FNSN 2 - Chapter 1

The static quark model

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AA 16-17

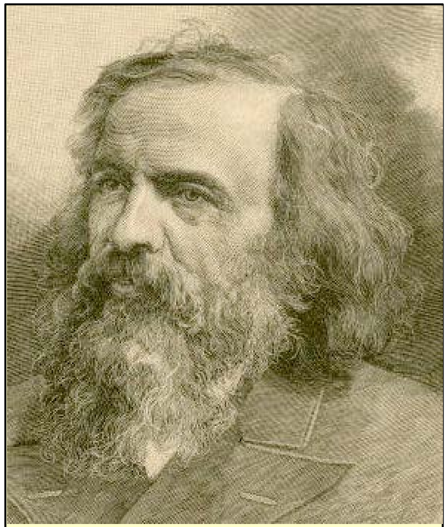
1 – The static quark model

Caveats for this chapter:

- arguments are presented in historical order; some of the results are incomplete, e.g. heavy flavors are only mentioned in next §;
- large overlap with [FNSN1, MQR, IE].

Quantum numbers : the Mendeleev way

- Many hadrons exist, with different quantum numbers (q.n.).
- Regularities (appear to) exist for some q.n. (spin, parity, …).
- Other q.n., like mass, are much more intriguing.
- A natural approach (à la D.I.M.):
  - investigate in detail the q.n.;
  - look for regularities;
  - the dynamics will actually follow.



Dmitri Ivanovich Mendeleev
(Дми́трий Ива́нович Менделе́ев)

an example from many years ago [+ antiparticles…]

the proliferation of hadrons started in the '50s – now they are few hundreds …

Name	$\pi^\pm$	π^0	$K^\pm$	K^0	η	p	n	Λ	$\Sigma^{\pm,0}$	Δ
Mass (MeV)	140	135	494	498	548	938	940	1116	1190	1232
Charge	±1	0	±1	0	0	1	0	0	±1,0	2,±1,0
Parity	–	–	–	–	–	+	+	+	+	+
Baryon n.	0	0	0	0	0	1	1	1	1	1
Spin	0	0	0	0	0	½	½	½	½	$^3/_2$

many other hadrons

other q.n. …

Quantum numbers : parity $\mathbb{P}$

for complete definitions and discussion, [FNSN1], [MQR], [BJ].

- Definition : $\mathbb{P} \, |\psi(x,t)\rangle = P|\psi(-x,t)\rangle$.
- Particles at rest (= in their own ref.sys.) are parity eigenstates.
- Their eigenvalue (± 1) is their intrinsic parity P.
- From Dirac equation, for spin ½ fermions, P(antiparticle) = -P(particle).
- By convention, quarks and leptons are defined with P=+1 :

  $P_{e-} = P_{\mu-} = P_{\tau-} = P_u = P_d = P_s = \ldots = +1$.
- Therefore

  $P_{e+} = P_{\mu+} = P_{\tau+} = P_{\bar{u}} = P_{\bar{d}} = P_{\bar{s}} = \ldots = -1$.
- From field theory, for spin-0 bosons P(antiparticle) = +P(particle) :

  $P_{\pi+} = P_{\pi^\circ} = P_{\pi-}$, etc.
- From gauge theories :

  $P_\gamma = P_g = -1$.

[W and Z do NOT conserve parity in their interactions, so their intrinsic parity is not defined].

(... is the sentence ok also for ν's ? well ...

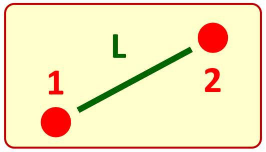
- For a many-body system, P is a multiplicative quantum number :

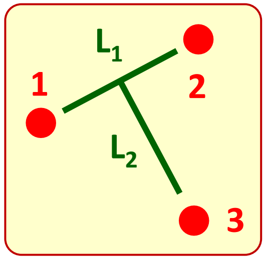
  $\mathbb{P}\psi(\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_n},t) =$
  $= P_1P_2\ldots P_n\psi(\mathbf{x_1},\mathbf{x_2},\ldots,\mathbf{x_n},t)$
- Particles in a well-defined state of orbital angular momentum are parity eigenstates :

  $Y_{km}(\theta,\phi) = (-1)^k \, Y_{km}(\pi-\theta,\phi+\pi)$
- Therefore, for a two- or a three-particle system

  $P_{sys(12)} = P_1P_2 \, (-1)^L$;

  $P_{sys(123)} = P_1P_2P_3(-1)^{L1+L2}$.





Quantum numbers : Charge conjugation $\mathbb{C}$

- The definition of $\mathbb{C}$ is to change a particle into its respective antiparticle, leaving untouched the space and time variables :

  $\mathbb{C}\ |a, \psi(\mathbf{x},t)\rangle = C|\bar{a}, \psi(\mathbf{x},t)\rangle$.

- Therefore, under $\mathbb{C}$:

  charge $q \to -q$;
  baryon n. $B \to -B$;
  lepton n. $L \to -L$;
  strang. $S \to -S$;
  position $\vec{x} \to \vec{x}$;
  momentum $\vec{p} \to \vec{p}$;
  spin $s \to s$.

- $\mathbb{C}$ is hermitian; its eigenvalues* are ± 1; they are multiplicatively conserved in strong and e.-m. interactions.

- However, almost NO particle is an eigenstate of $\mathbb{C}$; e.g.

  $\mathbb{C}\ |\pi^+\rangle = -\ |\pi^-\rangle$.

- Only particles (like π^0, unlike the K's) which are their own antiparticles, are eigenstates of $\mathbb{C}$:

  $C = +1$ for π^0, η, η';

  $C = -1$ for ρ^0, ω, ϕ;

  $C = -1$ for γ [*for Z, $\mathbb{C}$ and $\mathbb{P}$ are not defined*].

- E.g., use $\mathbb{C}$-conservation in e.-m. interactions for the following prediction :

  $\pi^0 \to \gamma\gamma$: $+1 \to (-1)\ (-1)$ ok;

  $\pi^0 \to \gamma\gamma\gamma$: $+1 \to (-1)\ (-1)\ (-1)$ no.

  $Br(\pi^0 \to \gamma\gamma\gamma)$ measured to be $\sim 10^{-8}$.

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* see next slides.

Quantum numbers : G-parity $\mathbb{G}$

proposed by Lee and Yang, 1956.

- charge conjugation $\mathbb{C}$ is defined as

  $\mathbb{C}\ |Q, B, L, S> = \pm\ |-Q, -B, -L, -S>$;
- therefore, only states Q = B = L = S = 0 may be $\mathbb{C}$ eigenstates (e.g. π^0, η, γ, $[\pi^+\pi^-]$);
- useful generalization [**G-parity**] :

  $\mathbb{G} \equiv \mathbb{C}\ \mathbb{R}_2$;

  where $\mathbb{R}_2$ = exp $(-i\pi\tau_2)$ is a rotation in the isospin space;
- remember :

  $\mathbb{C}\ |\pi^\pm> = -\ |\pi^\mp>$;

  $\mathbb{C}\ |\pi^0> = +\ |\pi^0>$;

  $\mathbb{R}_2 |I, I_3> = (-)^{I-I3}\ |I, -I_3>$;

  $\mathbb{R}_2 |\pi^\pm> = +\ |\pi^\mp>$;

  $\mathbb{R}_2 |\pi^0> = -\ |\pi^0>$;
- therefore, e.g.

  $\mathbb{G}\ |\pi^{\pm,0}> = \mathbb{C}\ \mathbb{R}_2\ |\pi^{\pm,0}> = -\ |\pi^{\pm,0}>$

- $\mathbb{G}$-parity is multiplicative :

  $\mathbb{G}\ |n\pi^+\ m\pi^-\ k\pi^0> =$

  $= (-)^{n+m+k}\ |n\pi^+\ m\pi^-\ k\pi^0>$;

  $\mathbb{G}\ |q\bar{q}> = (-)^{I+\ell-s}\ |q\bar{q}>$;
- G is a useful quantum number :
  - $\mathbb{G}$-parity is conserved in strong interactions (C and isospin are valid);
  - it produces selection rules (e.g. a decay in odd/even number of π's is allowed/forbidden).
- e.g. $\omega(782)$ is $I^G(J^{PC}) = 0^-(1^{--})$:

  BR $(\omega \to \pi^+\pi^-\pi^0)$ = $(89.2\pm0.7)\%$

  BR $(\omega \to \pi^+\pi^-)$ = $(\ 1.5\pm0.1)\%$

  opposite to the obvious phase-space predictions (more room for 2π than 3π decay).
- [*see also* J/ψ *decay*].

Quantum numbers : demonstrations

- Be |N, x, σ> the state, where
  - N : the "charges" (electric, baryon / lepton numbers, …);
  - x : the space vectors (position, momentum, …);
  - σ : the axial vectors (spin).

- Obviously, from their definition :

  $\mathbb{C}^2 = \mathbb{P}^2 = \mathbb{S}^2 = \mathbb{I}$;

  $\mathbb{C}^{-1} = \mathbb{C}$; $\mathbb{P}^{-1} = \mathbb{P}$; $\mathbb{S}^{-1} = \mathbb{S}$;

  [where $\mathbb{S}$ is the spin-flip operator].

- $\mathbb{C}$ has real eigenvalues, C=±1 :

  $\mathbb{C}$ |N, x, σ> = C |N, x, σ>;

  $\mathbb{C}^2$ |N, x, σ> = 1 |N, x, σ> = C^2 |N, x, σ>;

  $C^2 = 1 \rightarrow C = \pm 1$.

  (same for $\mathbb{P}$, $\mathbb{S}$, $\mathbb{G}$, $\mathbb{T}$).

- P(γ) = –1 [from Maxwell equations];

- For a q$\bar{q}$ (or particle-antiparticle) state, which is also a $\mathbb{C}$ eigenstate , $\mathbb{C} = \pm\mathbb{S}\,\mathbb{P}$:

  $\mathbb{S}\,\mathbb{P}\,\mathbb{C}$ |n,x,σ, -n,-x,-σ> =

  = C $\mathbb{S}\,\mathbb{P}$ |-n,x,σ, n,-x,-σ> =

  = C $\mathbb{S}$ |-n,-x,σ, n,x,-σ> =

  = C |-n,-x,-σ, n,x,σ> =

  = C |n,x,σ, -n,-x,-σ>.

  $\rightarrow \mathbb{S}\,\mathbb{P}\,\mathbb{C} = C = \pm 1$;

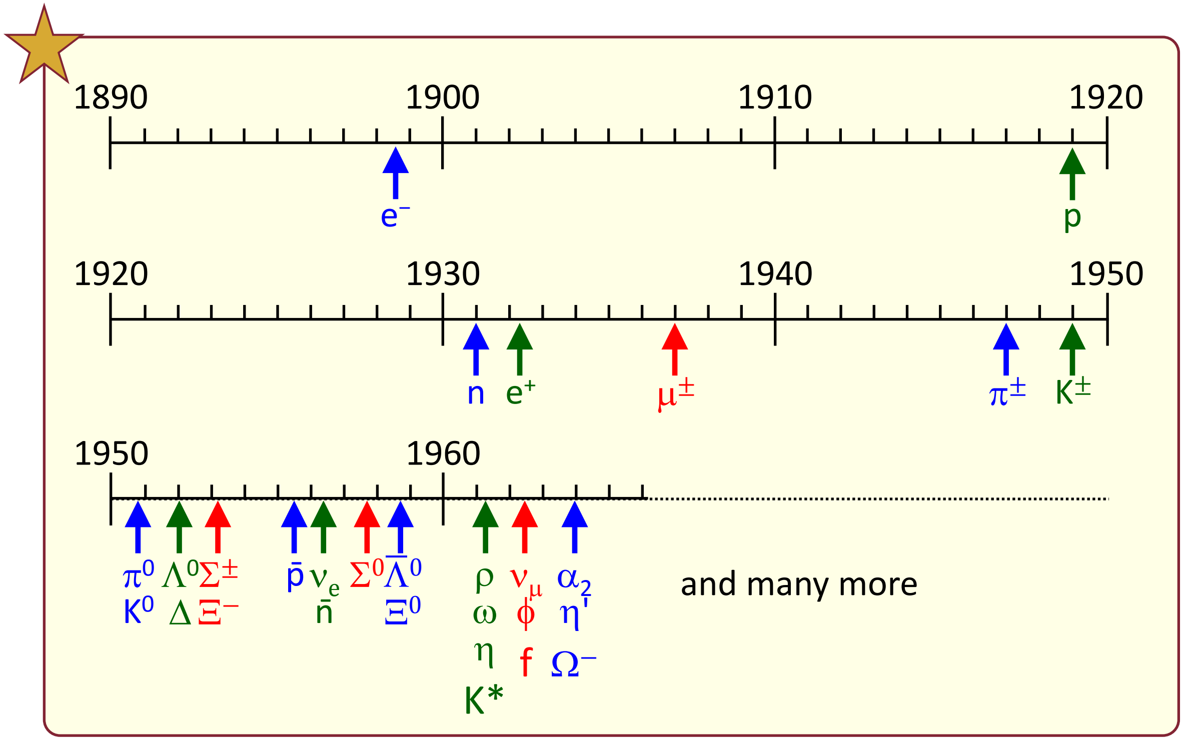
  $\rightarrow \mathbb{C} = \pm\mathbb{S}^{-1}\,\mathbb{P}^{-1} = \pm\mathbb{S}\,\mathbb{P}$. (q.e.d.)

