On scattering amplitudes in the presence of gravity

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PART I: Motivations

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Gravity is an EFT characterized by the effective interaction strength

$$\boxed{g_{\mathrm{eff}}^2 \equiv G_N E^2 = \frac{E^2}{M_{\mathrm{Pl}}^2}}$$

It becomes strongly coupled at energies $E \gtrsim M_{\mathrm{Pl}}$ thus demanding an ultraviolet (UV) completion.

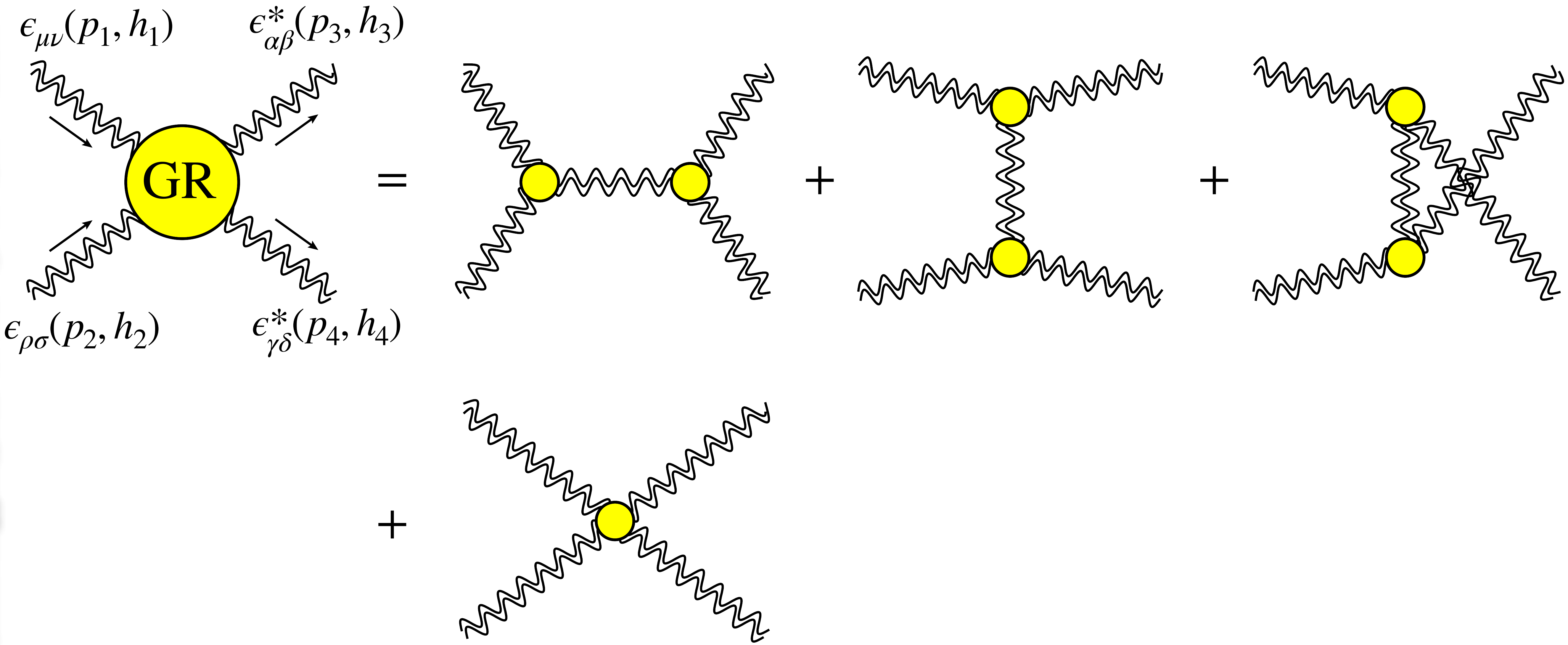
PART I: Motivations

What is the UV-completion of Gravity?

We try to answer this question by means of a "**bottom-up**" approach based on **on-shell spinor-helicity methods**.

PART I: Motivations

$$\mathscr{A}\left(1^{h_1}2^{h_2} \to 3^{h_3}4^{h_4}\right)$$



B. S. DeWitt, Phys. Rev. 162, 1239 (1967)

PART I: Motivations

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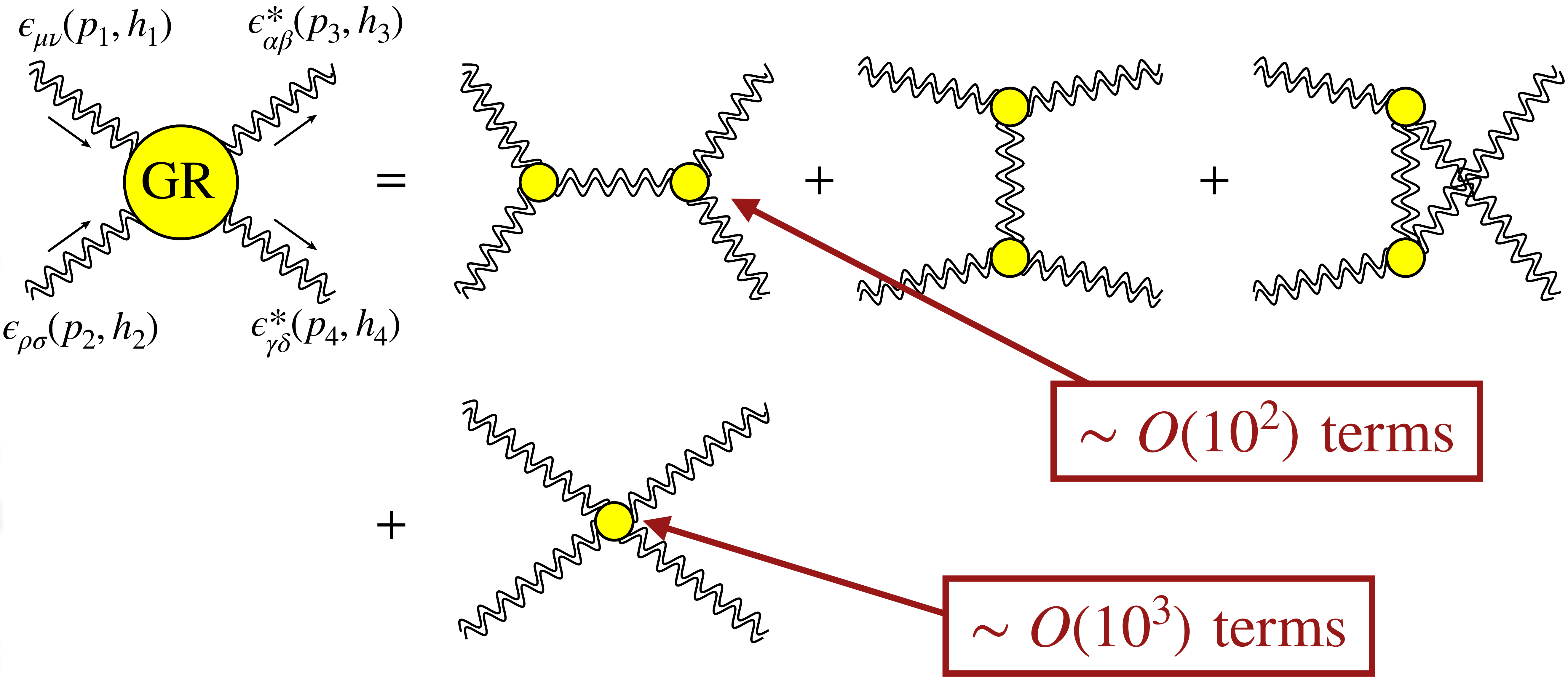


