Giovanni Tricella

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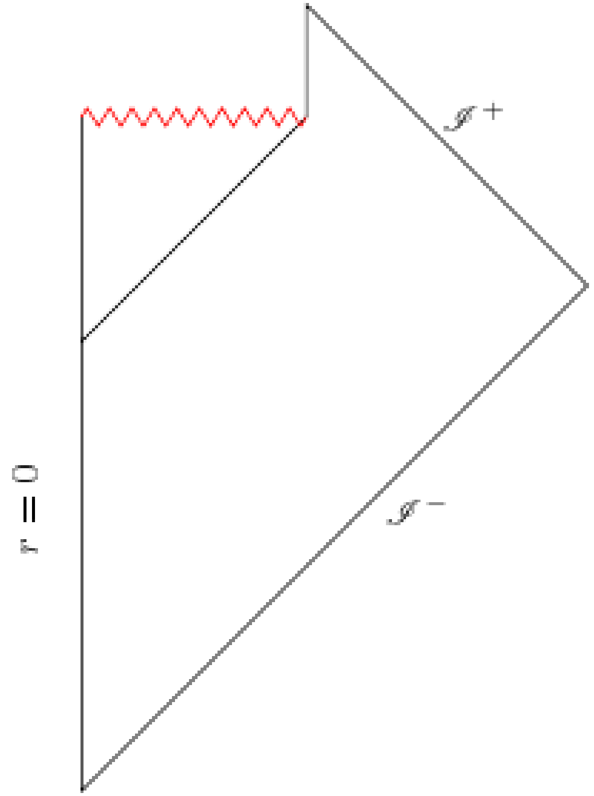
The information loss problem: an analogue gravity perspective

S. Liberati, G. Tricella, A. Trombettoni, Entropy 2019, 21(10), 940

SISSA

Motivation



Hawking Radiation raises the problem of information loss.

- The final state is Minkowski + thermal bath;
- HR requires UV completion (transplanckian problem);
- QFT in CS assumes separation of the Hilbert space (matter + geometry), QG requires unification.

Phenomenology simulated with analogue gravity.
In Bose-Einstein condensates, AG is an emergent theory of linearised excitations from atomic many-body theory.

AG inherits unitarity from the atomic nature of the substrate. Does it show correlation between the two sectors, when quantum features are retained?

Analogue Gravity

Non-relativistic BEC in Madelung representation

$$\phi = \langle\phi\rangle + \delta\phi \qquad \langle\phi\rangle = \rho_0^{1/2} e^{i\theta_0} \qquad \delta\phi = \rho_0^{1/2} e^{i\theta_0}\left(\frac{\rho_1}{2\rho_0} + i\theta_1\right)$$

Condensate: hydrodynamical equations for the perfect fluid (no vorticity, no viscosity)

$$\begin{aligned}\partial_t\rho_0 &= -\frac{1}{m}\nabla\left(\rho_0\left(\nabla\theta_0\right)\right)\\ \partial_t\theta_0 &= \frac{1}{2m}\frac{\nabla^2\sqrt{\rho_0}}{\sqrt{\rho_0}} - \frac{1}{2m}\left(\nabla\theta_0\right)\left(\nabla\theta_0\right) - \lambda\rho_0 - V_{\mathrm{ext}}\end{aligned}$$

Excited part: Klein-Gordon equation of an acoustic metric (for a negligible quantum pressure $\nabla\left(\rho_0^{-1}\left(\nabla\rho_1\right)\right) \ll 4m\partial_t\theta_1$)

$$\begin{aligned}\rho_1 &= -\frac{1}{\tilde{\lambda}}\left(\partial_t\theta_1 + \frac{1}{m}\left(\nabla\theta_0\right)\left(\nabla\theta_1\right)\right)\\ \partial_t\rho_1 &= -\frac{1}{m}\nabla\left(\rho_1\left(\nabla\theta_0\right) + \rho_0\left(\nabla\theta_1\right)\right)\end{aligned} \Rightarrow \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\left(\partial_\nu\theta_1\right)\right)$$

