Going Relativistic with Lattice Boltzmann Methods

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Outline

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1. Introduction
2. Relativistic Lattice Boltzmann Method
3. Numerical Validation
4. Transport Coefficients
5. Example of Application

Describing a fluid: from micro to macro scale

Scale	**Equations**	**Unknowns**
Microscopic (Å)	Newton	$\mathbf{x}(t)$, $\mathbf{v}(t)$
Mesoscopic ($nm - \mu m$)	Boltzmann (BE)	$f(\mathbf{x}, \mathbf{v}, t)$
Macroscopic ($> mm$)	Navier-Stokes	ρ, $\mathbf{u}$, T

Lattice Boltzmann Methods exploit the dynamic at the mesoscopic scale to provide a correct description of the fluid at the macroscopic level

Boltzmann Equation → Lattice Boltzmann Methods → Navier Stokes Eq.

Boltzmann equation

- Boltzmann equation with BGK collisional model:

$$\partial_t f + \mathbf{v} \cdot \nabla f = \Omega(f, f^{eq}) = -\frac{1}{\tau}(f - f^{eq})$$

- Relaxation to Maxwell-Boltzmann distribution:

$$f^{eq} \propto \exp\left(-\frac{m}{2k_B T}(\boldsymbol{v} - \boldsymbol{u})^2\right)$$

- Macroscopic quantities as the first moments of f

$$\rho(\mathbf{x}, t) = \int f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}$$

$$\mathbf{u}(\mathbf{x}, t) = \frac{1}{\rho(\mathbf{x}, t)} \int \mathbf{v} f(\mathbf{x}, \mathbf{v}, t) d\mathbf{v}$$

$\boldsymbol{v}$ particle velocity $\quad$ $\boldsymbol{u}$ macroscopic fluid velocity $\quad$ τ relaxation time $\quad$ ρ fluid density

Lattice Boltzmann Methods

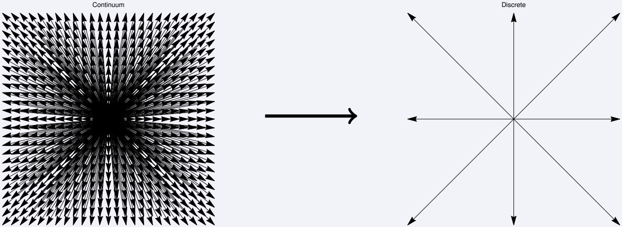
Velocity Discretization

Employ a Gauss Quadrature rule to replace $\mathbf{v}$ with a (small) set of velocities $\mathbf{c_i}$ such that:

- $\mathbf{c_i}$ connects two sites in a Cartesian grid (perfect streaming)
- Equality holds for the following:

$$\rho(\boldsymbol{x}, t) = \int f(\boldsymbol{x}, \boldsymbol{v}, t) d\boldsymbol{v} = \sum_i f_i(\boldsymbol{x}, t)$$

$$\rho(\boldsymbol{x}, t)\boldsymbol{u}(\boldsymbol{x}, t) = \int f(\boldsymbol{x}, \boldsymbol{v}, t)\boldsymbol{v} d\boldsymbol{v} = \sum_i f_i(\boldsymbol{x}, t)\boldsymbol{c_i}$$

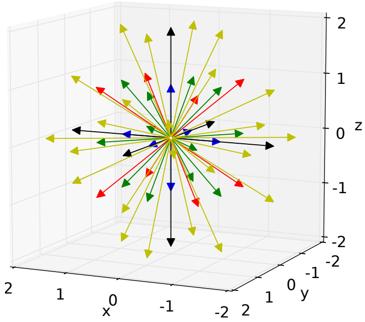


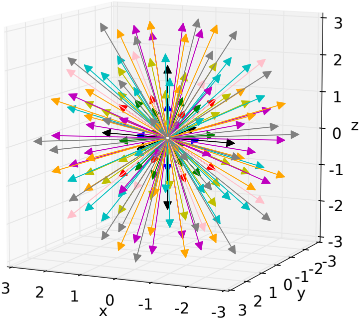
Outline

RLBM: Relativistic Lattice Boltzmann Method

Relativistic extension of Lattice Boltzmann Methods to simulate relativistic hydrodynamics

- Preserves all advantages of classic LB methods
- Perfect streaming: measures not affected by interpolation schemes
- Equipped to simulate fluids in 1, 2, 3 spatial dimensions





Relativistic Boltzmann Equation

- Relativistic Boltzmann Equation with Anderson-Witting collisional model:

$$p^\alpha \frac{\partial f}{\partial x^\alpha} = \Omega(f, f^{eq}) = -\frac{p^\mu U_\mu}{c^2 \tau}(f - f^{eq})$$

