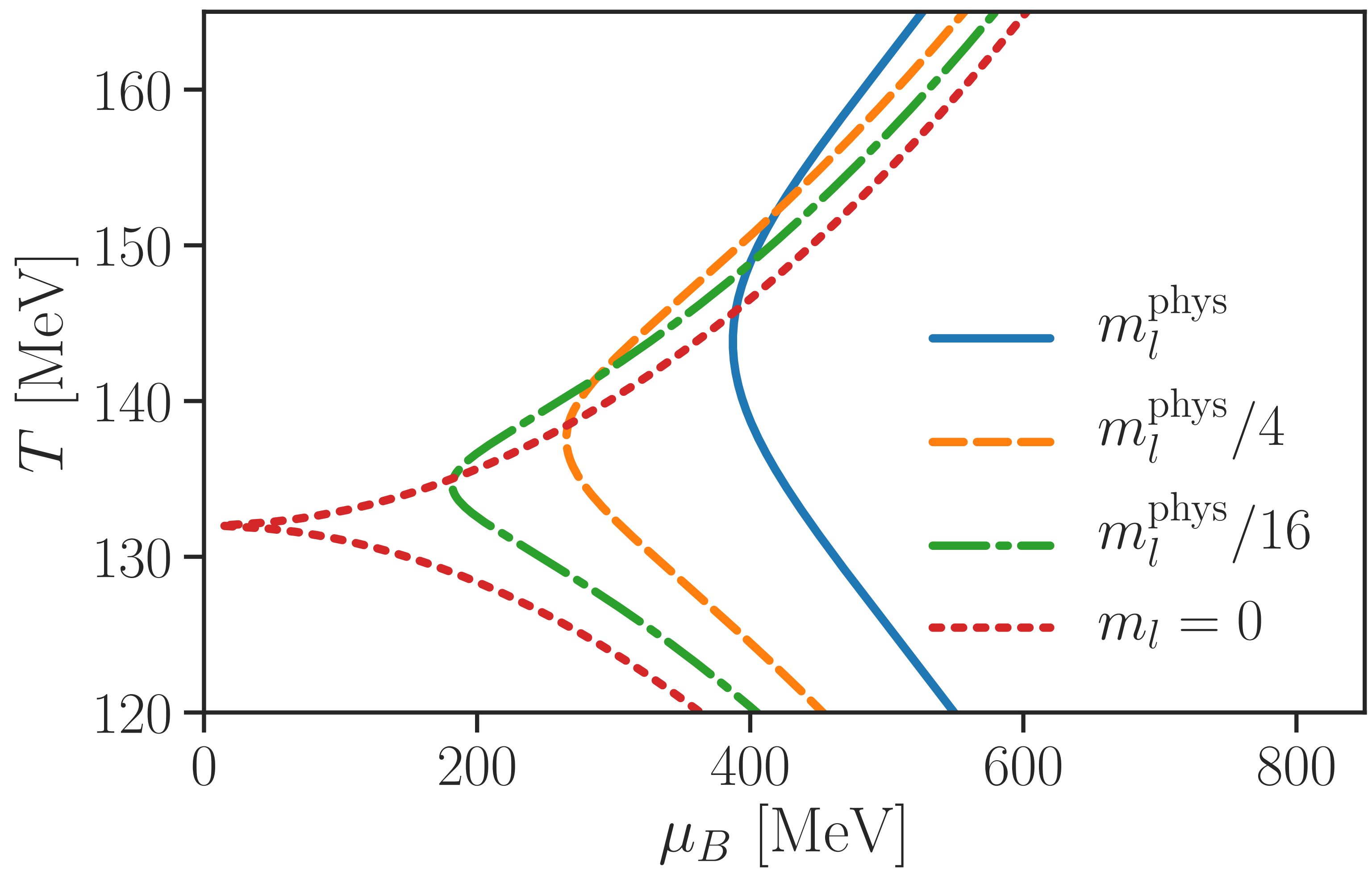
Effect of chiral transition

1909.04639
Mukherjee & Dkokov

- Chiral *phase transition* at $(T \approx 132\ \mathrm{MeV}, \mu = 0)$ for *massless* u,d