- see also [FNSN1, §7]

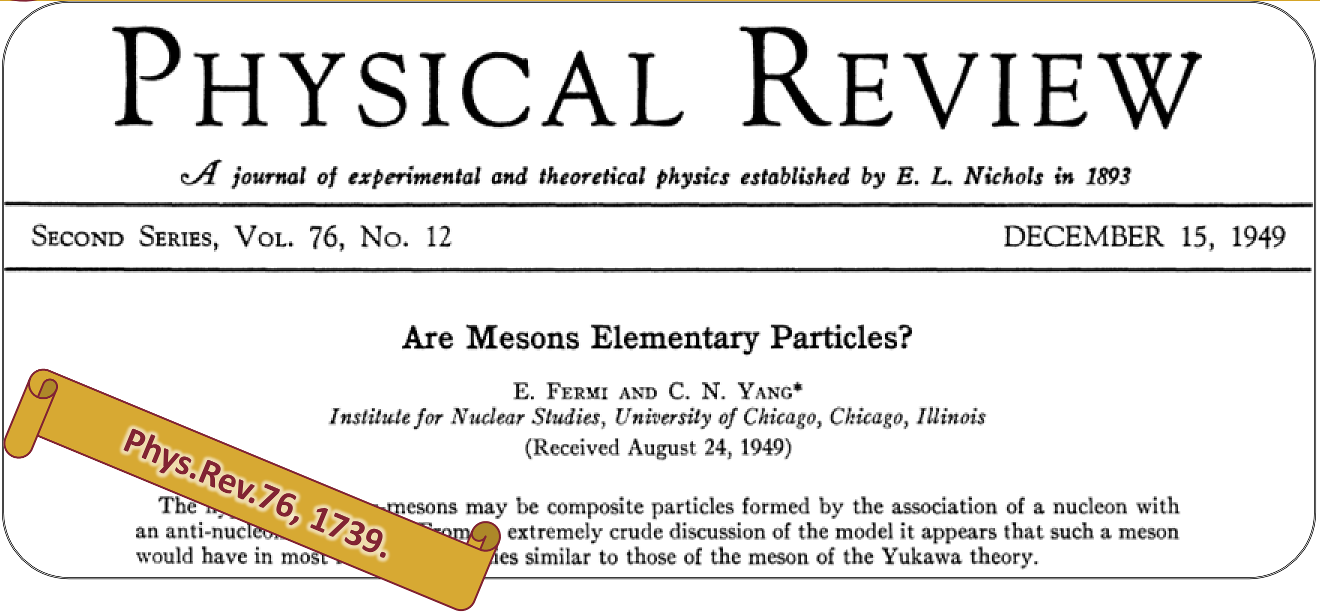
Hadrons : “elementary” or composite ?

too many hadronic resonances :

the figure shows the particle discoveries from 1898 to the '60s; their abundance and regularity, as a function of quantum numbers like charge and strangeness, were suggesting a possible regularity, similar to the Mendeleev table [FNSN1].



Hadrons : “elementary” or composite ?



PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 76, NO. 12 DECEMBER 15, 1949

Are Mesons Elementary Particles?

E. FERMI AND C. N. YANG*
Institute for Nuclear Studies, University of Chicago, Chicago, Illinois

(Received August 24, 1949)

The hy... mesons may be composite particles formed by the association of a nucleon with an anti-nucleo... From ... extremely crude discussion of the model it appears that such a meson would have in most ...ies similar to those of the meson of the Yukawa theory.

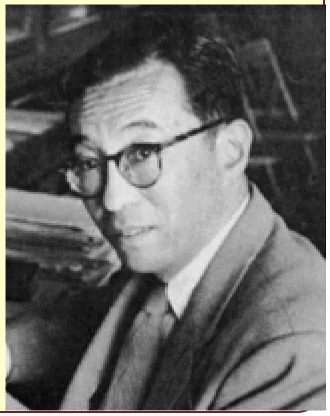
1949 : E.Fermi and C.N. Yang proposed that ALL the resonances were bound state p-n.

1956 : Sakata extended the Fermi-Yang model including the Λ, to account for strangeness : all hadronic states were then composed by (p, n, Λ) and their antiparticles.

Enrico Fermi

Chen-Ning Yang
(杨振宁 - 楊振寧,
Yáng Zhènníng)

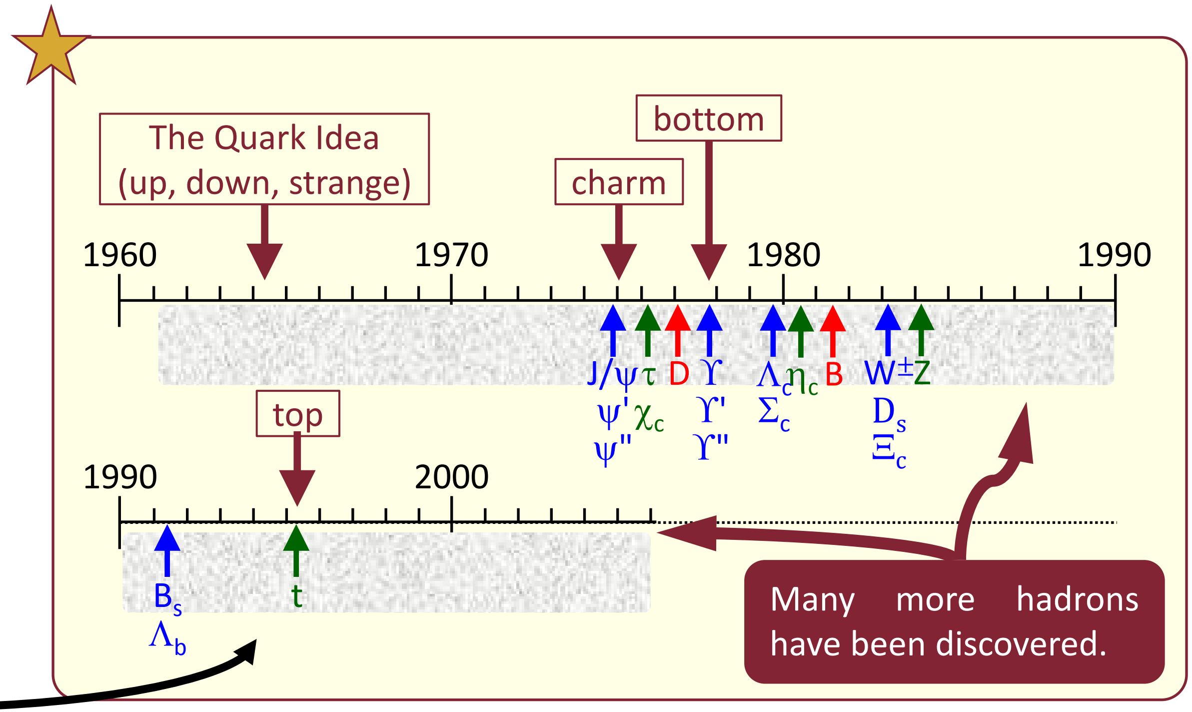
Shoiki Sakata
(坂田 昌一,
Sakata Shōichi)



Hadrons : elementary or composite ?

1961 : M. Gell-Mann and Y. Ne’eman (independently) proposed a new classification, the **Eightfold Way**, based on the symmetry group SU(3). The classification did **NOT** explicitly mention an **internal structure**. The name was invented by Gell-Mann and comes from the “eight commandments” of the Buddhism.

Murray Gell-Mann | Yuval Ne'eman (יובל נאמן)



Warning : "*t*" is a quark, not a hadron (in modern language).

The Eightfold Way (1961-1964)

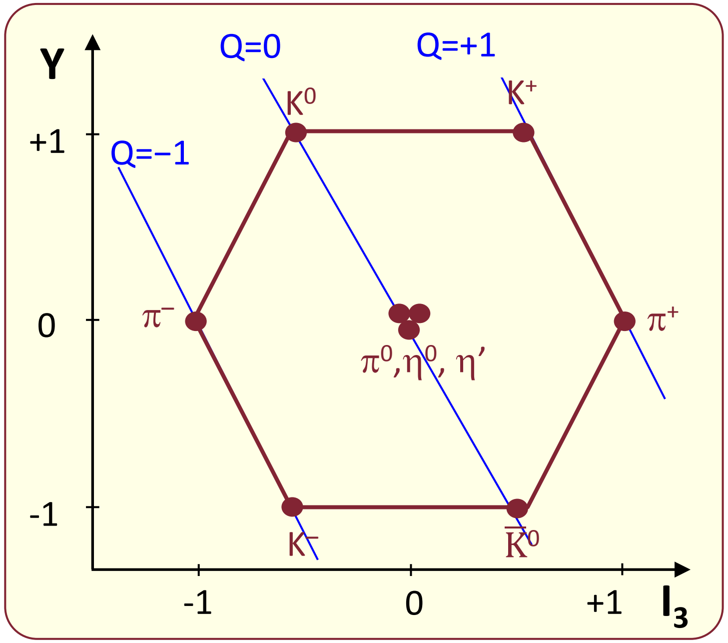
All hadrons are classified in the plane **(I_3 – Y)**, (Y = strong hypercharge), where

$I_3 = I_z$ = third component of isospin;

$Y = B + S$ **[baryon number + strangeness]**.

The strangeness ***S***, which contributes to ***Y***, had the effect to enlarge the isospin symmetry group **SU(2)** to the larger **SU(3): Special Unitarity group**, with dimension=3.

The Gell-Mann – Nishijima formula (1956) is :



$Q = I_3 + ½(B+S)$

including heavy flavors [B:baryon, B':bottom] :

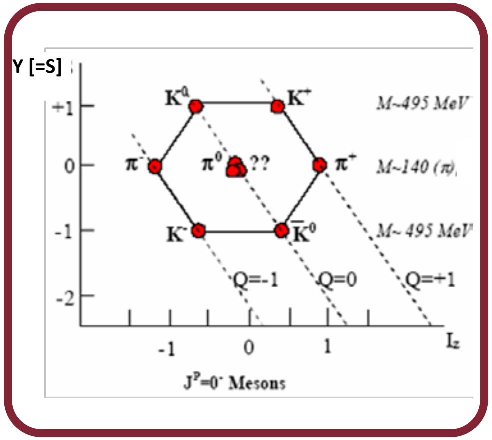
$Q = I_3 + ½(B+S+C+B'+T)$

This symmetry is now called "**flavor SU(3) [SU(3)$_F$]**", to distinguish it from the "**color SU(3) [SU(3)$_C$]**", which is the exact symmetry of the strong interactions QCD (see later).

The Eightfold Way (1961-1964)

The particles form the multiplets of **SU(3)$_F$**; each multiplet contains particles that must have the same spin and intrinsic parity. The basic multiplicity for mesons is nine (3×3), which splits in two SU(3) multiplets: (octet + singlet). Instead, for baryons we have both octects and decuplets (see later in this chapter).

The gestation of SU(3) was long and difficult. It was a triumph because it both explained the multiplets of known particles/resonances, and (more exciting) predicted new particles/resonances, before they were actually discovered .

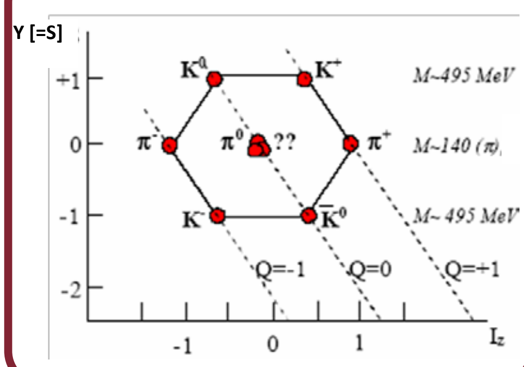


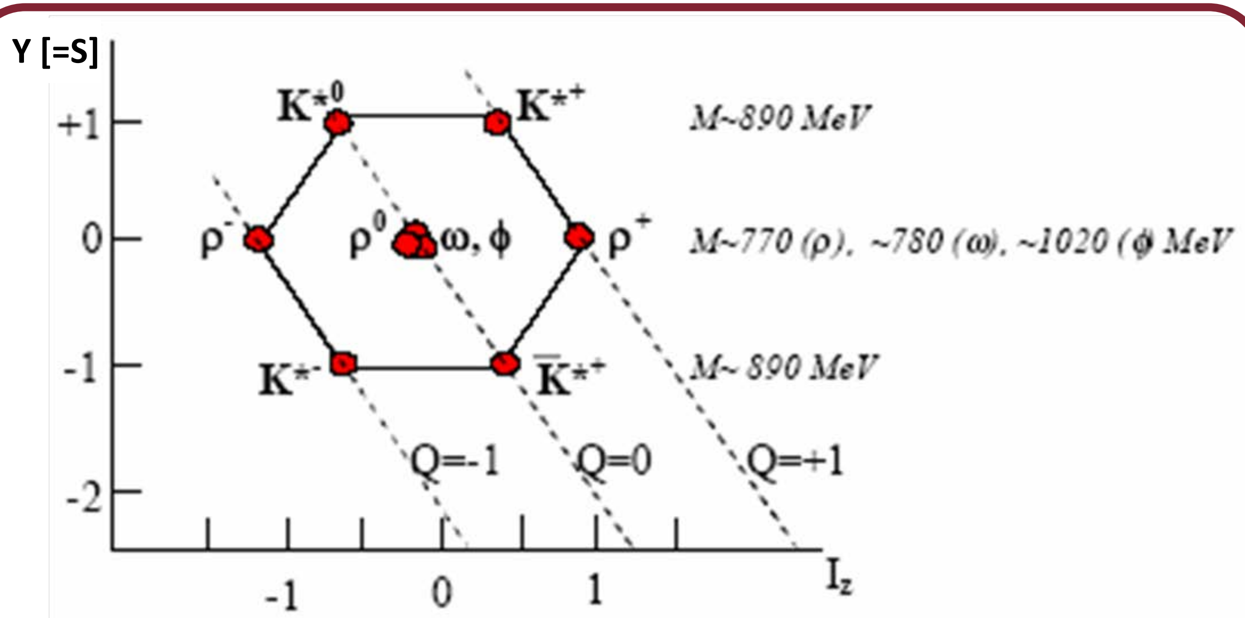
However, proton and neutron (or $\pi^\pm$ and π^0) have nearly identical mass. Therefore, while the isospin symmetry **SU(2)** is almost exact, the symmetry **SU(3)$_F$**, grouping together strange and non-strange particles, appears to be substantially violated.

