Search and Discovery Statistics in HEP Lecture 3

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This presentation would have not been possible without the tremendous help of
the following people throughout many years

Louis Lyons, Alex Read, Bob Cousins Glen Cowan ,Kyle Cranmer
Ofer Vitells & Jonathan Shlomi

What can you expect from the Lectures

- Lecture 1: Basic Concepts
  Histograms, PDF, Testing Hypotheses,
  LR as a Test Statistics, p-value, POWER, CLs
  Measurements
- Lecture 2: Wald Theorem, Asymptotic Formalism, Asimov Data Set, Feldman-Cousins, PL & CLs, Asimov Significance
- Lecture 3: Look Elsewhere Effect
  1D LEE the non-intuitive thumb rule
  (upcrossings, trial #~Z)
  2D LEE (Euler Characteristic)

Look Elsewhere Effect

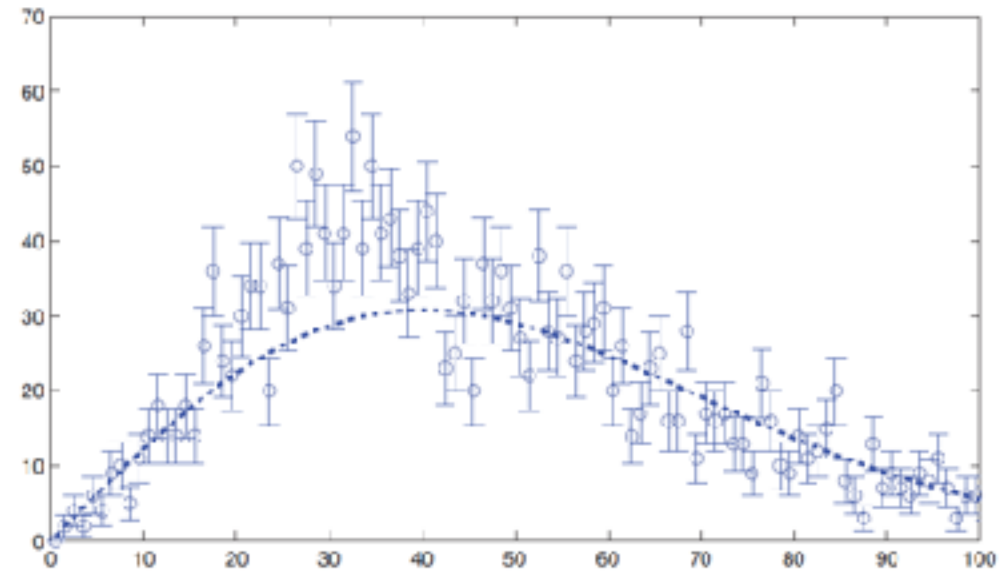
Look Elsewhere Effect

E.G., O. Vitells "Trial factors for the look elsewhere effect in high energy physics",
Eur. Phys. J. C 70 (2010) 525

O. Vitells and E. G., Estimating the significance of a signal in a multi-dimensional search,
1669 Astropart. Phys. 35 (2011) 230, arXiv:1105.4355

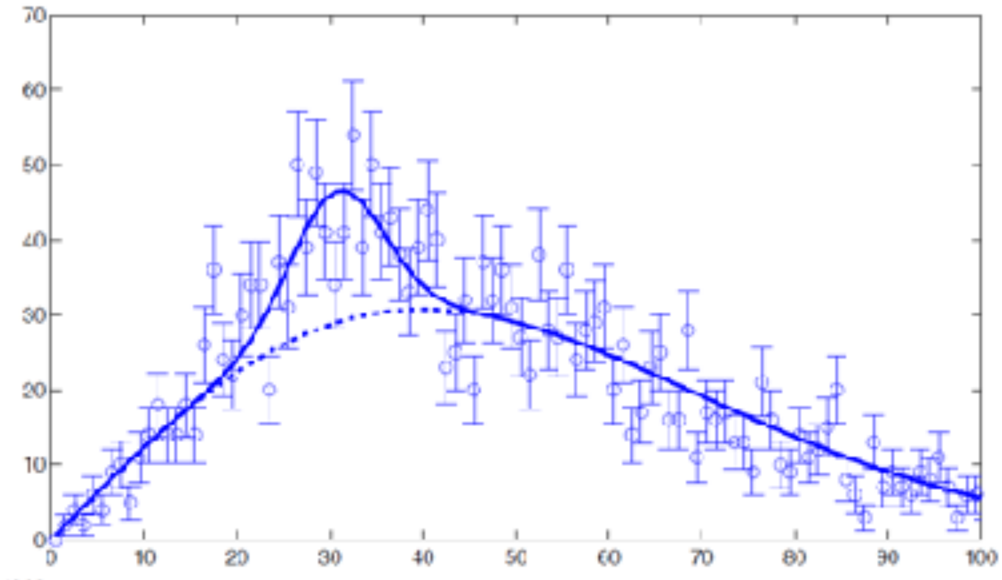
Look Elsewhere Effect

- Is there a signal here?



Look Elsewhere Effect

- Looks like a signal at m=30
- What is its significance?