- Exploit *universal scaling* (3d O(4)) to predict at physical point:

convergence radius

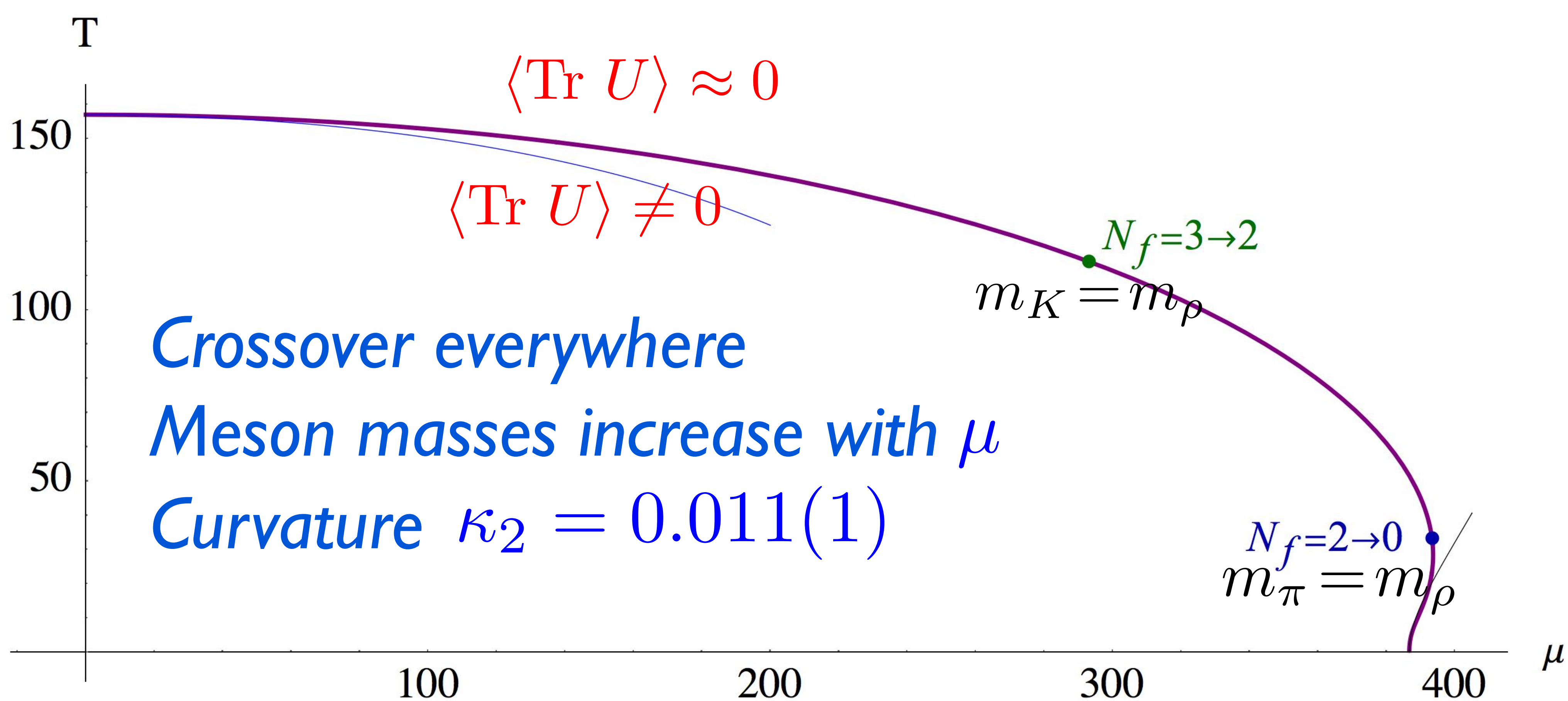


Crossover: No Critical Point within scaling region

Chiral effective theory

1903.11652
Zarembo

- Chiral Lagrangian $\mathcal{L} = \frac{F^2}{4} \mathrm{Tr}\, \partial_\mu U^\dagger \partial^\mu U,\; U = e^{iT^a \pi_a}$ insufficient
- Need quarks: $\mathcal{L} = \bar{\psi}\left(\not{\partial} + M e^{i\gamma_5 T^a \pi_a}\right)\psi$, M constituent mass
- Integrate quarks out $\rightarrow \log\det\left[\not{\partial} + M\left(\frac{1+\gamma_5}{2}U + \frac{1-\gamma_5}{2}U^\dagger\right)\right]$
- Compute $\langle \mathrm{Tr}\, U\rangle(T,\mu)$