- Maxwell Juettner distribution:

$$f^{eq} \propto \exp\left(-\frac{p^\mu U_\mu}{k_B T}\right)$$

- Macroscopic quantities as the first moments of f

$$N^\alpha = c \int f p^\alpha \frac{d\mathbf{p}}{p_0} \qquad\qquad T^{\alpha\beta} = c \int f p^\alpha p^\beta \frac{d\mathbf{p}}{p_0}$$

- Relativistic Parameter $\zeta = \frac{mc^2}{k_B T}$:

ultra-relativistic	mildly relativistic	non-relativistic
$\zeta \ll 1$	$\zeta \sim 1$	$\zeta \gg 1$

p^α particle four-momentum $\quad U^\alpha$ macroscopic four-velocity $\quad \tau$ relaxation time

RLBM: Relativistic Lattice Boltzmann Method

1. Expand f^{eq} in a basis of orthonormal polynomials $\mathbf{J}^{(k)}$ up to an order N capable of recovering the hydrodynamic quantities of interest:

$$f^{eq}(p^\mu, U^\mu, t) = \omega \sum_{k=0}^{N} \mathbf{a}^{(k)}(U^\mu, t) \cdot \mathbf{J}^{(k)}(p^\mu)$$

2. Using the following Gauss Quadrature rule

$$\int \omega(p) \mathbf{J}^{(k)}(p^\mu) \mathbf{J}^{(\ell)}(p^\mu) \frac{d\mathbf{p}}{p_0} = \delta_{\ell k} = \sum_i w_i \mathbf{J}^{(k)}(p_i^\mu) \mathbf{J}^{(\ell)}(p_i^\mu)$$

determine a discrete set of momenta p_i^μ that preserves the hydrodynamic quantities of interest:

$$N^\alpha = c \int f p^\alpha \frac{d\mathbf{p}}{p_0} = \sum_i w_i f_i p_i^\alpha \qquad T^{\alpha\beta} = c \int f p^\alpha p^\beta \frac{d\mathbf{p}}{p_0} = \sum_i w_i f_i p_i^\alpha p_i^\beta$$

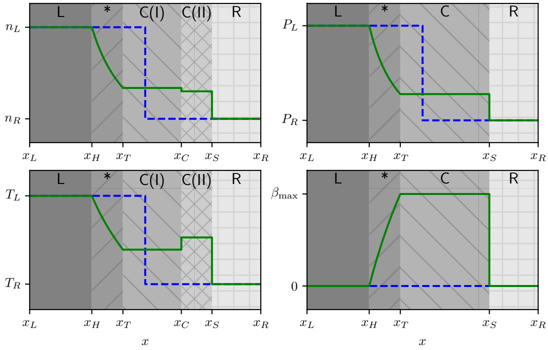
3. Derive the discrete relativistic Boltzmann equation via an Euler scheme:

$$f_i(\boldsymbol{x} + v_i \Delta t, t + \Delta t) - f_i(\boldsymbol{x}, t) = -\Delta t \, \frac{p_i^\mu U_\mu}{p^0 \tau} (f_i - f_i^{eq})$$

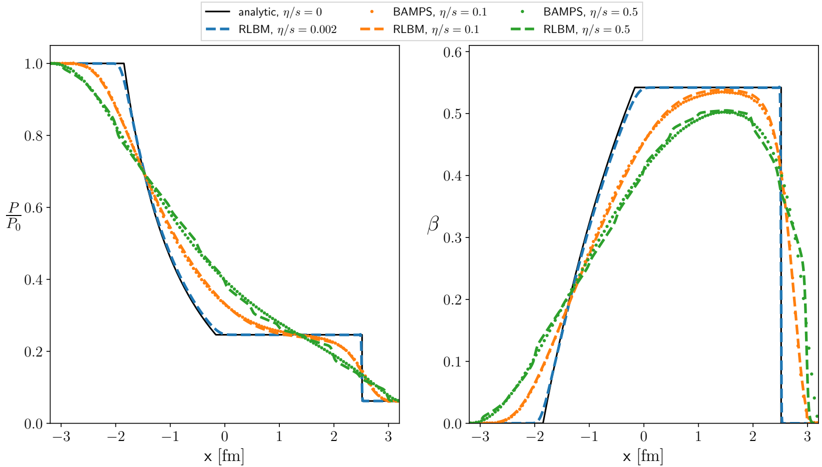
Outline

Numerical Validation: The Riemann Problem

- Commonly used benchmark in hydrodynamics
- It has a semi-analytical solution in the ultra-relativistic & non-dissipative case
- Possible to compare against other numerical methods in the mildly-relativistic / dissipative case