Test the BG hypothesis
At m=30

$$q_0(\theta) = \begin{cases} -2\log\dfrac{L(\mu=0)}{L(\hat{\mu},\theta)} \\ 0 \end{cases}$$

$$q_{fix,obs} = -2\ln\frac{L(b)}{L(\hat{\mu}s(m=30)+b)}$$

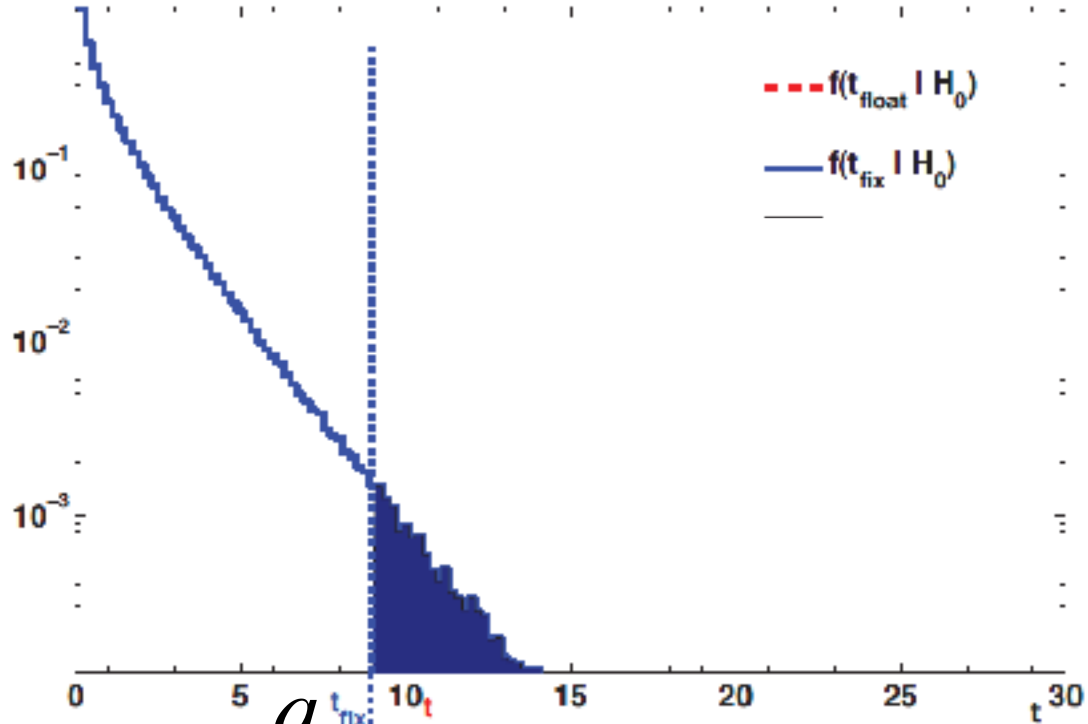
$$Z = \sqrt{q_{0,fix,obs}}$$

Look Elsewhere Effect

$$q_{0,fix} = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(30)+b)}$$

$$f(q_{0,fix} \mid H_0) \sim \chi^2$$

$$p_{fix} = \int_{q_{fix,obs}}^{\infty} f(q_0 \mid H_0)dq_0$$



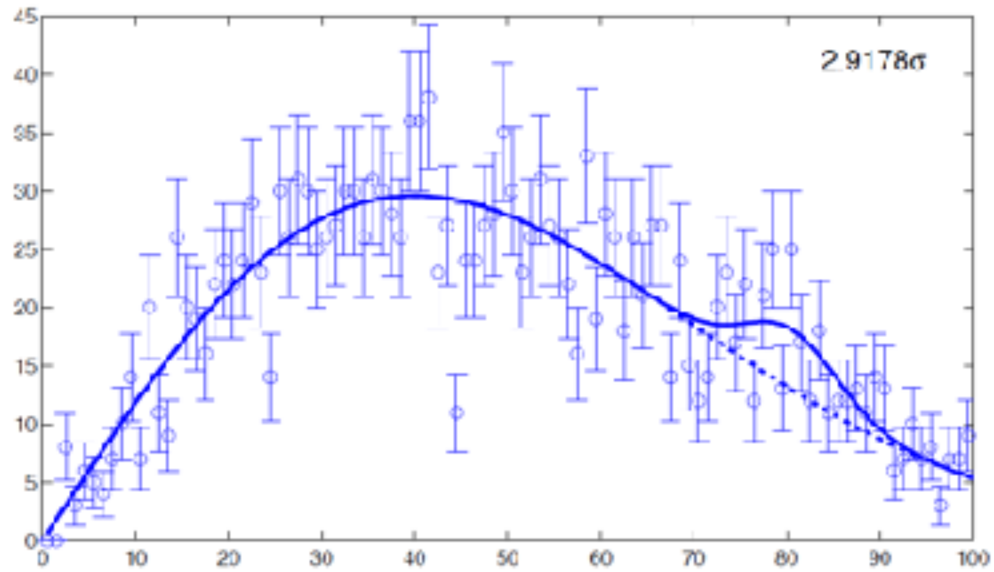
$q_{0,fix,obs}$

p_{fix} *answers the question :*

What is the probability to have a fluctuation as or bigger than the observed one?