Crossover everywhere

Meson masses increase with μ

Curvature $\kappa_2 = 0.011(1)$

Effect of nuclear liquid-gas transition

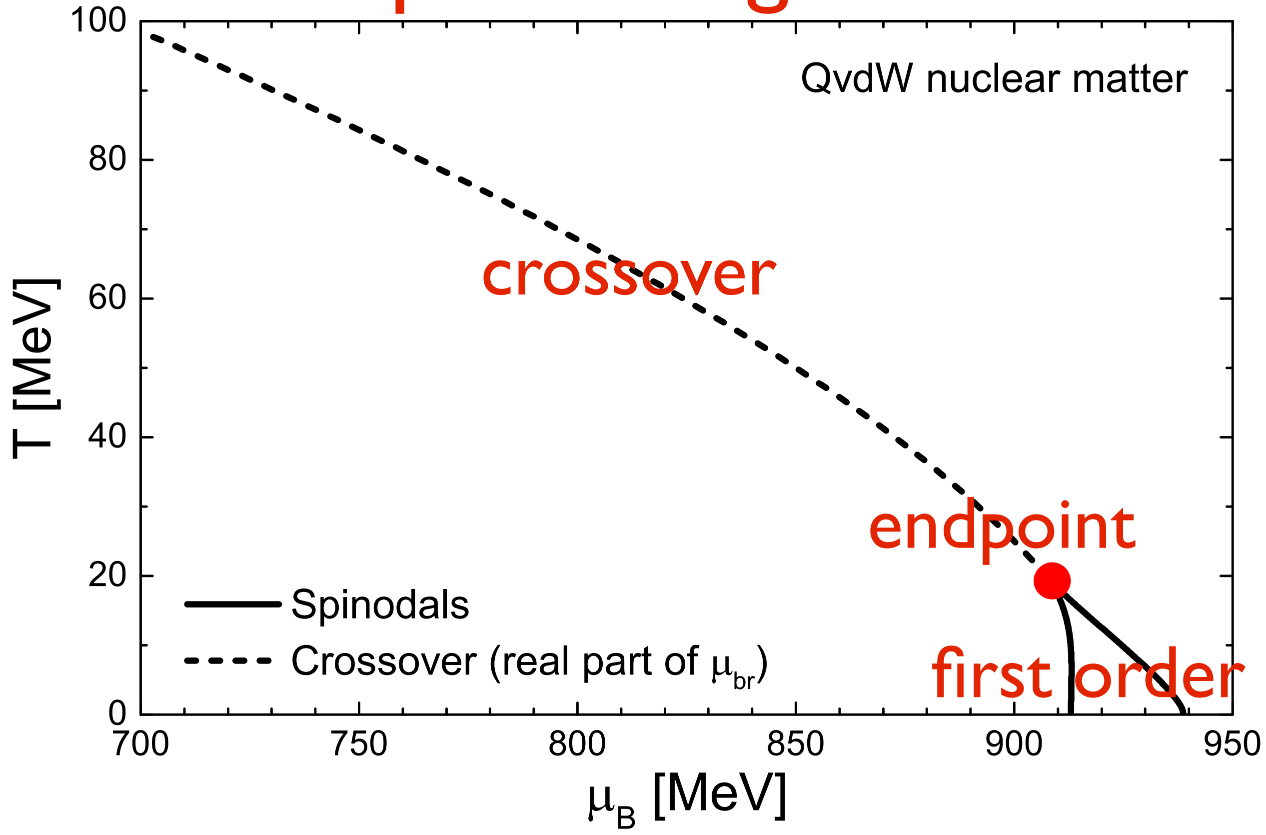
1909.04461
Vovchenko et al.