In principle, in a similar way, the discovery of heavier flavors could be interpreted with higher groups (e.g. **SU(4)** to incorporate charm, and so on). However, these higher symmetries are broken even more, as demonstrated by the mass values. Therefore, **SU(6)$_F$** for all known mesons $J^P = 0^-$ is (almost) never used.

The Eightfold Way (1961-1964)

Few other multiplets :



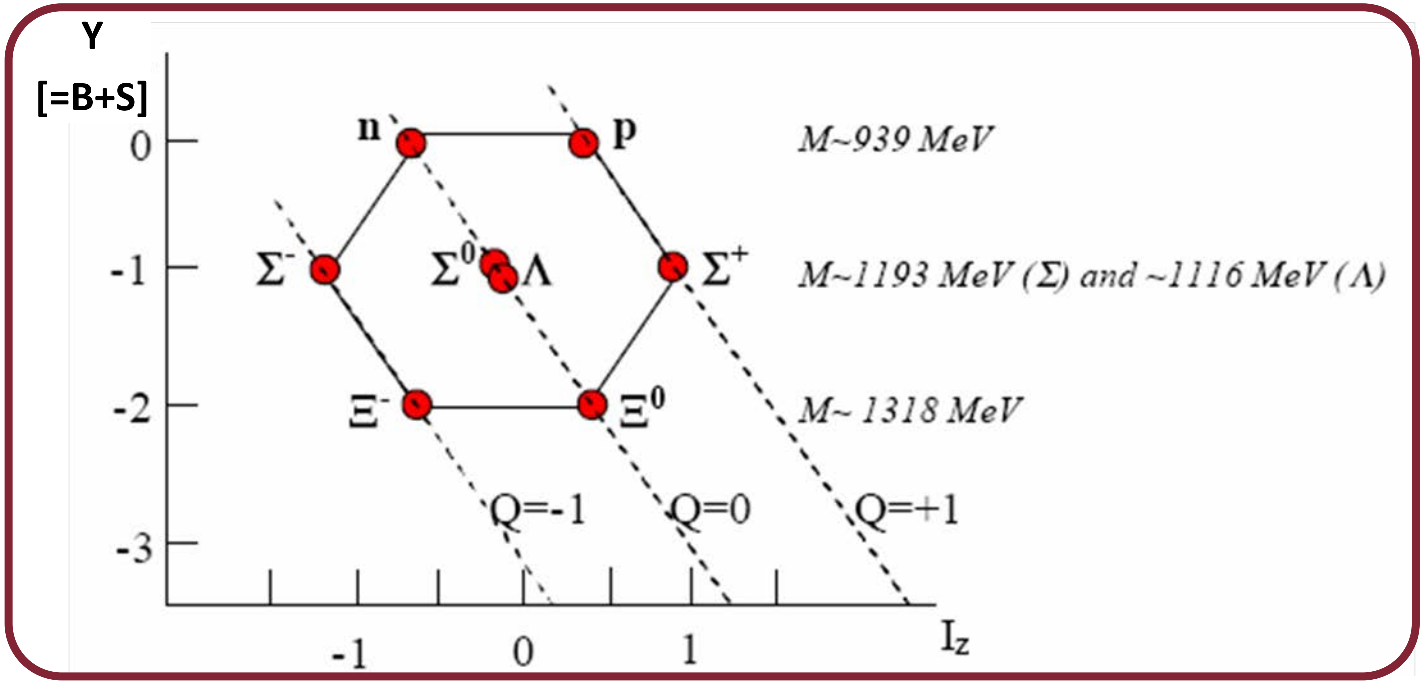


[Mesons $J^P = 0^-$]

Meson resonances $J^P = 1^-$
(all discovered by 1961).

The Eightfold Way (1961-1964)

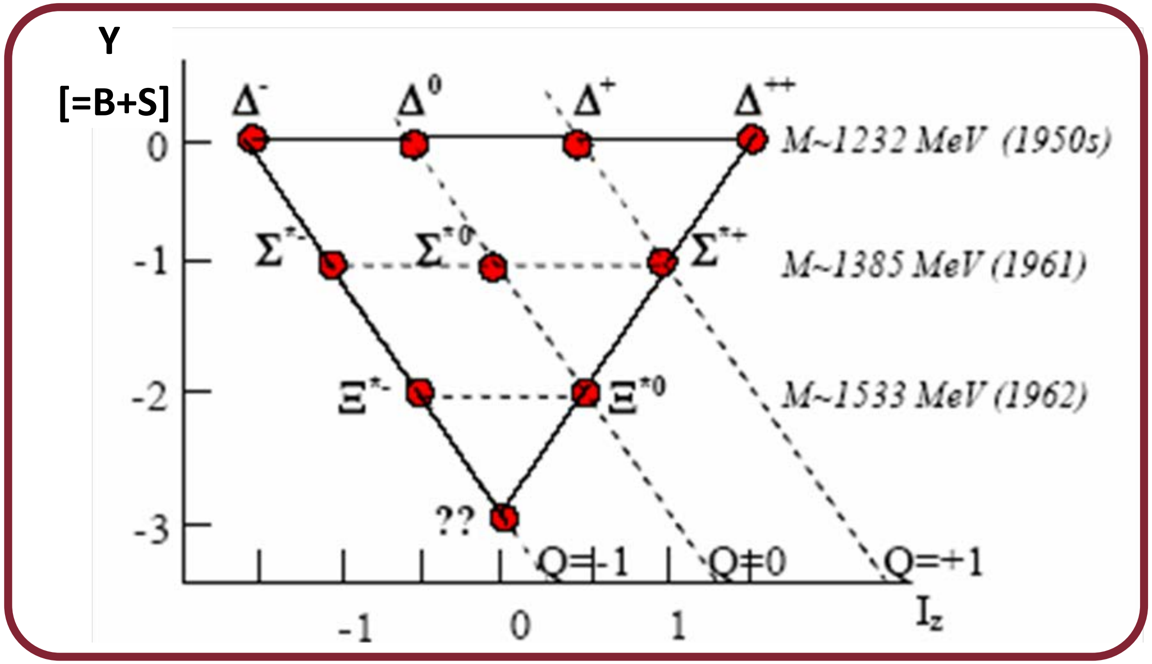
Other multiplets :



Octet of spin ½ baryons

The Eightfold Way (1961-1964)

Other multiplets :



Decuplet of spin $^3/_2$ baryon resonances

Baryon resonances $(^3/_2)^+$

As shown, these resonances exhibit a symmetry respect to the isospin I_3 and hypercharge **Y**.

Therefore, a new multiplet with Δ^* Σ^* Ξ^* with S = 0, –1,–2 and a new resonance with **S = –3** must exist.

The particle, called Ω^-, predicted (★) in 1962, was discovered in 1964 by N.Samios et al., using the 80-inch hydrogen bubble chamber at Brookhaven (Fig.).

The Ω^-, having S = -3, can only decay weakly (*) :

$\Omega^- \to \Xi^0 \pi^-$; $\to \Xi^- \pi^0$; $\to \Lambda^0 K^-$;

$\tau_{\Omega^-} \cong 0.82 \times 10^{-10}$ s.

____

(*) Since even the electromagnetic interactions conserve the strangeness, no electromagnetic decay would be possible; the simplest non-weak S-conserving decay is :

$\Omega^- \to \Lambda^0 K^0 K^-$,

which is impossible, because

$m(\Omega) \cong 1700$ MeV < $m(\Lambda) + 2\, m(K) \cong 2100$ MeV.

Therefore the Ω^- must decay via strangeness-violating weak interactions : the Ω^- is NOT a resonance.

From a 1962 report:

Discovery of Ξ^* resonance with mass ~1530 MeV is announced […].

Gell-Mann and Ne'eman […] predicted a new particle and wrote down all its properties:

- Name = Ω^- (*Omega* because this particle is the last in the decuplet);
- Mass ≈ 1680 MeV (the masses of Δ, Σ^* and Ξ^* are about equidistant ~150 MeV);
- Charge = –1;
- Spin = 3/2;
- Strangeness = –3;
- Lifetime ~ 10^{-10} s, because of its weak decay, since strong decay is forbidden(*);
- Decay modes: $\Xi^0 \pi^-$ and $\Xi^- \pi^0$;
- Isospin = 0 (no charge-partners of similar mass).

Baryon resonances $(^3/_2)^+$

$K^- + p \rightarrow \Omega^- + K^+ + K^0$

└→ $\Xi^0 + \pi^-$ ($\Delta S = 1$ weak decay)

└→ $\pi^0 + \Lambda$ ($\Delta S = 1$ w.d.)

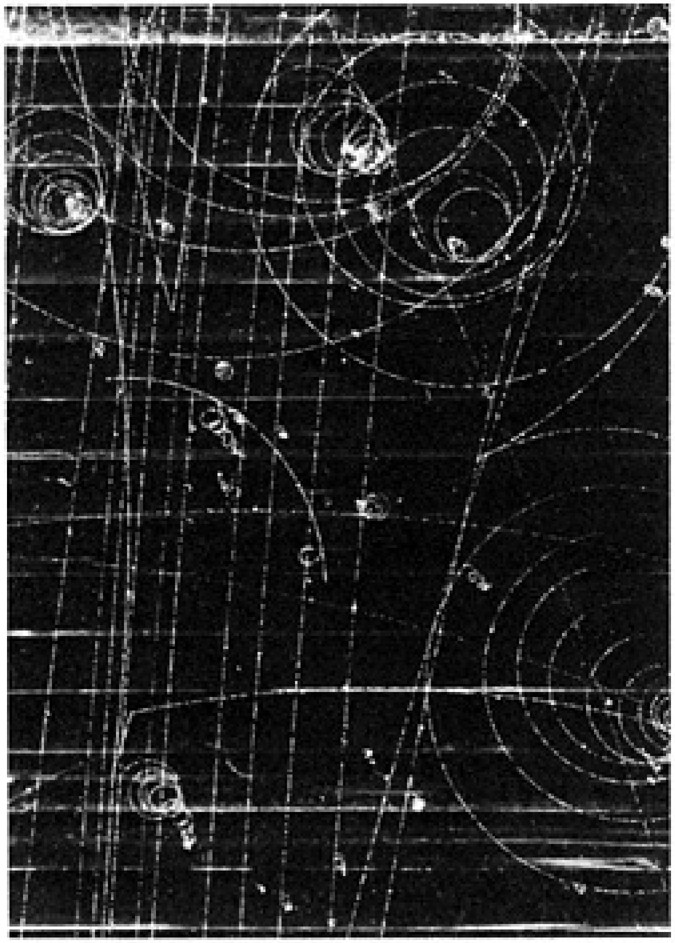
└→ $\pi^- + p$ ($\Delta S = 1$ w.d.)