Analogue Gravity

Analogue metric does not follow Einstein's equations, but there is hierarchy in the analogy:

Mean field + Bogoliubov	QFT in Curved Spacetime
Mean field + Bogoliubov + backreaction (anomalous mass and density)	Semiclassical Gravity
BEC (2-point correlation function) + Quasiparticles	Quantum Gravity + matter
Many-body atomic theory	Theory of Everything

Natural orbitals

Definition of condensation from the 2-point correlation function, diagonalised by time-dependent natural orbitals, a basis of the 1-particle Hilbert space.

$$\left\langle \phi^{\dagger}(x)\, \phi(y) \right\rangle = \quad \textstyle\sum_I \langle N_I \rangle\, \overline{f_I}(x)\, f_I(y) \qquad \displaystyle\sum_I \overline{f_I}(x)\, f_I(y) = \quad \delta(x, y)$$

The condensate eigenfunction differs slightly from the solution of the Gross-Pitaevskii equation.

$$i\partial_t f_0(x) = -\frac{i}{2}\frac{\partial_t \langle N_0 \rangle}{\langle N_0 \rangle} f_0(x) + \left(-\frac{\nabla^2}{2m} f_0(x) \right) + \frac{1}{\langle N_0 \rangle} \left\langle a_0^{\dagger} \left[\phi(x), V \right] \right\rangle +$$

$$+ \sum_{I \neq 0} \frac{\langle N_I \rangle \left\langle a_0^{\dagger} \left[a_I, V \right] \right\rangle + \langle N_0 \rangle \left\langle \left[a_0^{\dagger}, V \right] a_I \right\rangle}{\langle N_0 \rangle \left(\langle N_0 \rangle - \langle N_I \rangle \right)} f_I(x)$$

Approximations

It is necessary to consider approximations:

$$\left\langle a_0^\dagger \left[\phi\left(x\right), V\right]\right\rangle \approx \lambda \left\langle N_0\right\rangle^2 \overline{f_0}\left(x\right) f_0\left(x\right) f_0\left(x\right)$$

$$i\partial_t \left\langle N_0\right\rangle = \left\langle \left[a_0^\dagger a_0, V\right]\right\rangle \approx 0$$

$$\sum_{I\neq 0} \frac{\left\langle N_I\right\rangle \left\langle a_0^\dagger \left[a_I, V\right]\right\rangle + \left\langle N_0\right\rangle \left\langle \left[a_0^\dagger, V\right] a_I\right\rangle}{\left\langle N_0\right\rangle - \left\langle N_I\right\rangle} \approx 0$$

Hierarchy provided by linearisation of the theory, reproducing functionally the depletion in the Bogoliubov description.

Strongest case: homogeneous number density.

$$\left\langle a_0^\dagger \left[a_I, V\right]\right\rangle \approx \lambda \left(\int \mathrm{d}x \overline{f_I}\left(x\right) \overline{f_0}\left(x\right) f_0\left(x\right) f_0\left(x\right)\right) \left\langle N_0\right\rangle^2$$

Number conserving formalism

We retain the quantum nature of the atoms in the condensate, making use of the definition of condensed 1-particle state from the natural orbitals

$$a_I = \int \mathrm{d}x \overline{f_I}(x)\,\phi(x) \qquad N_I = a_I^\dagger a_I$$

$$\alpha_I = a_0^\dagger (N_0 + 1)^{-1/2} a_I$$

$$\left[\alpha_I, \alpha_J^\dagger\right] = \left[a_I, a_J^\dagger\right] = \delta_{IJ} \quad \forall I, J \neq 0$$

$$\left[\alpha_I, \alpha_J\right] = \left[a_I, a_J\right] = 0 \quad \forall I, J \neq 0$$