- Take 4 different models describing nuclear liquid-gas transition:

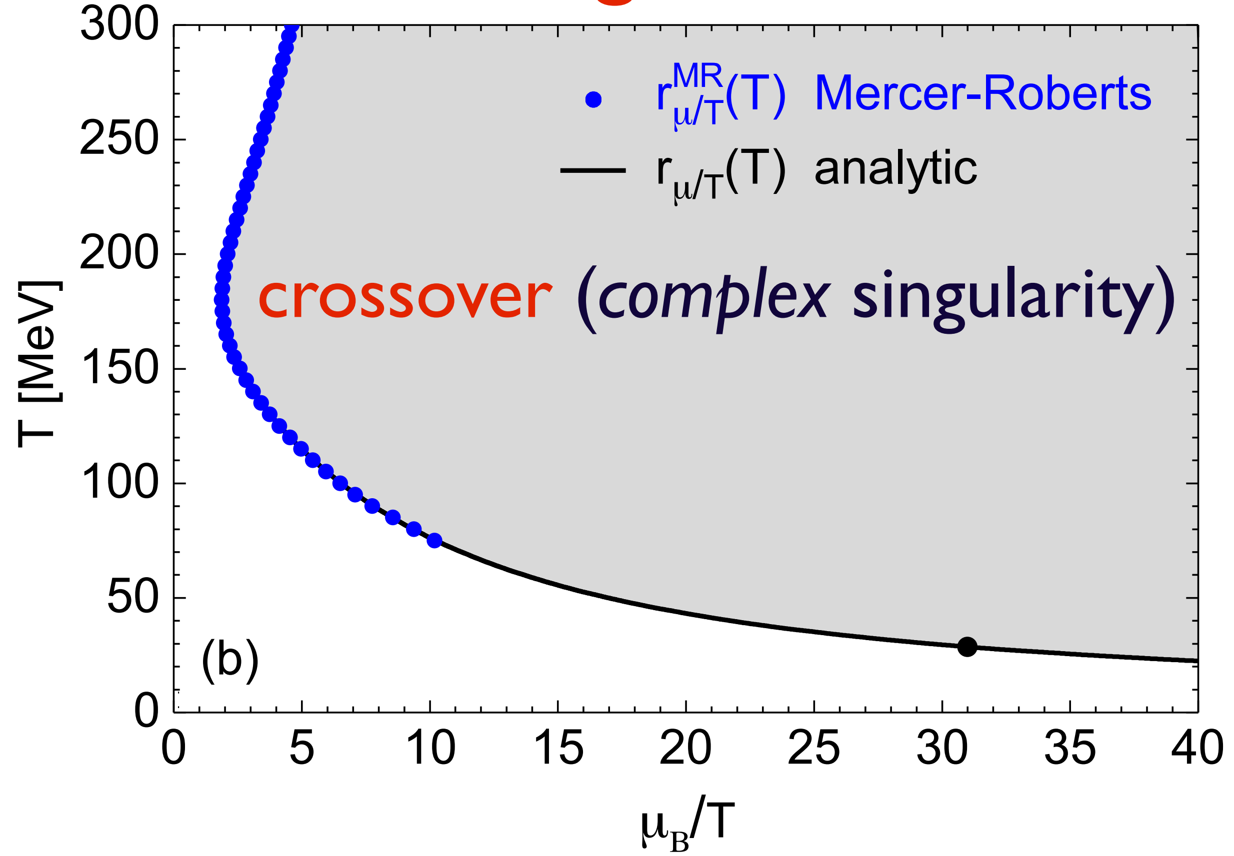
  van der Waals classical/quantum, Skyrme, Walecka

- Fit each model to liquid-gas transition. What happens at small μ?

phase diagram



convergence radius



No need for chiral d.o.f. ??

Outlook

- Re-engineered Taylor expansion approach

$\longrightarrow$ "turbo mode"