$\sim O(10^2)$ terms

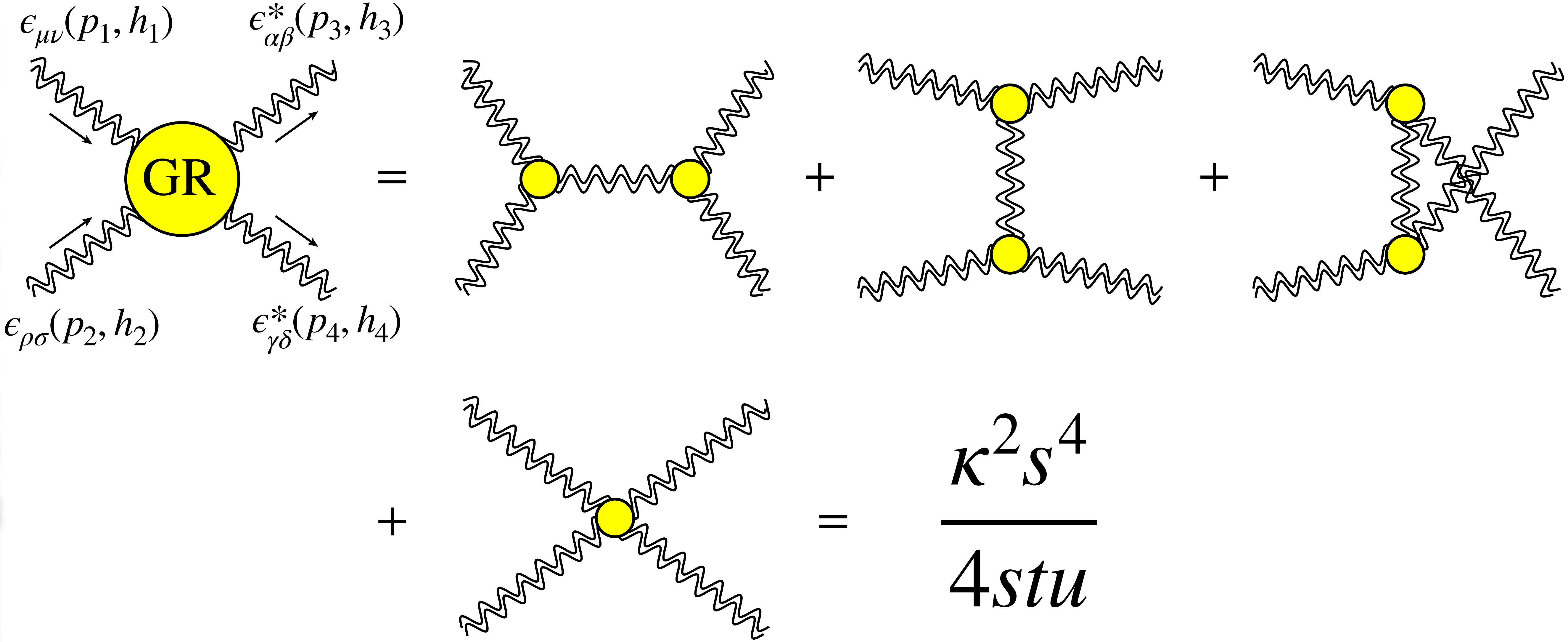
$\sim O(10^3)$ terms

B. S. DeWitt, Phys. Rev. 162, 1239 (1967)

$$\kappa^2 \equiv 32\pi G_N = \frac{32\pi}{M_{\mathrm{Pl}}^2}$$

PART I: Motivations

$$\mathscr{A}\left(1^{+2}2^{+2} \to 3^{+2}4^{+2}\right)$$



$$= \frac{\kappa^2 s^4}{4stu}$$

B. S. DeWitt, Phys. Rev. 162, 1239 (1967)

PART I: Motivations

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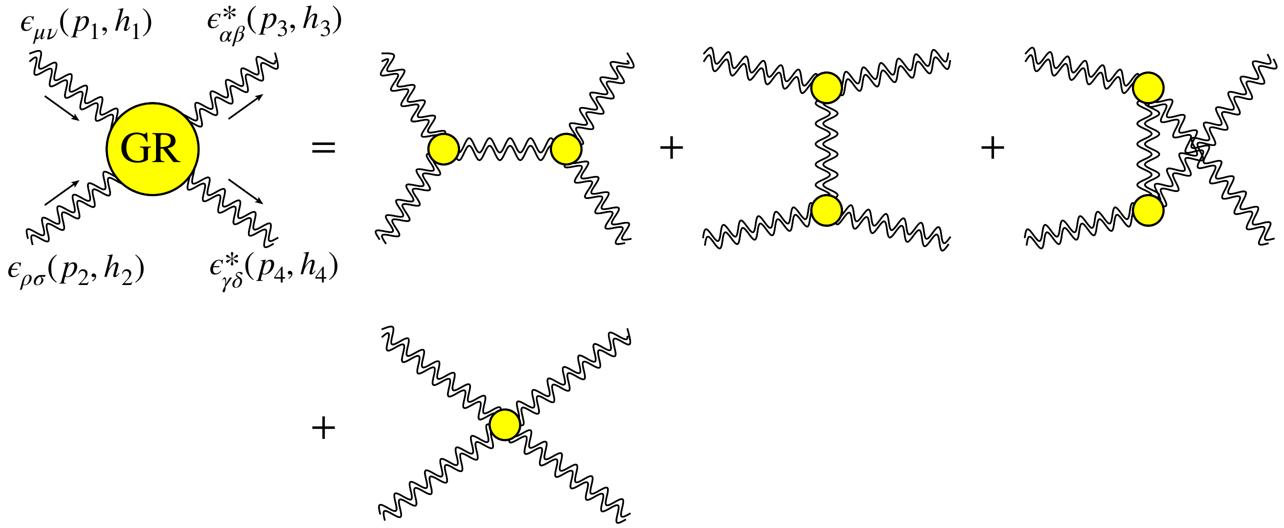
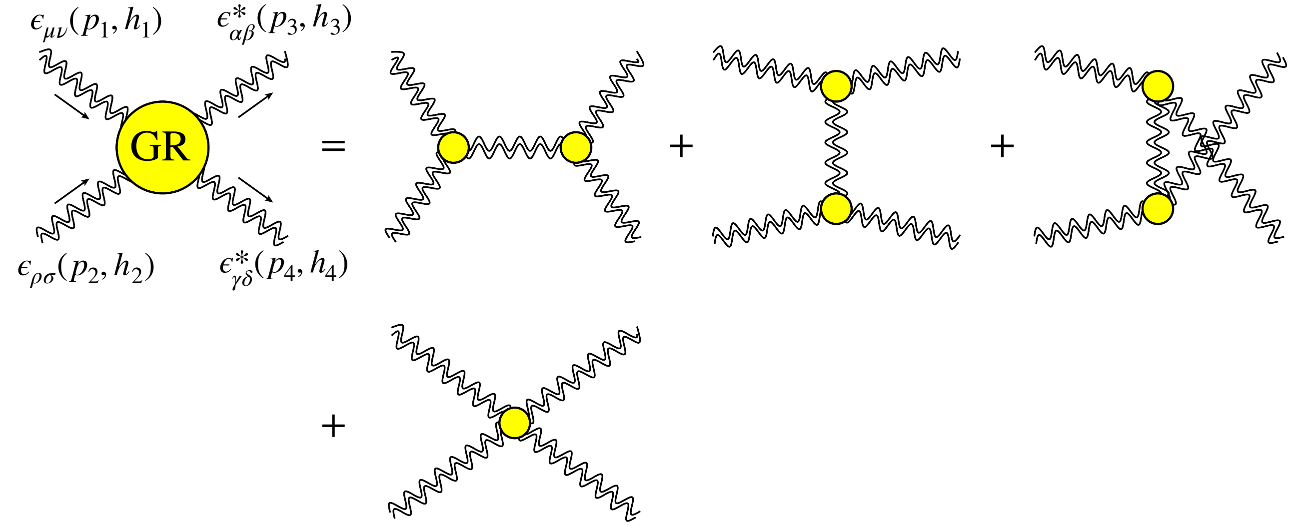


PART I: Motivations

$$\mathscr{A}\left(1^{+2}2^{+2} \to 3^{+2}4^{+2}\right)$$

$$(\langle 34\rangle[12])^4 = s^4$$

$$\mathscr{A} = \frac{s^4}{M_{\rm Pl}^2} \times \left[\qquad ? \qquad \right]$$

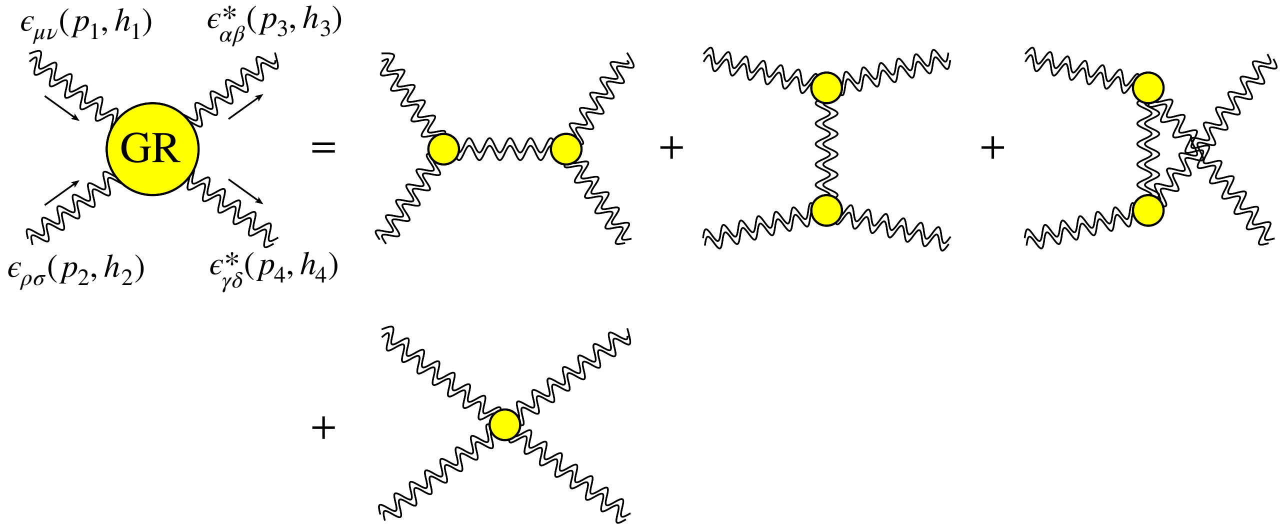


PART I: Motivations

$$\mathscr{A}\left(1^{+2}2^{+2}\rightarrow 3^{+2}4^{+2}\right)$$

$$(\langle 34\rangle[12])^4 = s^4$$

$$\mathscr{A} = \frac{s^4}{M_{\rm Pl}^2}\times\left[\frac{1}{stu} \quad + \quad ? \quad\right]$$