Approximate linearised dynamics for these operators, more accurate description

Ladder operators relative to the orbitals define a new set of number conserving ladder operators

Analogue Gravity with number conservation

New definition for the excited part: not given by translation of the field, but number conserving fluctuations + projection

$$\begin{aligned} \delta\phi(x) &= \sum_{I\neq 0} f_I(x)\,\alpha_I \\ \rho_0^{1/2} e^{i\theta_0} &= \langle N_0\rangle^{1/2} f_0(x) \end{aligned}$$

There is the same functional dependence from the solution of the Gross-Pitaevskii equation

$$i\partial_t\delta\phi = -\frac{\nabla^2}{2m}\delta\phi + 2\lambda\rho_0\delta\phi + \lambda\rho_0 e^{2i\theta_0}\delta\phi^\dagger$$

and therefore the same acoustic metric, following the steps of the Madelung representation

$$\theta_1 = -\frac{i}{2}\left(\frac{N_0^{-1/2}\phi_0^\dagger\phi_1 - \phi_1^\dagger\phi_0 N_0^{-1/2}}{\langle N_0\rangle^{1/2}\,\overline{f_0} f_0}\right)$$

$$\rho_1 = \langle N_0\rangle^{1/2}\left(N_0^{-1/2}\phi_0^\dagger\phi_1 + \phi_1^\dagger\phi_0 N_0^{-1/2}\right)$$

$$[\theta_I, \rho_J] = -i\overline{f_I} f_I \delta_{JI}$$

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Cosmological particle creation

Analogue cosmology best satisfies the approximations due to condensate homogeneity (description in the basis of momenta). Time dependence of the coupling: simulation of the expanding spacetime. Quasi-particle creation is analogous to cosmological particle creation.

$$\theta_k = -\frac{i}{2}\sqrt{V}\left(\delta\phi_k - \delta\phi^\dagger_{-k}\right) \qquad \partial_t\theta_k = -\frac{1}{2}\left(\frac{k^2}{2m} + 2\lambda\rho_0\right)\frac{\rho_k}{\rho_0}$$

Quantum pressure gives a Lorentz breaking dispersion relation

$$\theta_k\left(t\to-\infty\right) = \frac{1}{\mathcal{N}_k}\left(e^{-i\omega_k t}c_k + e^{i\omega_k t}c^\dagger_{-k}\right) \quad \theta_k\left(t\to+\infty\right) = \frac{1}{\mathcal{N}'_k}\left(e^{-i\omega'_k t}c'_k + e^{i\omega'_k t}c'^\dagger_{-k}\right)$$

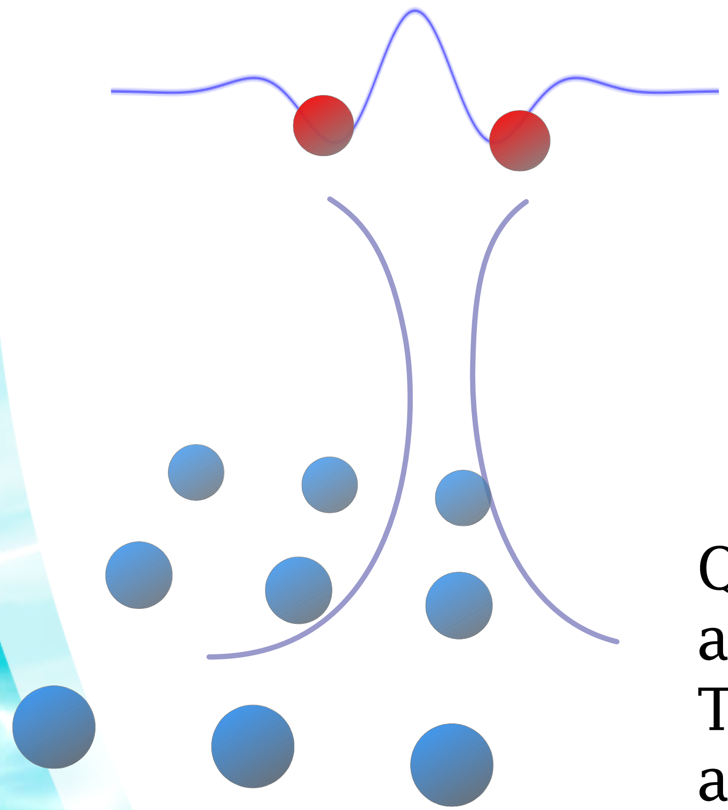
Number conserving atom operators and quasi-particle operators are related through Bogoliubov transformations