A few comparisons



Outline

Connection between scales

- **Chapman-Enskog's** (CE) and **Grad's** expansions connect mesoscopic to macroscopic scale
- They link macroscopic transport coefficients λ, μ, η to the kinetic ones, in our case the relaxation time τ

$$\lambda = c^2 k_B n f_1(\zeta)\tau \qquad \eta = P f_2(\zeta)\tau \qquad \mu = P f_3(\zeta)\tau$$

- They lead to divergent results in relativistic theories

CE analysis

Multi-scale expansion based on weak departure from local equilibrium:

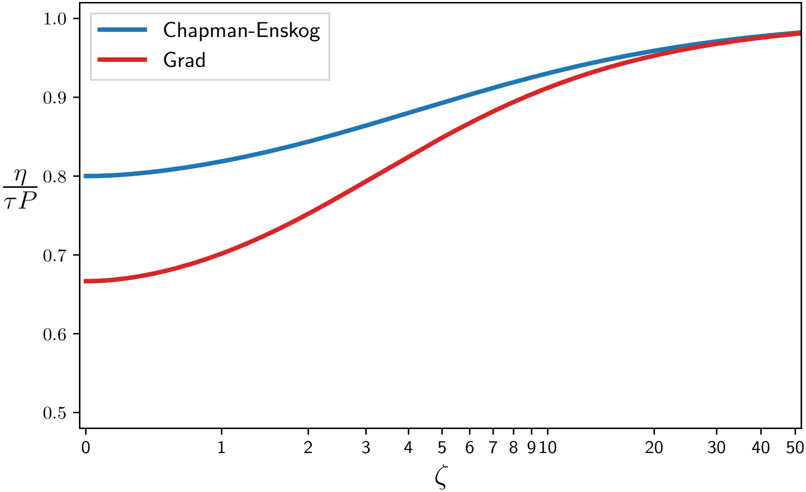
$$f = f^{eq} + \phi(\mathbf{x}, t) f^{eq} \quad \text{with} \quad \phi(\mathbf{x}, t) \sim O(Kn)^1 \qquad Kn = \frac{\lambda}{L}$$

Grad's method

Expansion of the particle distribution f in Hilbert space of the momentum vector p^α around the equilibrium f^{eq}

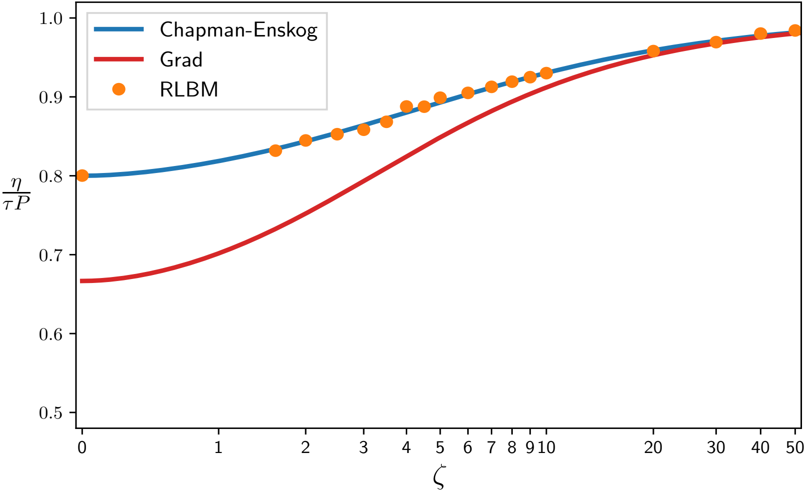
$$f = f^{eq}(1 + a_\alpha p^\alpha + a_{\alpha\beta} p^{\alpha\beta} + \dots)$$

Analytical comparison between the two methods: η



$$\lim_{\zeta\to 0} \eta_{CE} = \frac{4}{5} P\tau \qquad \lim_{\zeta\to 0} \eta_{GR} = \frac{2}{3} P\tau$$

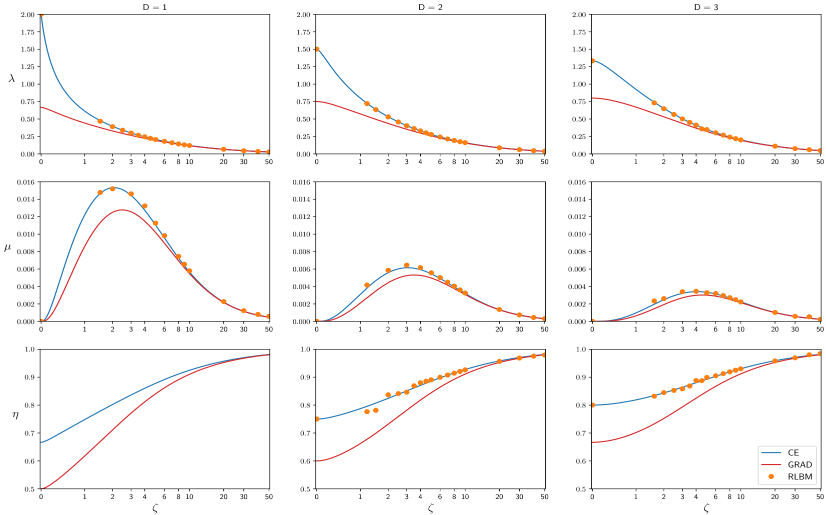
Numerical comparison between the two methods: η