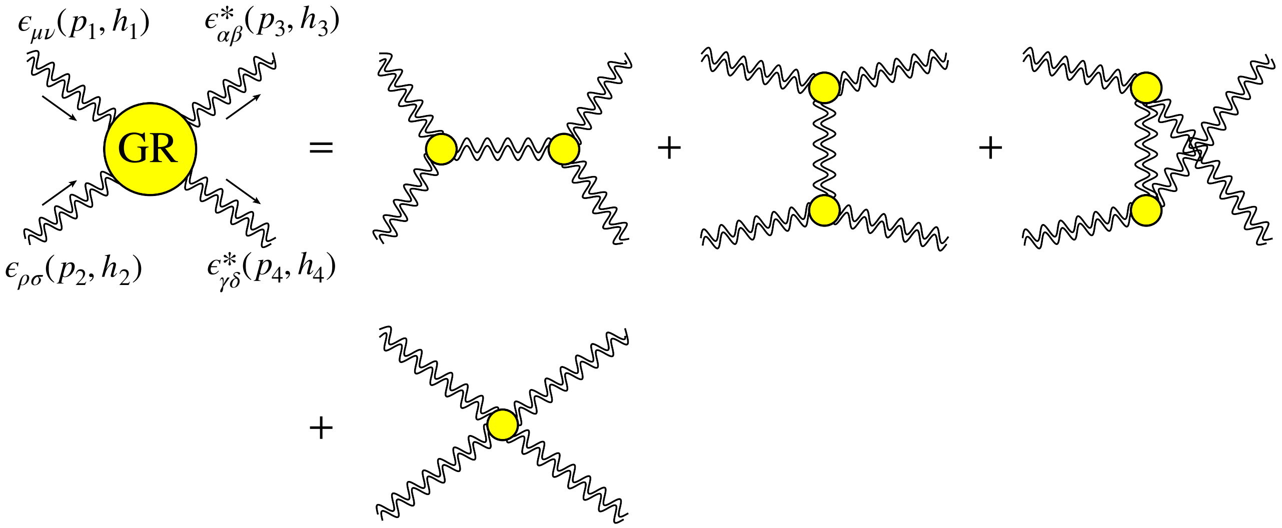
PART I: Motivations

$$\mathscr{A}\left(1^{+2}2^{+2}\to 3^{+2}4^{+2}\right)$$

$$(\langle 34\rangle[12])^4 = s^4$$

$$\mathscr{A} = \frac{s^4}{M_{\rm Pl}^2\, stu} \sim \frac{s}{M_{\rm Pl}^2}$$

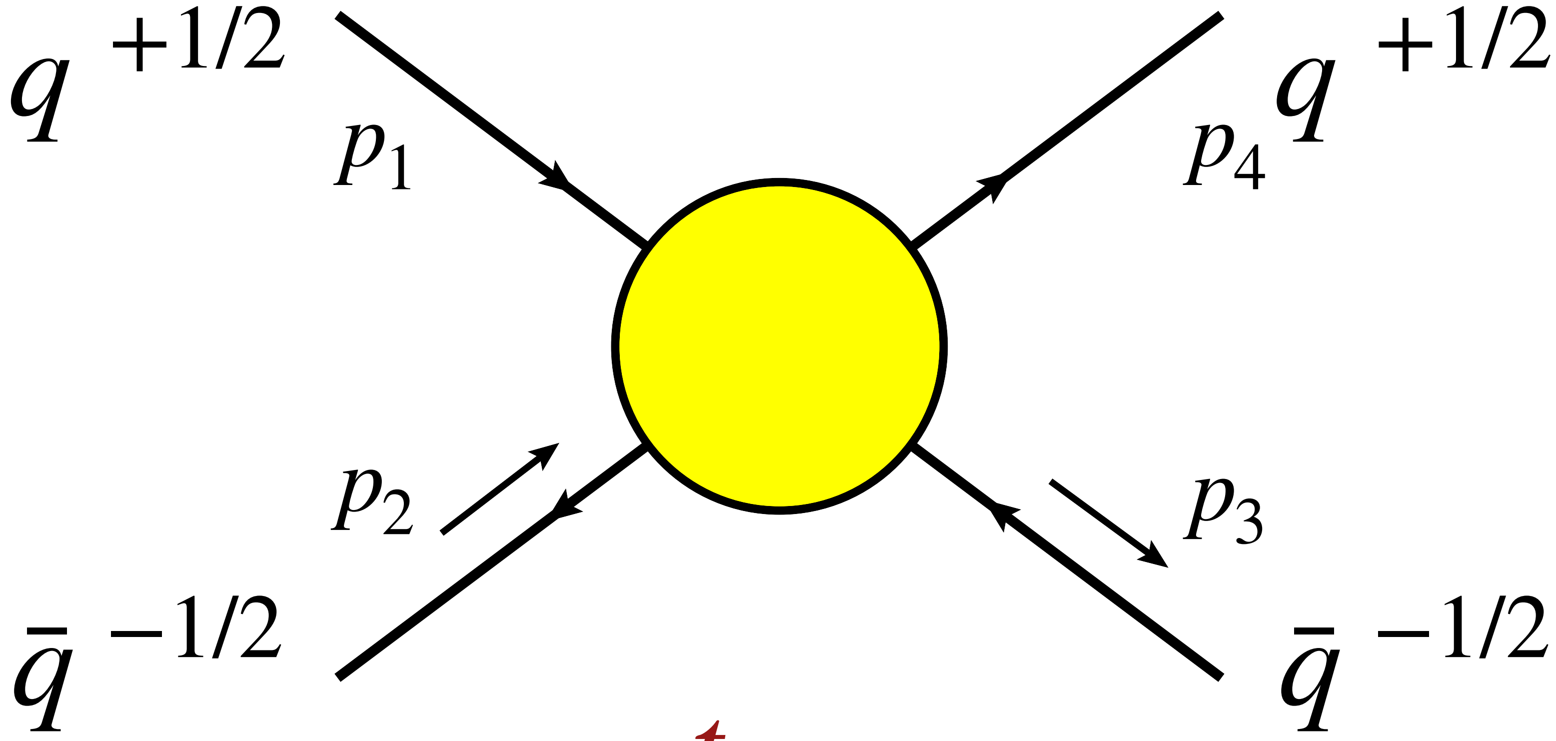
poles, no contact terms



PART I: Motivations

Example: fermion/fermion scattering

“Unitarity channel”



$$\mathscr{A} = \frac{\overbrace{\langle 24\rangle[13]}^{t}}{M_{\mathrm{Pl}}^2} \times \left[\qquad ? \qquad \right]$$

PART I: Motivations

Example: fermion/fermion scattering

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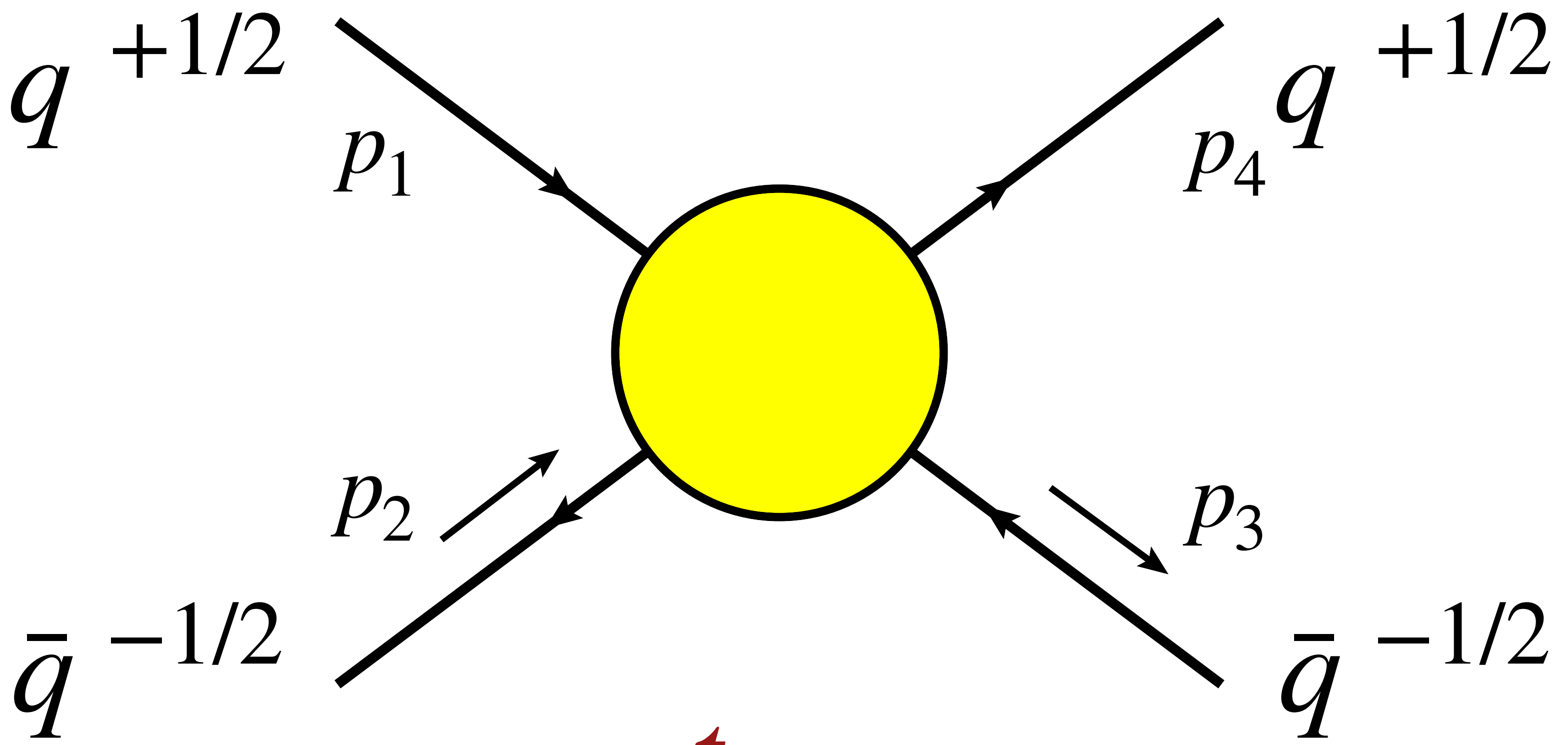


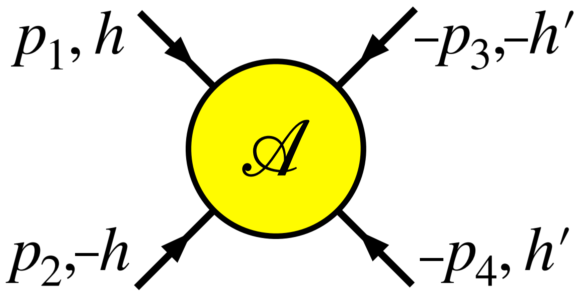
$$\mathscr{A} = \frac{\overbrace{\langle 24\rangle[13]}^{t}}{M_{\mathrm{Pl}}^2} \times \left[\frac{t}{s} + \frac{s}{t} + b\right]$$

