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Vortex mass in the three-dimensional $O(2)$ scalar theory

Gesualdo Delfino

SISSA – Trieste

Based on:

G. Delfino, W. Selke and A. Squarcini, PRL 122 (2019) 050602

G. Delfino, J. Phys. A 47 (2014) 132001

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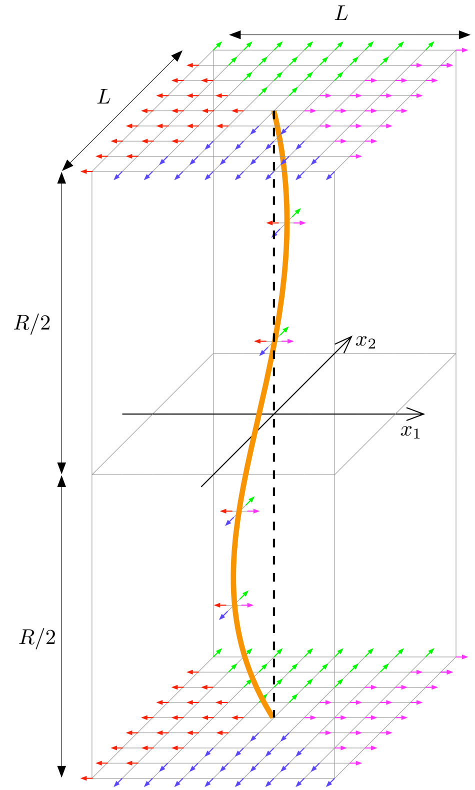
is there a vortex particle in 3D scalar quantum theory?

3D XY model slightly below T_c [GD, '14]

$$\mathcal{H} = -\frac{1}{T} \sum_{<i,j>} \mathbf{s}_i \cdot \mathbf{s}_j, \qquad \mathbf{s}_i = (\cos\theta_i, \sin\theta_i)$$



- scaling limit is 3D O(2) scalar theory in imaginary time (vertical direction)

- radial boundary conditions enforce vortex line

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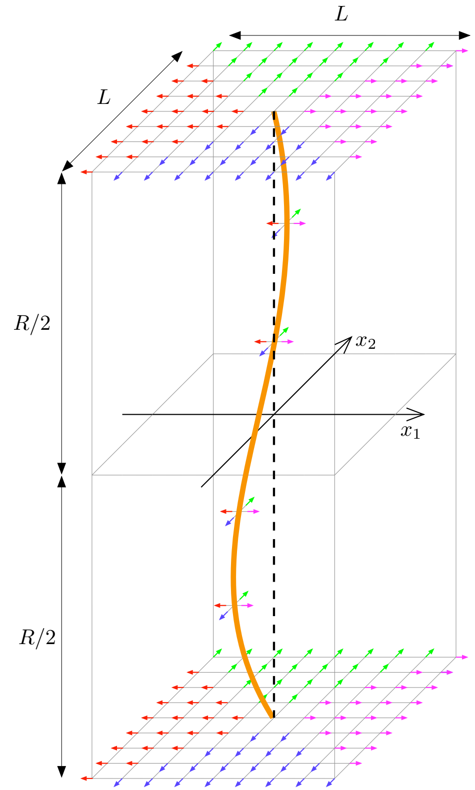

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- in field theory (with $L = \infty$) magnetization is

$$\langle \mathbf{s}(\mathbf{x},0)\rangle = \frac{\langle B(\frac{R}{2})|\mathbf{s}(\mathbf{x},0)|B(-\frac{R}{2})\rangle}{\langle B(\frac{R}{2})|B(-\frac{R}{2})\rangle}$$

$$|B(\pm R/2)\rangle = \int \frac{d\mathbf{p}}{(2\pi)^2 E}\, a_{\mathbf{p}}\, e^{\pm\frac{R}{2}E}\, |V(\mathbf{p})\rangle + \cdots \qquad E = \sqrt{\mathbf{p}^2 + m_V^2}$$

