BH RINGDOWN AS A PROBE FOR DARK ENERGY

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DARK ENERGY

DARK MATTER

Today's perspective: put aside our prejudices in favor of empirical verification



On cosmological scales, Dark Energy acts like a "medium" with a homogeneous and isotropic stress energy tensor that breaks spontaneously Lorentz invariance

Gravitational perturbations that travel on this background carry information about the underlying microscopic theory, already at the level of the quadratic action (speed of propagation can be different from *c*, damping)

$$\ddot{\gamma}_{ij} + H(3+\alpha)\dot{\gamma}_{ij} + c_T^2 k^2 \gamma_{ij} = 0$$

GW170817 = GRB170817A



Abbott et al. '17

$$c_T^2 - 1 \lesssim 10^{-15}$$















Inspiral



Merger



Ringdown

The observed accelerated expansion of the Universe can be given by new DOF that either:

1) have a background value that produce a sizable stress-energy tensor

2) affect the propagation of gravity, without any large contribution to T

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The dynamics is made structurally robust by symmetries (exact & approximate) Well defined low-energy EFTs

Allows for the largest variety of potentially observable signatures

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Assumption: exact shift symmetry $\ \pi \to \pi + c$

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Non pathological higher derivative (HD) operators are enhanced $\Lambda_2 \gg \Lambda_3$ robust due to the approximate global symmetry $\partial_\mu \pi \to \partial_\mu \pi + b_\mu$

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No gravitational coupling to the Standard Model

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Observational Constraints

Some HD operators modifies the speed of gravitational waves

$$\frac{(\partial \pi)^2 (\partial^2 \pi)^2}{\Lambda_3^6} \to \frac{(\partial \pi_0)^4}{\Lambda_3^6 M_{\rm P}^2} (\partial h)^2$$

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implies either

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$$c_T^2 - 1 \lesssim 10^{-15} \qquad \text{implies either}$$

1) the coefficients of those operators must be very small

2) the UV cutoff of the theory is higher

$$M_{Pl}$$

$$\Lambda_2 = (H_0^2 M_{Pl}^2)^{1/4}$$

$$M_{\Lambda_3} = (H_0^2 M_{Pl})^{1/3}$$

$$H_0$$

1. It explains why the speed of gravitational waves is the speed of light

$$\frac{(\partial \pi)^2 (\partial^2 \pi)^2}{M^6} \supset \alpha^6 \frac{(\partial \pi)^4 (\partial h)^2}{\Lambda_3^6 M_{\rm Pl}^2} \qquad \alpha \equiv \frac{\Lambda_3}{M} \lesssim 10^{-3}$$

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2. Positivity bounds from scattering amplitudes suggest that the symmetry breaking operator $(\partial \pi)^4$ cannot be to small compared to the invariant ones

Positivity bounds

The general properties of the S-matrix (unitarity, analyticity) imply dispersion relations for forward elastic scattering amplitudes positivity bounds for amplitudes in the IR.

$$-(\partial \pi)^2 + a \frac{(\partial \pi)^4}{\Lambda_2^4} - \frac{(\partial \pi)^2 \Box \pi}{\Lambda_3^3} \qquad a > 0$$

Adams, Arkani-Hamed, Dubovsky, Nicolis, Rattazzi '06

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the scales Λ_2 and Λ_3 cannot be arbitrarily separated while keeping the UV cutoff fixed

$$a^{\pi\pi} = \frac{1}{\Lambda_2^4} > \frac{2}{\pi} \int^{\Lambda_{UV}^2} \frac{ds}{s^3} \mathrm{Im}\mathcal{M}^{\pi\pi}(s) \propto \frac{1}{16\pi^2} \frac{\Lambda_{UV}^8}{\Lambda_3^{12}}$$

$$\Lambda_{UV} < \left(H^3 m_{\rm Pl}\right)^{1/4} \left(\frac{16\pi^2}{c}\right)^{1/8} \sim \frac{1}{10^7 \,\rm km}$$

Bellazzini, Lewandowski, Serra '19

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- 2. Positivity bounds from scattering amplitudes suggest that the symmetry breaking operator $(\partial \pi)^4$ cannot be to small compared to the invariant ones
- Extends the regime of validity at short length scales:
 20 solar masses black holes are within the EFT range (still above tabletop experiments but ...)

