Muon-electron Scattering @NNLO in QED

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MUonE meeting

Bologna, September 12-13, 2018

In collaboration with:

- Di Vita, Laporta, Passera, Primo, Schubert, Torres-Bobadilla

Physics and Astronomy Department
Galileo Galilei
University of Padova - Italy

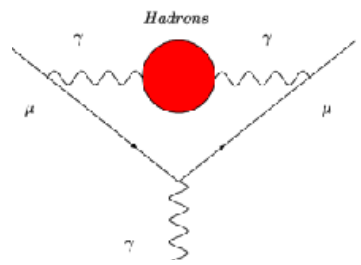
Outline

- Motivations
- Adaptive Integrand Decomposition
  - Improved reduction @ 1- and 2-loop
  - Automated two-loop corrections for generic processes
- Feynman Integrals in Dimensional Regularization
  - Integration-by-parts identities, Master Integrals & Differential Equations
  - Magnus Exponential Matrix and Canonical Forms
- Results
- Conclusions/Outlook

Motivations

New space-like proposal for HLO

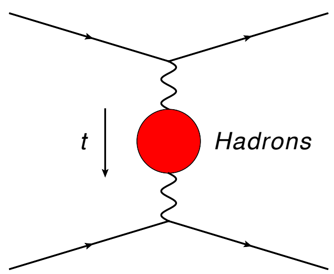
- At present, the leading hadronic contribution a_μ^{HLO} is computed via the time-like formula:



$$a_\mu^{\rm HLO} = \frac{1}{4\pi^3}\int_{4m_\pi^2}^{\infty} ds\, K(s)\, \sigma^0_{\rm had}(s)$$

$$K(s) = \int_0^1 dx\, \frac{x^2\,(1-x)}{x^2 + (1-x)\,(s/m_\mu^2)}$$

- Alternatively, exchanging the x and s integrations in a_μ^{HLO}



$$a_\mu^{\rm HLO} = \frac{\alpha}{\pi}\int_0^1 dx\,(1-x)\,\Delta\alpha_{\rm had}[t(x)]$$

$$t(x) = \frac{x^2 m_\mu^2}{x-1} < 0$$

which involves $\Delta\alpha_{had}(t)$, the hadronic contribution to the running of α in the space-like region. It can be extracted from scattering data!

Muon-electron scattering

Abbiendi, Carloni Calame, Marconi, Matteuzzi, Montagna, Nicrosini, MP, Piccinini, Tenchini, Trentadue, Venanzoni

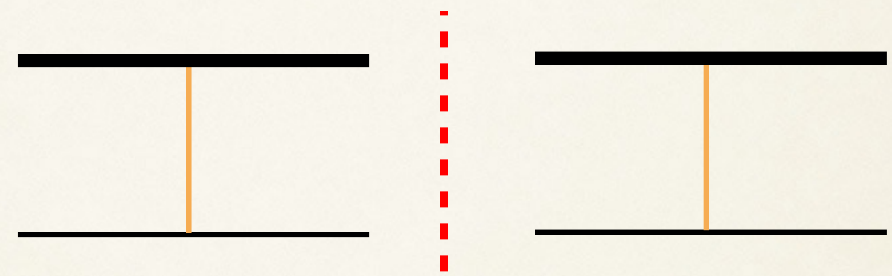
EPJC 2017 - arXiv:1609.08987

>> see Trentadue & Marconi's talk

What we *know* :: Anatomy of NLO

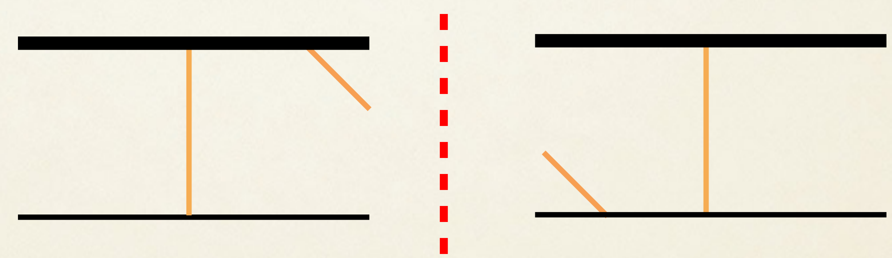
- **Born matrix-element**

  $d\sigma_{LO}$



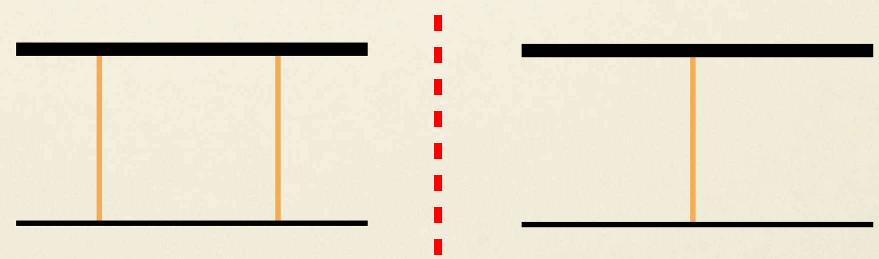
- **Real Radiation**
  tree-level matrix elements

  $d\sigma^{R}_{NLO}$



- **1-loop matrix elements**

  $d\sigma^{V}_{NLO}$



☑ Known since long
vanNieuwenhuizen

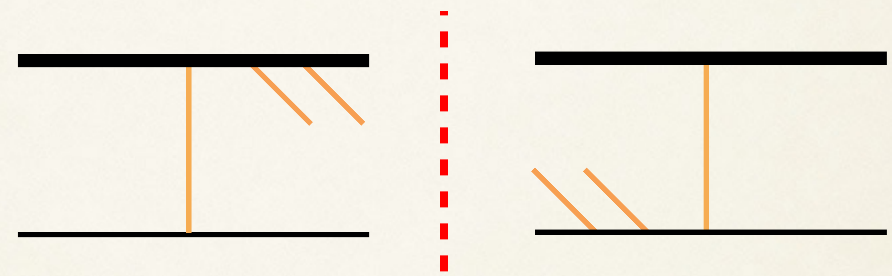
- double checked
Fael Passera

$$\sigma_{NLO} \sim \int_{d\Phi_{m+1}} d\sigma^{R}_{NLO} + \int_{d\Phi_{m}} d\sigma^{V}_{NLO}$$

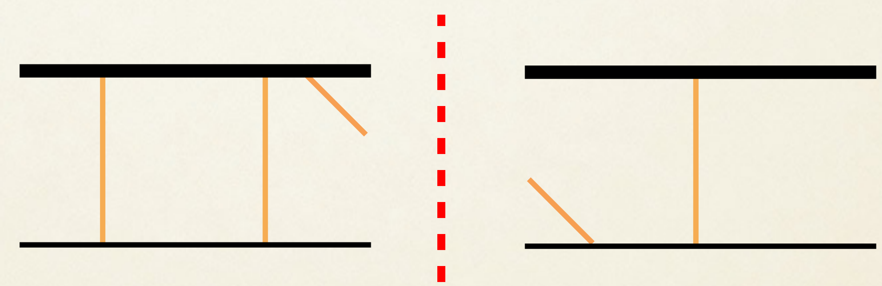
- **Subtractions and MC-integration** ☑

What we *need* :: Anatomy of NNLO

- **Double-real Radiation**
  tree-level matrix elements
  $\mathrm{d}\hat{\sigma}^{RR}_{NNLO}$

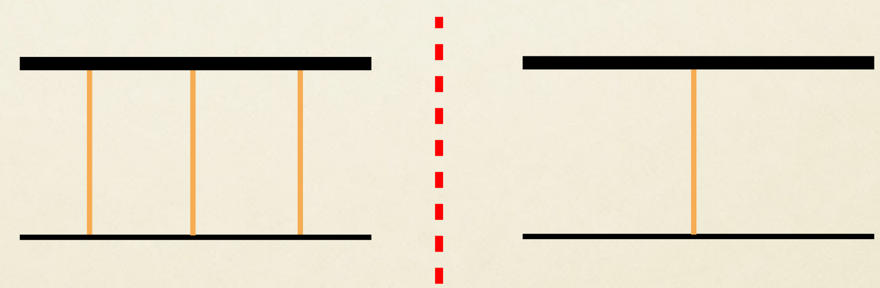


- **Single-real Radiation**
  1-loop matrix elements
  $\mathrm{d}\hat{\sigma}^{RV}_{NNLO}$



☑ GOSAM
Recent improvements
for QED/EW corrections
Chiesa Greiner Tramontano

- **Virtual 2-loop matrix element**
  $\mathrm{d}\hat{\sigma}^{VV}_{NNLO}$

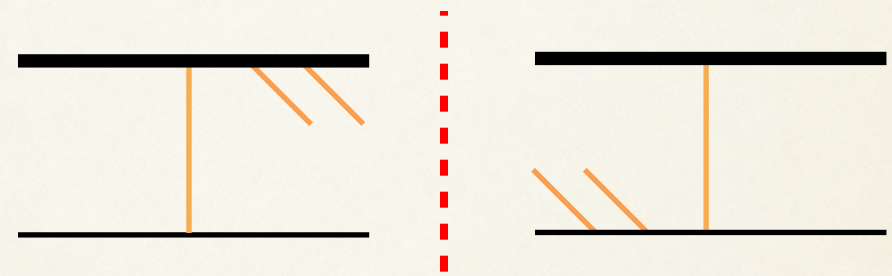


☑ this talk!

What we *need* :: Anatomy of NNLO

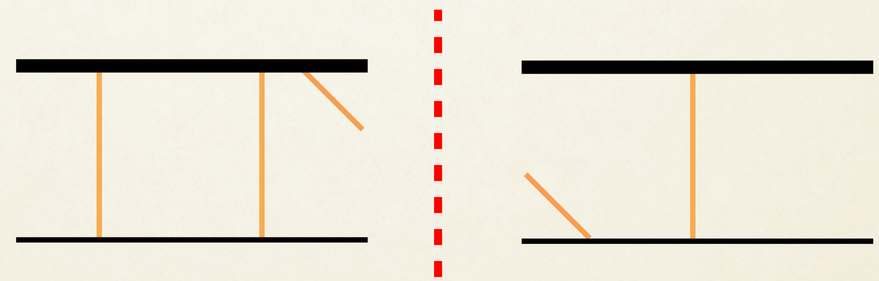
- **Double-real Radiation tree-level matrix elements**

  $\mathrm{d}\hat{\sigma}^{RR}_{NNLO}$



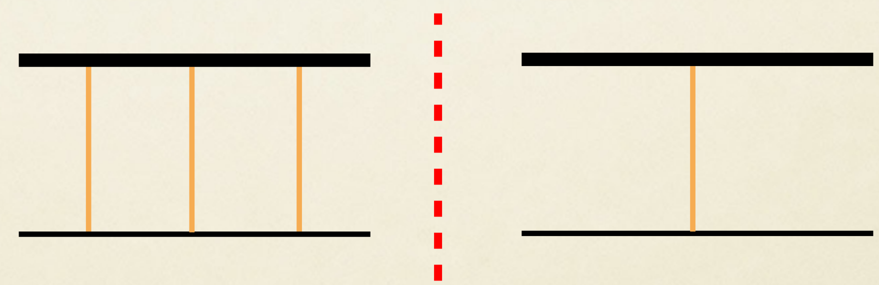
- **Single-real Radiation 1-loop matrix elements**

  $\mathrm{d}\hat{\sigma}^{RV}_{NNLO}$



- **Virtual 2-loop matrix element**

  $\mathrm{d}\hat{\sigma}^{VV}_{NNLO}$

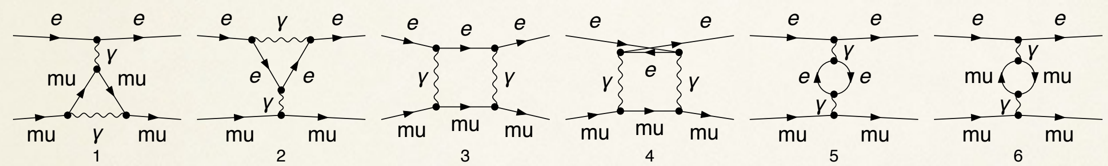


$$\hat{\sigma}_{NNLO} \quad \sim \quad \int_{\mathrm{d}\Phi_{m+2}} \mathrm{d}\hat{\sigma}^{RR}_{NNLO} + \int_{\mathrm{d}\Phi_{m+1}} \mathrm{d}\hat{\sigma}^{RV}_{NNLO} + \int_{\mathrm{d}\Phi_{m}} \mathrm{d}\hat{\sigma}^{VV}_{NNLO}$$