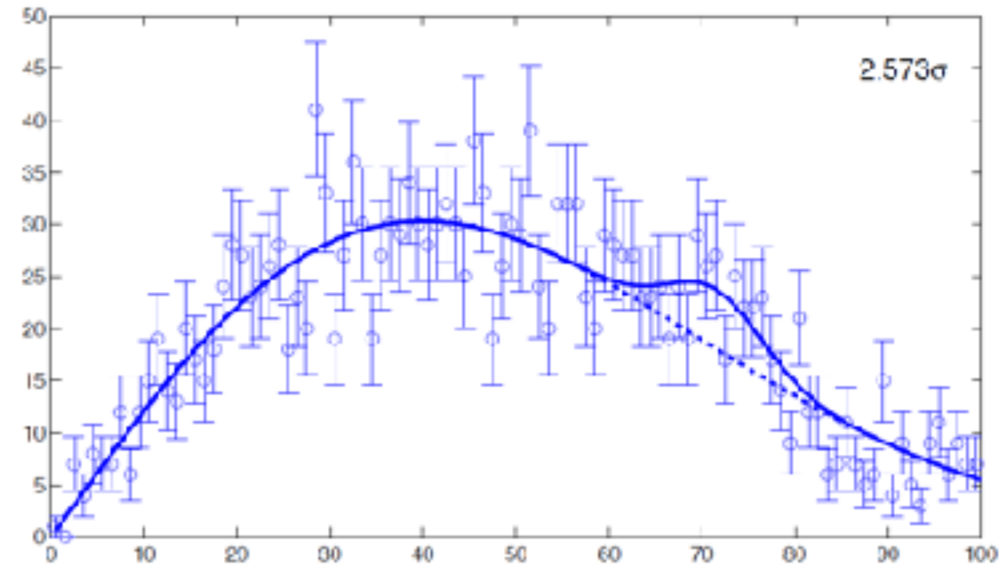
Look Elsewhere Effect

- Would you ignore this signal, had you seen it?



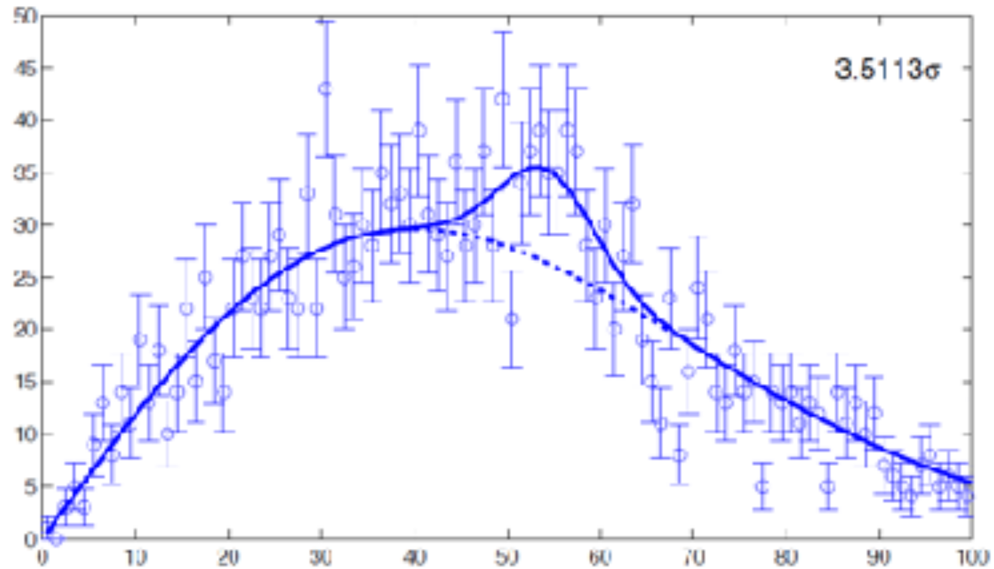
Look Elsewhere Effect

- Or this?



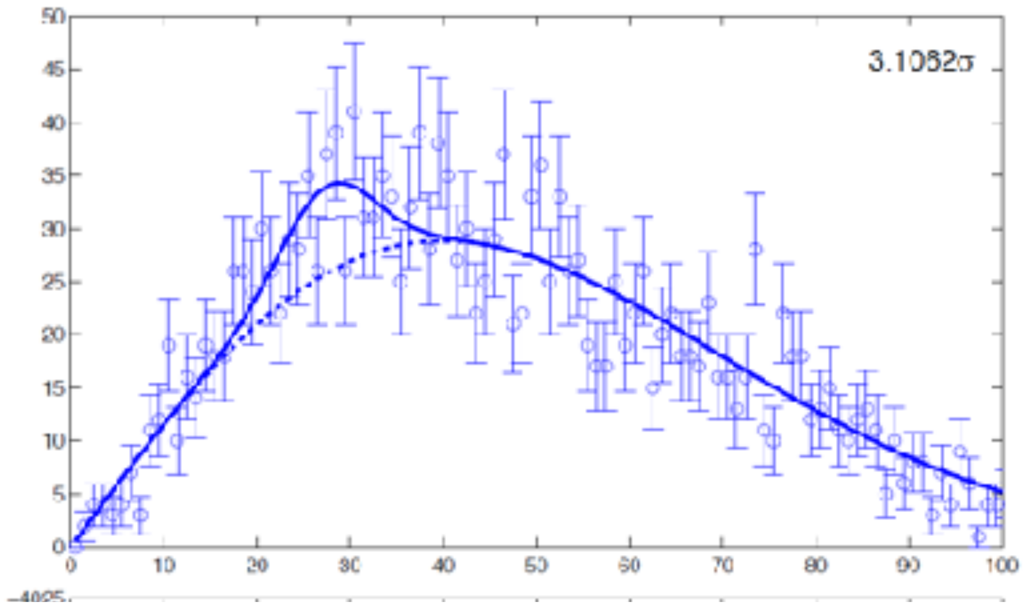
Look Elsewhere Effect

- Or this?