leading large R contribution from vortex particle V with mass m_V

exact large R result:

$$\langle \mathbf{s}(\mathbf{x},0)\rangle = \frac{\langle B(\frac{R}{2})|\mathbf{s}(\mathbf{x},0)|B(-\frac{R}{2})\rangle}{\langle B(\frac{R}{2})|B(-\frac{R}{2})\rangle}$$

$$\sim \frac{R}{2\pi^3 m_V^2}\int d\mathbf{p}_1 d\mathbf{p}_2\, F_{\mathbf{s}}(\mathbf{p}_1|\mathbf{p}_2)\, e^{-\frac{R}{4m_V}(\mathbf{p}_1^2+\mathbf{p}_2^2)+i\mathbf{x}\cdot(\mathbf{p}_1-\mathbf{p}_2)}$$

$$F_{\mathbf{s}}(\mathbf{p}_1|\mathbf{p}_2) = \langle V(\mathbf{p}_1)|\mathbf{s}(0,0)|V(\mathbf{p}_2)\rangle, \qquad \mathbf{p}_1,\mathbf{p}_2 \to 0$$

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$$\langle \mathbf{s}(\mathbf{x},0)\rangle = \begin{cases} 0\,, & \mathbf{x}=\mathbf{0} \\ v\,\hat{\mathbf{x}}\,, & |\mathbf{x}|\to\infty \end{cases} \qquad \text{requires} \qquad F_{\mathbf{s}}(\mathbf{p}_1|\mathbf{p}_2) \propto \frac{\mathbf{p}_1-\mathbf{p}_2}{|\mathbf{p}_1-\mathbf{p}_2|^3}$$

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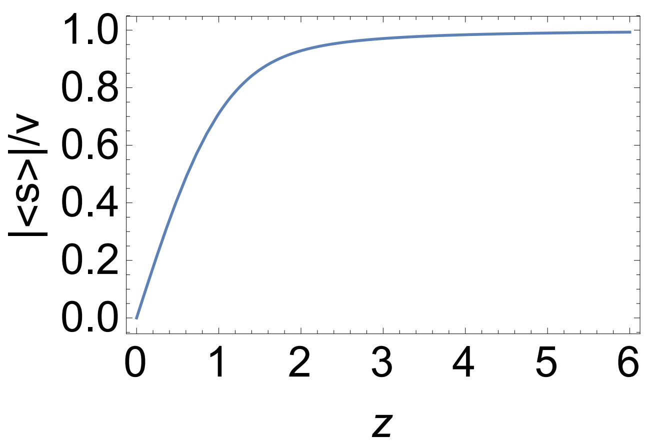
$$\Rightarrow \quad \langle \mathbf{s}(\mathbf{x},0)\rangle \sim v\,\frac{\sqrt{\pi}}{2}\,{}_1F_1\left(\tfrac{1}{2},2;-z^2\right) z\,\hat{\mathbf{x}}$$

$$z \equiv \sqrt{\frac{2m_V}{R}}\,|\mathbf{x}|$$



Comparison with Monte Carlo data [GD, Selke, Squarcini, '19]

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$$v \simeq v_0(T_c - T)^\beta, \qquad m_V \simeq m_V^0(T_c - T)^\nu$$

$\beta = 0.3486(1)$, $\nu = 0.6717(1)$, $v_0 = 0.945(5)$ [Hasenbusch et al, '06]