- **Subtractions and MC-integration?**

MUonE @ 1- and 2-loop

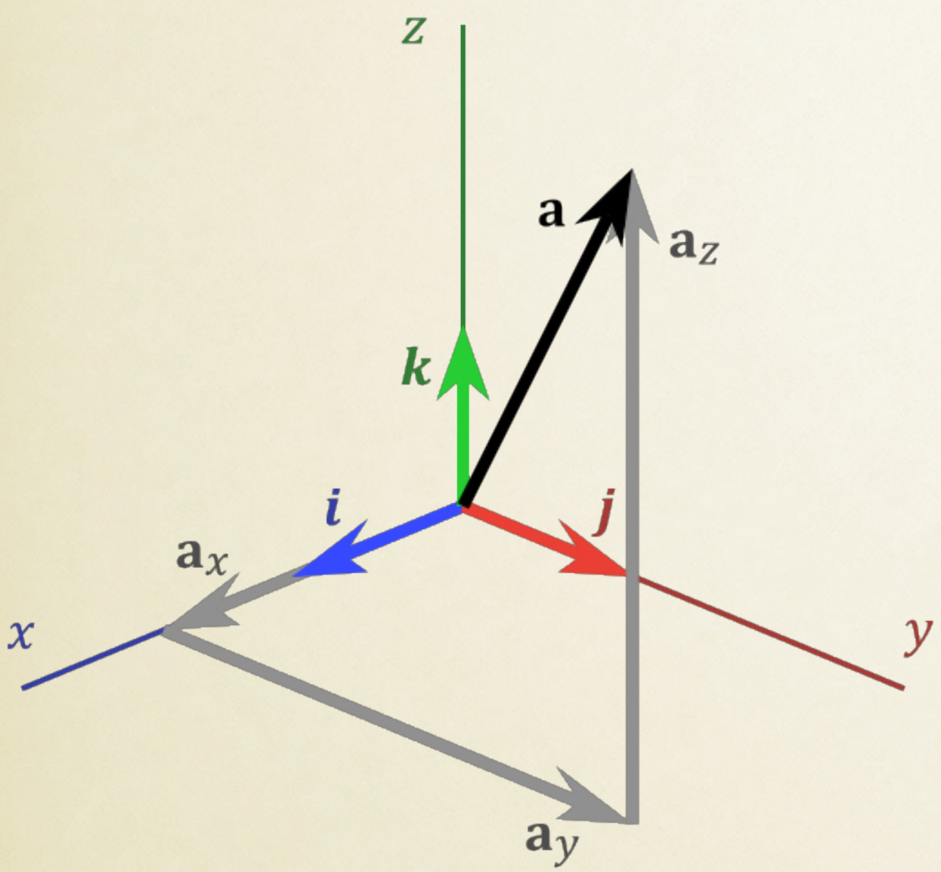
- 6 diagrams @1-loop



- 69 diagrams @2-loop

Amplitudes Decomposition:

the algebraic way



$\mathbf{a} = a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k}$

- **Basis**: {**i** **j** **k**}

- **Scalar product**/**Projection**:
  to extract the components

$a_x = \mathbf{a}.\mathbf{i}$

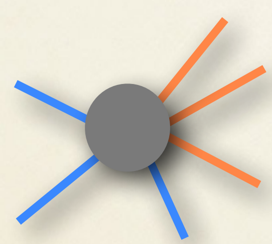
$a_y = \mathbf{a}.\mathbf{j}$

$a_z = \mathbf{a}.\mathbf{k}$

Amplitudes Decomposition:

the algebraic way

Master functions

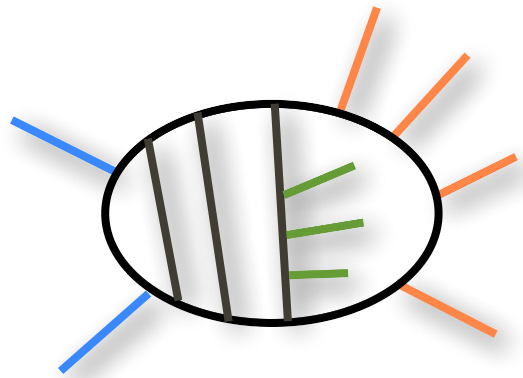


$$= \sum_i c_i \times f_i$$

coefficients

Dimensionally Regulated Integrals

Graph Topology & Integrals



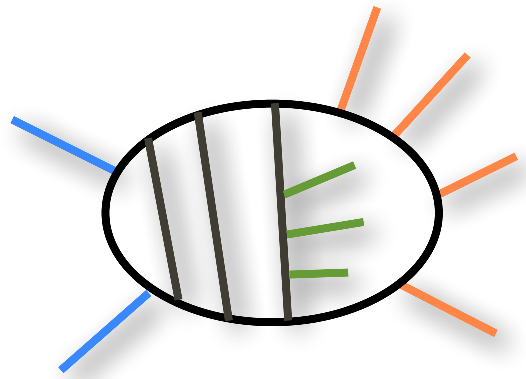
$e = \# \text{ legs} :: p_i, \quad (i = 1, \ldots, e);$
$\ell = \# \text{ loops} :: q_i \quad (i = 1, \ldots, \ell);$
$n = \# \text{ denominators} :: D_i \quad (i = 1, \ldots, n);$

$N = \#$ scalar products (of types $q_i \cdot p_j$ and $q_i \cdot q_j$) $\qquad N = \ell(e-1) + \frac{\ell(\ell+1)}{2}$

$n = \#$ **reducible** scalar products (expressed in terms of denominators);

$m = \#$ **irreducible** scalar products $= N - n :: S_i \quad (i = 1, \ldots, m)$

Graph Topology & Integrals



$e = \#\ \text{legs} :: p_i, \quad (i = 1, \ldots, e);$

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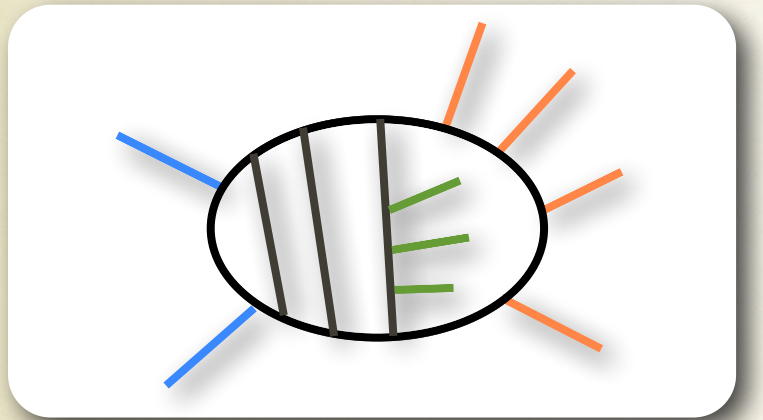
Graph Topology & Integrals



$e = \#\ \text{legs} :: p_i, \quad (i = 1, \ldots, e);$

$\ell = \#\ \text{loops} :: q_i \quad (i = 1, \ldots, \ell);$

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$n = \#$ **reducible** scalar products (expressed in terms of denominators);

$m = \#$ **irreducible** scalar products $= N - n :: S_i \quad (i = 1, \ldots, m)$

Associated Integrals ::

$$F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) \equiv \int_{q_1 \ldots q_\ell} f_{n,m}(\mathbf{x}, \mathbf{y}) \ , \qquad \int_{q_1 \ldots q_\ell} \equiv \int \frac{\mathrm{d}^d q_1}{(2\pi)^d} \cdots \frac{\mathrm{d}^d q_\ell}{(2\pi)^d}$$

$$f_{n,m}(\mathbf{x}, \mathbf{y}) = \frac{S_1^{y_1} \cdots S_m^{y_m}}{D_1^{x_1} \cdots D_n^{x_n}}$$

Integration-by-parts Identities (IBPs)

Tkachov; Chetyrkin Tkachov; Laporta;

$$\int_{q_1 \ldots q_\ell} \frac{\partial}{\partial q_i^\mu} \Big(v^\mu \; f_{n,m}(\mathbf{x}, \mathbf{y}) \Big) = 0 \; , \qquad v = q_1, \ldots, q_\ell, \; p_1, \ldots, p_{e-1}.$$

$\forall (n,m)$, $N_{\rm IBP}$ = # of IBP relations $= \ell(\ell + e - 1)$

Relations between integrals associated to the same topology (or subtopologies)

$$c_0 \; F^{[d]}_{n,m}(\mathbf{x}, \mathbf{y}) + \sum_{i,j} c_{i,j} \; F^{[d]}_{n,m}(\mathbf{x_i}, \mathbf{y_j}) = 0 \; ,$$

$$\mathbf{x_i} = \{x_1, \ldots, x_i \pm 1, \ldots, x_n\}$$

$$\mathbf{y_j} = \{y_1, \ldots, y_j \pm 1, \ldots y_n\}$$

public codes :: AIR; Reduze2; FIRE; LiteRed;
private codes :: ... many authors ... Laporta, Sturm ...

Master Integrals (MIs)

Independent set of integrals $M_i^{[d]}$,
$$M_i^{[d]} \equiv \int_{q_1 \ldots q_\ell} m_i(\bar{\mathbf{x}}, \bar{\mathbf{y}}) \ ,$$

with a definite set of powers $\bar{\mathbf{x}}, \bar{\mathbf{y}}$ such that

$$F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_k c_k M_k^{[d]} \ , \qquad \forall (n,m)$$

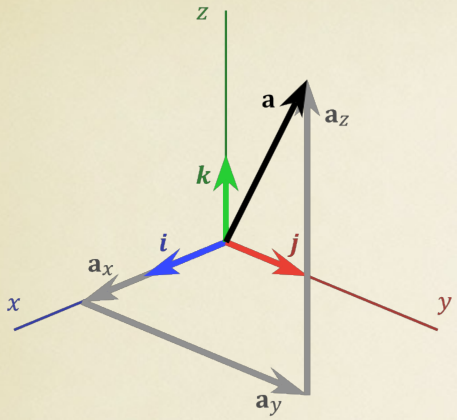
They form a *basis* for the integrals of the corresponding topology.