Look Elsewhere Effect

- Or this?
- Obviously NOT!
- ALL THESE "SIGNALS" ARE BG FLUCTUATIONS



The right question :

What is the probability to have a fluctuation

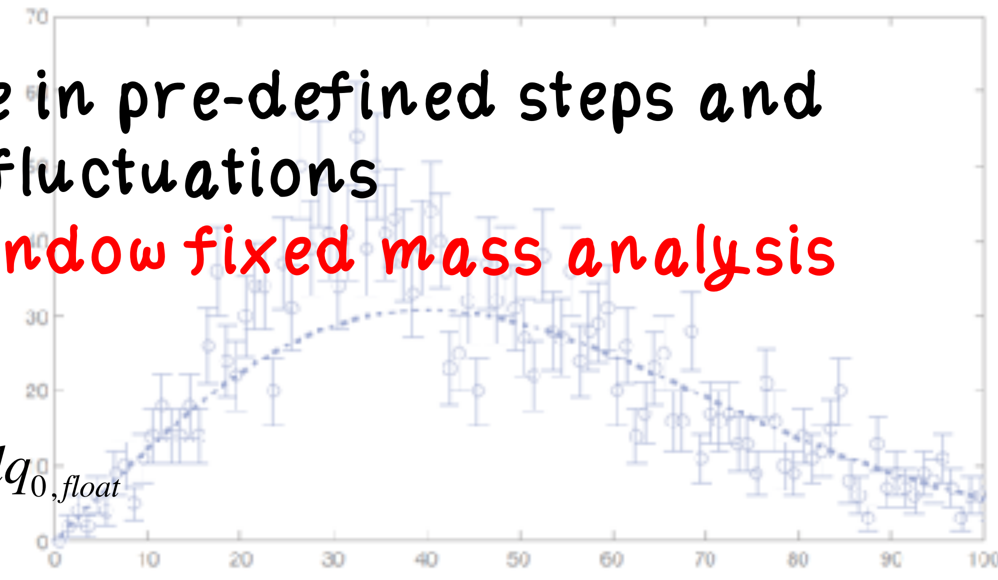
as or bigger than the observed one

ANYWHERE in the mass search range?

Look Elsewhere Effect

- Having no idea where the signal might be there are two equivalent options
- OPTION I:

scan the mass range in pre-defined steps and test any disturbing fluctuations



Perform a sliding window fixed mass analysis

$$q_{0,float} = \max_m \left(q_0(m) \right)$$

$$p_{float} = \int_{q_{float,obs}}^{\infty} f(q_{0,float} \mid H_0)\, dq_{0,float}$$

- OPTION II:

Perform a floating mass analysis

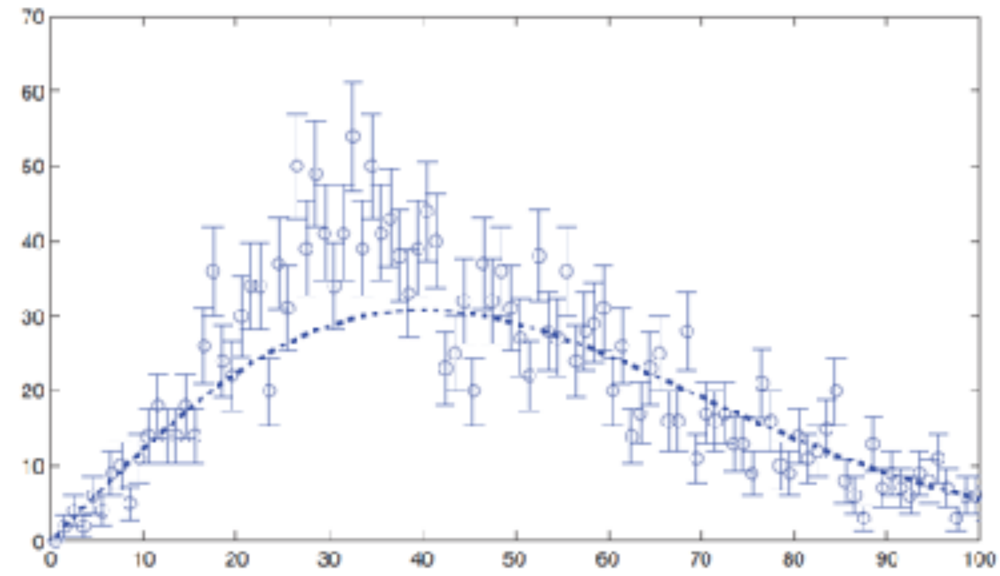
$$q_{0,float} = q_0(\hat{m}) = -2 \ln \frac{L(b)}{L(\hat{\mu} s(\hat{m}) + b)}$$

$$p_{float} = \int_{q_{float,obs}}^{\infty} f(q_{0,float} \mid H_0)\, dq_{0,float}$$

Sliding Window

- Scan and perform a fixed mass analysis at each point

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



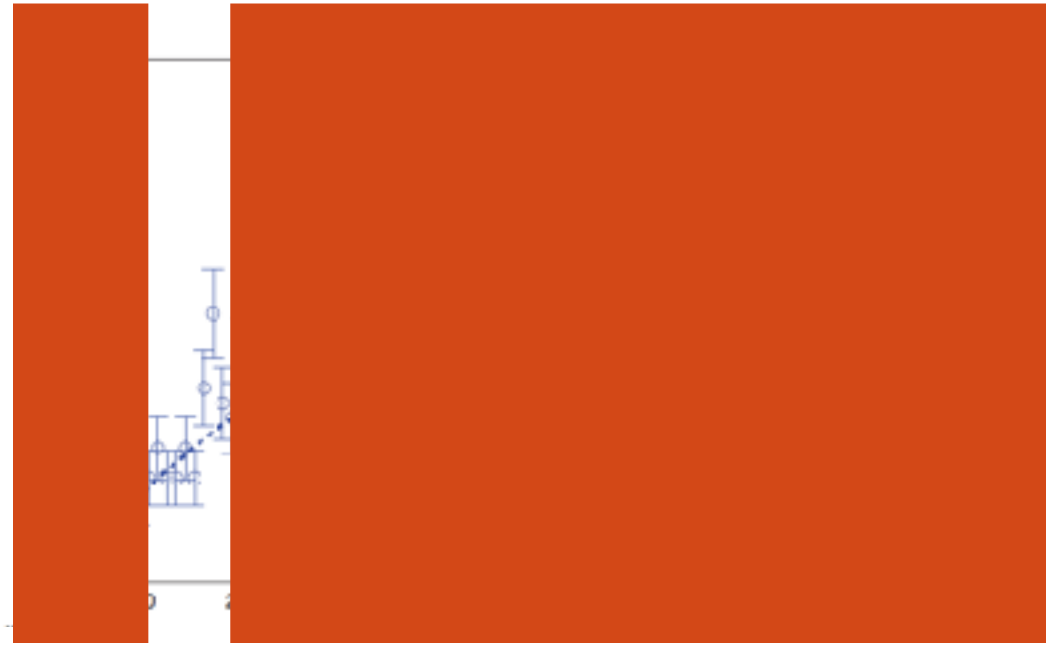
- The scan resolution must be less than the signal mass resolution

Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$

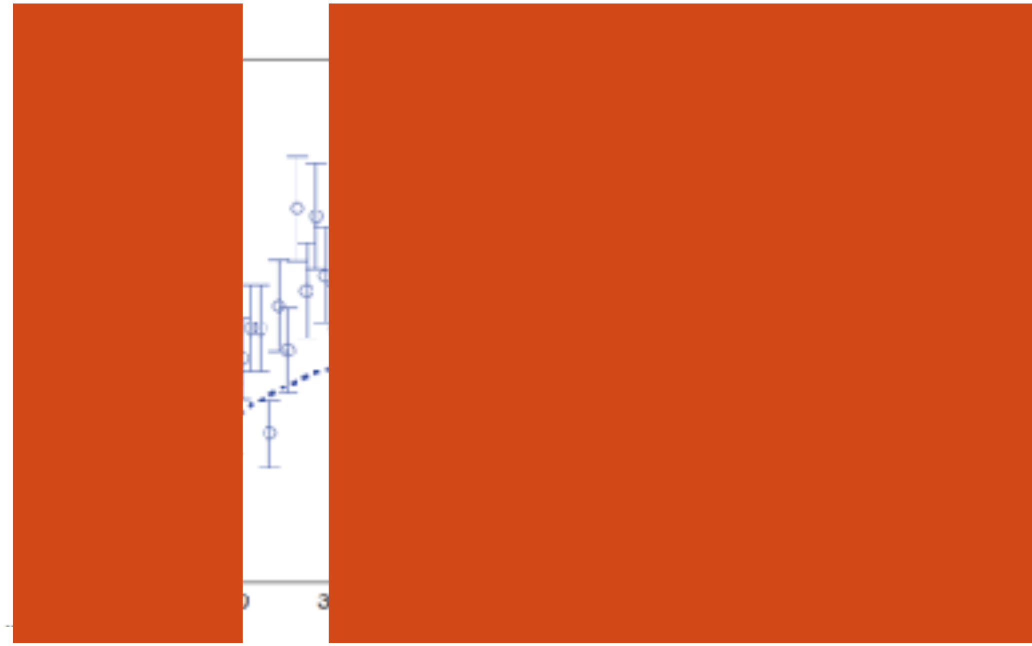
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



Sliding Window

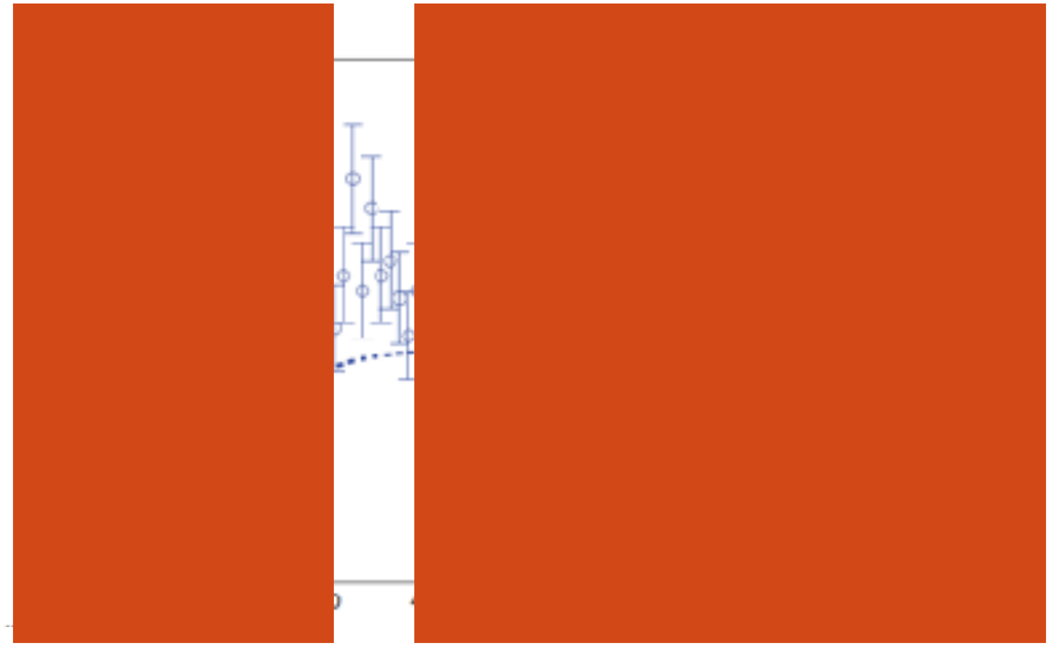
$$q_0 = -2 ln \frac{L(\mu = 0)}{L(\hat{\mu} s(m) + b)}$$