$$c_k = e^{-i\alpha_k}\cosh\Lambda_k\delta\phi_k + e^{i\beta_k}\sinh\Lambda_k\delta\phi^\dagger_{-k}$$

$$c'_k = \cosh\Theta_k c_k + \sinh\Theta_k e^{i\varphi_k}c^\dagger_{-k}$$

Cosmological particle creation

The quasi-particle creation is an extraction mechanism affecting the atomic substructure. At late times:



$$\left\langle \delta\phi_k^\dagger(t)\,\delta\phi_k(t)\right\rangle = \frac{1}{2}\frac{\frac{k^2}{2m}+\lambda'\rho_0}{\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m}+2\lambda'\rho_0\right)}}\cosh(2\Theta_k)-\frac{1}{2}+$$
$$+\frac{1}{2}\frac{\lambda'\rho_0}{\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m}+2\lambda'\rho_0\right)}}\sinh(2\Theta_k)\cdot$$
$$\cdot\cos\left(2\sqrt{\frac{k^2}{2m}\left(\frac{k^2}{2m}+2\lambda'\rho_0\right)}t-\varphi_k\right)$$

Quasi-particles are created exciting condensate atoms.
The process is defined on the whole Fock space, and does not require coherent states.

Scattering Operator

The Bogoliubov transformation is produced by the unitary operator relating late and early times, squeezing quasi-particle states.

$$
\begin{aligned}
S^\dagger c_k S = c'_k &= \cosh\Theta_k c_k + \sinh\Theta_k e^{i\varphi_k} c^\dagger_{-k} \\
&\Downarrow \\
S &= \exp\left(\frac{1}{2} \sum_{k\neq 0} \left(-e^{-i\varphi_k} c_k c_{-k} + e^{i\varphi_k} c^\dagger_k c^\dagger_{-k} \right) \Theta_k \right)
\end{aligned}
$$

It is an operatorial expression valid for all condensed states (in the sense of 2-p c.f.), approximating the exact operator of the full theory.
This operator is unique, being unique the linearisation providing the Bogoliubov transformation, and must satisfy the conservation laws of the Hamiltonian.
It is a unitary transformation: preserves purity of the state.

Entanglement and correlations

The quasi-particle vacuum can be redefined in terms of atom degrees of freedom

$$\left|\emptyset\right\rangle_{qp} = \mathcal{N}^{-1} \prod_k \exp\left(-\frac{1}{2} e^{i(\alpha_k+\beta_k)} \tanh \Lambda_k \delta\phi_k^\dagger \delta\phi_{-k}^\dagger\right) \left|\emptyset\right\rangle_a$$

The creation of coupled quasi-particles acts differently depending on the number of condensate atoms

$$\left(\delta\phi_k^\dagger \delta\phi_{-k}^\dagger\right)^n a_0^\dagger = a_0^\dagger \left(\delta\phi_k^\dagger \delta\phi_{-k}^\dagger\right)^n \left(\frac{N_0+1}{N_0+1-2n}\right)^{1/2}$$