Two special cases

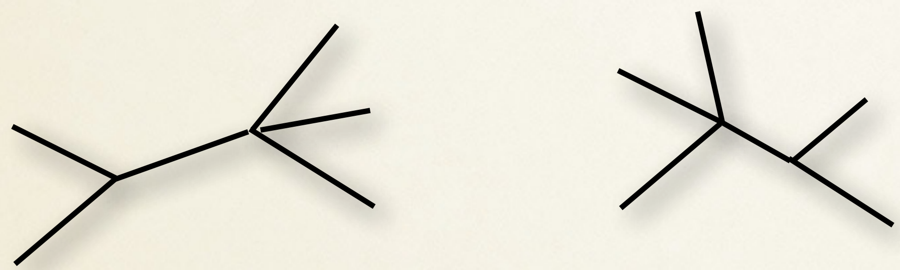
Two types of integrals generated from the **master integrands**

- Polynomial insertion: $$\int_{q_1 \ldots q_\ell} P(q_i \cdot p_j, q_i \cdot q_j) \ m_i(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \alpha_{n,m} \ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_i c_i \ M_i^{[d]}$$
- External-leg derivatives: $$p_i^\mu \frac{\partial}{\partial p_j^\mu} M_k^{[d]} = \int_{q_1 \ldots q_\ell} p_i^\mu \frac{\partial}{\partial p_j^\mu} \ m_k(\bar{\mathbf{x}}, \bar{\mathbf{y}}) = \sum_{n,m} \beta_{n,m} \ F_{n,m}^{[d]}(\mathbf{x}, \mathbf{y}) \stackrel{\text{IBP}}{=} \sum_i c_i \ M_i^{[d]}$$

Basis :: Master Functions

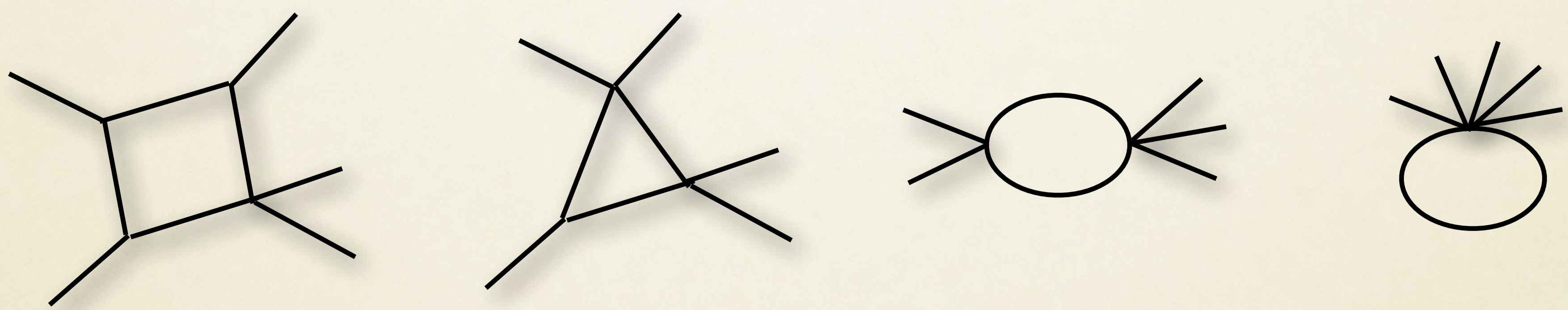


- **Tree level**



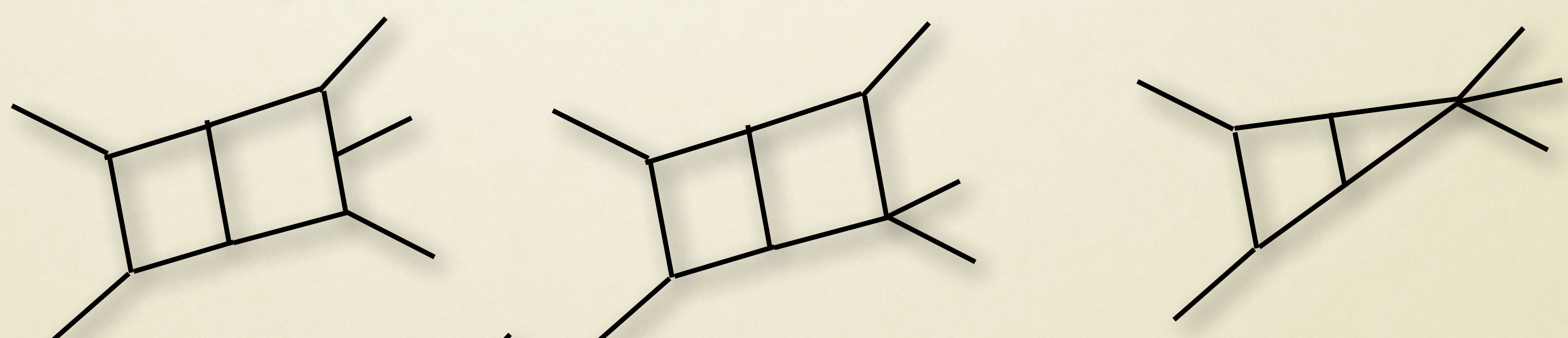
Known!

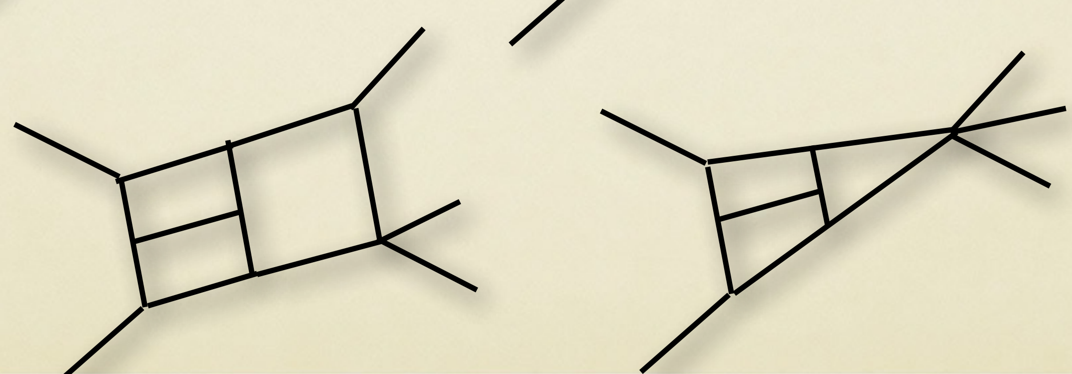
- **One Loop**



Known!

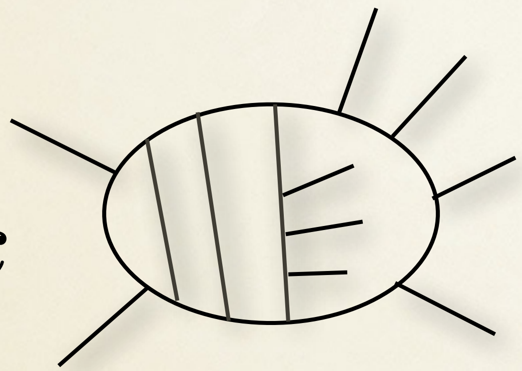
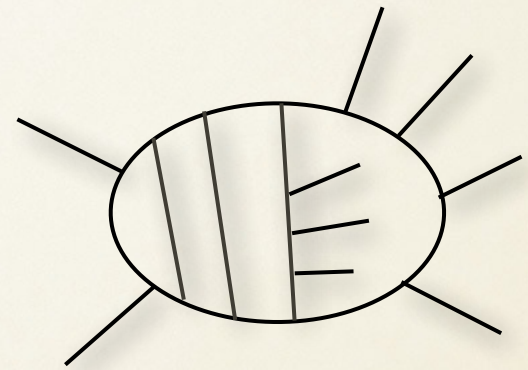
- **Higher Loops**





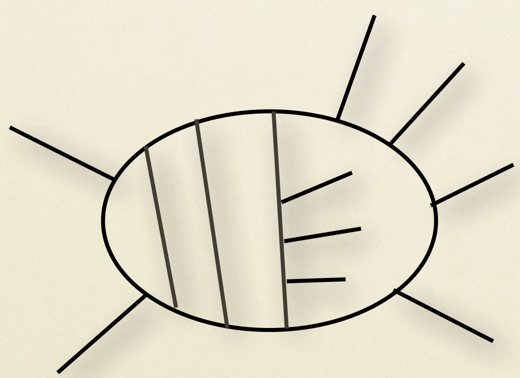
?Unknown?

Differential Equations for Master Integrals

∂_x  $= A(d, x)$ 

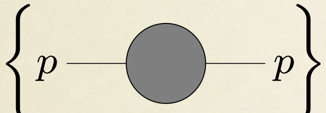
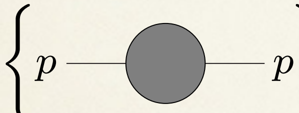
kinematic variable
(s,t,u, masses)

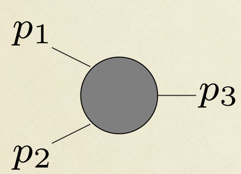
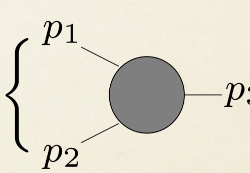
space-time
dimensions

 = ?

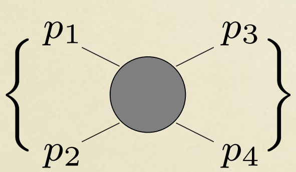
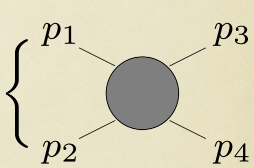
Differential Equations for Master Integrals

Kotikov; Remiddi;
Gehrmann Remiddi
Argeri Bonciani Ferroglia Remiddi **P.M.**
……
Henn; Henn Smirnov
Lee; Papadopoulos;
Argeri diVita Mirabella Schlenk Schubert Tancredi **P.M.**
diVita Schubert Yundin **P.M.**
Remiddi Tancredi; Primo Tancredi
Papadopoulos Frellesvig
Zheng
……

$$p^2 \frac{\partial}{\partial p^2} \Big\{$$  $$\Big\} = \frac{1}{2} p_\mu \frac{\partial}{\partial p_\mu} \Big\{$$  $$\Big\}$$

$$P^2 \frac{\partial}{\partial P^2} \Big\{$$  $$\Big\} = \left[A \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} \right) + B \left(p_{1,\mu} \frac{\partial}{\partial p_{2,\mu}} + p_{2,\mu} \frac{\partial}{\partial p_{1,\mu}} \right) \right] \Big\{$$  $$\Big\}$$

$$P = p_1 + p_2,$$

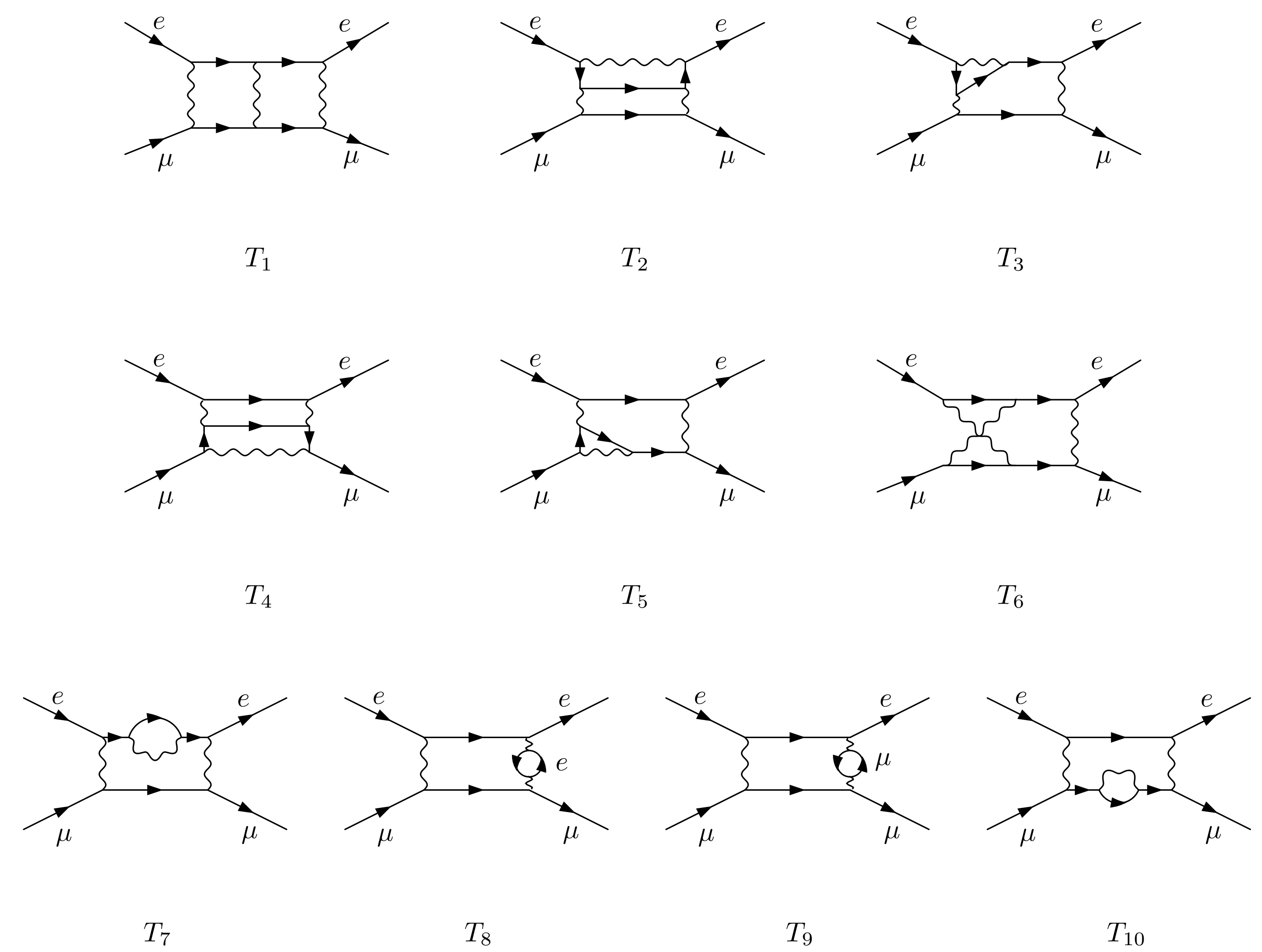
$$P^2 \frac{\partial}{\partial P^2} \Big\{$$  $$\Big\} = \left[C \left(p_{1,\mu} \frac{\partial}{\partial p_{1,\mu}} - p_{3,\mu} \frac{\partial}{\partial p_{3,\mu}} \right) + D p_{2,\mu} \frac{\partial}{\partial p_{2,\mu}} + E (p_{1,\mu} + p_{3,\mu}) \left(\frac{\partial}{\partial p_{3,\mu}} - \frac{\partial}{\partial p_{1,\mu}} + \frac{\partial}{\partial p_{2,\mu}} \right) \right] \Big\{$$  $$\Big\}.$$