1/30/18

Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



1/30/18

Sliding Window

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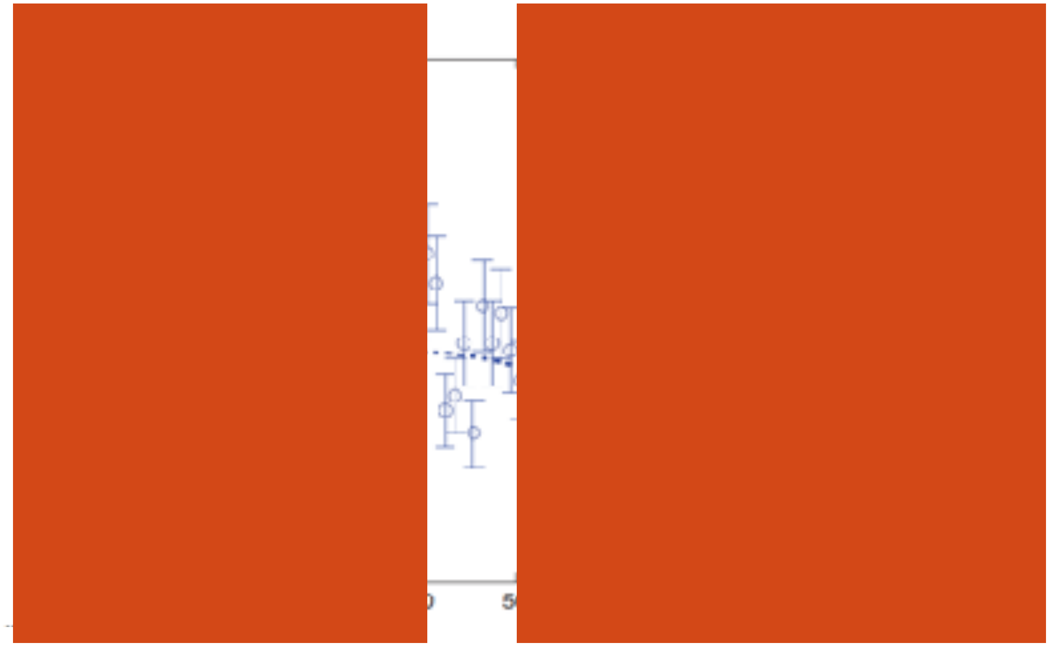
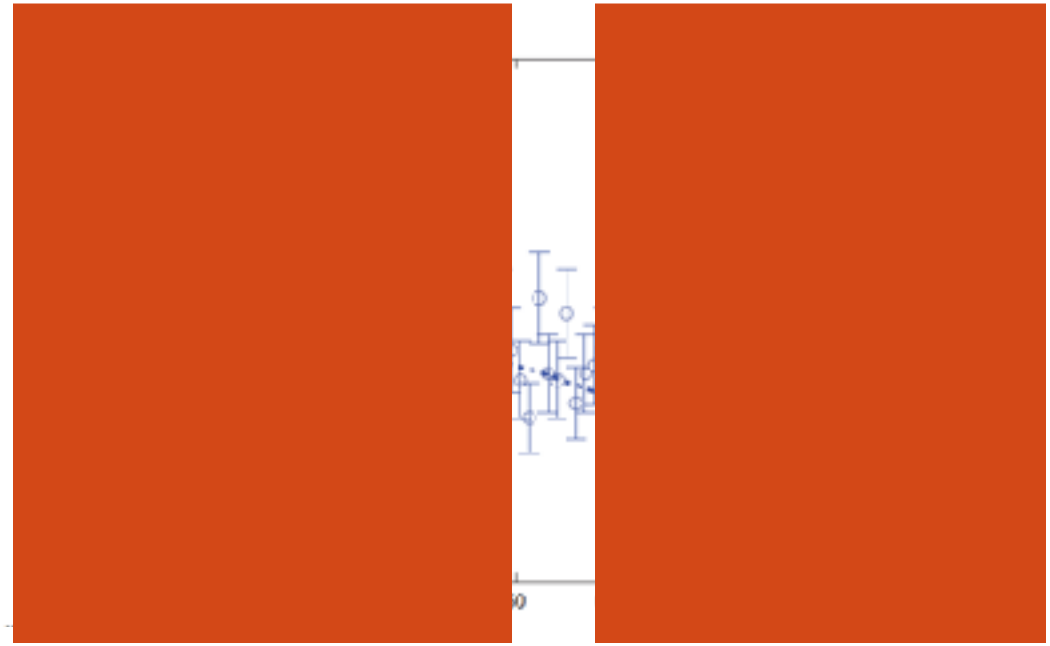