Correlation between quasiparticles and background is built

$$\langle c_k c_{k'} N_0\rangle - \langle c_k c_{k'}\rangle \langle N_0\rangle = -\sum_q \left(\left\langle c_k c_{k'} \delta\phi_q^\dagger \delta\phi_q\right\rangle - \langle c_k c_{k'}\rangle \left\langle \delta\phi_q^\dagger \delta\phi_q\right\rangle\right) + \\ + \langle c_k c_{k'} N\rangle - \langle c_k c_{k'}\rangle \langle N\rangle$$

Factorisation of the state

Factorisation cannot be preserved along the evolution

$$\left|\langle N\rangle\right\rangle_{mf}\otimes\sum_{lr}a_{lr}\left|l,r\right\rangle_{a\ \ Bog}\Rightarrow\left|\langle N\rangle\right\rangle_{mf}\otimes\sum_{lr}a'_{lr}\left|l,r\right\rangle_{a\ \ Bog}$$

$$\sum_{lr}a_{lr}\left|N,l,r\right\rangle_{a}\Rightarrow\sum_{lr}a'_{lr}\left(1+\mathcal{O}\left(N^{-1}\right)\right)\left|N-l-r,l,r\right\rangle_{a}$$

Factorisation, separation of the state between two sectors, is a requirement of QFT in CS and of semiclassical gravity.

Loss of factorisation therefore implies that (within analogue gravity) these assumption is insufficient to take into account the complexity of the full evolution.

State structure: density matrix

Direct consequence on the density matrix

$$\mathrm{Tr}\left[\rho_{fin}\right] = \mathrm{Tr}\left[\rho_{in}\right]$$

$$\mathrm{Tr}\left[\rho_{fin}^{reduced}\right] = \mathrm{Tr}_0\left[\rho_{fin}\right]$$

$$\mathrm{Tr}\left[\rho_{fin}^2\right] = 1 \nRightarrow \mathrm{Tr}\left[\left(\rho_{fin}^{reduced}\right)^2\right] = 1$$

In summary:

- Quasi-particle production implies depletion
- Depletion implies loss of factorisation (i.e. loss of coherence in the mean-field approximation)
- Loss of factorisation implies correlation

Conclusions

- AG can be extended beyond coherent states;
- AG can take into account the quantum nature of the condensate atoms, showing correlation between quasi-particles and analogue geometry (condensate);
- The unitarity of the evolution can be reconstructed, but only knowing the properties of both the parts of the system: condensate and excitations;
- The condensate is affected by the quasi-particle creation (atom extraction) which entangles the two parts (factorisation is not preserved);
- AG let us argue for QG to be the only environment where to look for the solution of the information loss paradox and loss of unitarity, even for regular spacetimes.

Analogue Gravity

Bogoliubov-de Gennes equation for the fluctuations in phase and density

$$\partial_t \theta_1 = -\left(\lambda + \frac{1}{4m}\rho_0^{-1}\nabla\left(\rho_0^{-1}\left(\nabla\rho_0\right)\right)\right)\rho_1 - \frac{1}{m}\left(\nabla\theta_0\right)\left(\nabla\theta_1\right) + \frac{1}{4m}\nabla\left(\rho_0^{-1}\left(\nabla\rho_1\right)\right)$$

$$\partial_t \rho_1 = -\frac{1}{m}\nabla\left(\rho_1\left(\nabla\theta_0\right) + \rho_0\left(\nabla\theta_1\right)\right)$$

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$$+\frac{1}{4m}\nabla\left(\rho_0^{-1}\left(\nabla\rho_1\right)\right)$$

When this **quantum pressure** is negligible

$$\rho_1 = -\frac{1}{\tilde{\lambda}}\left(\partial_t\theta_1 + \frac{1}{m}\left(\nabla\theta_0\right)\left(\nabla\theta_1\right)\right)$$

Analogue Gravity

The quantum fluctuation of the phase satisfies a Klein-Gordon equation for an acoustic metric given by functions of the condensate and of the coupling