In general, n MIs obey a **system of 1st ODE**

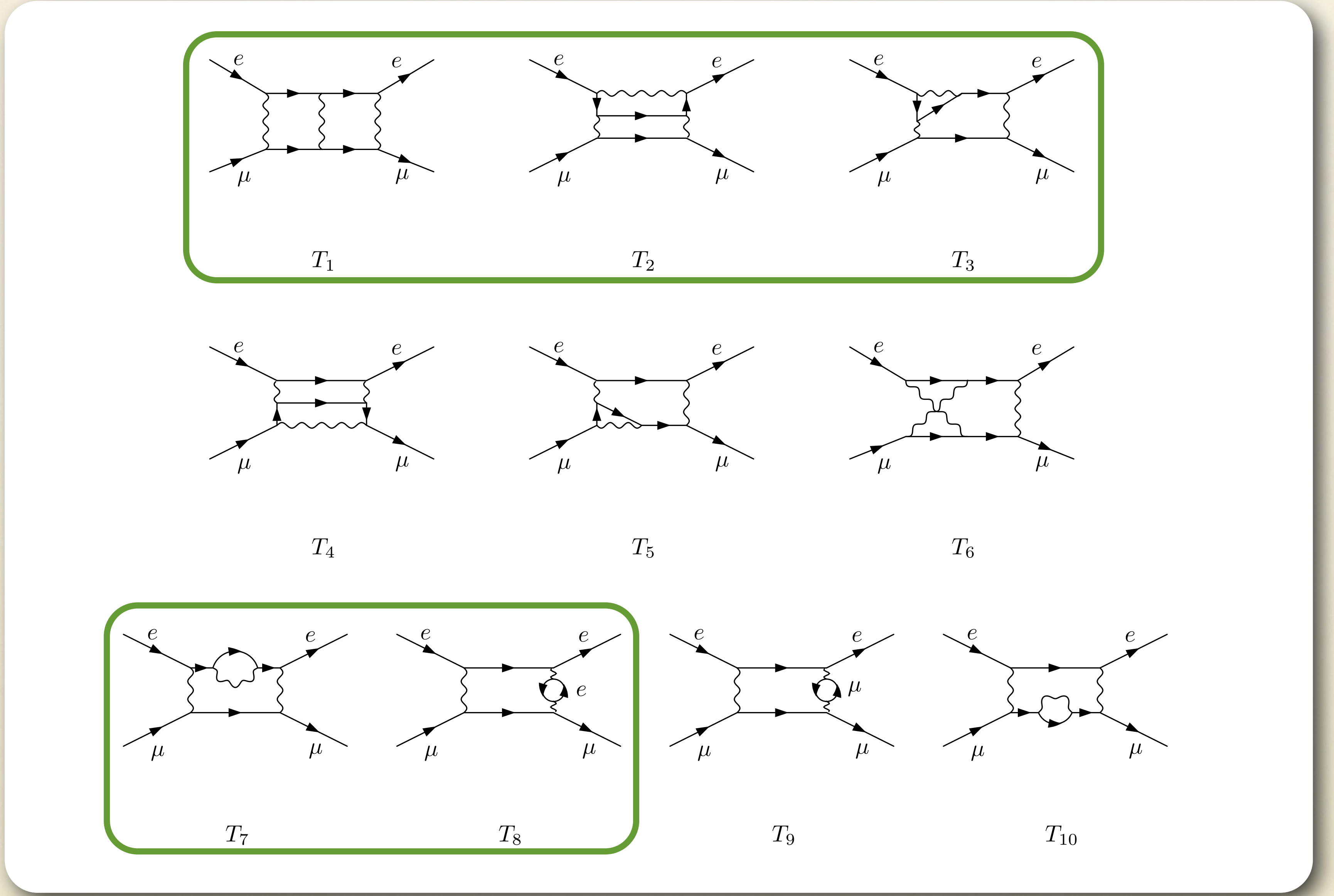
$$\partial_z \mathbf{M}^{[d]} = \mathbb{A}(d, z)\ \mathbf{M}^{[d]}$$