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Black Hole Background



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Deviation from GR predictions:

- 1. the scalar non-trivial profile deforms the BH geometry
- 2. there is a quadratic mixing between scalar and gravitational perturbations

Both will affect the QNM spectrum

No Hair Theorem

Hui, Nicolis '12

Scalar EOM
$$\nabla_{\mu}J^{\mu} = 0$$
 $ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + \rho^2(r)d\Omega^2$

If the solution is static and spherical symmetric only $J^r \neq 0$

 $J^{\mu}J_{\mu} = (J^{r})^{2}/f$ should be regular at the horizon $\implies J^{r} = 0$ at the horizon

Using the conservation of the current $\implies J^r(r) = 0$

One last crucial step is need to conclude that a vanishing current implies a constant scalar

$$J^r = f \cdot \pi' \cdot F(\pi'; g, g', g'')$$
 F is a polynomial
 π' vanishes at infinity
 F asymptotes to a constant at infinity

Then $\pi'(r) = 0$

Scalar coupled to Gauss-Bonnet

$$aM_{\rm Pl} \pi \mathcal{G} \qquad \qquad \mathcal{G} \equiv R^{\mu\nu\lambda\kappa}R_{\mu\nu\lambda\kappa} - 4R^{\mu\nu}R_{\mu\nu} + R^2$$

The GB invariant is a total derivative: the coupling is shift invariant

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What is aM_{Pl} ? The GB coupling breaks explicitly the galileon symmetry

 $\frac{1}{M_{\rm Pl}M} < a < \frac{M_{\rm Pl}}{M^3}$ ~~ is consistent with the power counting of the EFT

When R is linearized it can be rewritten as $\partial^2\pi\partial h\partial h$

Hairy BH



Hairy BH



Solution of the scalar EOM with spherical symmetry

$$X_0(r)^{n+(m-1)/2} \sim aM_{\rm Pl} \frac{r_s^2}{\Lambda_2^2 r^5} \left(\frac{r\Lambda^3}{\Lambda_2^2}\right)^m$$

$$(\partial \pi)^2 + \Lambda_2^4 \frac{(\partial \pi)^{2n}}{\Lambda_2^{4n}} \frac{(\partial \partial \pi)^m}{M^{3m}} + aM_{\rm Pl}\pi R^2$$

There is a large kinetic mixing

$$a \frac{r_s}{r^3} \partial h_c \partial \pi \equiv \mathcal{Z}_{mix}^{GB} \partial h_c \partial \pi$$

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Not so simple...
$$\mathcal{Z}_{\pi}(\partial\pi)^2 + \mathcal{Z}_{mix}^{GB} \partial h_c \partial \pi$$

Deviations in the QNM spectrum depends on $\varepsilon_{mix}(r) \equiv \frac{Z_{mix}^{GB}}{\sqrt{Z_{\pi}}}$ evaluated at the BH light ring $r = \frac{3}{2}r_s$

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The same coupling gives rise to the emission of scalar wave in the inspiral phase It is already constrained: $a_{insp} = \frac{a}{\sqrt{Z_{\pi}(r_{insp})}} \lesssim 2 \,\mathrm{km}^2$

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m	n	Operator	$\varepsilon_{mix}(r=r_s=10\mathrm{km})$	$\varepsilon_{mix}(E-M)$	$\Lambda_{min}[{ m km}^{-1}]$	$lpha[{ m km}^2]$
0	1	X	3×10^{-2}	10^{-21}	10^{4}	10
0	2	X^2	$3 imes 10^{-3}$	3×10^{-11}	10^{2}	3×10^9
1	1	XZ	$3 imes 10^{-3}$	3×10^{-13}	10^{2}	$3 imes 10^{10}$
1	2	X^2Z	10^{-3}	10^{-9}	10^{2}	$3 imes 10^{14}$
2	1	XZ^2	10^{-3}	3×10^{-10}	10^{2}	10^{20}