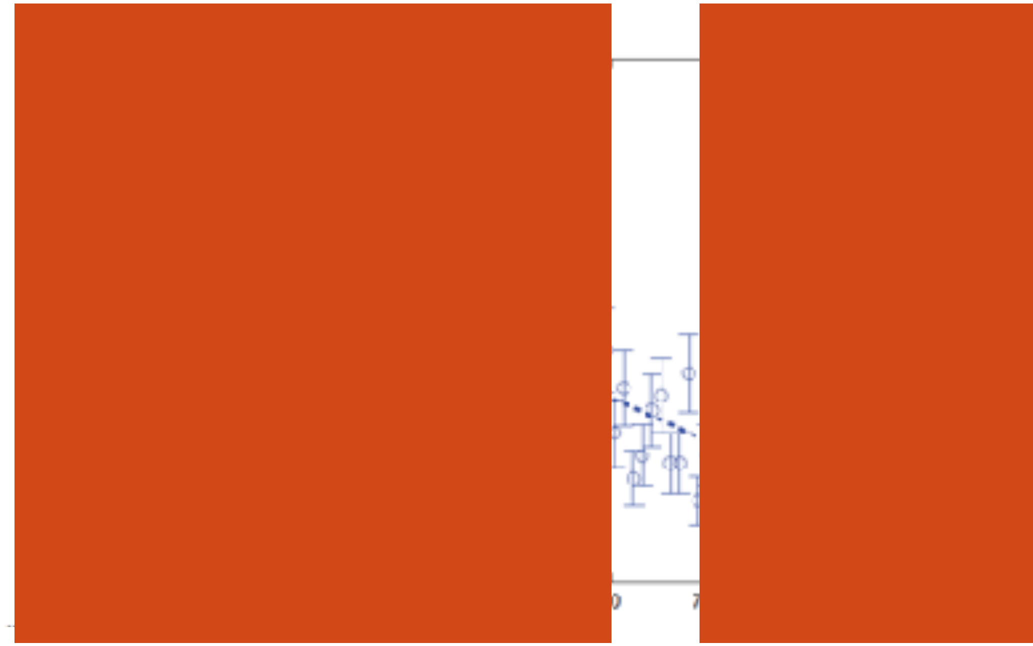
Sliding Window

$$q_0 = -2ln\frac{L(\mu = 0)}{L(\hat{\mu}s(m) + b)}$$



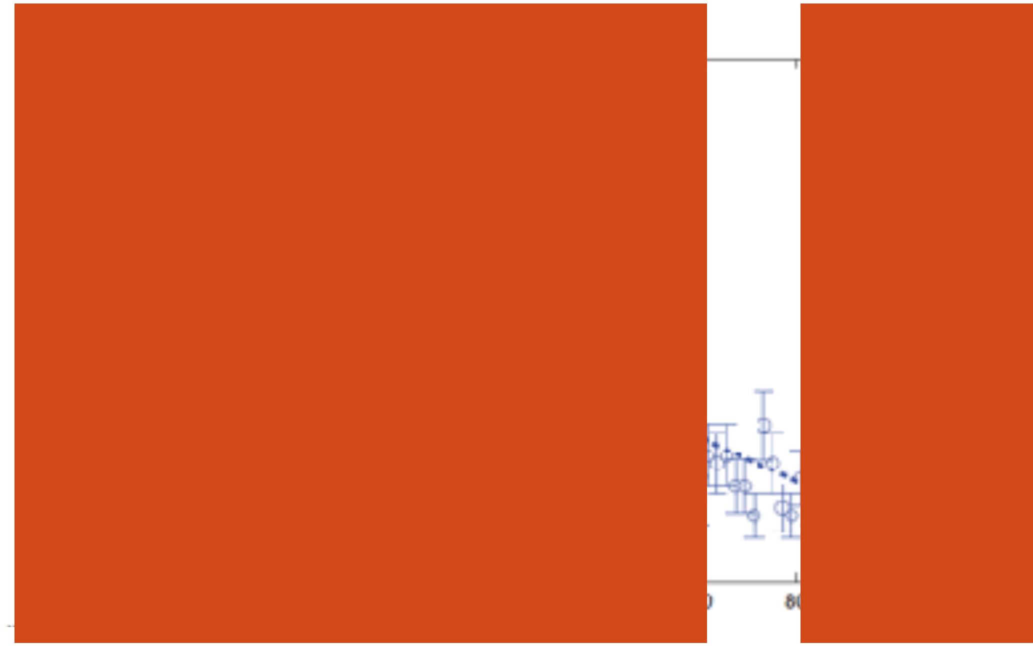
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



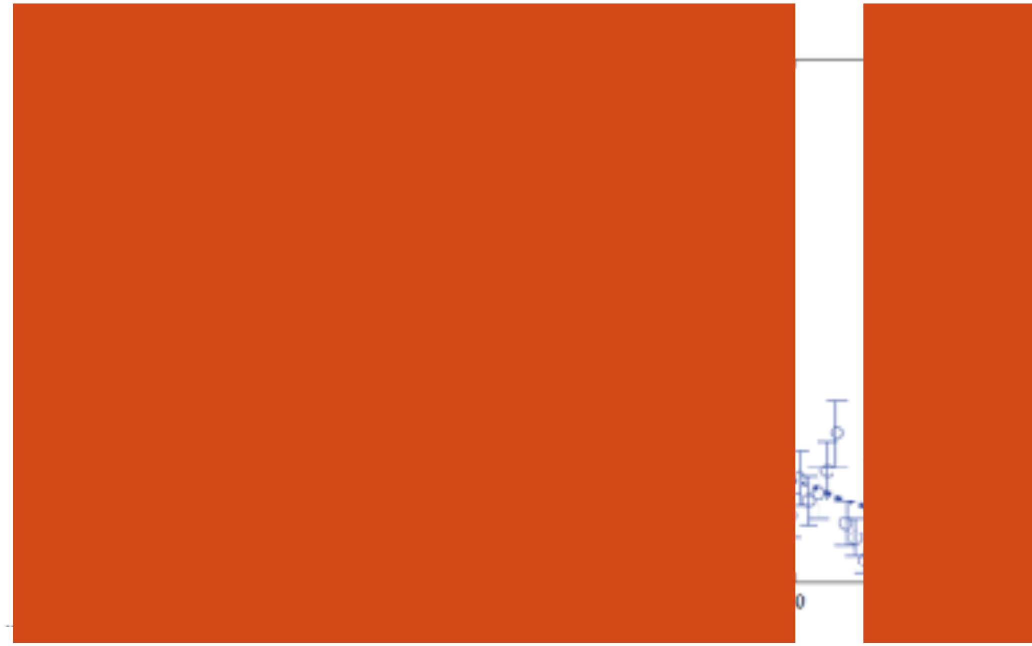
Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