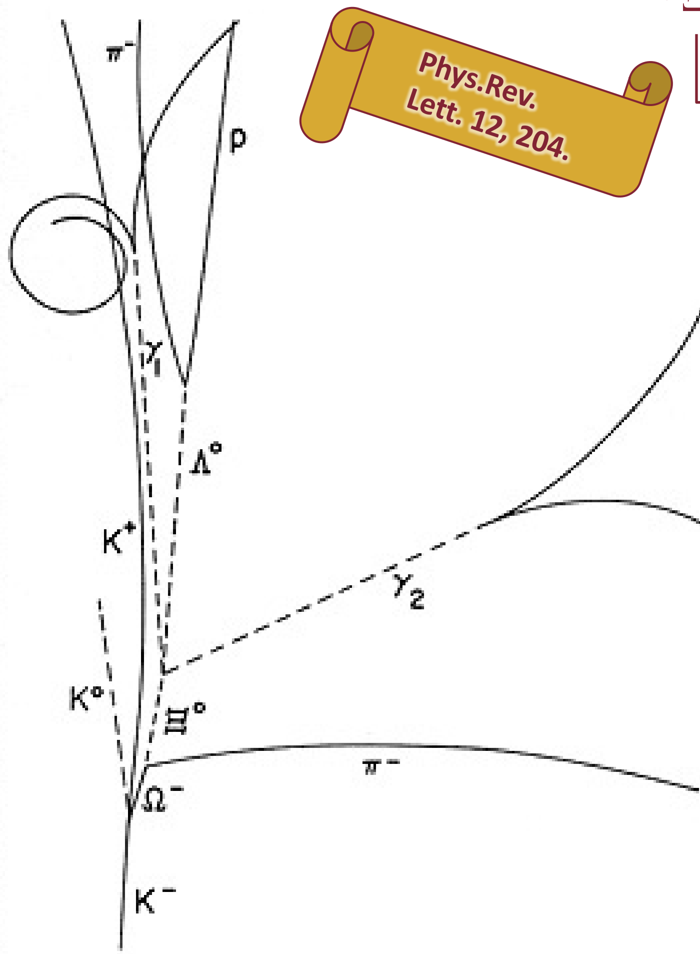
└→ $\gamma + \gamma$ (e.m. decay)

└→ e^+e^-

└→ e^+e^-

Phys.Rev. Lett. 12, 204.





Brookhaven National Laboratory 80-inch hydrogen bubble chamber - 1964

The static quark model

A deeper understanding of SU(3) and its success in describing the properties of hadrons was achieved in 1964, when M. Gell-Mann and G. Zweig proposed independently the hypothesis that all the hadrons are composed of three basic constituents, that Gell-Mann called(*) quarks.

This model, enriched by both extensions (e.g. new quarks) and dynamics (e.g. electroweak interactions and QCD) is still the basic constituent of our understanding of the elementary particles, the Standard Model.

In this chapter we consider only the static part, in the world of the '50s and '60s. Sometimes, in the literature, it is referred as the *naïve quark model*.

1969 : Gell-Mann is awarded Nobel Prize *"for his contributions and discoveries concerning the classification of elementary particles and their interactions"*.

However, at that time, it was not at all clear whether all that was just "a mathematical convenience", or reality.

(*) The name so whimsical was taken from the (now) famous quote "Three quarks for Muster Mark", from James Joyce's novel "Finnegans Wake".

The static quark model

today also **c**, **b**, **t** have been discovered !!!

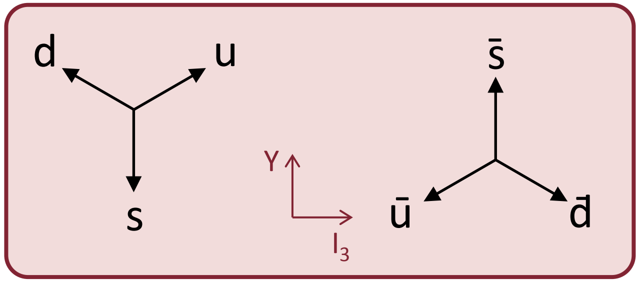
In this schema :

- there exists three quarks **u**, **d**, and **s** (up, down, strange);
- quarks are fermions with spin ½ and fractional charge (±⅓, ±⅔);
- the baryons are made from three quarks (e.g. uds, uud);
- the mesons from quark-antiquark pairs (e.g $u\bar{u}$, $u\bar{d}$, $s\bar{u}$);

- Therefore, according to the Dirac theory :
- charged-conjugate states exists, called *antiquarks*, which are the quarks antiparticles with opposite charges;
- the *antibaryons* are made of three antiquarks (e.g $\bar{u}\bar{u}\bar{d}$);
- The *antimesons* are made from an "antiquark-quark" pair (i.e. the mesons are their own antiparticles).

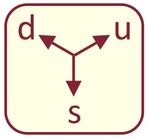
		u	d	s
B	baryon	⅓	⅓	⅓
J	spin	½	½	½
I	isospin	½	½	0
$\mathbf{I_3}$	3rd isospin	½	−½	0
S	strang.	0	0	−1
Y	B+S	⅓	⅓	−⅔
Q	I_3 + ½Y	⅔	−⅓	−⅓

The three quarks form a triplet, which is a basic representation of the group. Quarks may be represented in a vector shape in the plane I_3 / Y; then their combinations are the sums of such vectors.

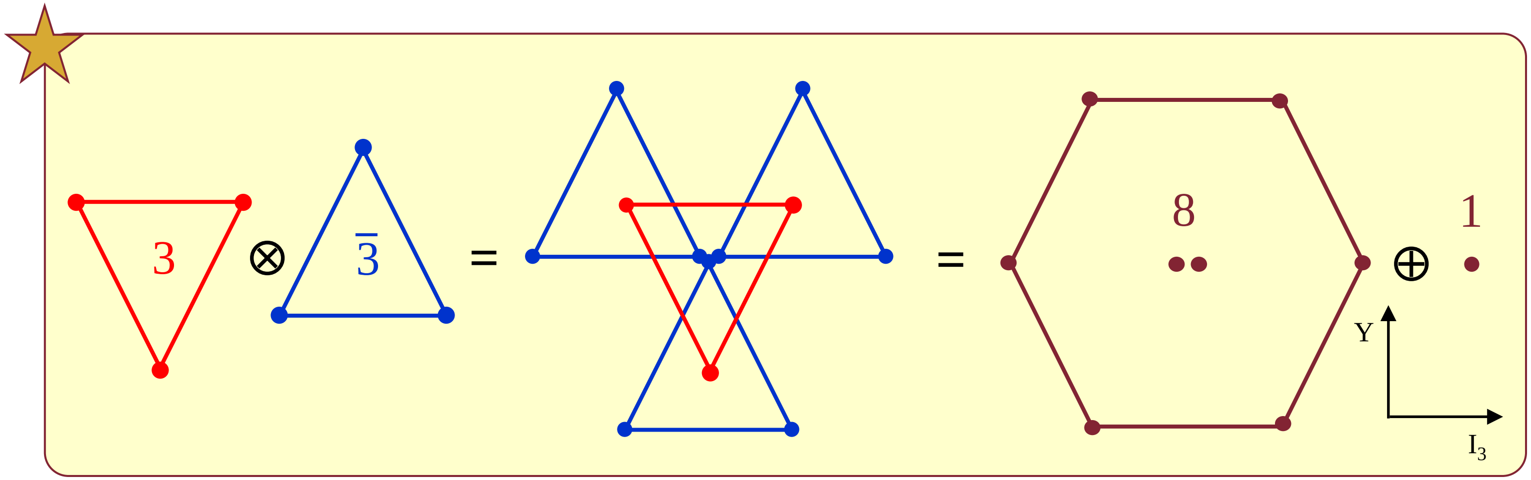


Graphical construction of the mesons

Rules :

- in the space I_3 / Y, sum "vectors" (i.e. add quarks 

, or the corresponding antiquarks) to produce "states";
- all the combinations are allowed.

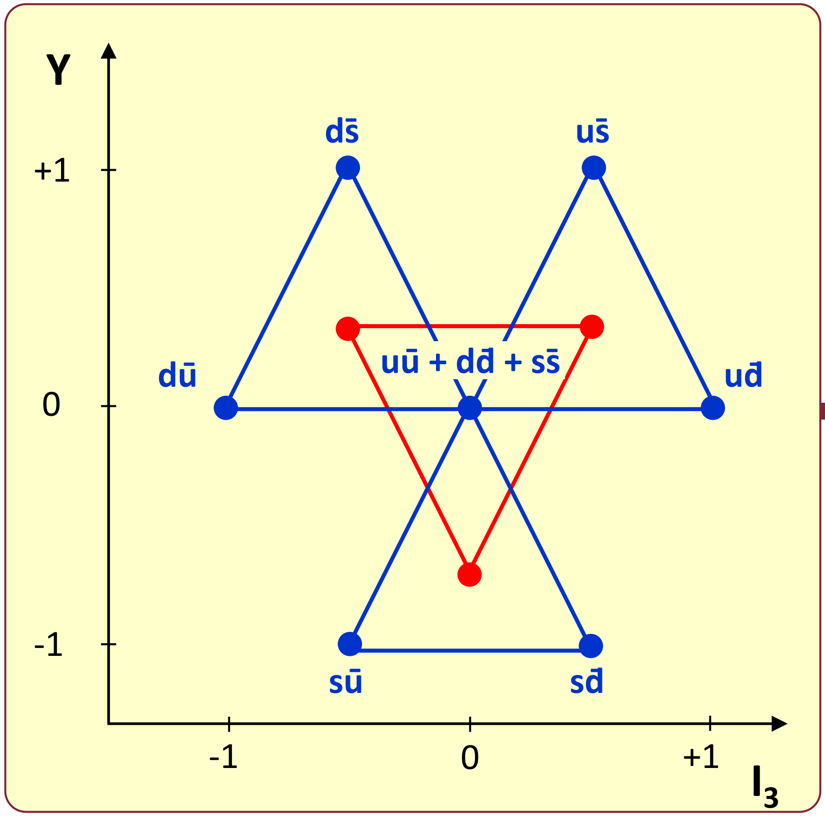


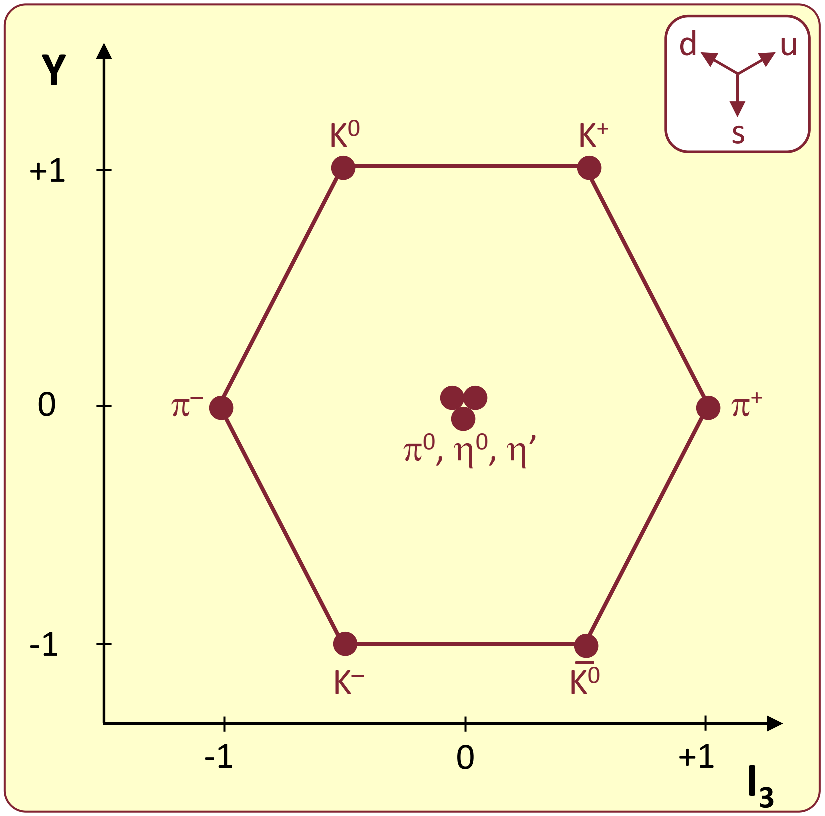
- the pseudoscalar mesons ($J^P=0^-$) are $q\bar{q}$ states in s-wave with opposite spins (⇑⇓).