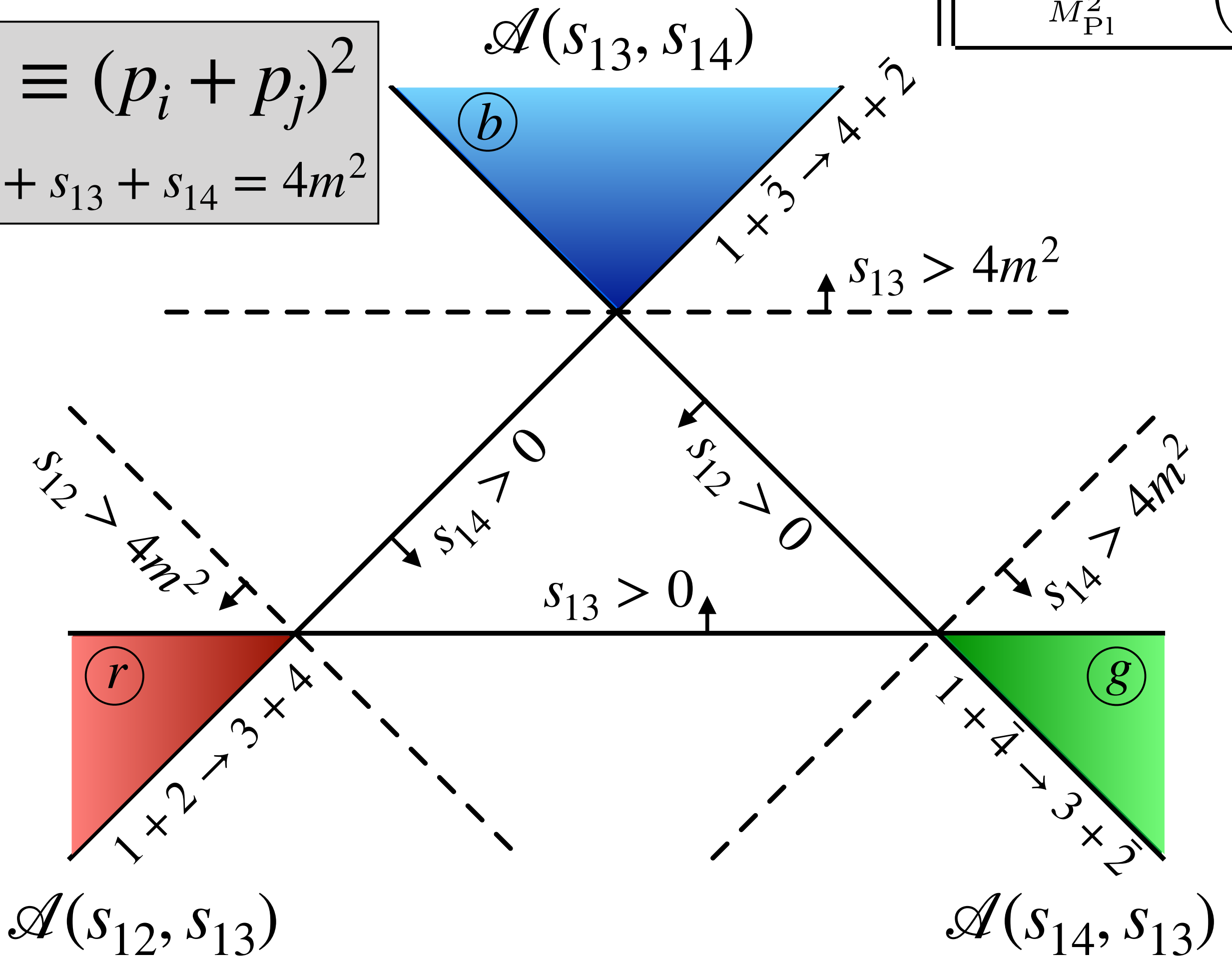
PART I: Motivations

$\left\|\mathcal{A}_{1^h 2^{-h}\to 3^{h'} 4^{-h'}}\right\|$	**Scalar**	**Fermion**	**Vector**	**Graviton**
Scalar	$\frac{8\pi}{M_{\rm Pl}^2}\left(\frac{s_{13}s_{14}}{s_{12}}+bs_{12}\right)$ $\frac{8\pi}{M_{\rm Pl}^2}\left(\frac{s_{13}s_{14}}{s_{12}}+\frac{s_{12}s_{14}}{s_{13}}+\frac{s_{13}s_{12}}{s_{14}}\right)$	$\frac{2\pi\left(\lambda_3 P_{12}\tilde{\lambda}_4\right)}{M_{\rm Pl}^2}\left(\frac{s_{13}}{s_{12}}-\frac{s_{14}}{s_{12}}\right)$	$\frac{2\pi\left(\lambda_3 P_{12}\tilde{\lambda}_4\right)^2}{M_{\rm Pl}^2 s_{12}}$	$\frac{\pi\left(\lambda_3 P_{12}\tilde{\lambda}_4\right)^4}{2M_{\rm Pl}^2 s_{12}s_{13}s_{14}}$
Fermion		$\frac{8\pi\langle 2\,3\rangle[1\,4]}{M_{\rm Pl}^2}\left(\frac{s_{13}}{s_{12}}+b\right)$ $\frac{8\pi\langle 2\,3\rangle[1\,4]}{M_{\rm Pl}^2}\left(\frac{s_{13}}{s_{12}}+\frac{s_{12}}{s_{13}}+b\right)$	$\frac{4\pi\langle 2\,3\rangle[1\,4]\left(\lambda_3 P_{12}\tilde{\lambda}_4\right)}{M_{\rm Pl}^2 s_{12}}$	$\frac{\pi\langle 2\,3\rangle[1\,4]\left(\lambda_3 P_{12}\tilde{\lambda}_4\right)^3}{M_{\rm Pl}^2 s_{12}s_{13}s_{14}}$
Vector			$\frac{8\pi\langle 2\,3\rangle^2[1\,4]^2}{M_{\rm Pl}^2 s_{12}}$ $\frac{8\pi\langle 2\,3\rangle^2[1\,4]^2}{M_{\rm Pl}^2}\left(\frac{1}{s_{12}}+\frac{1}{s_{13}}\right)$	$\frac{2\pi\langle 2\,3\rangle^2[1\,4]^2\left(\lambda_3 P_{12}\tilde{\lambda}_4\right)^2}{M_{\rm Pl}^2 s_{12}s_{13}s_{14}}$
Graviton				$\frac{8\pi\langle 2\,3\rangle^4[1\,4]^4}{M_{\rm Pl}^2 s_{12}s_{13}s_{14}}$

$$s_{ij} \equiv (p_i + p_j)^2$$

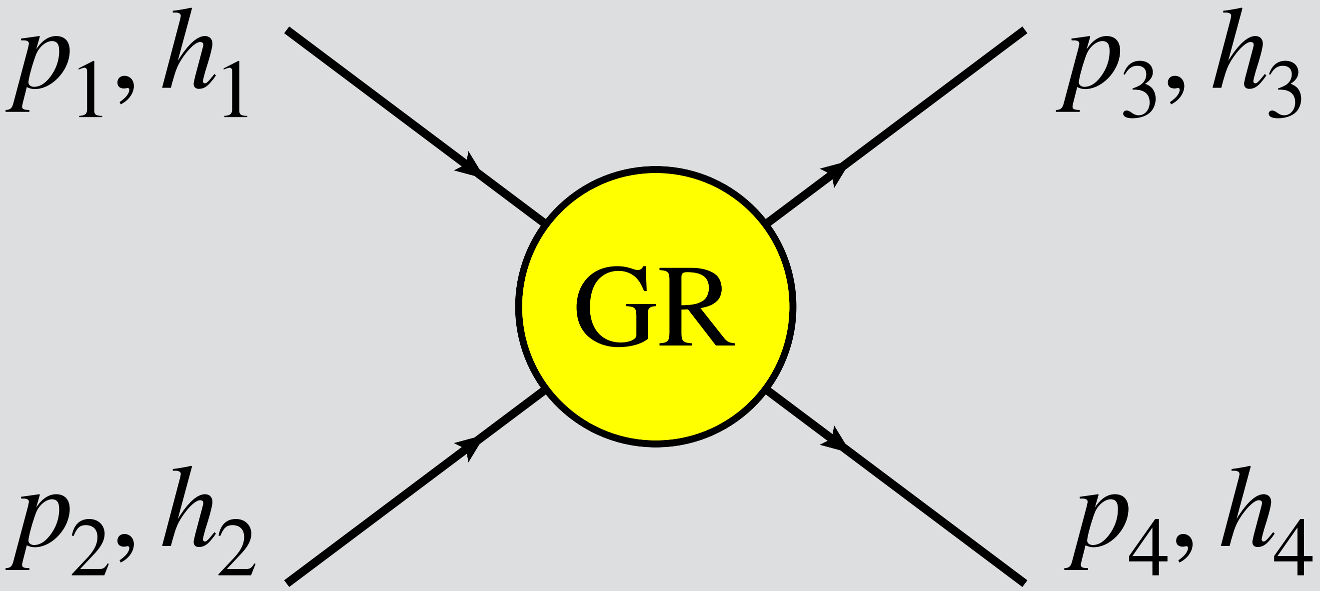
$$s_{12} + s_{13} + s_{14} = 4m^2$$





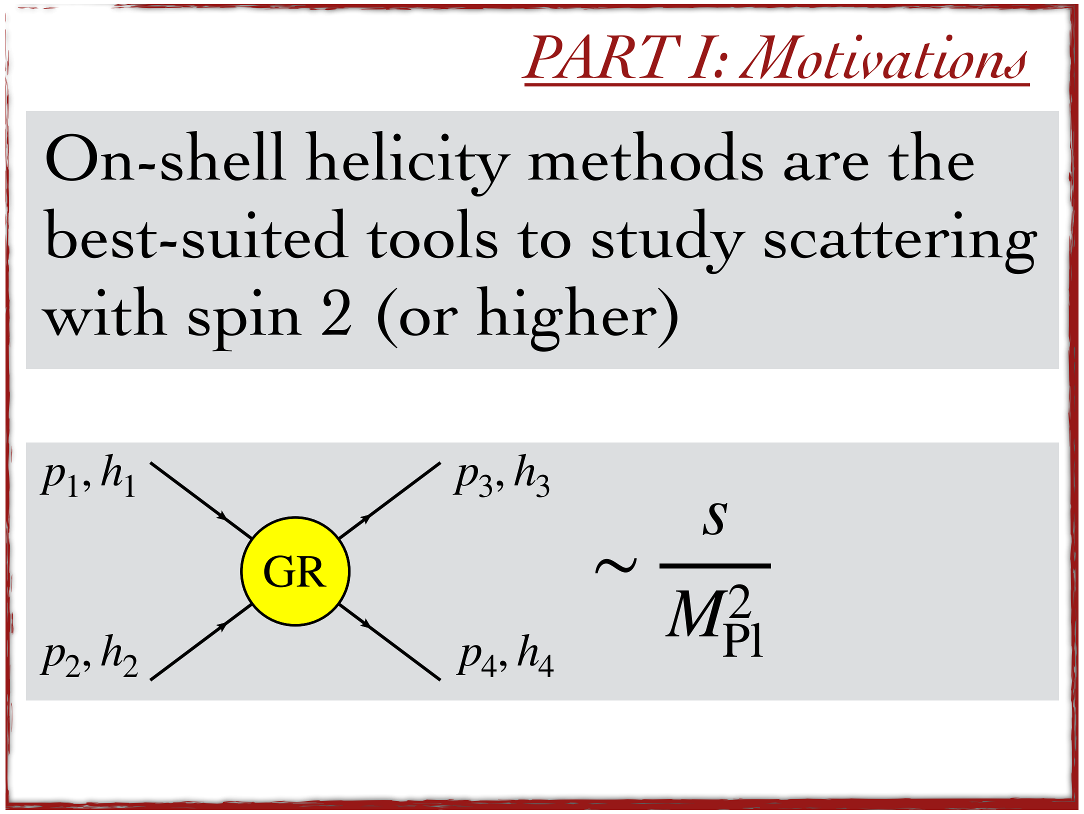
PART I: Motivations

On-shell helicity methods are the best-suited tools to study scattering with spin 2 (or higher...)



$$\sim \frac{s}{M_{\mathrm{Pl}}^2}$$

PART II: Analysis

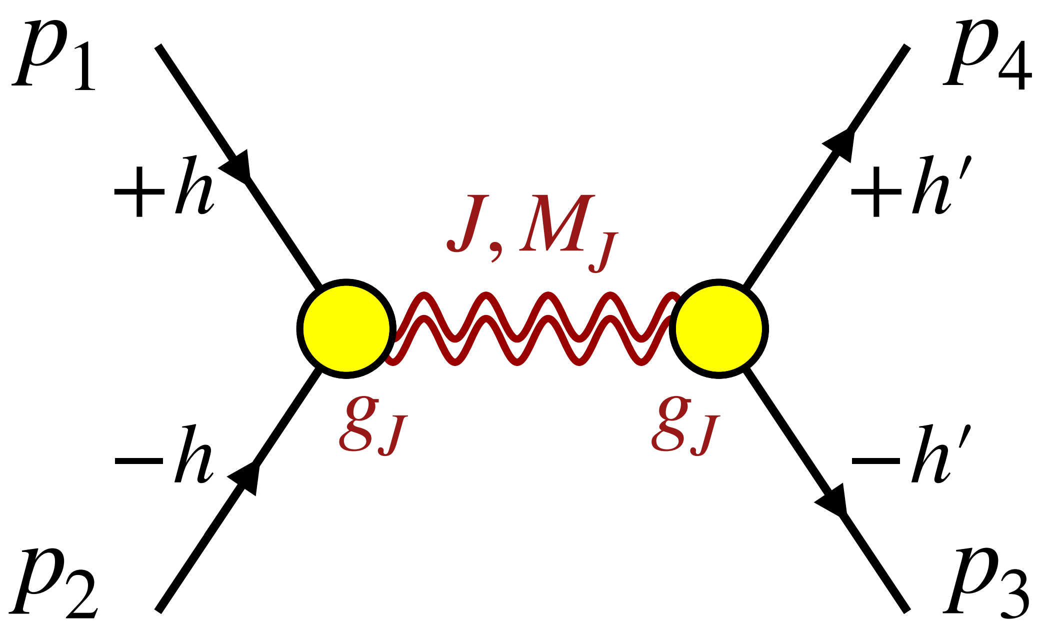


PART I: Motivations

On-shell helicity methods are the best-suited tools to study scattering with spin 2 (or higher)