Two-Loop Integrals for Mu-E Scattering

Feynman diagrams @ 2-loop

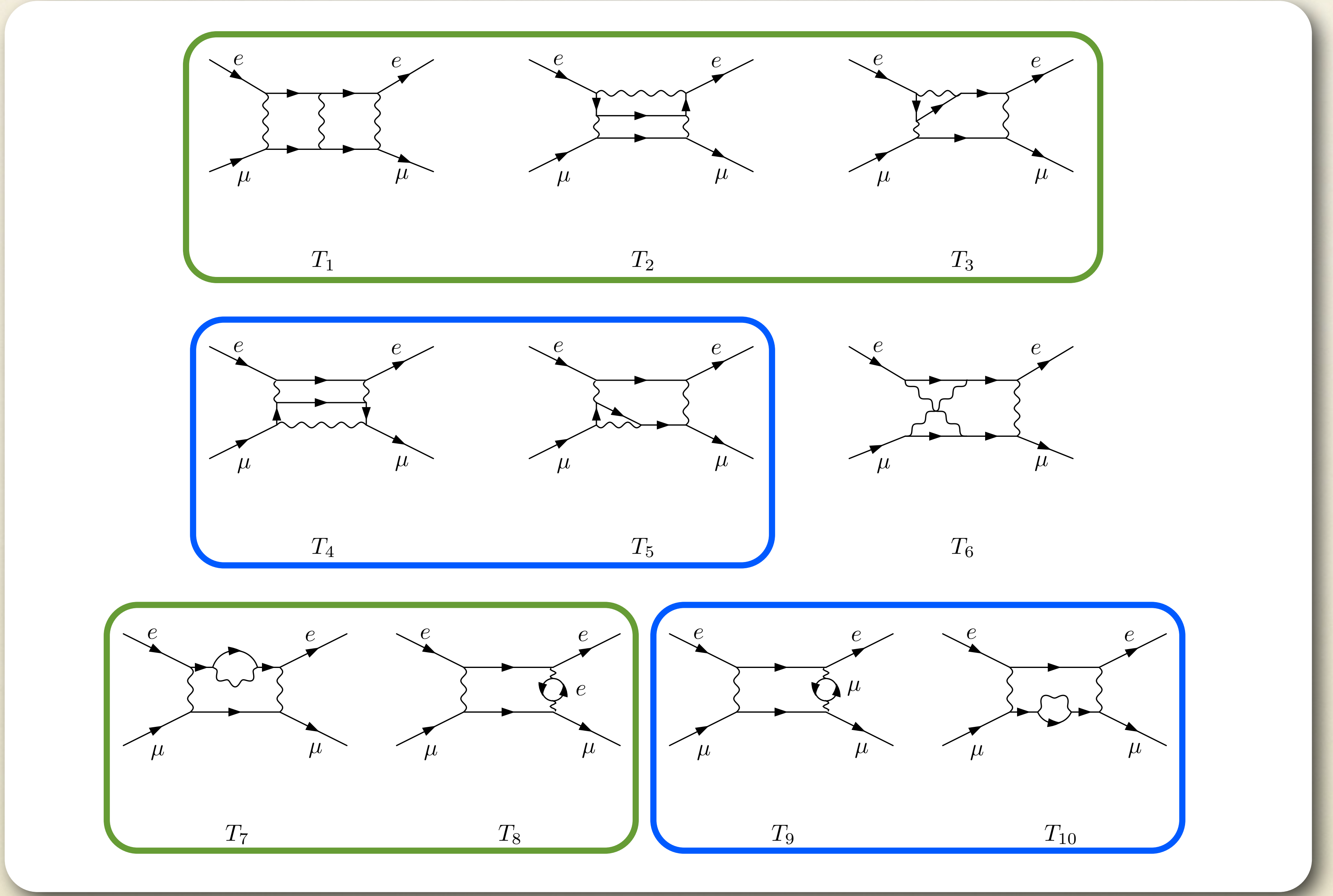


Feynman diagrams @ 2-loop



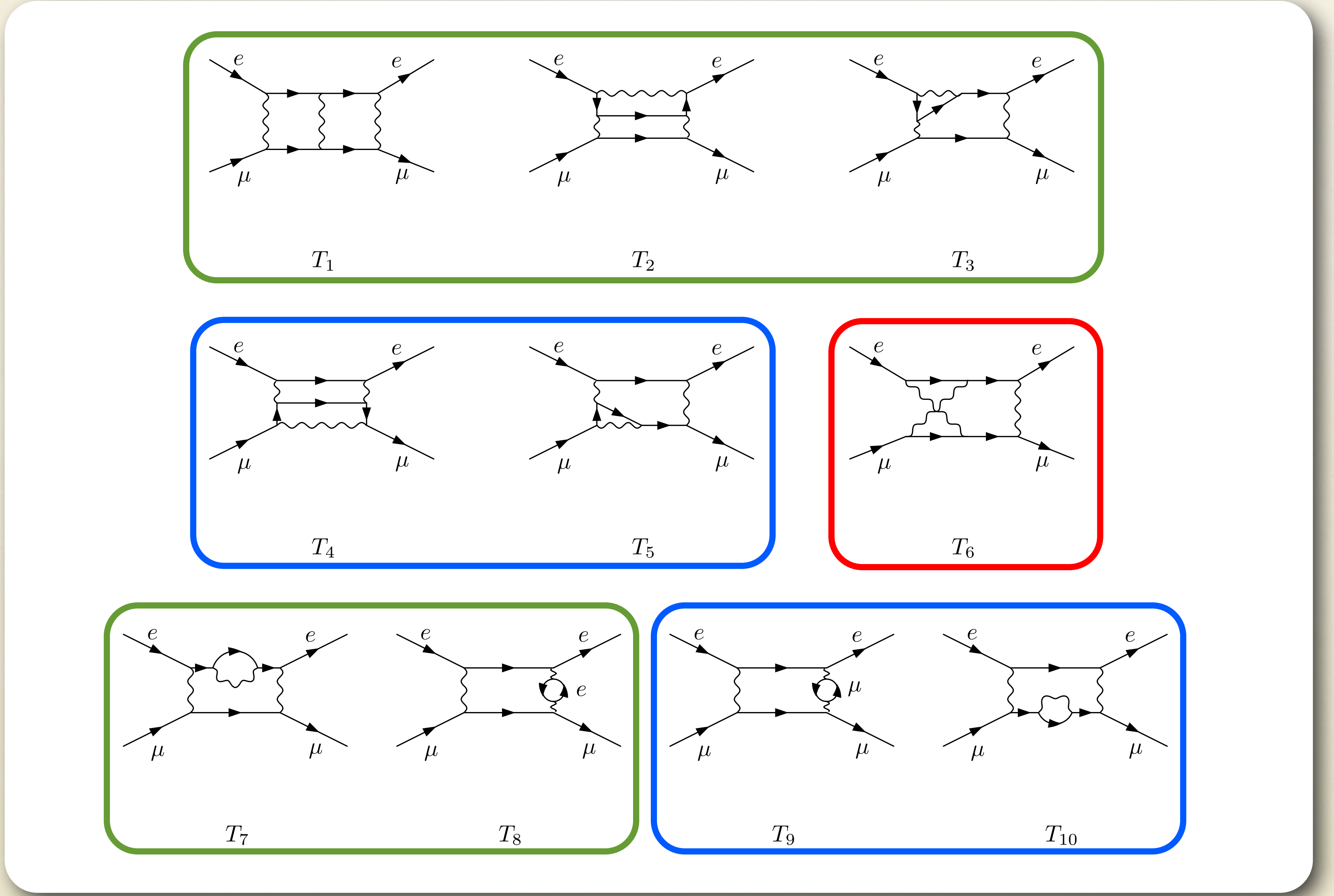
- **Planar Integrals :: Family-1**

Feynman diagrams @ 2-loop



- Planar Integrals :: Family-1
- Planar Integrals :: Family-2

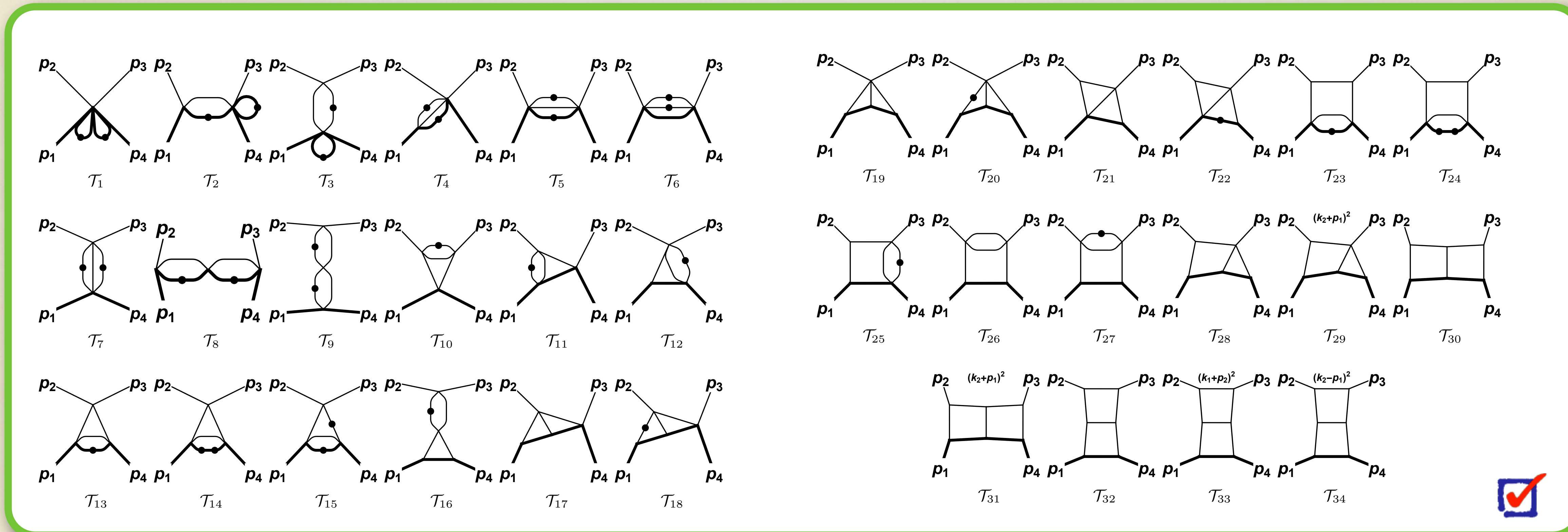
Feynman diagrams @ 2-loop



- Planar Integrals :: Family-1
- Planar Integrals :: Family-2
- Non-Planar Integrals

- massless ***electron***
- **massive *muon***

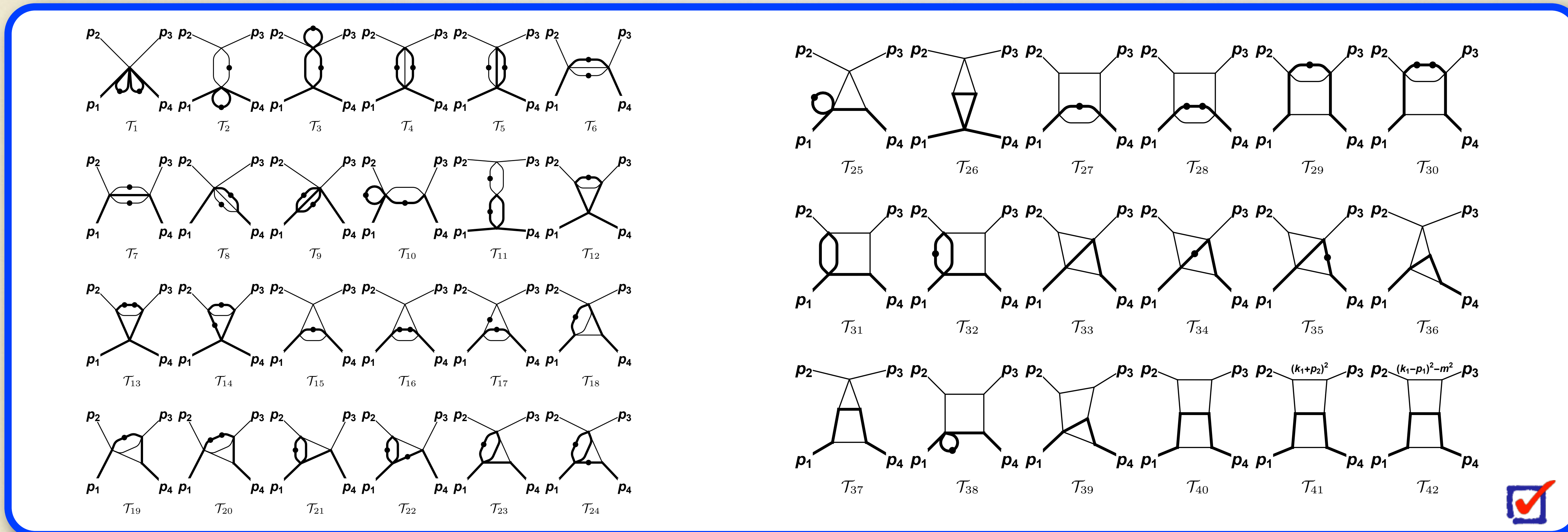
MIs for Family-1 Passera Primo Schubert & **P.M.** (2017)



- **A different set of MIs already known** Bonciani Ferroglia Gehrmann vonManteuffel
- **We recomputed them with an alternative/simpler way**

- **31 MIs**
- **alphabet:** 8 rational letters
- **solution:** GPL's
- **numerical checks** using **GiNac** vs **SecDec**

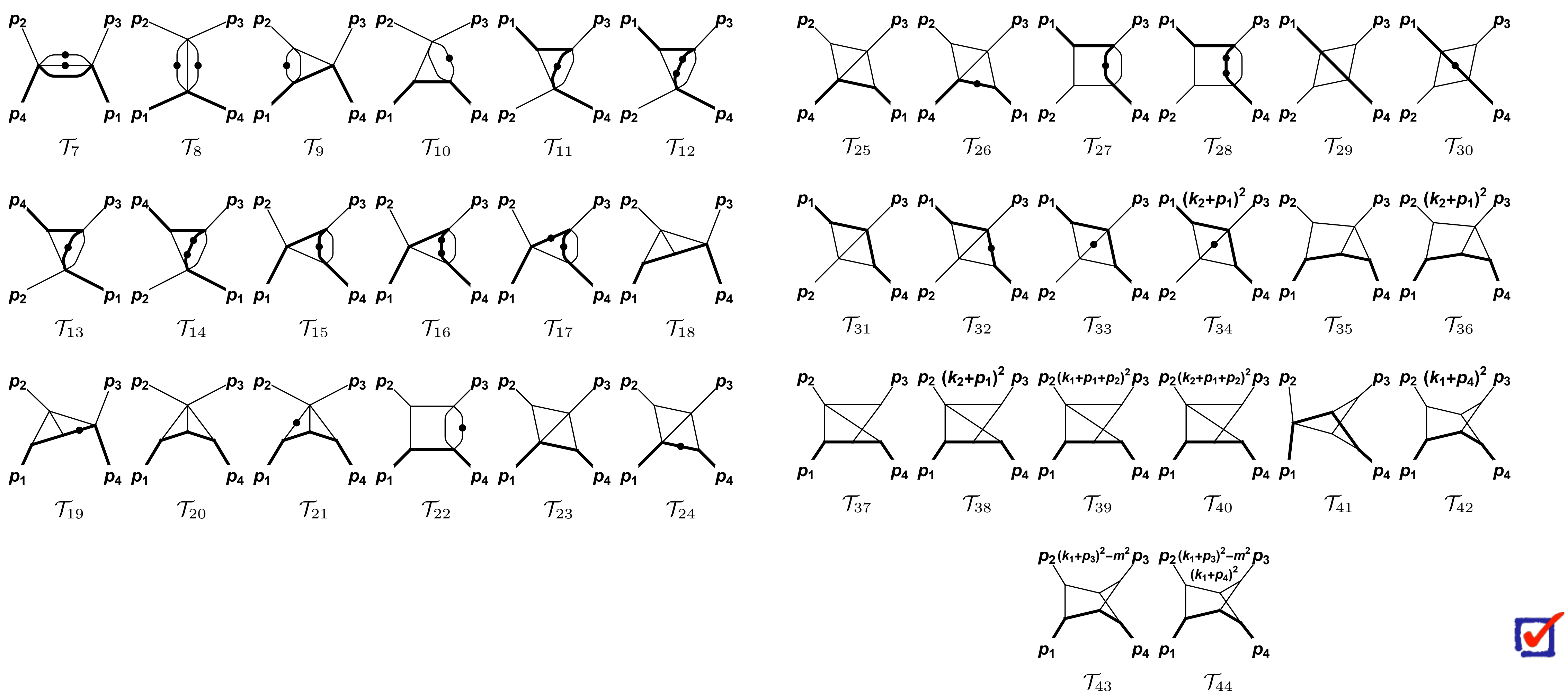
MIs for Family-2 Passera Primo Schubert & **P.M.** (2017)



- **42 MIs**
- **alphabet:** 9 rational letters
- **solution:** GPL's
- **numerical checks** using **GiNac** vs **SecDec**

44 MIs for non-planar diagrams

DiVita Laporta Primo Schubert & **P.M.** (2018)



- **44 MIs**
- **alphabet:** 12 rational letters
- **solution:** GPL's
- **numerical checks** using **GiNac** vs **inHouse code**

NEW

Quantum Mechanics

Schroedinger Eq'n (*eps-linear Hamiltonian*)

$$i\hbar\,\partial_t|\Psi(t)\rangle = H(\epsilon,t)|\Psi(t)\rangle\ , \qquad H(\epsilon,t) = H_0(t) + \epsilon H_1(t)$$

Interaction Picture

$$H_{i,I}(t) = B^\dagger(t)\ H_i(t)\ B(t)$$

Matrix Transform

$$i\hbar\,\partial_t B(t) = H_0(t)B(t) \qquad B(t) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} d\tau H_0(\tau)}$$

Schroedinger Eq'n (*canonical form*)

$$i\hbar\,\partial_t|\Psi_I(t)\rangle = \epsilon\, H_{1,I}(t)|\Psi_I(t)\rangle\,,$$

Magnus Expansion

Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

System of 1st ODE

$$\partial_x Y(x) = A(x) Y(x) \ , \quad Y(x_0) = Y_0 \ .$$

A(x) non-commutative

Magnus' series :: *Matrix Exponential*

$$Y(x) = e^{\Omega(x,x_0)} \ Y(x_0) \equiv e^{\Omega(x)} \ Y_0 \, ,$$

$$\Omega(x) = \sum_{n=1}^{\infty} \Omega_n(x) \ .$$

$$\Omega_1(x) = \int_{x_0}^{x} d\tau_1 A(\tau_1) \ ,$$

$$\Omega_2(x) = \frac{1}{2} \int_{x_0}^{x} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \ [A(\tau_1), A(\tau_2)] \ ,$$

$$\Omega_3(x) = \frac{1}{6} \int_{x_0}^{t} d\tau_1 \int_{x_0}^{\tau_1} d\tau_2 \int_{x_0}^{\tau_2} d\tau_3 \ [A(\tau_1), [A(\tau_2), A(\tau_3)]] + [A(\tau_3), [A(\tau_2), A(\tau_1)]] \ .$$

....