$$\lim_{\zeta \to 0} \eta_{CE} = \frac{4}{5} P\tau \qquad \lim_{\zeta \to 0} \eta_{GR} = \frac{2}{3} P\tau$$

Comparison between the two methods: η, μ, λ

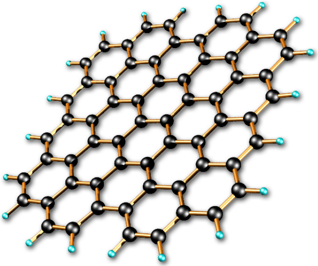
The analysis has been extended to all transport coefficients in all spatial dimensions



Outline

Applications: Graphene

- Free electrons in graphene behave like a plasma of charged particles with a *photon-like* relativistic dispersion relation: $E_k = \hbar v_f k$



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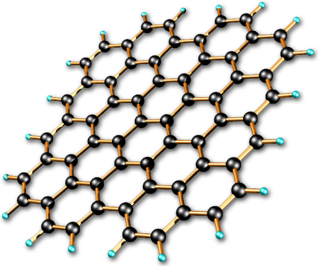
- Electron-electron collisions dominant over electron-phonon scattering even at room temperature



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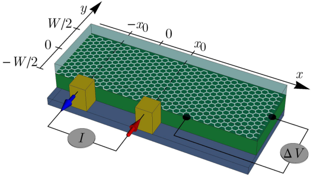
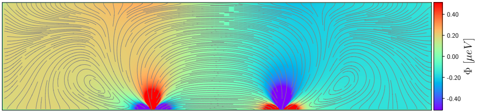
- Experimental observation of anomalous voltage drop near current injectors which originate the formation of whirlpools in the electron flow





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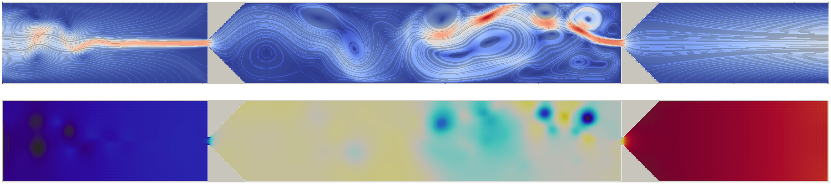
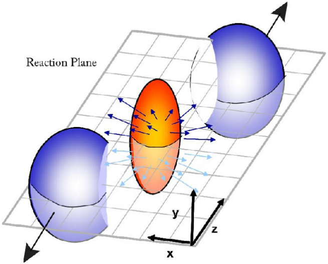
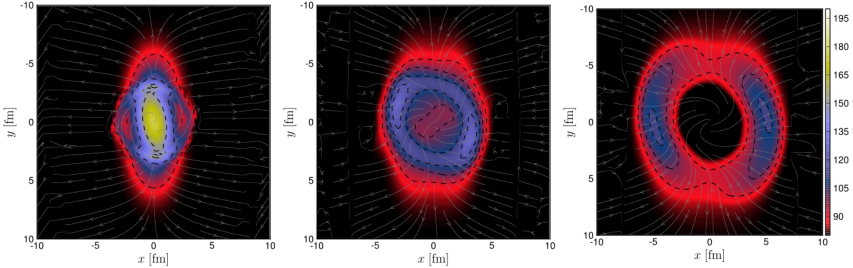
- Experimental observation of anomalous voltage drop near current injectors which originate the formation of whirlpools in the electron flow
- Theoretical study of pre-turbulent regimes at low Reynolds



Outlook: Quark Gluon Plasma





Sum up and Conclusions

Sum up:

- Developed a fast numerical scheme for the simulation of Relativistic Hydrodynamics
- Validated the code with the Riemann problem Benchmark
- Created a firm connection between the kinetic and the hydrodynamic layer via the Chapmann-Enskog expansion
- Correctly reproduced experiments of electron flows in graphene

Future steps:

- Introduce external magnetic field in graphene simulations to check for early occurrence of turbulent regimes
- Start using RLBM for Quark Gluon Plasma simulations

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Thank you for your attention!

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