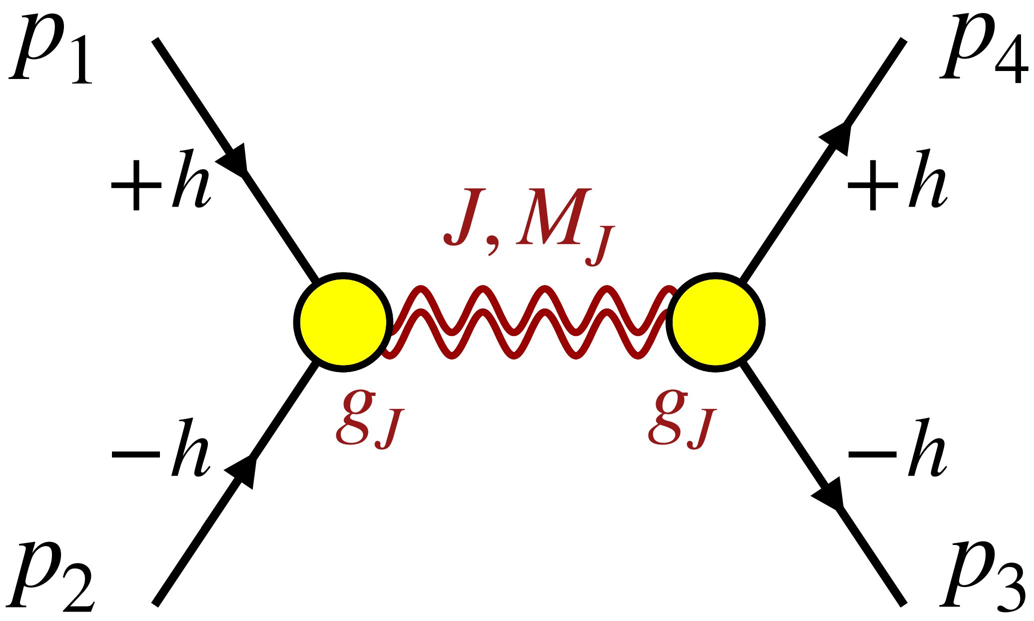
$$\sim \frac{s}{M_{\rm Pl}^2}$$

PART II: Analysis



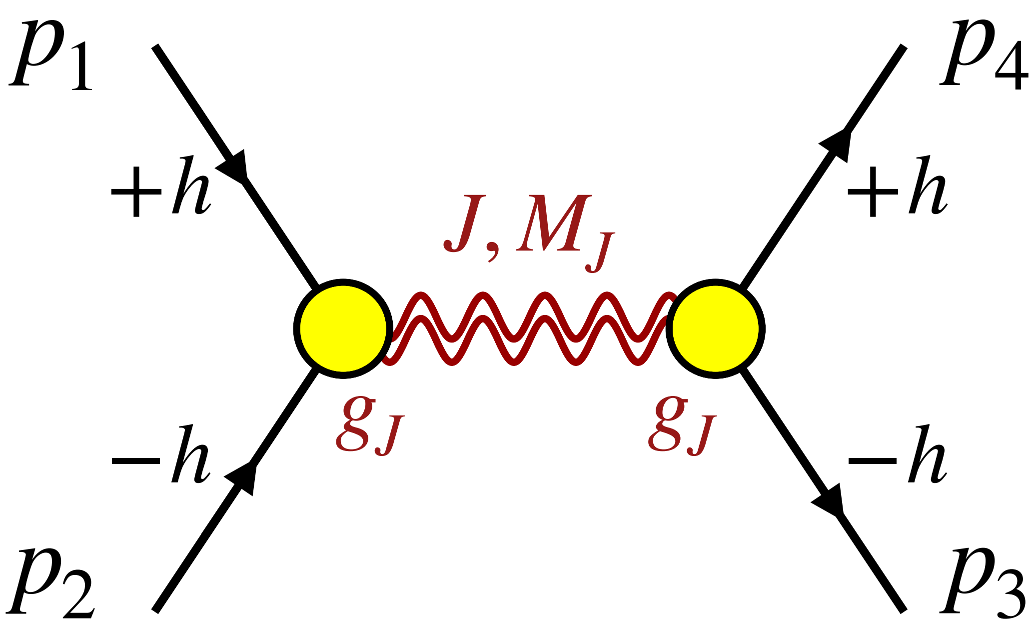
N. Arkani-Hamed, T. C. Huang and Y. t. Huang,
"Scattering Amplitudes For All Masses and Spins,"
arXiv:1709.04891

PART II: Analysis



$$\mathscr{A} = \frac{g_J^2 (2J)!!}{2^{J+2h}(2J-1)!!}\binom{J+2h}{J}\binom{J}{2h}^{-1}\frac{M_J^2}{s_{12}-M_J^2}\left(\frac{\langle 24\rangle[13]}{M_J^2}\right)^{2h} P_{J-2h}^{(0,4h)}(x)$$

PART II: Analysis



$$P_n^{(a,b)} = \sum_k \binom{n+a}{n-k}\binom{n+b}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}$$

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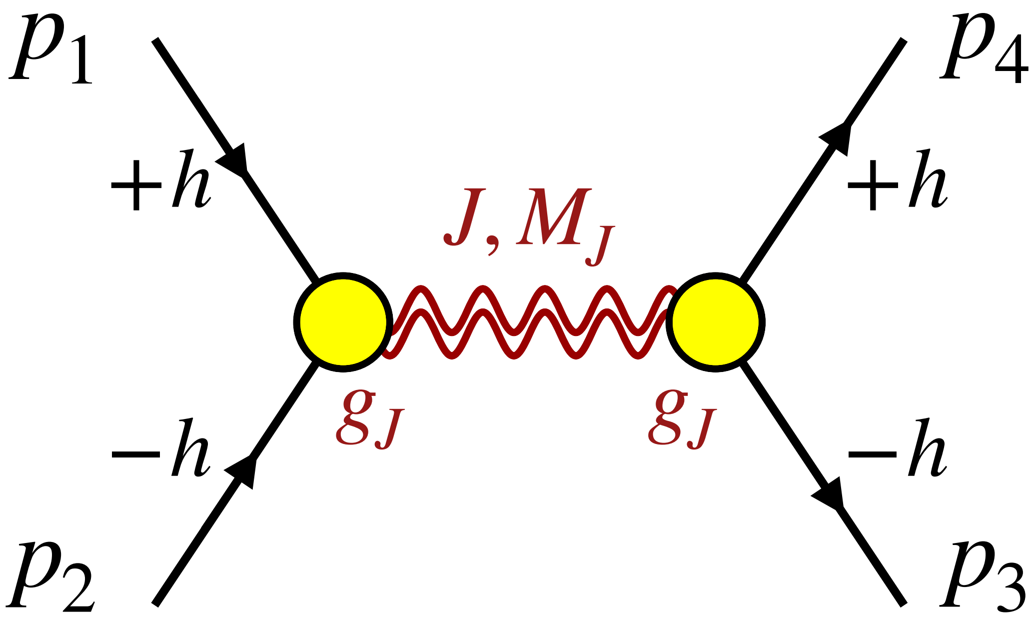
If h = 0 (scalar case) we recover the Legendre polynomials

$P_J^{(0,0)}(x) = \mathscr{P}_J(x)$

PART II: Analysis



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Jacobi polynomials

helicity structure

$$\mathscr{A} = \underbrace{\frac{g_J^2(2J)!!}{2^{J+2h}(2J-1)!!}\binom{J+2h}{J}\binom{J}{2h}^{-1}} \frac{M_J^2}{s_{12}-M_J^2}\overbrace{\left(\frac{\langle 24\rangle[13]}{M_J^2}\right)^{2h}}\overbrace{P_{J-2h}^{(0,4h)}(x)}$$

12 = 34

$$\frac{g_J^2(2J)!!}{2^{J+2h}(2J-1)!!}\binom{J+2h}{J}\binom{J}{2h}^{-1} = 16\pi(2J+1)\frac{\Gamma_{J\to 1^h 2^{-h}}}{M_J}$$

PART II: Analysis

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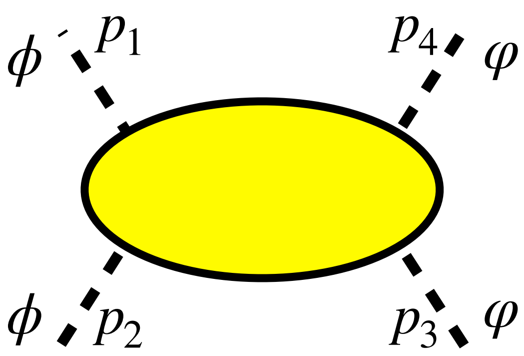
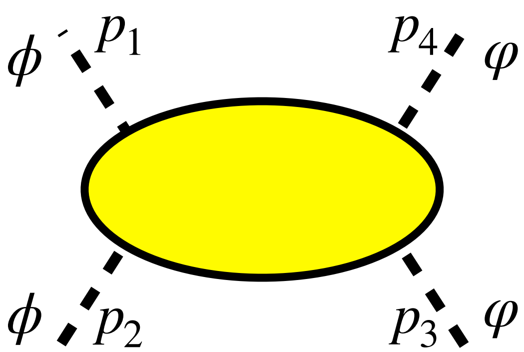


$$= 16\pi \sum_{J=0}^{\infty} (2J+1)\, a_J(s)\, \mathscr{P}_J(\cos\theta)$$

$$\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\rm Pl}^2}\left(-\frac{tu}{s} + bs\right)$$

$$\begin{cases} a_0(s) = \dfrac{s(-1+6b)}{96\pi M_{\rm Pl}^2} \\ a_1(s) = 0 \\ a_2(s) = \dfrac{s}{480\pi M_{\rm Pl}^2} \end{cases}$$

PART II: Analysis



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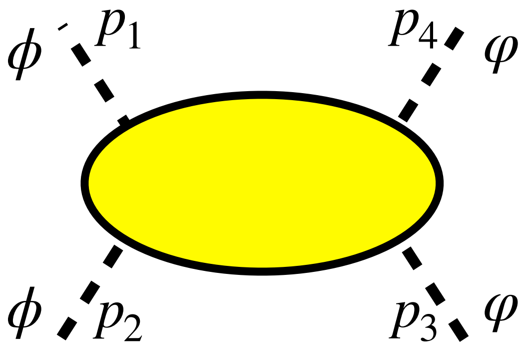
$$\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\text{Pl}}^2}\left(-\frac{tu}{s} + bs\right)$$

$$+ \frac{g_\phi g_\varphi}{s - M_2^2}\left[\underbrace{M_2^2 \mathscr{P}_2\left(1 + \frac{2t}{M_2^2}\right)}_{\text{on−shell}} + \underbrace{(s - M_2^2)\mathscr{G}_{(1,1)}\left(\frac{s}{M_2^2}, \frac{t}{M_2^2}\right)}_{\text{off−shell}}\right]$$