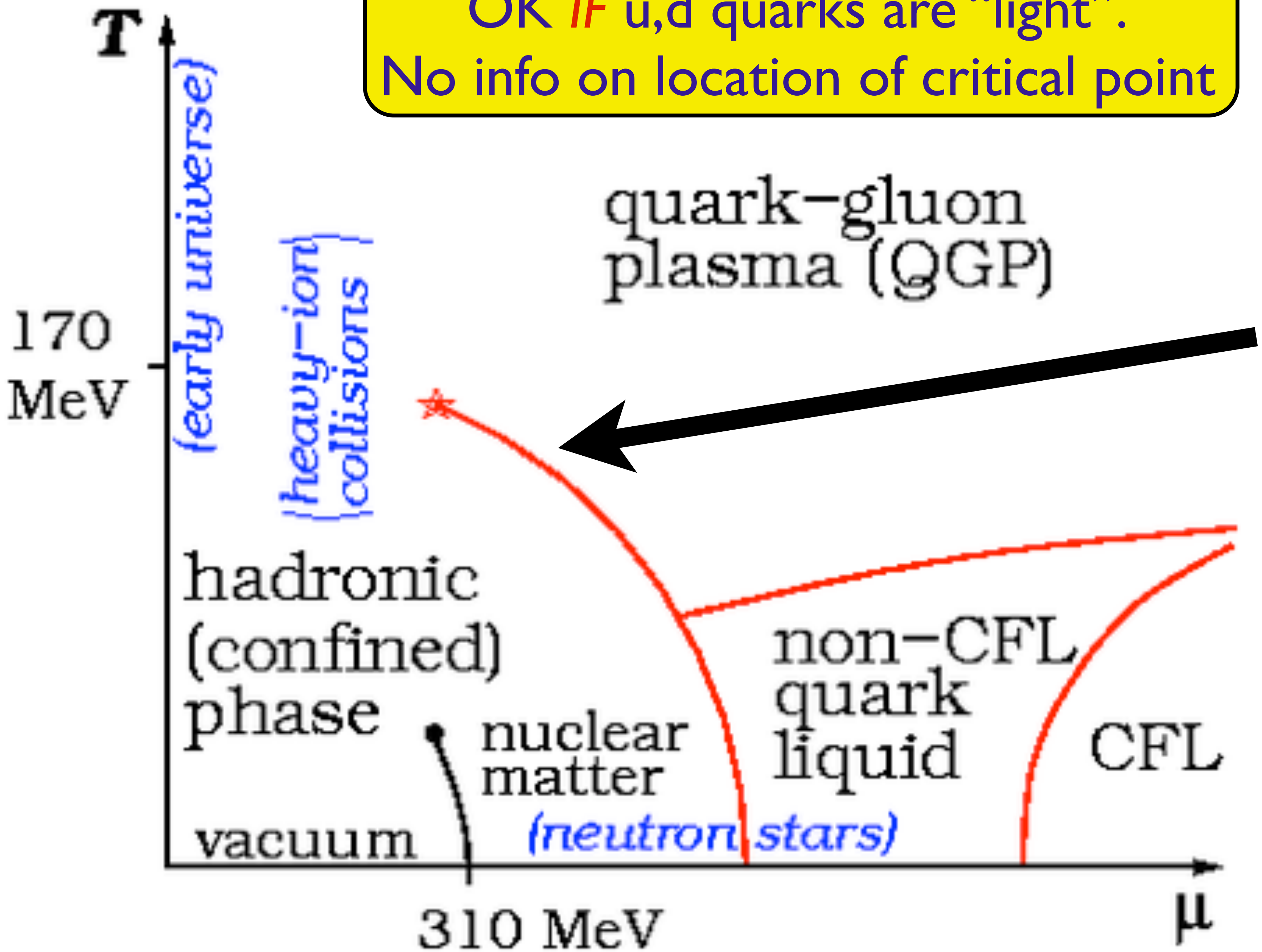
- More, better data coming to constrain models

- Qualitatively different models all suggest

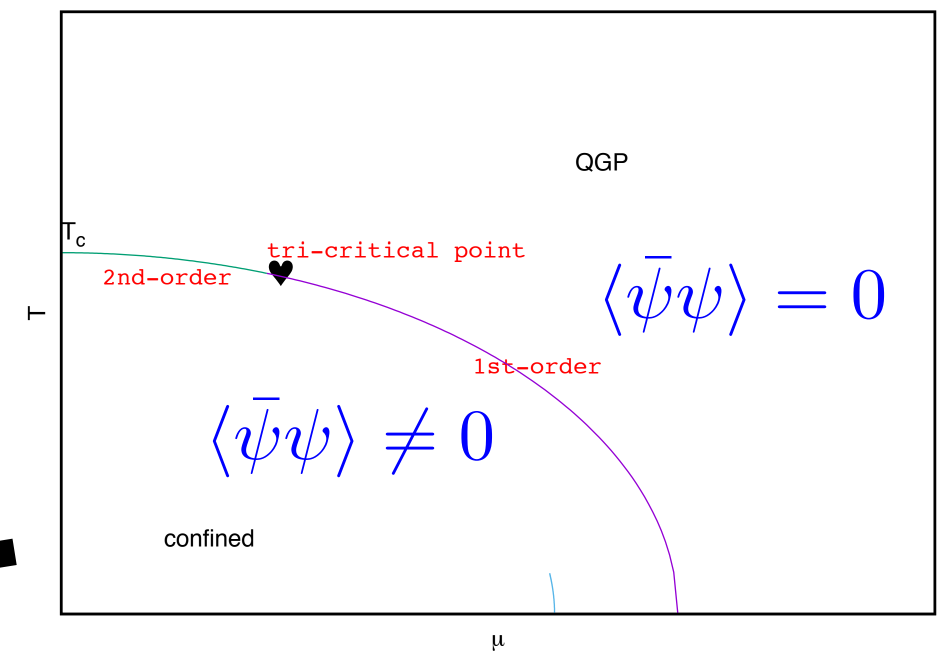
QCD critical point at large μ, or not there at all