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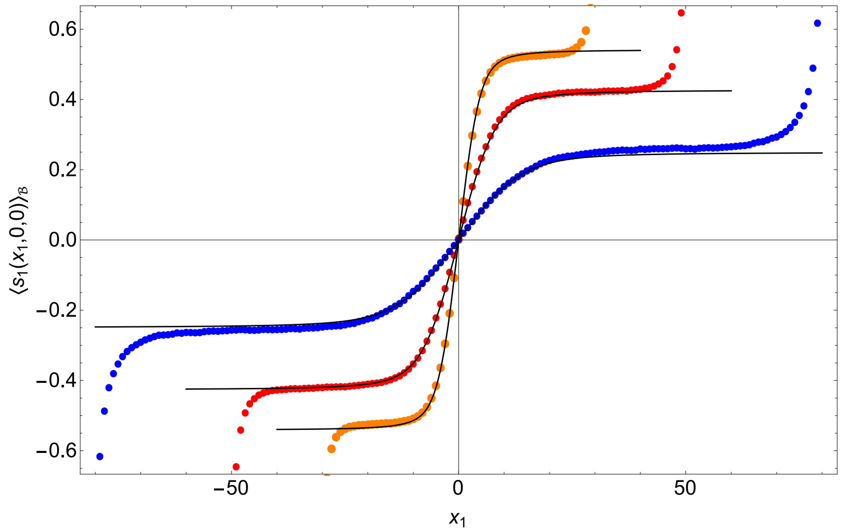
theory and simulations match with $m_V^0 \simeq 2.5$:

- $T = 2.0, \; R = 31, \; L = 61$
- $T = 2.1, \; R = 61, \; L = 101$
- $T = 2.18, \; R = 61, \; L = 161$

$T_c \simeq 2.2018$



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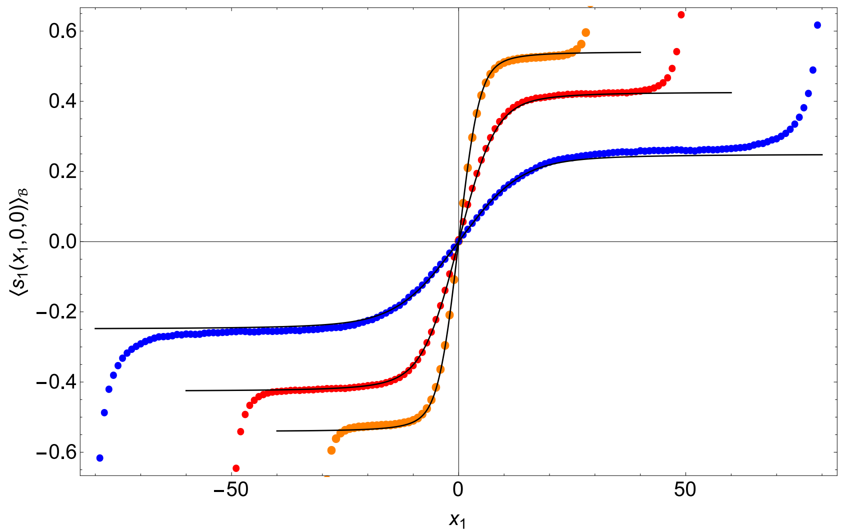


universal ratio with lightest mass above T_c: $\frac{m_V}{m_+} \approx 2.1 \quad (T \to T_c)$

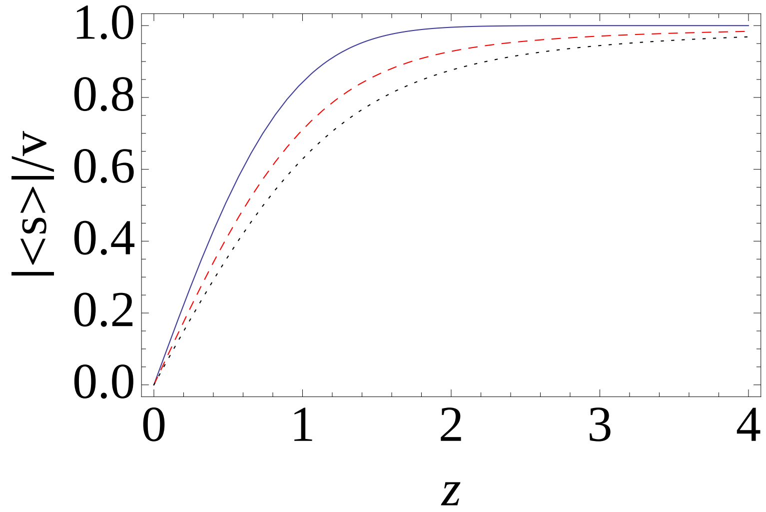
Generalization to $(n+1)$D $O(n)$ model [GD, '14]