PART II: Analysis



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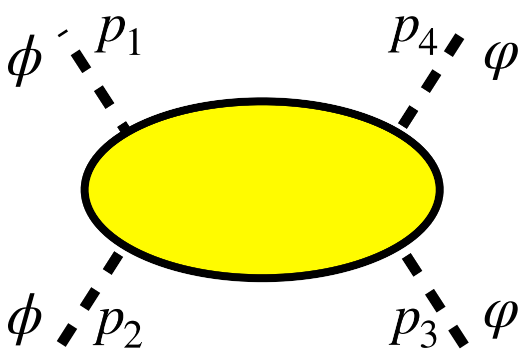
$$+ \frac{\tilde{g}_\phi \tilde{g}_\varphi}{s - M_3^2}\left[\underbrace{M_3^2 \mathscr{P}_3\left(1 + \frac{2t}{M_3^2}\right)}_{\text{on}-\text{shell}} + \underbrace{(s - M_3^2)\mathscr{G}_{(2,2)}\left(\frac{s}{M_3^2}, \frac{t}{M_3^2}\right)}_{\text{off}-\text{shell}}\right]$$

PART II: Analysis



$$= 16\pi \sum_{J=0}^{\infty} (2J+1)\, a_J(s)\, \mathcal{P}_J(\cos\theta)$$

$$1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots = e^{-x}$$

$$\mathcal{A}_{\phi\varphi} = \frac{1}{M_{\rm Pl}^2}\left(-\frac{tu}{s} + bs\right)$$

$$+ \frac{g_\phi g_\varphi}{s - M_2^2}\left[\underbrace{M_2^2 \mathcal{P}_2\left(1 + \frac{2t}{M_2^2}\right)}_{\text{on−shell}} + \underbrace{(s - M_2^2)\mathcal{G}_{(1,1)}\left(\frac{s}{M_2^2}, \frac{t}{M_2^2}\right)}_{\text{off−shell}}\right]$$

$$+ \ \ldots$$

$$= \frac{1}{M_{\rm Pl}^2}\left(-\frac{tu}{s} + bs\right)\frac{N(s,t)}{\prod_k^{\infty}(s - M_k^2)}$$

PART II: Analysis

$$\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\mathrm{Pl}}^2}\left(-\frac{tu}{s} + bs\right)\frac{\prod_n^\infty [t - f_n(s)]}{\prod_k^\infty (s - M_k^2)}$$

PART II: Analysis

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Unitarity + Locality

$$\downarrow$$

PART II: Analysis

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Unitarity + Locality

$$\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\rm Pl}^2}\left(-\frac{tu}{s} + bs\right)\frac{\prod_n^\infty [t - f_n(s)]}{\prod_m^\infty [t - g_m(s)]\prod_k^\infty (s - M_k^2)}$$

PART II: Analysis

$$\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\mathrm{Pl}}^2}\left(-\frac{tu}{s} + bs\right)\frac{\prod_n^\infty [t - f_n(s)]}{\prod_k^\infty (s - M_k^2)}$$

$$\Big\downarrow$$

Unitarity + Locality

$M_k^2 = kM^2$ with $k \in \mathbb{Z}^+$

$$\boxed{\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\mathrm{Pl}}^2}\left(-\frac{tu}{s} + bs\right)\frac{\prod_k^\infty M_k^2\left(-t - s + M_k^2\right)}{\prod_k^\infty \left(t - M_k^2\right)\left(s - M_k^2\right)}}$$

PART II: Analysis

$$\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\rm Pl}^2}\left(-\frac{tu}{s} + bs\right)\frac{\prod_n^\infty [t - t_n(s)]}{\prod_k^\infty (s - M_k^2)}$$

Unitarity + Locality

$M_k^2 = kM^2$ with $k \in \mathbb{Z}^+$

$$\boxed{\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\rm Pl}^2}\left(-\frac{tu}{s} + bs\right)\frac{\prod_k^\infty \left(s + M_k^2\right)\left(t + M_k^2\right)\left(u + M_k^2\right)}{\prod_k^\infty \left(s - M_k^2\right)\left(t - M_k^2\right)\left(u - M_k^2\right)}}$$

[See also N. Arkani-Hamed, talk @String '16]

PART II: Analysis

$$\mathscr{A}_{\phi\varphi} = \frac{1}{M_{\mathrm{Pl}}^2}\left(-\frac{tu}{s} + bs\right)\frac{\prod_n^\infty [t - t_n(s)]}{\prod_k^\infty (s - M_k^2)}$$

Unitarity + Locality

$M_k^2 = kM^2$ with $k \in \mathbb{Z}^+$

$\tilde{s} = s/M^2$ $\qquad g^2 = M^2/M_{\mathrm{Pl}}^2$

$$\boxed{\mathscr{A}_{\phi\varphi} = g^2\left(-\frac{\tilde{t}\tilde{u}}{\tilde{s}} + b\tilde{s}\right)\frac{\Gamma(1-\tilde{s})\,\Gamma\left(1-\tilde{t}\right)\Gamma(1-\tilde{u})}{\Gamma(1+\tilde{s})\,\Gamma\left(1+\tilde{t}\right)\Gamma(1+\tilde{u})}}$$

PART II: Analysis

$$\mathscr{A} = \mathscr{A}_{\mathrm{GR}} \times \begin{cases} \dfrac{\Gamma(1-\tilde{s})\Gamma\left(1-\tilde{t}\right)\Gamma(1-\tilde{u})}{\Gamma(1+\tilde{s})\Gamma\left(1+\tilde{t}\right)\Gamma(1+\tilde{u})} \\ \\ \dfrac{\Gamma(1-\tilde{s})\Gamma\left(1-\tilde{t}\right)}{\Gamma\left(1-\tilde{s}-\tilde{t}\right)} \end{cases}$$

$\tilde{x} = x/M^2\,, \quad x = s, t, u$

Virasoro-Shapiro form factor

Veneziano form factor

poles at $1 - \tilde{s} = -n$

Virasoro-Shapiro form factor:

Compton scattering with gravity, scattering with scalars

Veneziano form factor:

Fermion/fermion, fermion/vector, vector/vector scattering

PART II: Analysis

$$\mathscr{A} = \mathscr{A}_{\mathrm{GR}} \times \begin{cases} \dfrac{\Gamma(1-\tilde{s})\Gamma\left(1-\tilde{t}\right)\Gamma(1-\tilde{u})}{\Gamma(1+\tilde{s})\Gamma\left(1+\tilde{t}\right)\Gamma(1+\tilde{u})} \\ \\ \dfrac{\Gamma(1-\tilde{s})\Gamma\left(1-\tilde{t}\right)}{\Gamma\left(1-\tilde{s}-\tilde{t}\right)} \end{cases}$$

$\tilde{x} = x/M^2\,, \quad x = s, t, u$

poles at $1 - \tilde{s} = -n$

$$\mathscr{A} = 16\pi(2J+1)\,\alpha^J_{h,h'}\,\frac{M_J^2}{s_{12}-M_J^2}\underbrace{\left(\frac{\langle 24\rangle[13]}{M_J^2}\right)^{2h}\left(\frac{\lambda_4 P_{12}\tilde{\lambda}_3}{M_J^2}\right)^{2h'-2h} P^{(2h'+2h,2h'-2h)}_{J-2h'}(x)}_{\equiv\, d^J_{2h,2h'}(\theta)}$$

PART IV: (some) Results

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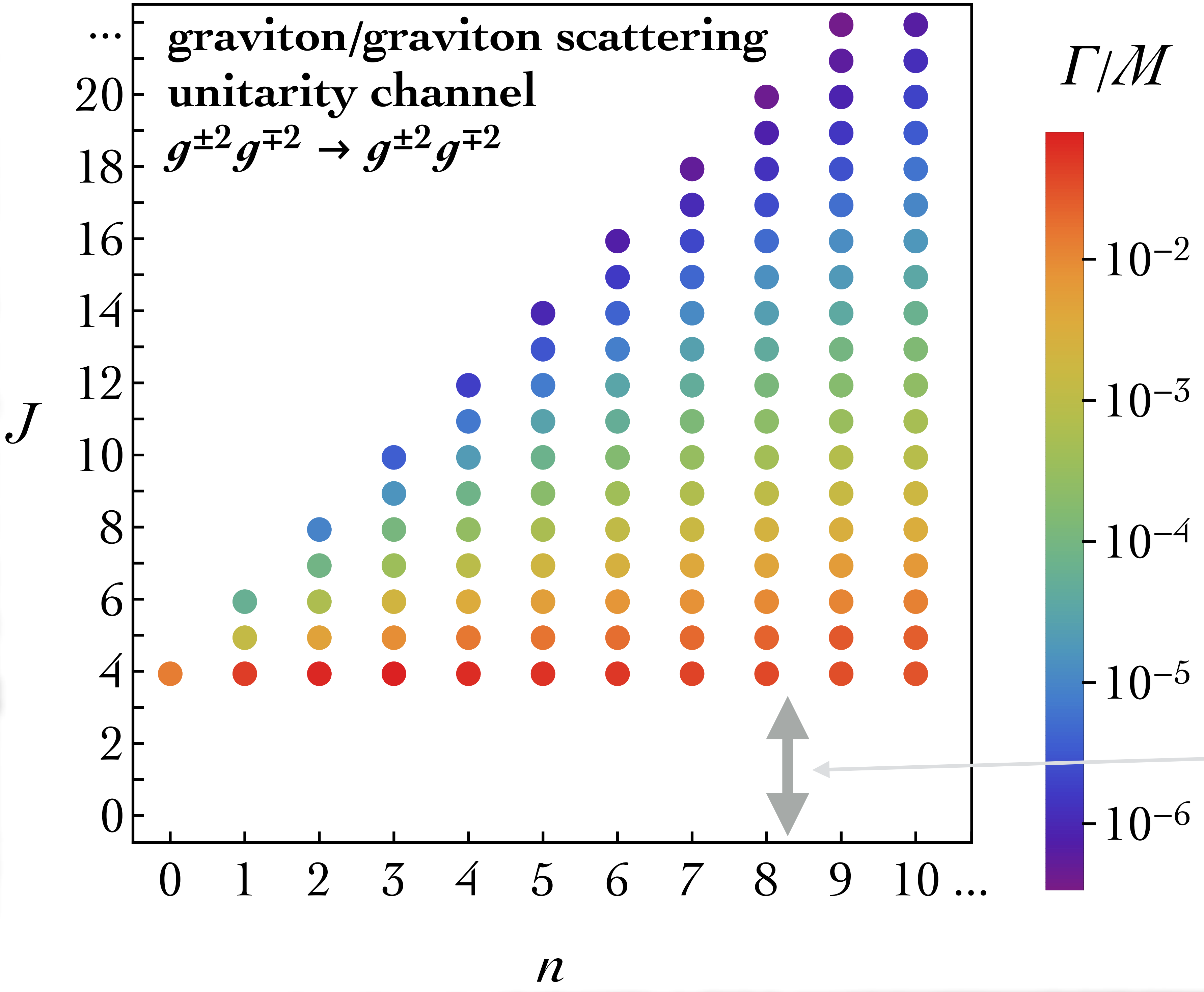
$$\mathscr{A}(g^{+2}g^{-2} \to g^{+2}g^{-2}) = \frac{t^4}{M_{\mathrm{Pl}}^2\, stu}\left[\frac{\Gamma(1-\tilde{s})\,\Gamma\left(1-\tilde{t}\right)\Gamma(1-\tilde{u})}{\Gamma(1+\tilde{s})\,\Gamma\left(1+\tilde{t}\right)\Gamma(1+\tilde{u})}\right]$$