Backup

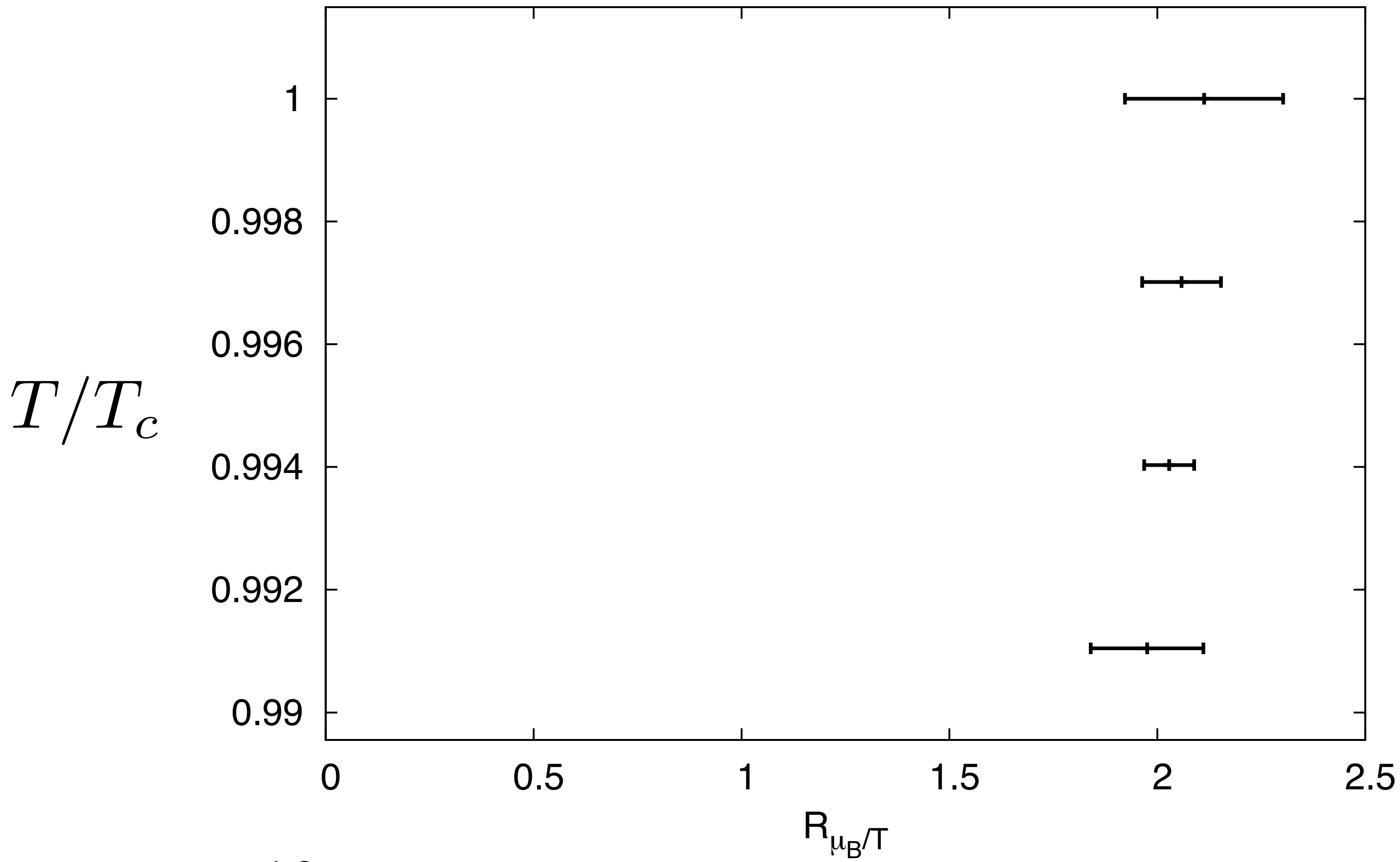
“Small” deformation of
two-flavor massless case:
OK *IF* u,d quarks are “light”.
No info on location of critical point