BCH-formula

Iterated Integrals
(generalized polylogs)

Dyson's series

$$Y(x) = Y_0 + \sum_{n=1}^{\infty} Y_n(x) \ ,$$

$$Y_n(x) \equiv \int_{x_0}^{x} d\tau_1 \dots \int_{x_0}^{\tau_{n-1}} d\tau_n \ A(\tau_1) A(\tau_2) \cdots A(\tau_n)$$

Chen
Goncharov
Remiddi Vermaseren
Gehrmann Remiddi
Bonciani Remiddi **P.M.**
Vollinga Weinzierl
Brown
Duhr Gangl Rhodes
....

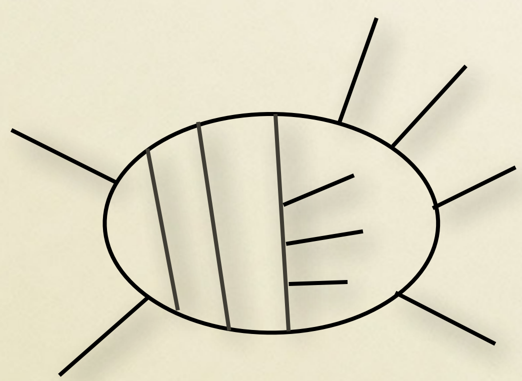
Argeri, Di Vita, Mirabella,
Schlenk, Schubert, Tancredi, **P.M.** (2014)

Quantum Mechanics

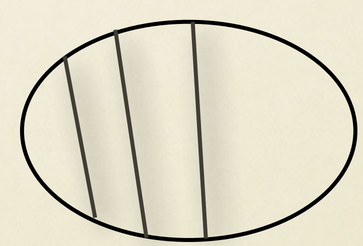
- Time-evolution in Perturbation Theory
- perturbation parameter: ε
- **Linear** Hamiltonian in ε
- **Unitary transform**
- Schroedinger Equation
  in the interaction picture (ε-factorization)
- solution: Dyson series

Feynman Integrals

- Kinematic-evolution in Dimensional Regularization
- space-time dimensional parameter: ε = (4-d)/2
- **Linear** system in ε
- non-Unitary **Magnus transform**
- System of Differential Equations
  in canonical form (ε-factorization) Henn (2013)
- solution: Dyson/Magnus series



$$= \mathrm{e}^{\Omega(d,x)}$$



boundary term
(simpler integral)

- Feynman integrals can be determined from differential equations that looks like **gauge transformations**

Argeri, Di Vita, Mirabella, Schlenk, Schubert, Tancredi, **P.M.** (2014)

- ***pre-Canonical form***
  Linear-**eps** Matrix

$$\partial_x f(\epsilon, x) = A(\epsilon, x)\; f(\epsilon, x)\;, \qquad A(\epsilon, x) = A_0(x) + \epsilon A_1(x)\;,$$

- change of basis :: Magnus #1

$$f(\epsilon, x) = B_0(x)\; g(\epsilon, x)\;, \qquad B_0(x) \equiv e^{\Omega[A_0](x,x_0)}\;. \qquad \partial_x B_0(x) = A_0(x) B_0(x)\;,$$

- ***Canonical form*** Henn (2013)

$$\partial_x g(\epsilon, x) = \epsilon \hat{A}_1(x) g(\epsilon, x) \qquad \hat{A}_1(x) = B_0^{-1}(x) A_1(x) B_0(x)\;.$$

(*)

- Solution :: Magnus #2 (or Dyson)

$$g(\epsilon, x) = B_1(\epsilon, x) g_0(\epsilon)\;, \qquad B_1(\epsilon, x) = e^{\Omega[\epsilon \hat{A}_1](x,x_0)}$$

Feynman integrals can be determined from differential equations that looks like gauge transformations

- Kinematic variables

$$-\frac{s}{m^2} = x, \quad -\frac{t}{m^2} = \frac{(1-y)^2}{y}$$

- eps-linear basis

$$\partial_x \; f(x,y,\epsilon) = \Big(A_{10}(x,y) + \epsilon A_{11}(x,y)\Big) \; f(x,y,\epsilon)$$
$$\partial_y \; f(x,y,\epsilon) = \Big(A_{20}(x,y) + \epsilon A_{21}(x,y)\Big) \; f(x,y,\epsilon)$$

- canonical form: Magnus #1

$$\partial_x \; g(x,y,\epsilon) = \epsilon \hat{A}_1(x,y) \; g(x,y,\epsilon)$$
$$\partial_y \; g(x,y,\epsilon) = \epsilon \hat{A}_2(x,y) \; g(x,y,\epsilon)$$

- Total Differential dLog-form

$$dg(x,y,\epsilon) = \epsilon \; d\hat{A}(x,y) \; g(x,y,\epsilon) \;, \qquad d\hat{A} \equiv \hat{A}_1 dx + \hat{A}_2 dy$$

$$dA = \sum_{i=1}^{n} M_i \; \mathrm{dlog}\, \eta_i$$

Alphabet

Constant matrices

Canonical systems and Iterated Integrals Henn (2013)

- Canonical system of DE $\quad d\mathbf{I} = \epsilon \, d\mathbb{A}\, \mathbf{I} \qquad dA = \sum_{i=1}^{n} \mathbb{M}_i \; d\log \eta_i \qquad \vec{x} = (x_1, x_2, \ldots, x_n)$

- Solution as path-ordered exponential

$$\mathrm{I}(\epsilon, \vec{x}) = \mathcal{P} \exp \left\{ \epsilon \int_\gamma dA \right\} \mathrm{I}(\epsilon, \vec{x}_0)\,, \qquad \mathcal{P} \exp \left\{ \epsilon \int_\gamma dA \right\} = \mathbb{1} + \epsilon \int_\gamma dA + \epsilon^2 \int_\gamma dA\, dA + \epsilon^3 \int_\gamma dA\, dA\, dA \ldots\,,$$

- Path invariance $\quad \gamma : [0,1] \ni t \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t))\,, \qquad \gamma(0) = \vec{x}_0 \text{ and } \gamma(1) = \vec{x}.$

- Taylor expansion and Dyson/Magnus series

$$\mathrm{I}(\epsilon, \vec{x}) = \mathrm{I}^{(0)}(\vec{x}) + \epsilon\, \mathrm{I}^{(1)}(\vec{x}) + \epsilon^2 \mathrm{I}^{(2)}(\vec{x}) + \ldots$$

$$\begin{aligned}
\mathrm{I}^{(0)}(\vec{x}) &= \mathrm{I}^{(0)}(\vec{x}_0)\,,\\
\mathrm{I}^{(1)}(\vec{x}) &= \mathrm{I}^{(1)}(\vec{x}_0) + \int_\gamma dA\, \mathrm{I}^{(0)}(\vec{x}_0)\,,\\
\mathrm{I}^{(2)}(\vec{x}) &= \mathrm{I}^{(2)}(\vec{x}_0) + \int_\gamma dA\, \mathrm{I}^{(1)}(\vec{x}_0) + \int_\gamma dA\, dA\, \mathrm{I}^{(0)}(\vec{x}_0)\,,\\
\mathrm{I}^{(3)}(\vec{x}) &= \mathrm{I}^{(3)}(\vec{x}_0) + \int_\gamma dA\, \mathrm{I}^{(2)}(\vec{x}_0)\,, + \int_\gamma dA\, dA\, \mathrm{I}^{(1)}(\vec{x}_0) \; + \int_\gamma dA\, dA\, dA\, \mathrm{I}^{(0)}(\vec{x}_0)\,,\\
\mathrm{I}^{(4)}(\vec{x}) &= \mathrm{I}^{(4)}(\vec{x}_0) + \int_\gamma dA\, \mathrm{I}^{(3)}(\vec{x}_0) + \int_\gamma dA\, dA\, \mathrm{I}^{(2)}(\vec{x}_0) \; + \int_\gamma dA\, dA\, dA\, \mathrm{I}^{(1)}(\vec{x}_0) + \int_\gamma dA\, dA\, dA\, dA\, \mathrm{I}^{(0)}(\vec{x}_0)\,.
\end{aligned}$$

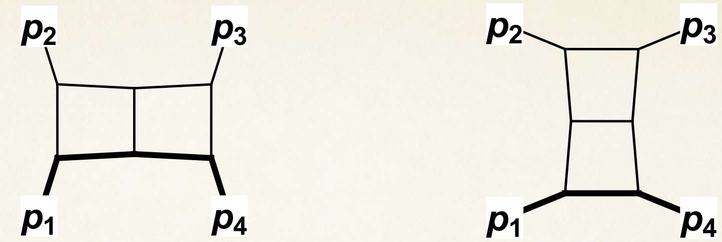
- Chen's Iterated integral

$$\int_\gamma \underbrace{dA \ldots dA}_{\text{k times}}, \longleftrightarrow \quad \mathcal{C}^{[\gamma]}_{i_k,\ldots,i_1} \equiv \int_\gamma d\log \eta_{i_k} \ldots d\log \eta_{i_1} \equiv \int_{0 \le t_1 \le \ldots \le t_k \le 1} g^\gamma_{i_k}(t_k) \ldots g^\gamma_{i_1}(t_1)\, dt_1 \ldots dt_k\,, \qquad g^\gamma_i(t) = \frac{d}{dt} \log \eta_i(\gamma(t))\,.$$