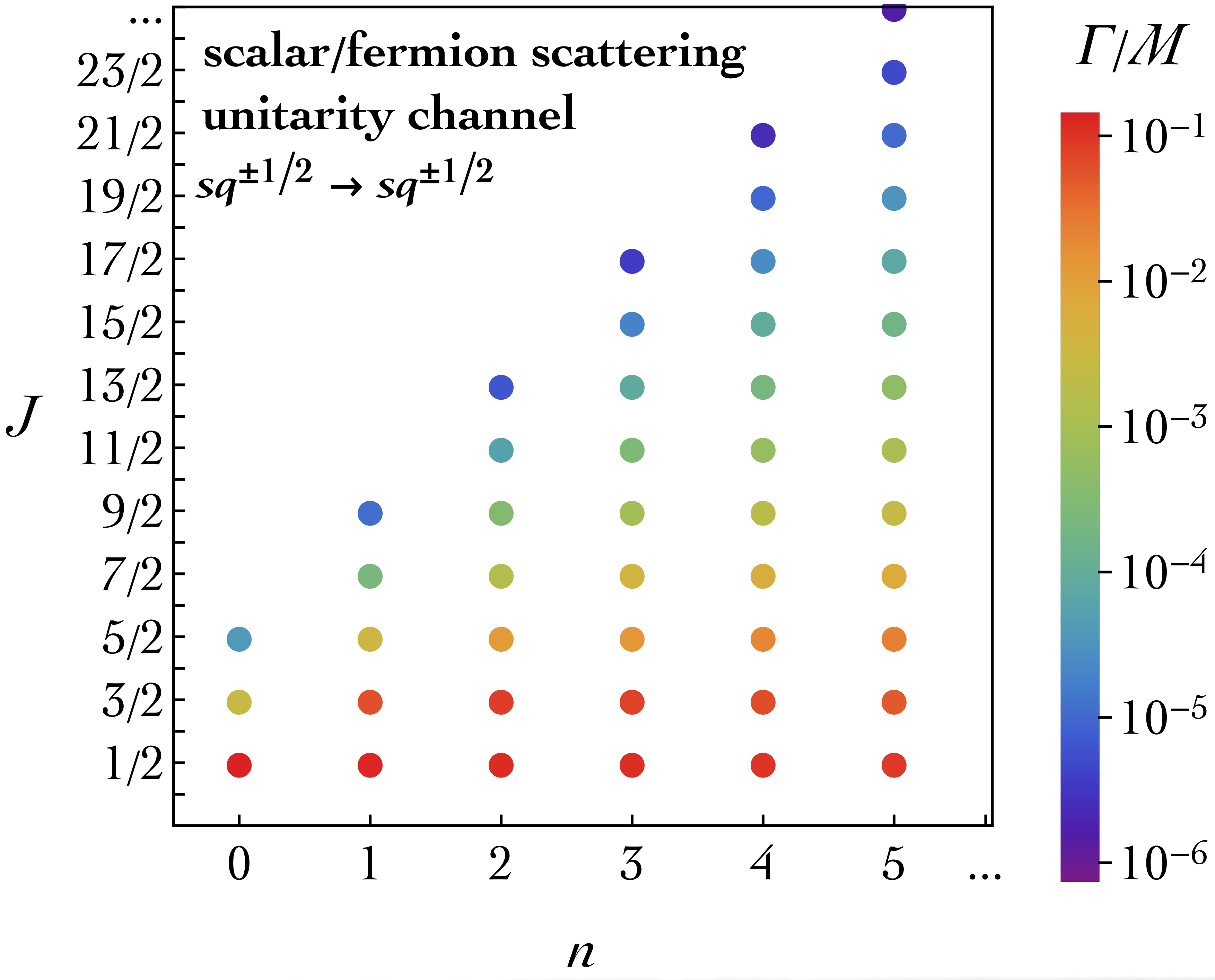
“Force carrier”
resonances
(integer spin)

Selection rule:
only resonances
with spin
$J \geqslant h_1 - h_2 = 4$

$M_n^2 = (1+n)M^2$

PART IV: (some) Results

$$\mathscr{A}(sq^{1/2} \to sq^{1/2}) = \frac{\lambda_4 P_{13} \tilde{\lambda}_2}{M_{\text{Pl}}^2} \left(\frac{s}{t} - \frac{u}{t} \right) \left[\frac{\Gamma(1-\tilde{s})\,\Gamma\left(1-\tilde{t}\right)\Gamma(1-\tilde{u})}{\Gamma(1+\tilde{s})\,\Gamma\left(1+\tilde{t}\right)\Gamma(1+\tilde{u})} \right]$$

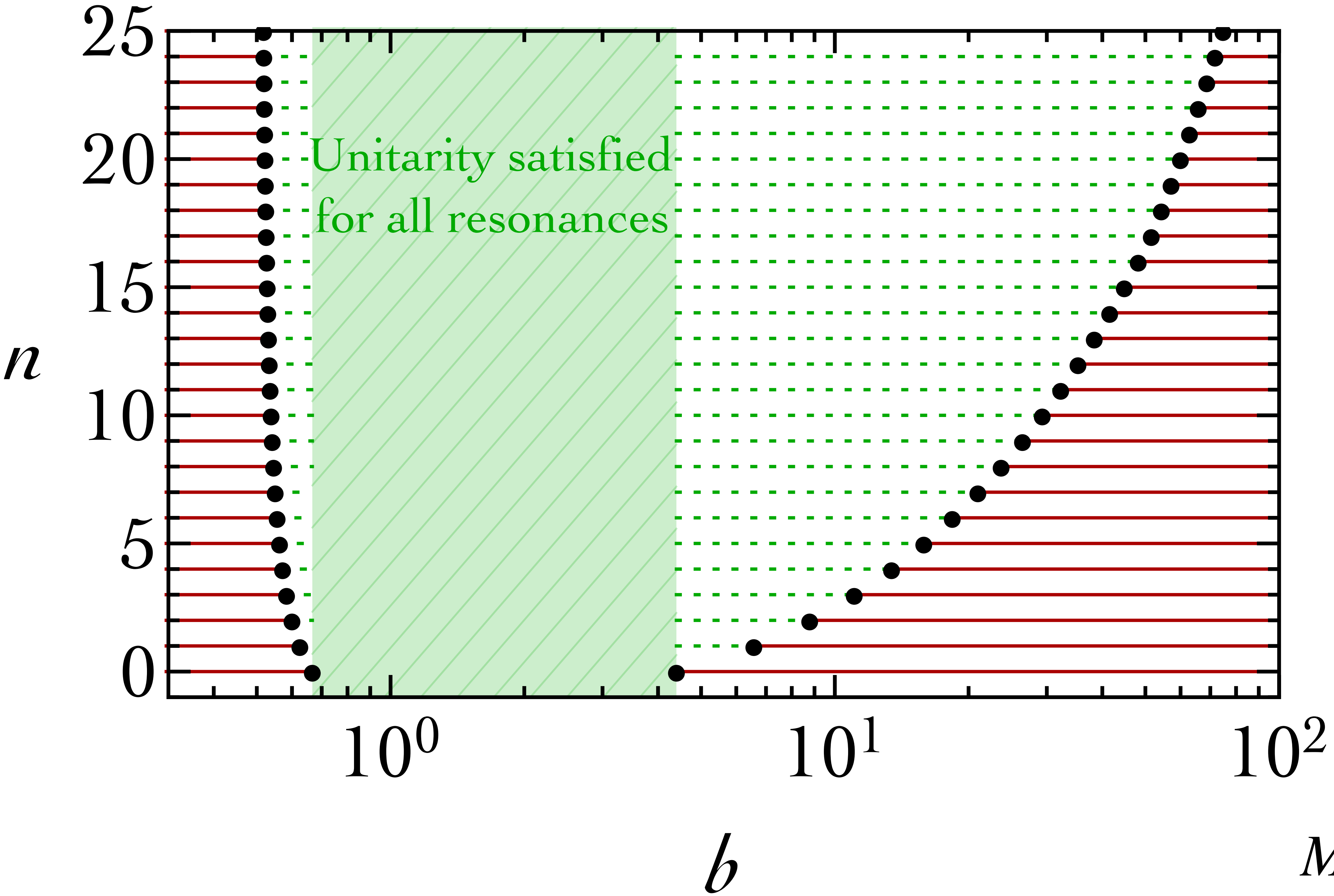


"Matter" resonances
(half-integer spin)