Graphical construction of the mesons

More specifically, with s-wave ($J^{PC}=0^{-+}$), we get the "pseudoscalar" nonet :

Notice that π^0, η, η' are combinations (mixing) of the three possible $q\bar{q}$ states (see later for the mixing parameters) :

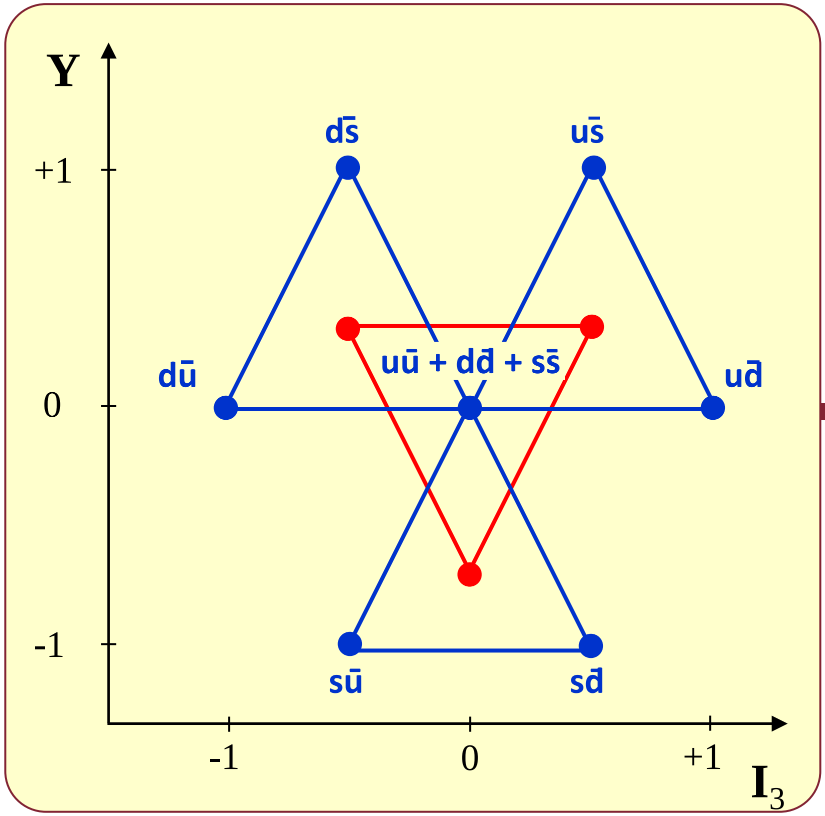


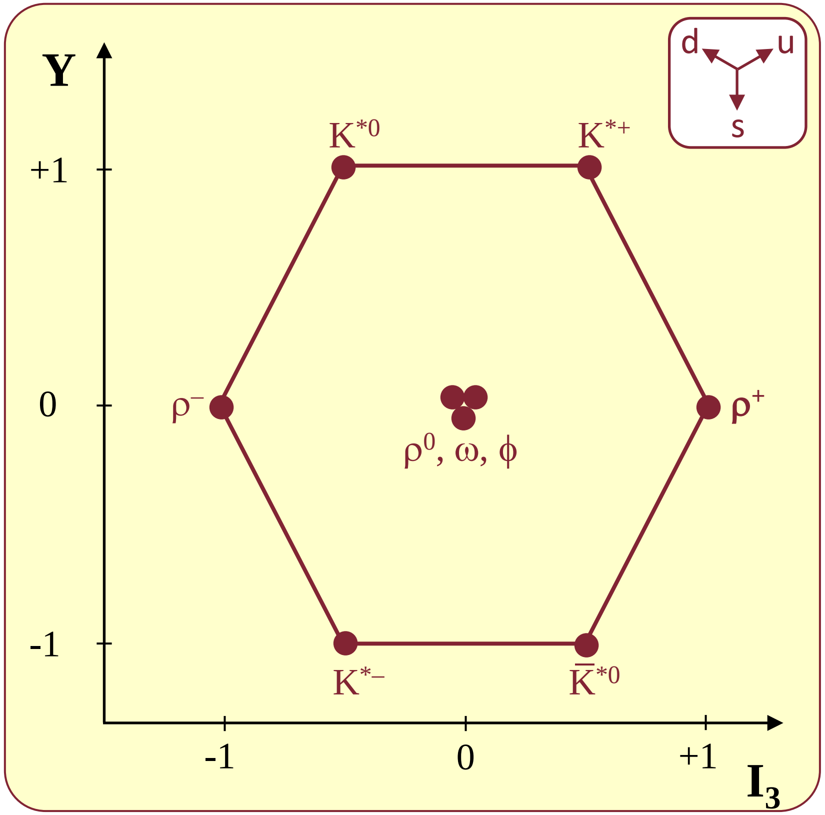


Graphical construction of the mesons

In addition, with $J^{PC} = 1^{--}$ (i.e. spin ⇈), the "vector" nonet :

Notice that ρ^0, ω, φ are combinations (mixing) of the three possible $q\bar{q}$ states :





Meson quantum numbers : J^{PC}

- Parity : the quarks and the antiquarks have opposite P :

  $P_{q\bar{q}} = P_1 P_2 (-1)^{\ell} = -1\,(-1)^{\ell} = (-1)^{\ell+1}$.

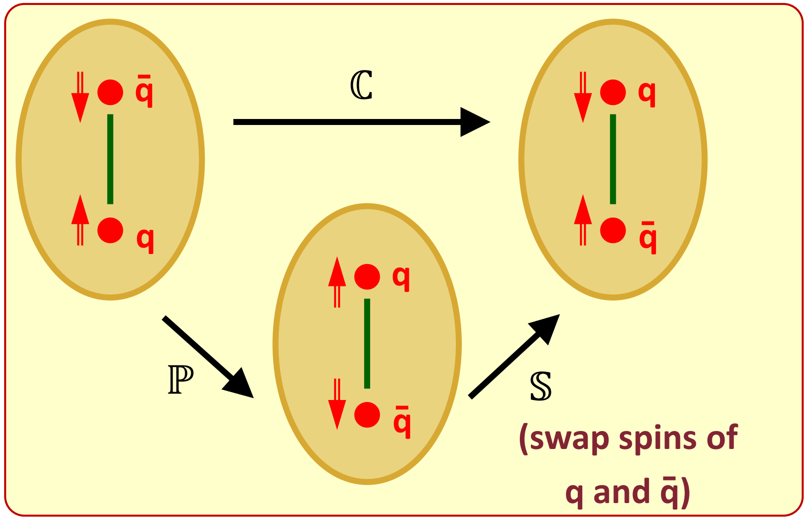
- Charge conjugation : for mesons, which are also $\mathbb{C}$ eigenstates, $\mathbb{C} = \mathbb{P}\mathbb{S}$, parity followed by spin swap (see before).

- therefore, C is the product :

  $P = (-1)^{\ell+1}$;

  $S = (-1)^{s+1}$; (Pauli principle, [BJ, 263])

  $C = P \times S = = (-1)^{\ell+s}$.



$$\underbrace{\frac{1}{\sqrt{2}}\left(\Uparrow\Downarrow - \Downarrow\Uparrow\right);}_{S=0} \qquad \underbrace{\Downarrow\Downarrow; \quad \frac{1}{\sqrt{2}}\left(\Uparrow\Downarrow + \Downarrow\Uparrow\right); \quad \Uparrow\Uparrow}_{S=1}$$

Meson quantum numbers : multiplets

- For the lowest state nonets, these are the quantum numbers :

L	S	J^{PC}	$^{2s+1}L_J$	I=1 state
0	0	0^{-+}	$^1S_0$	$\pi(140)$
	1	1^{--}	$^3S_1$	$\rho(770)$
1	0	1^{+-}	$^1P_1$	$b_1(1235)$
	1	0^{++}	$^3P_0$	$a_0(1450)$
		1^{++}	$^3P_1$	$a_1(1260)$
		2^{++}	$^3P_2$	$a_2(1320)$

- all these multiplets have the main q.n. n = 1;
- as of today ~20 meson multiplets have been (partially) discovered [PDG].
- important activity from the ’50 to the ’70; still some addict;

- method (mainly bubble chambers) :
  - measure (tons of) events; e.g. :

    $\bar{p}p \to \pi^+ \pi^+ \pi^- \pi^- \pi^0;$
  - look for “peaks” in final state combined mass, e.g. m($\pi^+ \pi^- \pi^0$);
  - the peaks are associated with high mass resonances, decaying via strong interactions (width $\to \Gamma \to$ strength);
  - the scattering properties (e.g. the angular distribution) identify the quantum numbers;
- result : an overall consistent picture;
- Great success !!!

“If I could remember the names of all these particles, I'd be a botanist.”
Enrico Fermi

Meson quantum numbers : $\rho^0 \not\to \pi^0\pi^0$

$\rho^0 \to \pi^0\pi^0$ is forbidden ?

a) C-parity

$C(\rho^0) = -1$; $C(\pi^0) = +1$

therefore, since the initial state is a C-eigenstate,

$-1 = (+1) \times (+1)$ **???? NO**

NB. A general rule : "a vector cannot decay into two equal (pseudo-)scalars".

But (a) and (b) do not hold for weak decays. Instead (c) is due to spin-statistics and angular momentum conservation, which holds for all interactions.

$Z \not\to HH$ is also forbidden.

b) Clebsch-Gordan coeff. in isospin space

$|\rho^0\rangle = |I=1, I_3=0\rangle$;

$|\pi^0\rangle = |1, 0\rangle$;

therefore the decay is

$$\langle\pi^0\pi^0|\rho^0\rangle = \langle j_1 j_2 m_1 m_2 | J\ M\rangle = $$
$$= \langle 1\ 1\ 0\ 0\ |\ 1\ 0\ \rangle = 0;$$

i.e. it does NOT happen.

[PDG, § 44 :

$1\otimes1$		…	1
		…	0
…	…	…	…
0	0	…	**0**

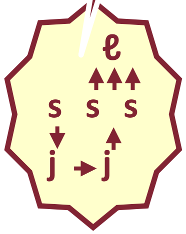
]

c) Spin-statistics

follow the →

(see [Povh, 199,374])

	ρ^0	$\to \pi^0$	π^0
L		1	
S	1	0	0
J	1	1	



ρ^0 is a boson → wave function symmetric;

the π^0's are two equal bosons → space wave function symmetric;

L=1 makes the function anti-symmetric →

no.

Meson mixing

Light mesons	$q\bar{q}$	J^{PC} (1)	I	I_3	S	Q (1)	mass (MeV)	note
π^+, π^0, π^-	$u\bar{d}$, $q\bar{q}^{(2)}$, $d\bar{u}$	0^{-+}	1	1, 0, -1	0	1, 0, -1	140	(2)
η	$q\bar{q}^{(2)}$	0^{-+}	0	0	0	0	550	(2)
η'	$q\bar{q}^{(2)}$	0^{-+}	0	0	0	0	960	(2)
K^+, K^0	$d\bar{s}$, $u\bar{s}$	0^-	½	½,-½	+1	1, 0	495	
$\bar{K}^0, K^-$	$s\bar{u}$, $s\bar{d}$	0^-	½	½, -½	-1	0, -1	495	
ρ^+, ρ^0, ρ^-	$u\bar{d}$, $q\bar{q}^{(3)}$, $d\bar{u}$	1^{--}	1	1, 0, -1	0	1, 0, -1	770	(3)
ω	$q\bar{q}^{(3)}$	1^{--}	0	0	0	0	780	(3)
ϕ	$q\bar{q}^{(3)}$	1^{--}	0	0	0	0	1020	(3)
K^{*+}, K^{*0}	$d\bar{s}$, $u\bar{s}$	1^-	½	½,-½	+1	1, 0	890	
$\bar{K}^{*0}, K^{*-}$	$s\bar{u}$, $s\bar{d}$	1^-	½	½, -½	-1	0, -1	890	

Notes :

(1) $P = (-)^{\ell+1} \rightarrow P = -$; $C = (-)^{\ell+s} \rightarrow C = (-)^s$ [only when $\mathbb{C}$ eigenstates]; $Q = I_3 + ½Y$; (B=0) $\rightarrow Q = I_3 + ½S$;

(2) The mesons π^0, η, η' are mixing of $u\bar{u} \oplus d\bar{d} \oplus s\bar{s}$, but essentially π^0 is only $u\bar{u} \oplus d\bar{d}$ (see next);

(3) The mesons ρ^0, ω, ϕ are mixing of $u\bar{u} \oplus d\bar{d} \oplus s\bar{s}$, but essentially ϕ is only $s\bar{s}$ (see next).

Meson mixing: 0^- and 1^-

Mesons are bound states $q\bar{q}$. Consider only uds quarks (+ $\bar{u}\bar{d}\bar{s}$) in the nonets ($J^P = 0^-\ 1^-$, the *pseudo-scalar* and *vector* nonets) :

- the states ($u\bar{d}$, $u\bar{s}$, $d\bar{u}$, $d\bar{s}$, $s\bar{u}$, $s\bar{d}$) have no ambiguity : (K^{0+}, $\bar{K}^{0-}$, $\pi^{\pm}$);
- but ($u\bar{u}$ $d\bar{d}$ $s\bar{s}$) have the same quantum numbers and the three states ($\psi_{8,0}$ $\psi_{8,1}$ ψ_1) mix together ($\rightarrow$ 2 angles per nonet);
- the physical particles (π^0, η, η' for 0^-, ρ^0, ω, ϕ for 1^-) are linear combinations $q\bar{q}$;
- ($\psi_{8,1}$) decuples (π^0 ρ^0) ($\rightarrow$ 1 angle only);
- θ_{ps} and θ_v are computed from the mass matrices [PDG, §15.2];
- notice: the vector mixing $\theta_v \approx 36° \approx \tan^{-1}(1/\sqrt{2})$, i.e. the ϕ meson is almost $s\bar{s}$ only [i.e. $\phi \rightarrow K\bar{K}$, see KLOE exp.];

(... continue)