- a special case :: Goncharov's polylogs

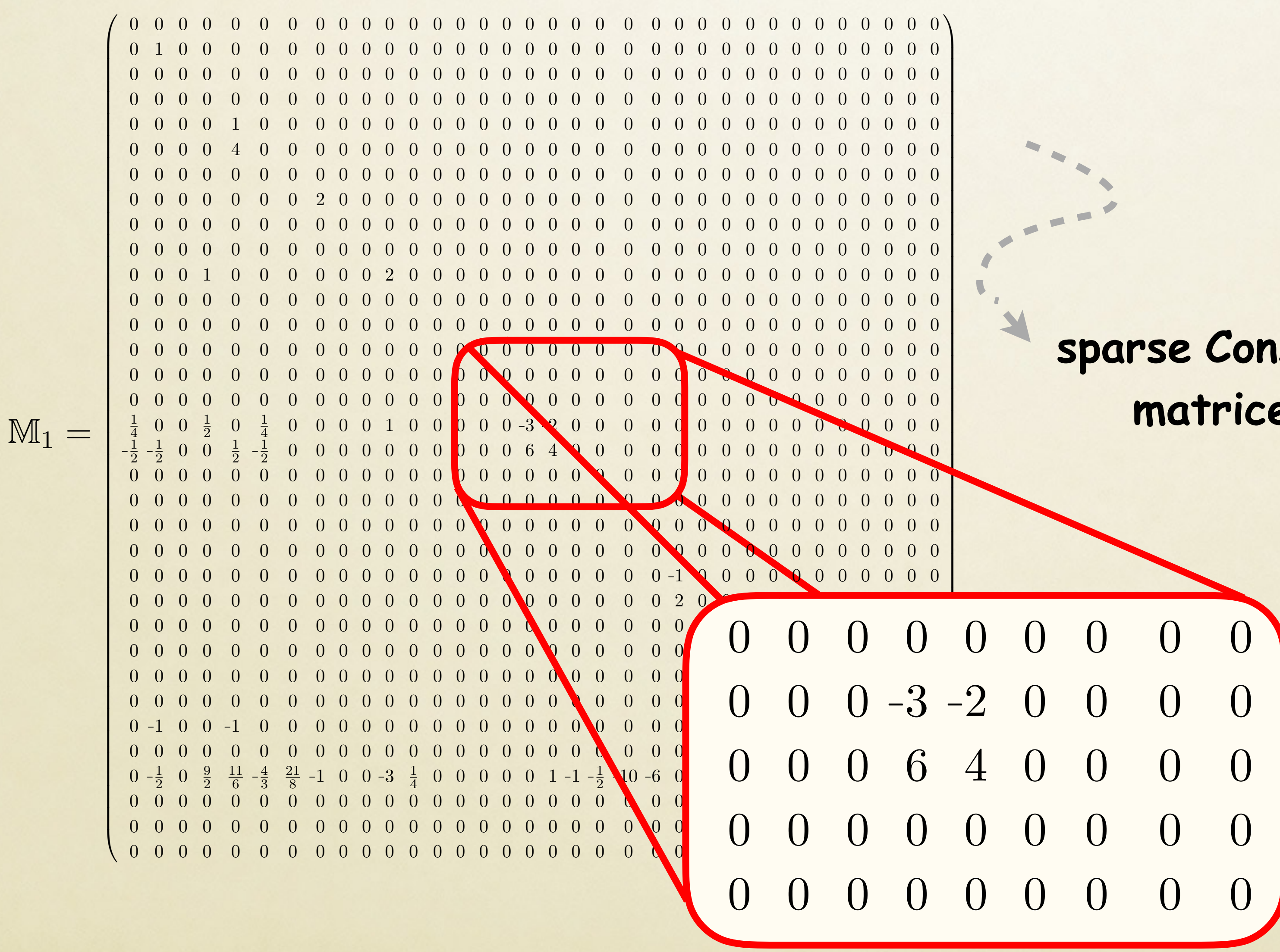
$$G(\vec{w}_n; x) \equiv G(w_1, \vec{w}_{n-1}; x) \equiv \int_0^x dt \frac{1}{t - w_1} G(\vec{w}_{n-1}; t), \qquad G(\vec{0}_n; x) \equiv \frac{1}{n!} \log^n(x), \qquad \partial_x G(\vec{w}_n; x) = \partial_x G(w_1, \vec{w}_{n-1}; x) = \frac{1}{x - w_1} G(\vec{w}_{n-1}; x).$$

Ex :: Family-1



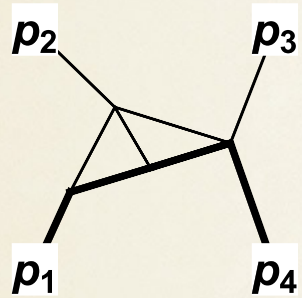
- Total Differential dLog-form

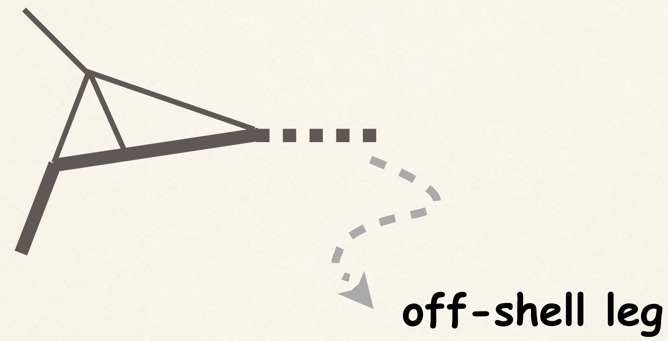
$$
\begin{aligned}
d\mathbb{A} = \mathbb{M}_1\ d\log(x) + \mathbb{M}_2\ d\log(1+x) + \mathbb{M}_3\ d\log(1-x) \\
+ \mathbb{M}_4\ d\log(y) + \mathbb{M}_5\ d\log(1+y) + \mathbb{M}_6\ d\log(1-y) \\
+ \mathbb{M}_7\ d\log(x+y) + \mathbb{M}_8\ d\log\left(1+x\,y\right) \\
+ \mathbb{M}_9\ d\log\left(1-y(1-x-y)\right)\ ,
\end{aligned}
$$

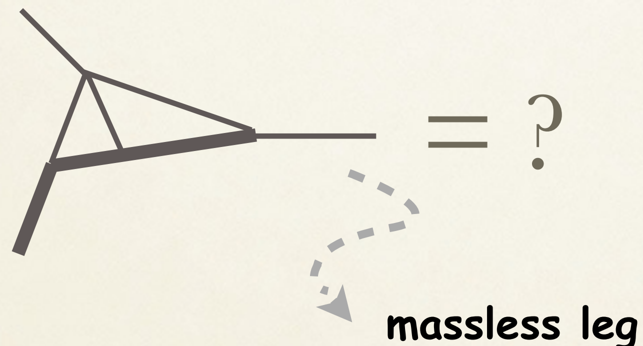


sparse Constant matrices

Ex :: Boundary conditions

$F(s) =$ 

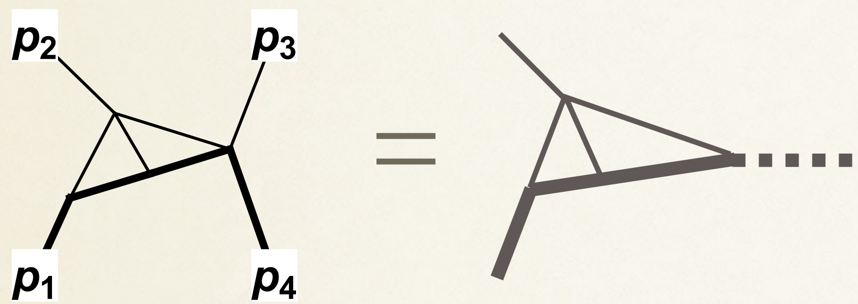
$=$ 

$\longrightarrow F(0) =$ 

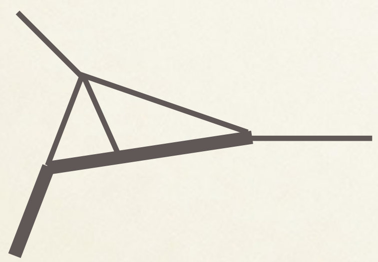
$= ?$

- regular @ s = 0
- trivial conditions !

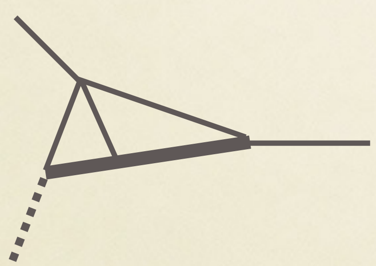
Ex :: Boundary conditions

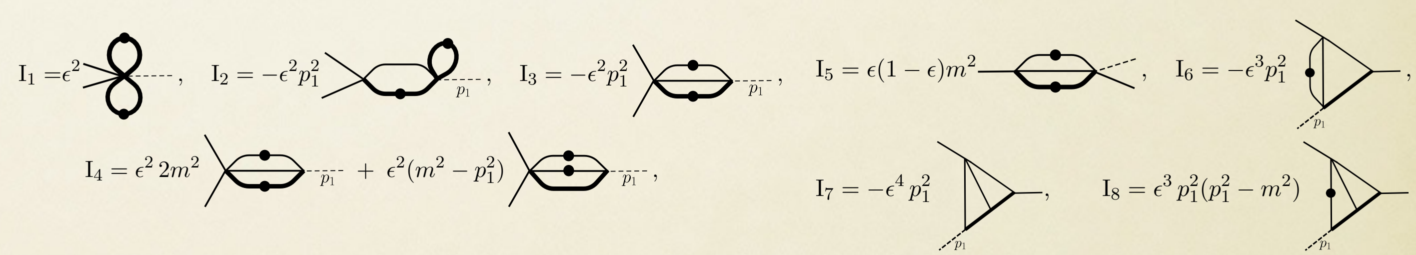
$F(s) =$ 

- regular @ s = 0
- trivial conditions !

$\rightarrow F(0) =$  $= ?$

- Auxiliary system of differential equation for:

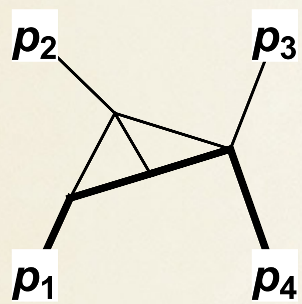
$I(p_1^2) =$ 

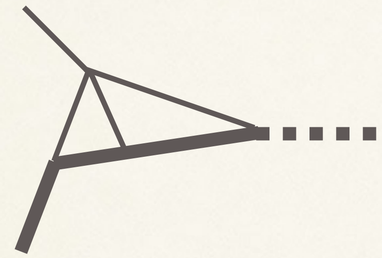


$$d\mathbb{A} = \mathbb{M}_0\ d\log x + \mathbb{M}_{-1}\ d\log(x+1)$$

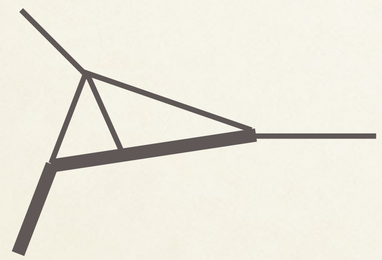
$$\mathbb{M}_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{4} & -\frac{1}{2} & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & -\frac{1}{2} & -1 & 0 & 0 & \frac{1}{2} & -1 & \frac{5}{2} \end{pmatrix}, \quad \mathbb{M}_{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & -\frac{1}{2} & -3 & 0 & -4 \end{pmatrix}.$$

Ex :: Boundary conditions

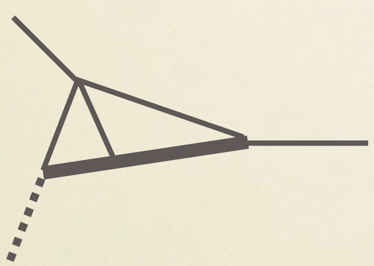
$F(s) =$ 

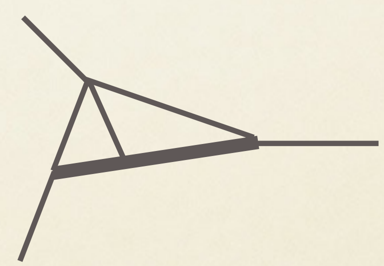
$=$ 

- **regular @ s = 0**
- trivial conditions !

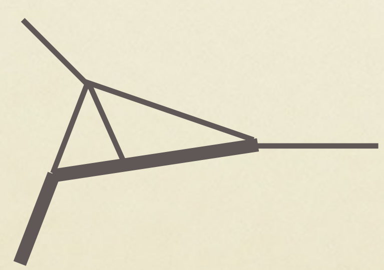
$\longrightarrow$ $F(0) =$  $= ?$

- Auxiliary system of differential equation for:

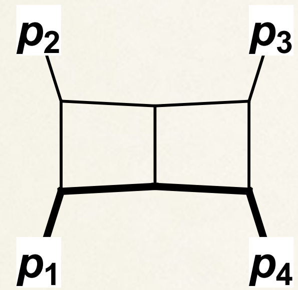
$I(p_1^2) =$ 

$I(0) =$ 

- **regular @** p_1^2 **= 0**
- **non-trivial** b.c. conditions !