$$\langle \mathbf{s}(\mathbf{x},0)\rangle \sim v\,\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1+\frac{n}{2}\right)}\,{}_1F_1\left(\frac{1}{2}, 1+\frac{n}{2}; -z^2\right) z\,\hat{\mathbf{x}} \qquad z \equiv \sqrt{\frac{2m_\tau}{R}}\,|\mathbf{x}|$$

—— $n = 1$, 2D Ising

\- - - $n = 2$, 3D XY

...... $n = 3$, 4D $O(3)$

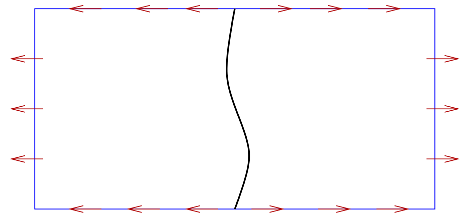


$n = 1$**:** Ising interface

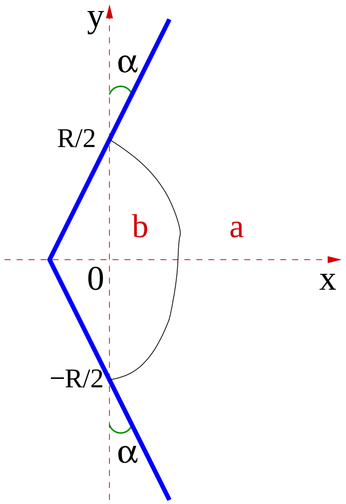
result reduces to $\langle s(x,0)\rangle \sim v\,\mathrm{erf}(z)$

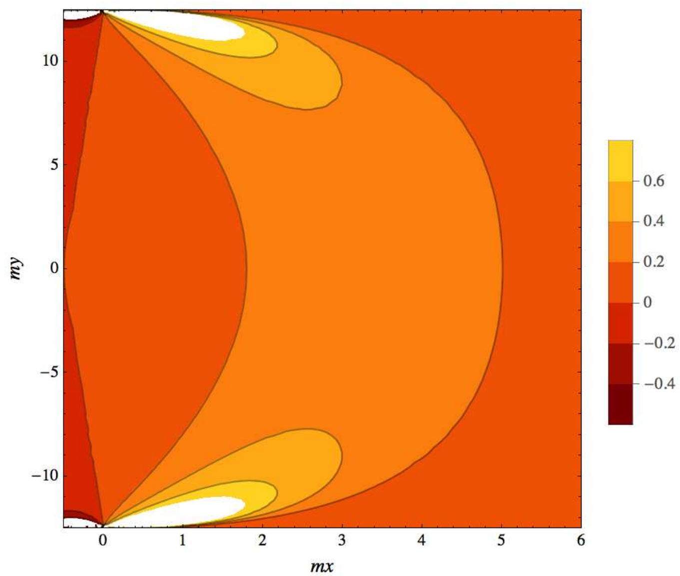
coincides with exact lattice solution
[Abraham, '81]

idea solves longstanding problems



2D Ising: kink





phase separation in a wedge [GD, Squarcini, '14]

Conclusion

- combination of exact analytic results and Monte Carlo simulations allows to check the presence of a vortex particle in the 3D $O(2)$ scalar theory

- its mass close to the XY critical temperature is numerically determined in a universal way

- first direct verification that Derrick's theorem is not an essential obstruction

- the results should be relevant for the controversial problem of defining a mass of vortices in superfluids

- would be interesting to perform the simulations for the 4D $O(3)$ case