Subluminality problem

The cutoff can be lower around non-linear backgrounds

 $rac{(\partial \pi)^2 (\partial \partial \pi)^2}{\Lambda_2^6}$

Around a point-like source $Z_0^2 (\partial_{\parallel} \pi)^2 + (\partial_{\perp} \pi)^2$

The energy cutoff is significantly lowered due to the scattering of the slow moving modes along the transverse directions

This is not true anymore in the BH background: there is no cancellation

$$Z_0^2 \left((\partial_{\parallel} \pi)^2 + (\partial_{\perp} \pi)^2 \right)$$

Conclusions

Dynamical DE is still a possibility

Maybe it leaves an imprint during BH merger

Not much is know about BH with non-trivial scalar backgrounds

Several ways to avoid the ho-hair theorems (non-trivial boundary conditions, timedependent solutions, breaking of the shift symmetry, HD interactions,...)

Useful to use an EFT framework to describe QNMs of hairy black hole

Several important missing step: how to generalize to Kerr, ...

Exponentially damped sinusoidal waves: Quasi Normal Modes (QNM)

Spectrum of characteristic (complex) frequencies ω_{nlm}



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Spectrum of characteristic (complex) frequencies ω_{nlm}



n	$2M_{\bullet}\omega (L=2)$	$2M_{\bullet}\omega (L=3)$	$2M_{\bullet}\omega$ (<i>L</i> = 4)
0	0.747 343 + 0.177 925i	1.198 887 + 0.185 406i	1.618 36 + 0.188 32i
1	0.693 422 + 0.547 830i	1.165 288 + 0.562 596i	1.593 26 + 0.568 86i
2	0.602 107 + 0.956 554i	1.103 370 + 0.958 186i	1.545 42 + 0.959 82i
3	0.503 010 + 1.410 296i	1.023 924 + 1.380 674i	1.479 68 + 1.367 84i

In GR black holes are characterized only by 3 parameters: M, J, Q

Nollert '99

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 $g_{\mu\nu} = g_{\mu\nu}^{\rm BH}(r) + h_{\mu\nu} \qquad \text{Schwarzschild: static, spherically symmetric background} \\ h(t, r, \theta, \phi) = \sum_{lm} h_{lm}(r) Y_{lm}(\theta, \phi) e^{i\omega t} \\ \text{Classified accordingly to the behavior under parity} \quad (\theta, \phi) \to (\pi - \theta, \phi + \pi) \end{cases}$

Axial (odd) perturbations

Polar (even) perturbations

Regge Wheeler '57

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Axial (odd) perturbations Regge Wheeler '57 Polar (even) perturbations

Fix the gauge + solve for the constraint

One propagating DOF in the odd sector

One propagating DOF in the even sector

$$\Big[\frac{d^2}{dr^2} + \omega^2\Big]h(r) = V^{(-)}(r)h(r)$$

$$V^{(-)}(r) = \frac{l(l+1)}{r^2} \left(1 - \frac{r_S}{r}\right) - 3\frac{r_S}{r^3} \left(1 - \frac{r_S}{r}\right)$$

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$$V^{(+)}(r) = \dots$$

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EFT for perturbations in spher symm bkgrd

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The linearized equations of motion are modified

More information than just the velocity: the whole QNM spectra are modified



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If the strength of the scalar-matter coupling is gravitational or bigger

the most prominent observational signal would be the scalar mode itself (the extra mode in the even sector)

EFT for perturbations in spher symm bkgrd

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introduce deviations from GR in the spectrum of even and odd modes while preserving isospectrality

break isospectrality

mix the even and odd modes if it is a pseudo-scalar

Quantum Corrections



$$\partial_{\mu}\pi \to \partial_{\mu}\pi + b_{\mu}$$

Only operators with al least $\partial^2 \pi$ are generated



Quantum Corrections



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