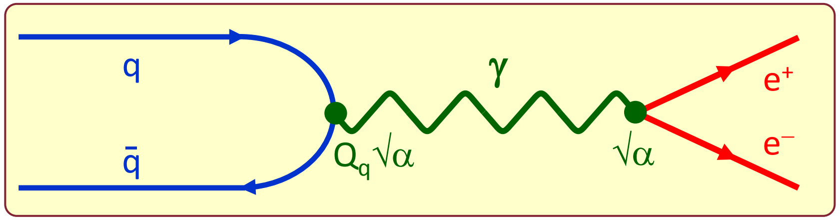
$$\left.\begin{aligned}\psi_{8,1}[\text{oct},I=1] &= (u\bar{u} - d\bar{d})/\sqrt{2}\\ \psi_{8,0}[\text{oct},I=0) &= (u\bar{u} + d\bar{d} - 2s\bar{s})/\sqrt{6}\\ \psi_1[\text{sing}] &= (u\bar{u} + d\bar{d} + s\bar{s})/\sqrt{3}\end{aligned}\right\} \begin{array}{c}\psi_{\text{multi},I}\\ \text{ideal case}\end{array}$$

$$\left.\begin{aligned}\pi^0(140) &\approx \psi^{ps}_{8,1} = (u\bar{u} - d\bar{d})/\sqrt{2}\\ \eta(550) &= \psi^{ps}_{8,0}\cos\theta_{ps} - \psi^{ps}_1 \sin\theta_{ps}\\ \eta'(960) &= \psi^{ps}_{8,0}\sin\theta_{ps} + \psi^{ps}_1\cos\theta_{ps}\end{aligned}\right\} \begin{array}{c}J^P = 0^-,\\ \theta_{\text{pseudo-scalar}} \approx -25°;\end{array}$$

$$\left.\begin{aligned}\rho^0(770) &\approx \psi^{v}_{8,1} = (u\bar{u} - d\bar{d})/\sqrt{2}\\ \phi(1020) &= \psi^{v}_{8,0}\cos\theta_v - \psi^{v}_1\sin\theta_v \approx s\bar{s}\\ \omega(780) &= \psi^{v}_{8,0}\sin\theta_v + \psi^{v}_1\cos\theta_v \approx\\ &\approx (u\bar{u} + d\bar{d})/\sqrt{2}\end{aligned}\right\} \begin{array}{c}J^P = 1^-,\\ \theta_{\text{vector}} \approx 36°.\end{array}$$

Meson mixing: 1^{-}

The decay amplitudes in the e.m. channels may be computed, up to a common factor, and compared to the experiment;



$$\left.\begin{aligned} \rho^0(770) &= \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}); \\ \phi(1020) &= s\bar{s}; \\ \omega(780) &= \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}); \end{aligned}\right\} \rightarrow \begin{aligned} &\mathcal{M}_{fi}(X \rightarrow e^+e^-) \\ &\propto \sum \alpha_q Q_q; \end{aligned}$$

$$\rightarrow \begin{cases} \Gamma(\rho^0 \rightarrow e^+e^-) \propto \left[\frac{1}{\sqrt{2}}\left(\frac{2}{3} - \frac{-1}{3}\right)\right]^2 = \frac{1}{2}; \\ \Gamma(\omega \rightarrow e^+e^-) \propto \left[\frac{1}{\sqrt{2}}\left(\frac{2}{3} + \frac{-1}{3}\right)\right]^2 = \frac{1}{18}; \\ \Gamma(\phi \rightarrow e^+e^-) \propto \left[\frac{1}{3}\right]^2 = \frac{1}{9}; \end{cases}$$

$$\rightarrow \Gamma_\rho : \Gamma_\omega : \Gamma_\phi = \begin{cases} 9 \quad : 1 : 2 \quad \text{(theo)} \\ 8.8 \pm 2.6 : 1 : 1.7 \pm 0.4 \quad \text{(exp).} \end{cases}$$

Few problems :

- the values are small, e.g. BR($\rho^0 \rightarrow e^+e^-$) $\cong$ 4.7×10^{-5};
- the phase-space factor is important, especially for ϕ, which is very close to the $s\bar{s}$ threshold.

However, the overall picture is clear: the theory explains the data ***very well***.

Graphical construction of the baryons

The construction looks complicated, but in fact is quite simple :

- add the three quarks one after the other;
- count the resultant multiplicity.

In group's theory language :

$3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8' \oplus 1$

i.e. two octets, a decuplet and a singlet.

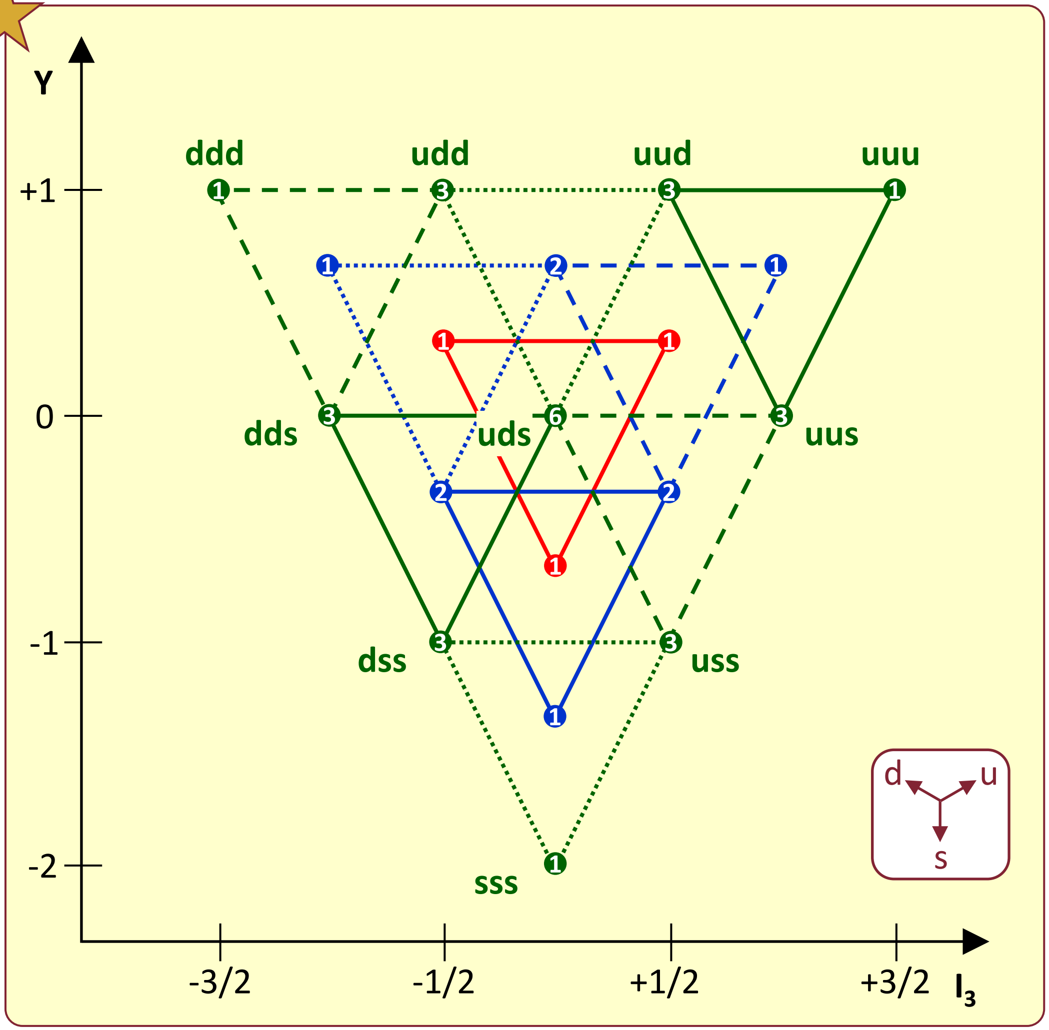
[dem. :

$3 \otimes 3 = 6 \oplus \bar{3};$

$6 \otimes 3 = 10 \oplus 8;$

$\bar{3} \otimes 3 = 8 \oplus 1.$ q.e.d.]

Both for 10, 8, 8' and 1 the three quarks have L = 0.



Graphical construction of the baryons

Baryons	qqq	J^P	I	I_3	S	$Q^{(1)}$	mass (MeV)
p, n	uud, udd	½⁺	½	½, -½	0	1, 0	940
Λ	uds	½⁺	0	0	-1	0	1115
$\Sigma^+, \Sigma^0, \Sigma^-$	uus, uds, dds	½⁺	1	1, 0, -1	-1	1, 0, -1	1190
Ξ^0, Ξ^-	uss, dss	½⁺	½	½,-½	-2	1, 0	1320
$\Delta^{++}, \Delta^+, \Delta^0, \Delta^-$	uuu, uud, udd, ddd	$^3/_2{}^+$	$^3/_2$	$^3/_2$, $^1/_2$, $-^1/_2$, $-^3/_2$	0	2, 1, 0, -1	1230
$\Sigma^{*+}, \Sigma^{*0}, \Sigma^{*-}$	uus, uds, dds	$^3/_2{}^+$	1	1, 0, -1	-1	1, 0, -1	1385
Ξ^{*0}, Ξ^{*-}	uss, dss	$^3/_2{}^+$	½	½,-½	-2	1, 0	1530
Ω^-	sss	$^3/_2{}^+$	0	0	-3	-1	1670

10 ⇑⇑⇑

8 ⇑⇑⇓

Notes :

(1) $Q = I_3 + ½Y = I_3 + ½(B + S)$; B = 1.

Graphical construction of the baryons: ½⁺

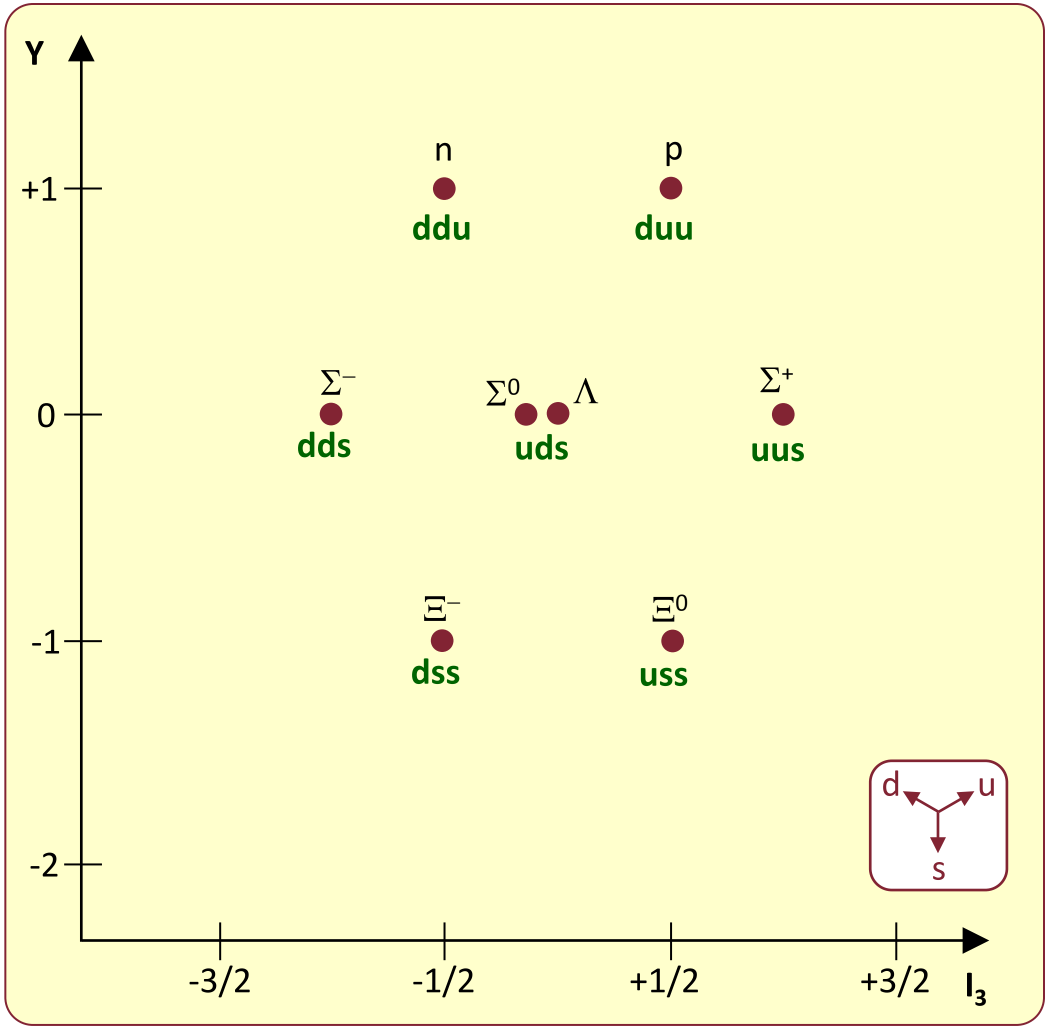
The lowest mass multiplet is an octet, which contains the familiar p and n, a triplet of S=-1 (the Σ's) a singlet S=-1 (the Λ) and a doublet of S=-2 (the Ξ's, sometimes called "cascade baryons").

The three quarks have ℓ = 0 and spin (⇑⇑⇓), i.e. a total spin of ½.

The masses are :

- ~ 940 MeV for p and n;
- ~1115 MeV for the Λ;
- ~1190 MeV for the Σ's;
- ~1320 MeV for the Ξ's;

(difference of < few MeV in the isospin multiplet, due to e-m interactions.



Graphical construction of the baryons: $^3/_2{}^+$

The decuplet is rather simple (but see the next slide). The spins are aligned (⥣ ⥣ ⥣), to produce an overall $J={}^3/_2$.