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\left(\partial_\nu\theta_1\right)\right) = 0$$

$$g_{tt} = -\sqrt{\frac{\rho_0}{m\widetilde{\lambda}}}\left(\frac{\widetilde{\lambda}\rho_0}{m} - v^2\right) \qquad g_{ij} = \sqrt{\frac{\rho_0}{m\widetilde{\lambda}}}\delta_{ij}$$

$$g_{it} = -\sqrt{\frac{\rho_0}{m\widetilde{\lambda}}}v_i \qquad \sqrt{-g} = \sqrt{\frac{\rho_0^3}{m^3\widetilde{\lambda}}}$$

$$\widetilde{\lambda} = \lambda + \frac{1}{4m}\rho_0^{-1}\nabla\left(\rho_0^{-1}\left(\nabla\rho_0\right)\right) \qquad v = \frac{\nabla\theta_0}{m}$$

Time-dependent natural orbitals

More general definition of condensation comes from the 2-point correlation function [Penrose-Onsager criterion]

$$\left\langle \phi^{\dagger}(x)\,\phi(y)\right\rangle = \sum_I \langle N_I\rangle\,\overline{f_I}(x)\,f_I(y)$$

Condensation for $\langle N_0\rangle / \langle N\rangle \approx 1$

Basis of eigenfunctions:

$$\begin{aligned} a_I &= \int \mathrm{d}x\,\overline{f_I}(x)\,\phi(x) \\ \left[a_I, a_J^{\dagger}\right] &= \delta_{IJ} \\ \left[a_I, a_J\right] &= 0 \\ N_I &= a_I^{\dagger} a_I \end{aligned}$$

Time-dependent natural orbitals

Under the following conditions, the Gross-Pitaevskii equation is retrieved:

$$\left\langle a_0^\dagger \left[\phi\left(x\right), V\right]\right\rangle \approx \lambda \left\langle N_0\right\rangle^2 \overline{f_0}\left(x\right) f_0\left(x\right) f_0\left(x\right)$$

$$i\partial_t \left\langle N_0\right\rangle = \left\langle \left[a_0^\dagger a_0, V\right]\right\rangle \approx 0$$

$$\sum_{I\neq 0} \frac{\left\langle N_I\right\rangle \left\langle a_0^\dagger \left[a_I, V\right]\right\rangle + \left\langle N_0\right\rangle \left\langle \left[a_0^\dagger, V\right] a_I\right\rangle}{\left\langle N_0\right\rangle - \left\langle N_I\right\rangle} f_I\left(x\right) \approx \quad 0$$

When the effects of the depletion are negligible

Time-dependent natural orbitals

To retrieve the same quadratic depletion terms, we need further approximations:
- negligible contributions from fully connected diagrams

$$\left\langle a_0^\dagger a_0^\dagger a_0 \phi_1 \right\rangle = \left\langle a_0^\dagger \phi_1 \left(N_0 - \langle N_0 \rangle\right) \right\rangle$$

- approximate homogeneity of the atomic density

$$\begin{aligned} \left\langle a_0^\dagger \left[a_I, V\right] \right\rangle &\approx \lambda \left(\int \mathrm{d}x \overline{f_I}\left(x\right) \overline{f_0}\left(x\right) f_0\left(x\right) f_0\left(x\right) \right) \langle N_0 \rangle^2 \\ &\Downarrow \\ \left| f_0\left(x\right) \right| &\approx V^{-1/2} \end{aligned}$$

Simulating Cosmology

Analogy to cosmological metric

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = \sqrt{\frac{\rho_0}{m\lambda}}\left(-\frac{\lambda\rho_0}{m}\mathrm{d}t^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\right)$$

$$g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu = -\mathrm{d}\tau^2 + a^2\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

$$\begin{aligned} a\left(\tau\left(t\right)\right) &= \left(\frac{2m}{\lambda\left(t\right)\rho_0 C^2}\right)^{1/4} \\ \mathrm{d}\tau &= \frac{\mathrm{d}t}{C a\left(\tau\left(t\right)\right)} \end{aligned}$$