$$M_n^2 = (1+n)M^2$$

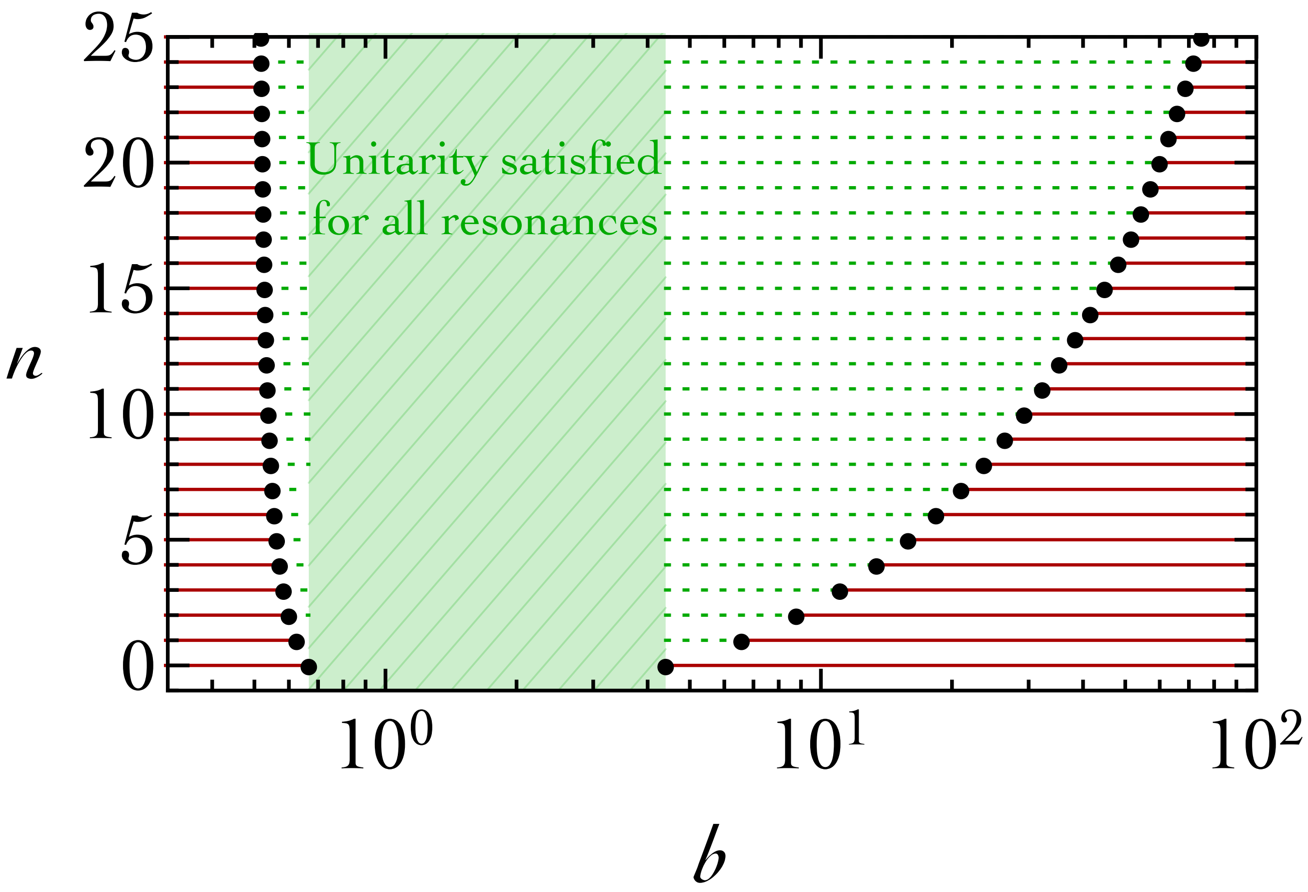
PART IV: (some) Results

$$\mathscr{A}(q^{1/2}\bar{q}^{-1/2} \to q^{1/2}\bar{q}^{-1/2}) = \frac{1}{M_{\rm Pl}^2}\left(\frac{s^2+t^2}{st}+b\right)\left[\frac{\Gamma(1-\tilde{s})\Gamma\left(1-\tilde{t}\right)}{\Gamma\left(1-\tilde{s}-\tilde{t}\right)}\right]$$



$M_n^2 = (1+n)M^2$

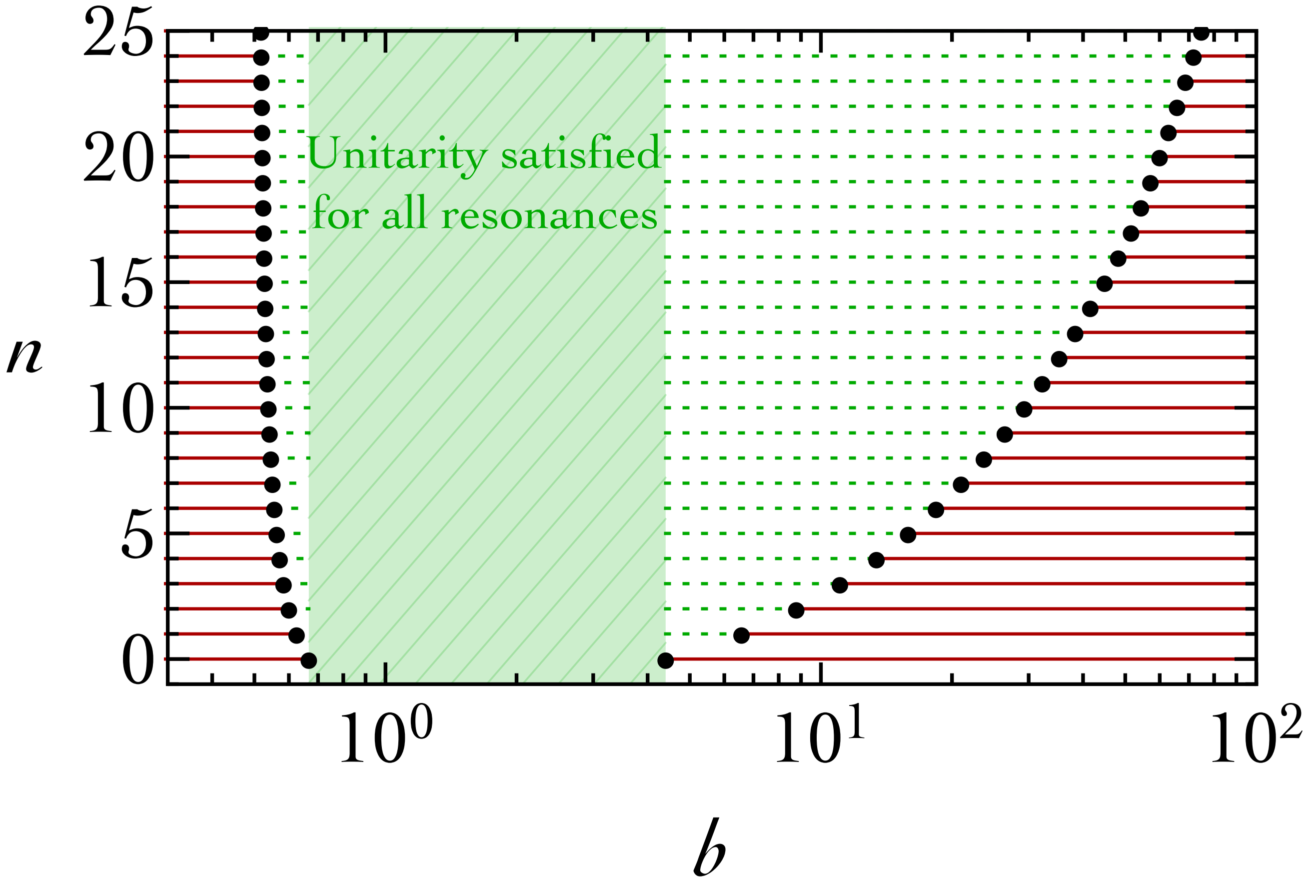
PART IV: (some) Results



$$b = \frac{1}{2}$$

Dirac fermion minimally coupled to gravity gives b = 1/2… what is going on?

PART IV: (some) Results



What if space-time has **torsion** in addition to curvature?

$$b = \frac{1}{2} + \frac{18g^2M_{\rm Pl}^2}{\pi M^2}$$

PART V: Conclusions

PART V: Conclusions

- We propose a **UV-completion of Gravity** based on the tree-level exchange of an infinite tower of **massive higher-spin resonances** (with both integer and half-integer spins).

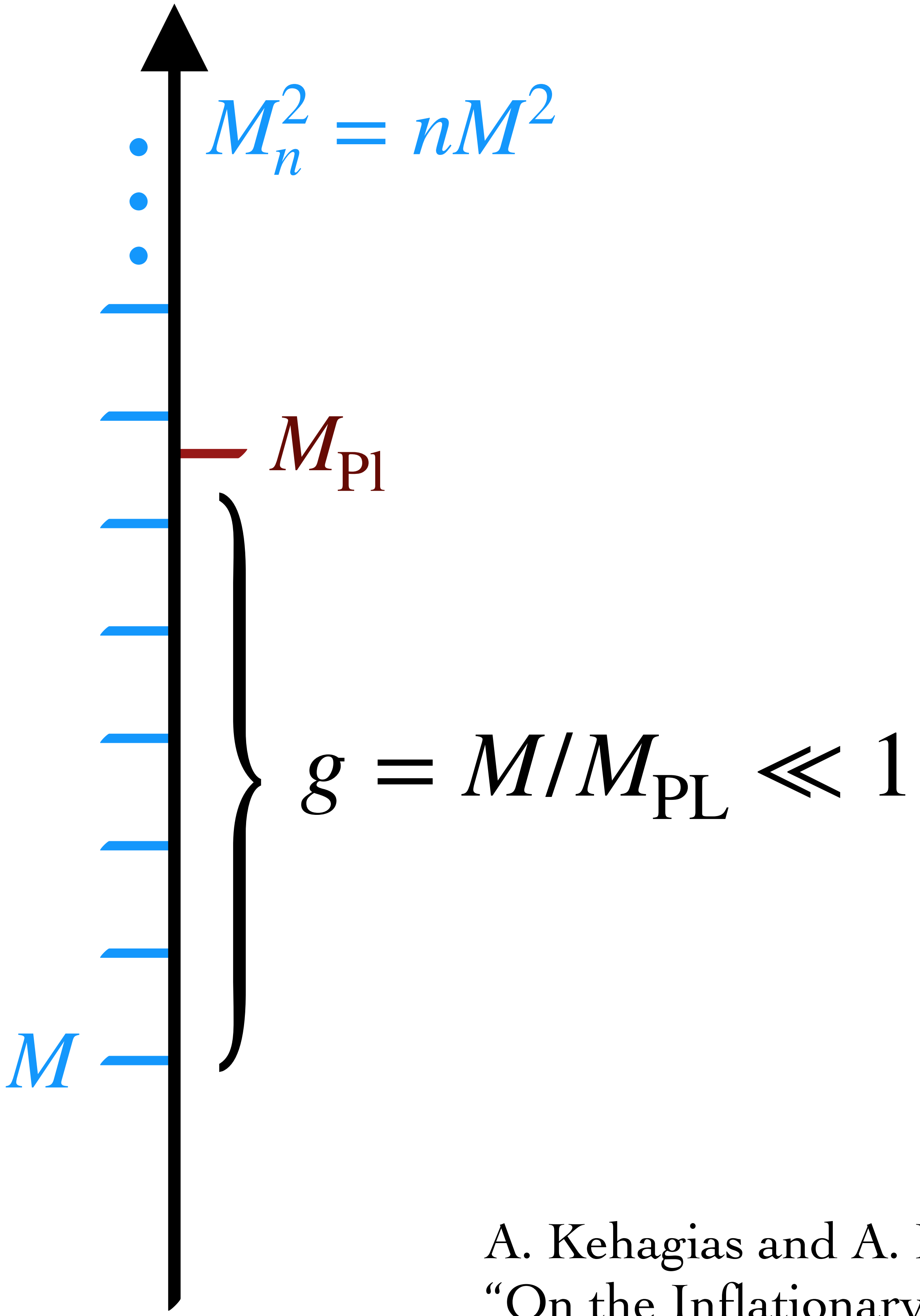
  Like what we expect in string theory but following a "bottom-up" approach.
  Amplitudes are UV-completed by either Virasoro-Shapiro or Veneziano form factors.

- We only assume the bedrock principles of **Locality**, **Causality** and **Unitarity** (thus Analyticity).

- Technically, the **on-shell spinor-helicity formalism** is of crucial importance to deal with massive higher spins.

  We cannot reconstruct the full amplitude but a number of crucial properties of the resonances can be extracted from the poles (spectrum, decay width,…). Furthermore, even if we have some unknown parameters unitarity puts non-trivial positivity constraint on them.

PART V: Conclusions



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