The masses, at percent level, are :

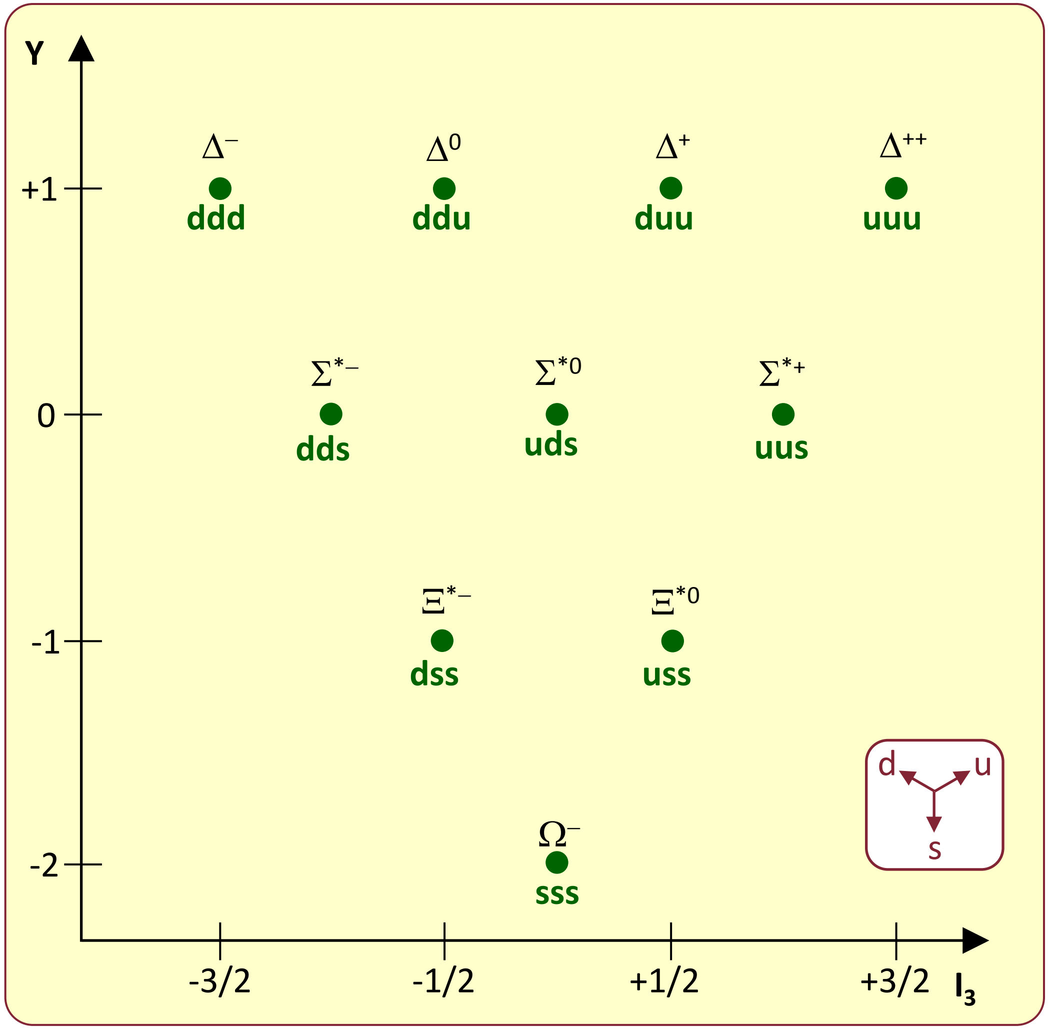
~ 1230 MeV for the Δ's;

~ 1385 MeV for the Σ^*'s,

~ 1530 MeV for the Ξ^*'s

~ 1670 MeV for the Ω^-.

Notice that the mass split among multiplets is very similar, ~150 MeV (lot of speculations, no real explanation).



SU(3)

- For the SU(2) symmetry, the generators are the Pauli matrices. The third one is associated to the conserved quantum number I_3.
- For SU(3), the Gell-Mann matrices T_j (j=1-8) are defined (next page).
- The two diagonal ones are associated to the operators of the third component of isospin (T_3) and hypercharge (T_8).
- The eigenvectors $|u\rangle$ $|d\rangle$ $|s\rangle$ are associated with the quarks (u, d, s).

in the following, some of the properties of SU(3) in group theory: no rigorous math, only results useful for our discussions.

Demonstrations (some trivial) may be found in [BJ 10] or [YK1 G]. A discussion of the group theory, applied to elementary particle physics, can be found in [IE, app. C].

And we have separate – optional – courses.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \psi_1^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \psi_1^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

$$\sigma_2 = i\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \quad \psi_2^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}; \quad \psi_2^- = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix};$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \psi_3^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \psi_3^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix};$$

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = -i\sigma_1\sigma_2\sigma_3 = I;$$

$$\left[\sigma_i, \sigma_j\right] = 2i\sum_k \varepsilon_{ijk}\sigma_k.$$

Pauli matrices and eigenvectors

SU(3) : Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_2 = i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_5 = i\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$$\lambda_7 = i\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Gell-Mann matrices λ_i

$$T_i = \frac{1}{2}\lambda_i$$

$$\sum_{j=1}^{8} \lambda_j^2 = \frac{16}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ diagonal.}$$

$$U = 1 + \frac{i}{2}\sum_{j=1}^{8} \varepsilon_j \lambda_i \text{ unitary matrix, } \det U = 1.$$

SU(3) : eigenvectors

Definition of I_3, Y, quark eigenvectors and related relations :

$$\hat{T}_3 = \frac{1}{2}\lambda_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \qquad Y = \frac{1}{\sqrt{3}}\lambda_8 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix};$$

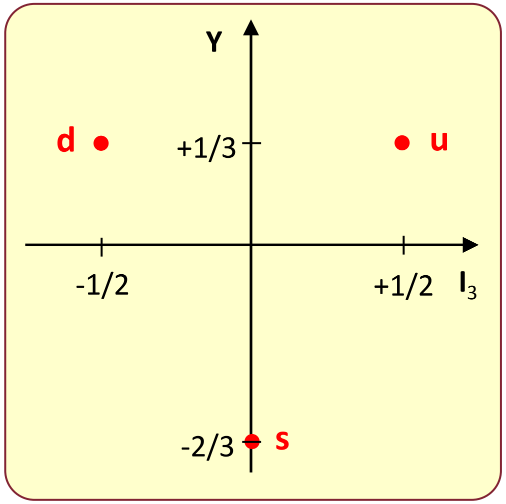
$$|u\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad |d\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad |s\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix};$$

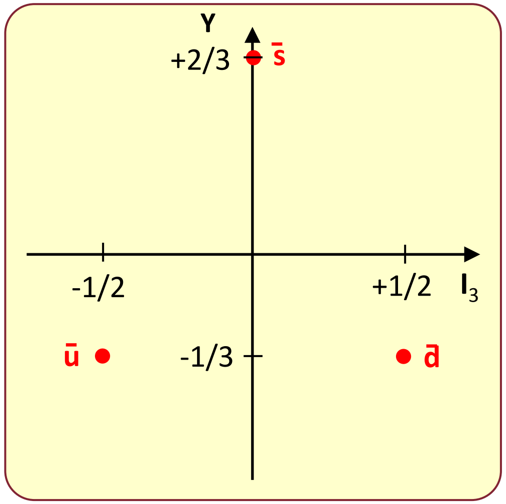
$$\hat{T}_3|u\rangle = +\frac{1}{2}|u\rangle; \quad \hat{T}_3|d\rangle = -\frac{1}{2}|d\rangle; \quad \hat{T}_3|s\rangle = 0;$$

$$\hat{Y}|u\rangle = +\frac{1}{3}|u\rangle; \quad \hat{Y}|d\rangle = +\frac{1}{3}|d\rangle; \quad \hat{Y}|s\rangle = -\frac{2}{3}|s\rangle;$$

$$\hat{T}_3|\bar{u}\rangle = -\frac{1}{2}|\bar{u}\rangle; \quad \hat{T}_3|\bar{d}\rangle = +\frac{1}{2}|\bar{d}\rangle; \quad \hat{T}_3|\bar{s}\rangle = 0;$$

$$\hat{Y}|\bar{u}\rangle = -\frac{1}{3}|\bar{u}\rangle; \quad \hat{Y}|\bar{d}\rangle = -\frac{1}{3}|\bar{d}\rangle; \quad \hat{Y}|\bar{s}\rangle = +\frac{2}{3}|\bar{s}\rangle.$$



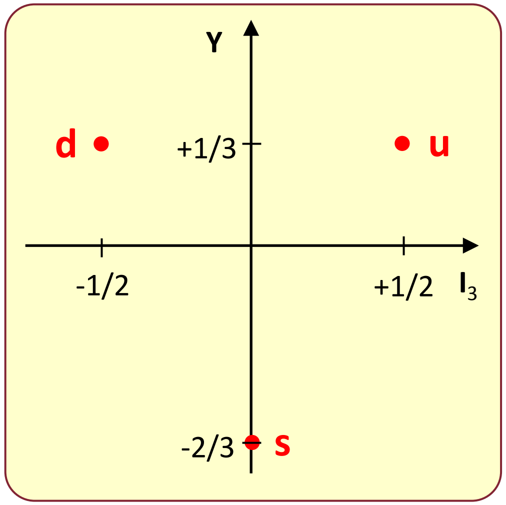


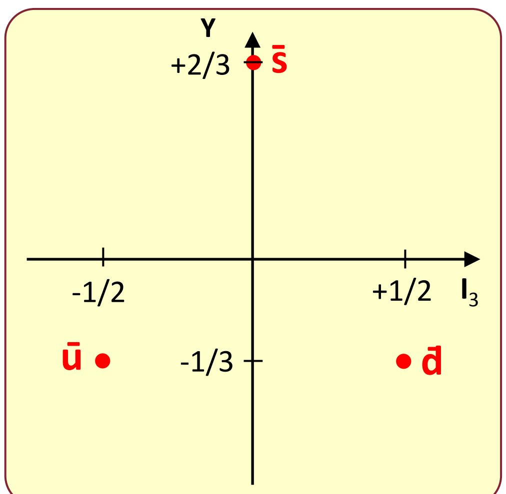
SU(3) : operators

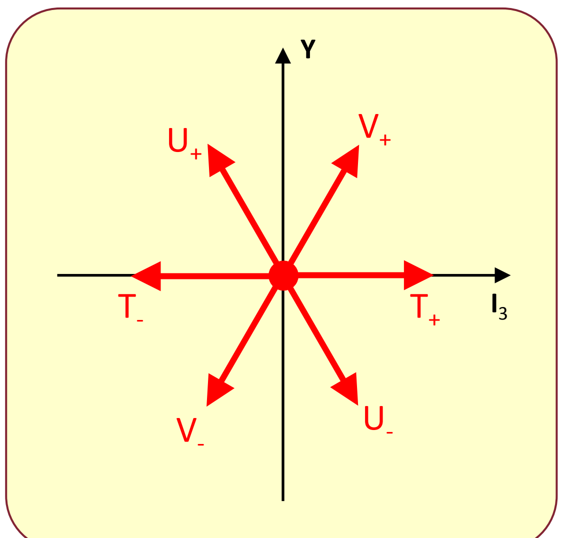
The ladder operators $T_\pm$, $U_\pm$, $V_\pm$: → $T_\pm = T_1 \pm iT_2; \quad U_\pm = T_6 \pm iT_7; \quad V_\pm = T_4 \pm iT_5;$

As an example, take V_+ :

$$V_+ = T_4 + iT_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \left\{ \begin{array}{l} V_+|\bar{u}\rangle = -|\bar{s}\rangle; \\ V_+|\bar{d}\rangle = 0; \\ V_+|\bar{s}\rangle = 0; \end{array} \right.$$