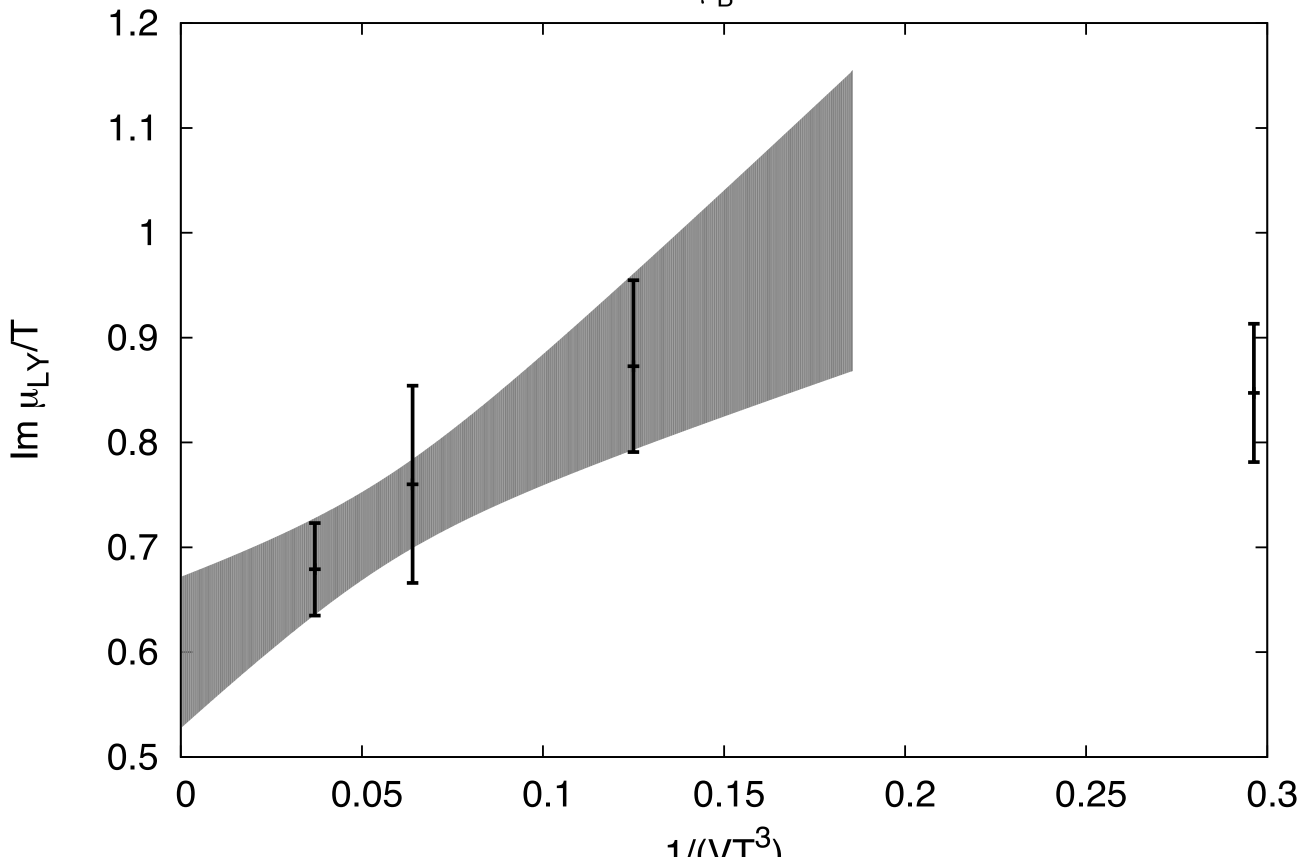
$N_f = 2, m_u = m_d = 0$



Test of approach, $N_t = 4$ (1911.00043)



radius of convergence



Imag. part of L-Y zero remains non-zero as $V \to \infty$

$\Longrightarrow$ crossover