$\longrightarrow$ $F(0) =$  $= I(m\hat{}2)$ ☑

Ex :: Family-1 :: MI I_{30}



$$= \sum_{k=0}^{4} I_{30}^{(k)}(x,y)\ \epsilon^k$$

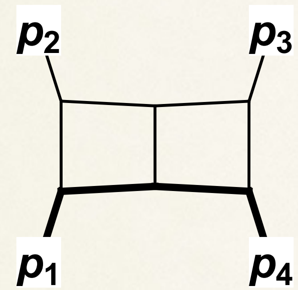
$$I_{30}^{(0)}(x,y) = -\frac{4}{3}$$

$$I_{30}^{(1)}(x,y) = \frac{7}{3}\,G[\{-1\},x] - \frac{3}{2}\,G[\{0\},y] + 3\,G[\{1\},y]$$

$$I_{30}^{(2)}(x,y) = \frac{11\,\pi^2}{9} + G[\{-1\},x]\,\left(4\,G[\{0\},y] - 8\,G[\{1\},y]\right) - \frac{4}{3}\,G[\{-1,-1\},x] - G[\{0,0\},y] + 2\,G[\{0,1\},y] + 2\,G[\{1,0\},y] - 4\,G[\{1,1\},y]$$

$$\begin{aligned}
I_{30}^{(3)}(x,y) = {} & \pi^2\left(-\frac{23}{9}\,G[\{-1\},x] + \frac{7}{6}\,G[\{0\},y] - \frac{11}{3}\,G[\{1\},y] + \frac{4}{3}\,G\left[\left\{-\frac{1}{y}\right\},x\right]\right) + \\
& G[\{-y\},x]\,\left(-2\,G[\{1,0\},y] + 4\,G[\{1,1\},y]\right) + \\
& G\left[\left\{-\frac{1}{y}\right\},x\right]\,\left(2\,G[\{0,0\},y] - 4\,G[\{0,1\},y] - 2\,G[\{1,0\},y] + 4\,G[\{1,1\},y]\right) + \\
& G[\{-1\},x]\,\left(4\,G[\{0,0\},y] - 8\,G[\{0,1\},y] - 8\,G[\{1,0\},y] + 16\,G[\{1,1\},y]\right) + \\
& G[\{1\},y]\,\left(16\,G[\{-1,-1\},x] - 4\,G\left[\left\{-\frac{1}{y},-1\right\},x\right] - 4\,G[\{-y,-1\},x]\right) + \\
& G[\{0\},y]\,\left(-8\,G[\{-1,-1\},x] + 2\,G\left[\left\{-\frac{1}{y},-1\right\},x\right] + 2\,G[\{-y,-1\},x]\right) - \\
& \frac{32}{3}\,G[\{-1,-1,-1\},x] + 4\,G[\{-1,0,-1\},x] - 2\,G[\{0,1,0\},y] + \\
& 4\,G[\{0,1,1\},y] - 2\,G[\{1,0,0\},y] + 4\,G[\{1,0,1\},y] + 4\,G[\{1,1,0\},y] - \\
& 8\,G[\{1,1,1\},y] + 4\,G\left[\left\{-\frac{1}{y},-1,-1\right\},x\right] - 2\,G\left[\left\{-\frac{1}{y},0,-1\right\},x\right] + \\
& 4\,G[\{-y,-1,-1\},x] - 2\,G[\{-y,0,-1\},x] + \frac{29\,\text{Zeta}[3]}{3}
\end{aligned}$$

Ex :: Family-1 :: MI I_{30}



$$= \sum_{k=0}^{4} I_{30}^{(k)}(x,y)\,\epsilon^k$$

$$I_{30}^{(4)}(x,y) =$$

$$-\frac{4051\,\pi^4}{7200}+$$
$$G[\{-1,-1\},x]\ (-16\,G[\{0,0\},y]+32\,G[\{0,1\},y]+32\,G[\{1,0\},y]-64\,G[\{1,1\},y])$$
$$G[\{0,0\},y]\ \Big(2\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y}\Big\},x\Big]+4\,G[\{-y,-1\},x]-2\,G\Big[\Big\{-y,-\frac{1}{y}\Big\},x\Big]\Big)+$$
$$\pi^2\ \Big(G[\{-1\},x]\ \Big(-\frac{14}{3}\,G[\{0\},y]+\frac{28}{3}\,G[\{1\},y]\Big)-$$
$$\frac{4}{3}\,G[\{-1\},y]\,G\Big[\Big\{-\frac{1}{y}\Big\},x\Big]+G[\{0\},y]\,G\Big[\Big\{-\frac{1}{y}\Big\},x\Big]+\frac{4}{3}\,G[\{-1\},y]\,G[\{-y\},x]+$$
$$G[\{1\},y]\ \Big(-\frac{11}{3}\,G\Big[\Big\{-\frac{1}{y}\Big\},x\Big]+\frac{5}{3}\,G[\{-y\},x]\Big)+\frac{8}{9}\,G[\{-1,-1\},x]+$$
$$\frac{4}{3}\,G[\{0,-1\},y]-G[\{0,1\},y]-G[\{1,0\},y]+\frac{22}{3}\,G[\{1,1\},y]-$$
$$3\,G\Big[\Big\{-\frac{1}{y},-1\Big\},x\Big]+\frac{4}{3}\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y}\Big\},x\Big]-\frac{1}{3}\,G[\{-y,-1\},x]-\frac{4}{3}\,G\Big[\Big\{-y,-\frac{1}{y}\Big\},x\Big]\Big)+$$
$$G[\{0,1\},y]\ \Big(-4\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y}\Big\},x\Big]-8\,G[\{-y,-1\},x]+4\,G\Big[\Big\{-y,-\frac{1}{y}\Big\},x\Big]\Big)+$$
$$G[\{1,0\},y]\ \Big(-4\,G\Big[\Big\{-\frac{1}{y},-1\Big\},x\Big]-2\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y}\Big\},x\Big]+2\,G\Big[\Big\{-\frac{1}{y},-y\Big\},x\Big]-$$
$$4\,G[\{-y,-1\},x]+2\,G\Big[\Big\{-y,-\frac{1}{y}\Big\},x\Big]-2\,G[\{-y,-y\},x]\Big)+$$
$$G[\{1,1\},y]\ \Big(8\,G\Big[\Big\{-\frac{1}{y},-1\Big\},x\Big]+4\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y}\Big\},x\Big]-4\,G\Big[\Big\{-\frac{1}{y},-y\Big\},x\Big]+$$
$$8\,G[\{-y,-1\},x]-4\,G\Big[\Big\{-y,-\frac{1}{y}\Big\},x\Big]+4\,G[\{-y,-y\},x]\Big)+G[\{-y\},x]$$
$$(2\,G[\{-1,0,0\},y]-4\,G[\{-1,0,1\},y]-6\,G[\{0,1,0\},y]+12\,G[\{0,1,1\},y]-$$
$$2\,G[\{1,0,0\},y]+4\,G[\{1,0,1\},y]+12\,G[\{1,1,0\},y]-24\,G[\{1,1,1\},y])+$$
$$G\Big[\Big\{-\frac{1}{y}\Big\},x\Big]\ (-2\,G[\{-1,0,0\},y]+4\,G[\{-1,0,1\},y]+6\,G[\{0,0,0\},y]-$$

$$12\,G[\{0,0,1\},y]-6\,G[\{0,1,0\},y]+12\,G[\{0,1,1\},y]-10\,G[\{1,$$
$$20\,G[\{1,0,1\},y]+12\,G[\{1,1,0\},y]-24\,G[\{1,1,1\},y])+$$
$$G[\{1\},y]\ \Big(8\,G\Big[\Big\{-\frac{1}{y},-1,-1\Big\},x\Big]-4\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y},-1\Big\},x\Big]+4\,G\Big[\Big\{-\frac{1}{y},-y$$
$$8\,G[\{-y,-1,-1\},x]+4\,G\Big[\Big\{-y,-\frac{1}{y},-1\Big\},x\Big]-4\,G[\{-y,-y,-1\},x]$$
$$G[\{0\},y]\ \Big(-4\,G\Big[\Big\{-\frac{1}{y},-1,-1\Big\},x\Big]+2\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y},-1\Big\},x\Big]-2\,G\Big[\Big\{-\frac{1}{y},-$$
$$4\,G[\{-y,-1,-1\},x]-2\,G\Big[\Big\{-y,-\frac{1}{y},-1\Big\},x\Big]+2\,G[\{-y,-y,-1\},x]$$
$$\frac{128}{3}\,G[\{-1,-1,-1,-1\},x]-16\,G[\{-1,-1,0,-1\},x]-$$
$$24\,G[\{-1,0,-1,-1\},x]+$$
$$12\,G[\{-1,0,0,-1\},x]+2\,G[\{0,-1,0,0\},y]-$$
$$4\,G[\{0,-1,0,1\},y]-10\,G[\{0,0,1,0\},y]+$$
$$20\,G[\{0,0,1,1\},y]-6\,G[\{0,1,0,0\},y]+$$
$$12\,G[\{0,1,0,1\},y]+20\,G[\{0,1,1,0\},y]-$$
$$40\,G[\{0,1,1,1\},y]-6\,G[\{1,0,0,0\},y]+$$
$$12\,G[\{1,0,0,1\},y]+20\,G[\{1,0,1,0\},y]-$$
$$40\,G[\{1,0,1,1\},y]+20\,G[\{1,1,0,0\},y]-$$
$$40\,G[\{1,1,0,1\},y]-40\,G[\{1,1,1,0\},y]+80\,G[\{1,1,1,1\},y]-$$
$$24\,G\Big[\Big\{-\frac{1}{y},-1,-1,-1\Big\},x\Big]+8\,G\Big[\Big\{-\frac{1}{y},-1,0,-1\Big\},x\Big]+$$
$$12\,G\Big[\Big\{-\frac{1}{y},0,-1,-1\Big\},x\Big]-6\,G\Big[\Big\{-\frac{1}{y},0,0,-1\Big\},x\Big]+$$
$$4\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y},-1,-1\Big\},x\Big]-2\,G\Big[\Big\{-\frac{1}{y},-\frac{1}{y},0,-1\Big\},x\Big]-$$
$$4\,G\Big[\Big\{-\frac{1}{y},-y,-1,-1\Big\},x\Big]+2\,G\Big[\Big\{-\frac{1}{y},-y,0,-1\Big\},x\Big]-$$
$$24\,G[\{-y,-1,-1,-1\},x]+8\,G[\{-y,-1,0,-1\},x]+$$
$$12\,G[\{-y,0,-1,-1\},x]-6\,G[\{-y,0,0,-1\},x]-$$
$$4\,G\Big[\Big\{-y,-\frac{1}{y},-1,-1\Big\},x\Big]+2\,G\Big[\Big\{-y,-\frac{1}{y},0,-1\Big\},x\Big]+$$
$$4\,G[\{-y,-y,-1,-1\},x]-2\,G[\{-y,-y,0,-1\},x]+$$
$$\Big(-\frac{62}{3}\,G[\{-1\},x]+9\,G[\{0\},y]-28\,G[\{1\},y]+10\,G\Big[\Big\{-\frac{1}{y}\Big\},x\Big]\Big)\ \mathrm{Zeta}[3]$$

Summary ...

- ☑ **Feasibility of Two-Loop QED Corrections *analytically***
  - Master Integrals via Differential Equations + Magnus Series
  - Amplitude decomposition via Adaptive Integrand Decomposition (AID)

    Mu-e scattering :: a first example of 2-loop automation for massive amplitudes
  - Progress with hadronic corrections Fael (2018) + Fael Passera (to appear)

    >> see Passera's talk

...Outlook

- Building the 2-loop amplitude (Form Factors and AID)
- Analytic continuation and Numerical Evaluation of 2-loop MIs
- The 1-Loop amplitude and 2-loop renorm. counterterms (GoSam, AID)
- Implementing a Subtraction Scheme for NNLO (*hinc sunt leones*)
- MonteCarlo Integration

  >> see Piccinini's talk

☑ Progress in **Mu-e Scattering** @ 2-loop QED
==> benefit for **e+ e- —> Mu+ Mu-** @ 2-loop QED
==> benefit for **p p —> t T** @ 2-loop QCD

Muon-electron scattering:
Theory kickoff workshop
4-5 September 2017
Padova

https://agenda.infn.it/internalPage.py?pageId=0&confId=13774

One year ago

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