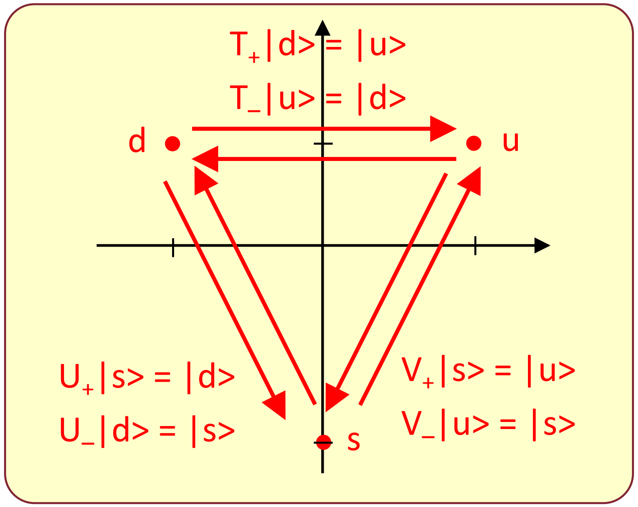


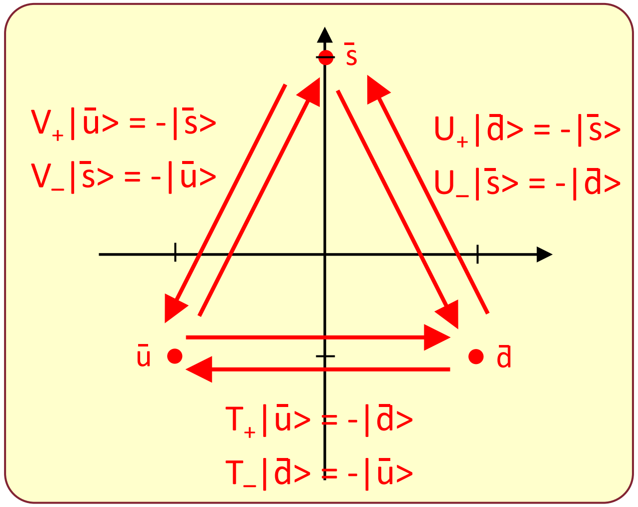
$$V_+|u\rangle = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0;$$

$$V_+|d\rangle = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0;$$

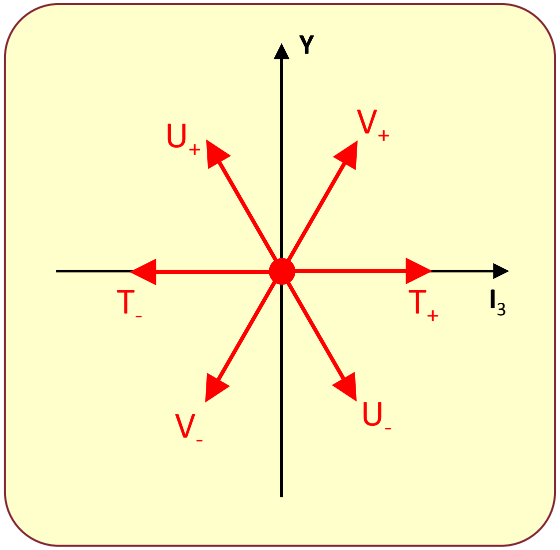
$$V_+|s\rangle = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = |u\rangle.$$

SU(3) : ladder operators





The ladder operators $T_{\pm}$, $U_{\pm}$, $V_{\pm}$.



Color : a new quantum number

Consider the Δ^{++} resonance:

- $J^P=3/2^+$ (measured);
- quark content: uuu state (no other possibility);
- wave function:

$$\psi(\Delta^{++}) = \psi_{space} \times \psi_{flavor} \times \psi_{spin}$$

It is lightest uuu state $\rightarrow \ell = 0 \rightarrow \psi_{space}$ be symmetric.

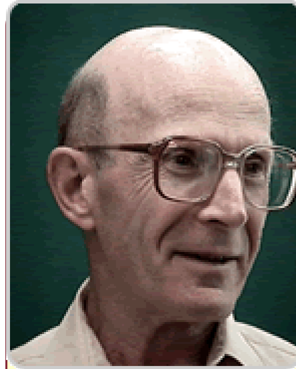
The overall spin comes from the alignment of the spins of the single quarks (⇑ ⇑ ⇑) :

$$|\Delta^{++}\rangle = | u\Uparrow u\Uparrow u\Uparrow \rangle$$

$\rightarrow \psi_{flavor}$ and ψ_{spin} are symmetric.

$\rightarrow \psi(\Delta^{++})$ = sym.×sym.×sym. = sym.

... but the Δ^{++} is a fermion ...


Oskar W. Greenberg


Moo-Young Han (한무영)


Yoichiro Nambu (南部 陽一郎, *Nambu Yōichirō*)

Anomaly : the Δ^{++} is a spin $^3/_2$ fermion and its function MUST be antisymmetric for the exchange of two quarks (Pauli principle). However, this function is the product of three symmetric functions, and therefore is symmetric → **???**.

The solution was suggested in 1964 by Greenberg, later also by Han and Nambu. They introduced a new quantum number for strongly interacting particles, starting from quarks : the **COLOR**.

Color : why's and how's

The idea (we will see later the algebra of the color, the following is quite naïve) :

1. quarks exist in three colors (say Red, Green and Blue, like the TV screen[1]);
2. they sum like in a TV-screen : e.g. when RGB are all present, the screen is white; the "anticolor" is such that, color + anticolor gives white (i.e. $\bar{R}$ = G + B);
3. anti-quarks are equipped with ANTI-colors (see previous point);
4. Mesons and Baryons, which are made of quarks, are white and have no color: they are a "color singlet".

Therefore, we have to include the color in the complete wave function; e.g. for Δ^{++} :

$$\psi(\Delta^{++}) = \psi_{space} \times \psi_{flavor} \times \psi_{spin} \times \psi_{color}$$

$$\psi_{color} = (1/\sqrt{6})\ (u_r u_g u_b + u_g u_b u_r + u_b u_r u_g - u_g u_r u_b - u_r u_b u_g - u_b u_g u_r)$$

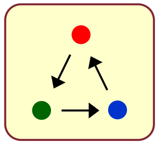
[where u_r, u_g, u_b are the color functions for u quarks of red, green, blue type]

Then ψ_{color} is antisymmetric for the exchange of two quarks and so is the global wave function.

The introduction of the color has many other experimental evidences and theoretical implications, which we will discuss in the following.

____

[1] *however, these colors are in no way similar to the popular colors; therefore their name is totally irrelevant.*



Symmetries and Multiplets

for a complete discussion, [BJ 10].

1. Since the strong interactions conserve isotopic spin ("$\mathbb{I}$"), hadrons gather in multiplets. Within each multiplet, the states are identified by the value of I_3.
2. In the absence of effects that break the symmetry, the members of each multiplet would be degenerate in mass. The electromagnetic interactions, which do not respect the isospin symmetry, remove the mass degeneration (at few %) in isospin multiplets.
3. Since the strong interactions conserve $\mathbb{I}$, $\mathbb{I}$-operators must commute with the strong interactions Hamiltonian ("$\mathbb{H}_s$") and with all the operators which in turn commute with $\mathbb{H}_s$.
4. Among these operators, consider the angular momentum $\mathbb{J}$ and the parity $\mathbb{P}$. As a result, all the members of an isospin multiplet must have the same spin and the same parity.
5. $\mathbb{H}_s$ is also invariant with respect to unitary representations of SU(2). The quantum numbers which identify the components of the multiplets are as many as the number of generators, which can be diagonalized simultaneously, because are mutually commuting. This number is the *rank* of the Group. In the case of SU(2) the rank is 1 and the operator is I_3.
6. Since $[I_j;I_k] = i\varepsilon_{jkm}I_m$, each of the generators commutes with $\mathbb{I}^2$:
   $\mathbb{I}^2 = \mathbb{I}_1{}^2 + \mathbb{I}_2{}^2 + \mathbb{I}_3{}^2$.

(continue …)

Symmetries and Multiplets

7. Therefore $\mathbb{I}^2$, obviously hermitian, can be diagonalized at the same time as $\mathbb{I}_3$, and its eigenvalues, together with those from $\mathbb{I}_3$, can "tag" the eigenvectors and the particles.

8. This fact gives the possibility to regroup the states into multiplets with a given value of *I*. Within each multiplet the operators are represented by matrices $(2\ell + 1) \bullet (2\ell + 1)$. In the language of group theory they realize "irreducible representations" of dimensions $(2\ell + 1) \bullet (2\ell + 1)$ of the Group of transformations.

9. We can generalize this mechanism from the isospin case to any operator : if we can prove that the hamiltonian is invariant for a given kind of transformations, then:
   a. look for an appropriate symmetry group;
   b. identify its irreducible representations and derive the possible multiplets,
   c. verify that they describe physical states which actually exist.

10. This approach suggested the idea that Baryons and Mesons are grouped in two octets, composed of multiplets of isotopic spin.

11. In reality, since the differences in mass between the members of the same multiplet are ~20%, the symmetry is “broken” (i.e. approximated).

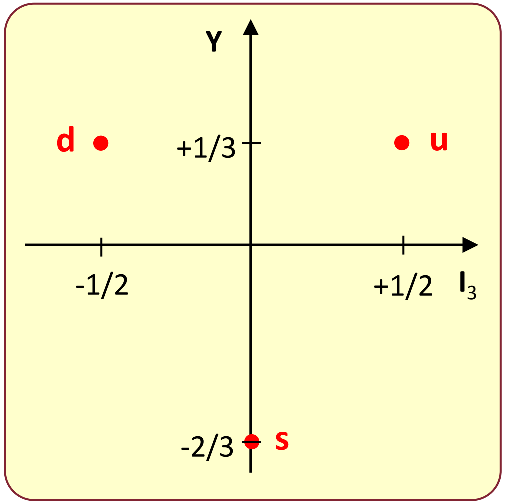
(… continue …)

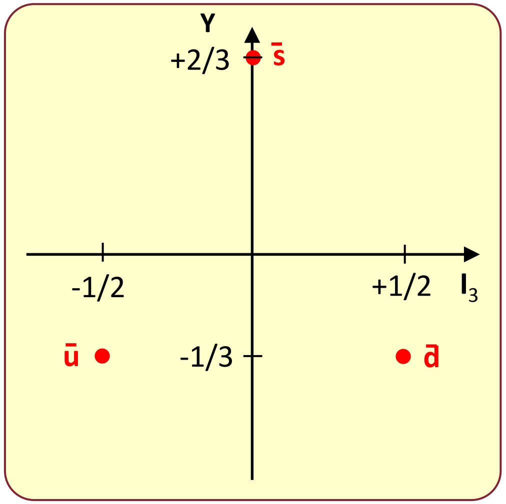
Symmetries and Multiplets

12. Since the members of the octet are characterized by two quantum numbers, both additives (I_3 and Y), the symmetry group must be found among those of rank = 2. I.e. two of the generators commute between them. We are interested in the "irreducible representations" of the group, such that we get any member of a multiplet from everyone else, using the transformations.
13. The non-trivial representation (non-trivial = other than the Singlet) of lower dimension is called "Fundamental representation".
14. In our case, it is SU(3) [NB "flavor SU(3)" in modern jargon, shortly $SU(3)_F$].
15. In SU(3) there are eight symmetry generators. Two of them are diagonal and associated to I_3 and Y.
16. The fundamental representations are triplets ($\rightarrow$ quarks), from which higher multiplets ($\rightarrow$ hadrons) are derived :

mesons: $3 \otimes \bar{3} = 1 \oplus 8$;

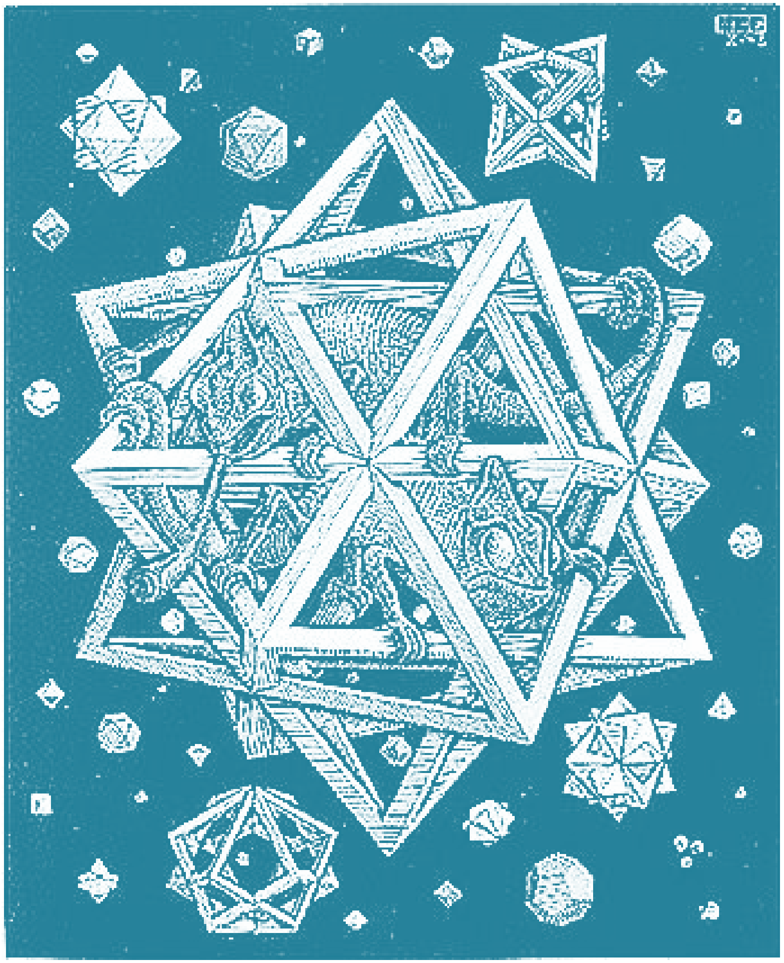
baryons: $3 \otimes 3 \otimes 3 = 1 \oplus 8 \oplus 8 \oplus 10$.





References

1. e.g. [BJ, 8];
2. large overlap with [FNSN1, 7]
3. isospin and SU(3) : [IE, 2];
4. group theory : [IE, app C];
5. color + eightfold way : [IE, 7-8]
6. G.Salmè – appunti.



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End of chapter 1