Sliding Window

$$q_0 = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



Sliding Window

- Assuming the signal can be only at one place
- pick the one with the MAXIMUM SIGNIFICANCE

$$q_{0,float} = \max_{m}\left(q_0(m)\right)$$

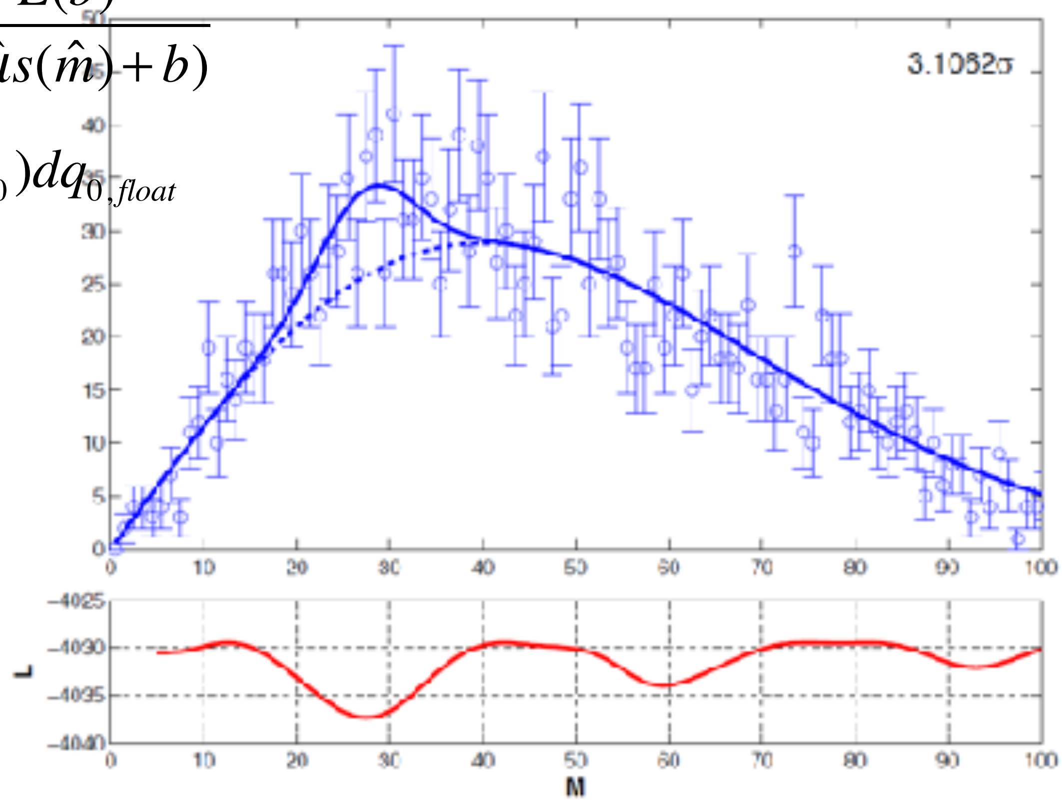
1/30/18

Look Elsewhere Effect: Floating Mass

OPTION II

$$q_{0,float} = q_0(\hat{m}) = -2\ln\frac{L(b)}{L(\hat{\mu}s(\hat{m})+b)}$$

$$p_{float} = \int_{q_{float,obs}}^{\infty} f(q_{0,float} \mid H_0)\,dq_{0,float}$$

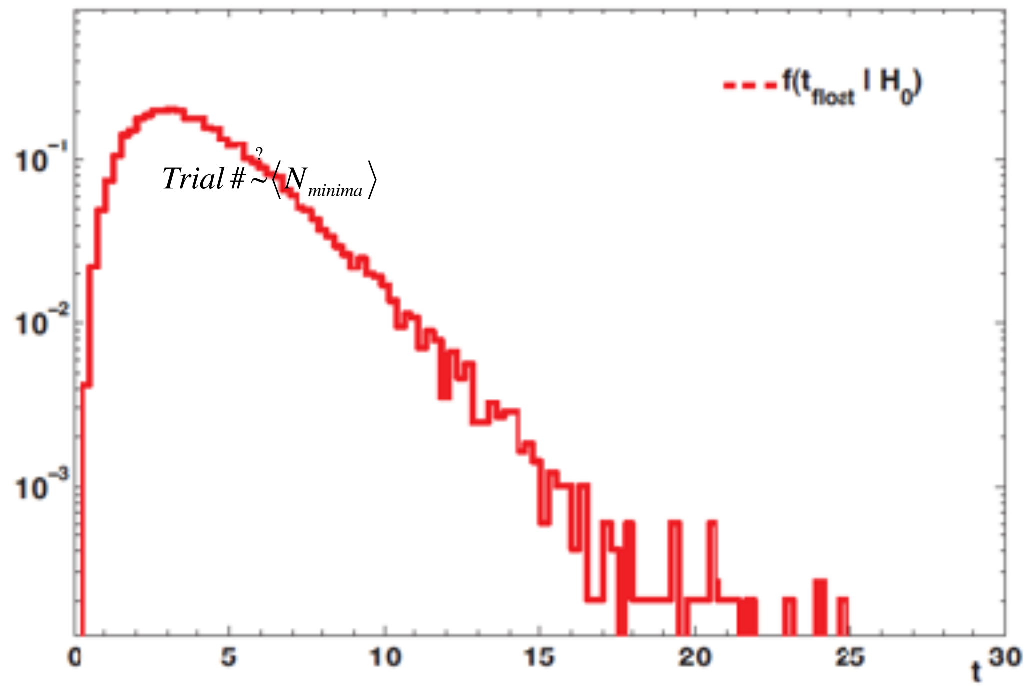


Look Elsewhere Effect

- The distribution $f(q_{float}|H_0)$ does not follow a chi-squared with 2dof because the mass parameter is not defined under the null hypothesis

$$\text{for any } m_{fix} \quad q_0(\hat{m}) \geq q_0(m_{fix})$$

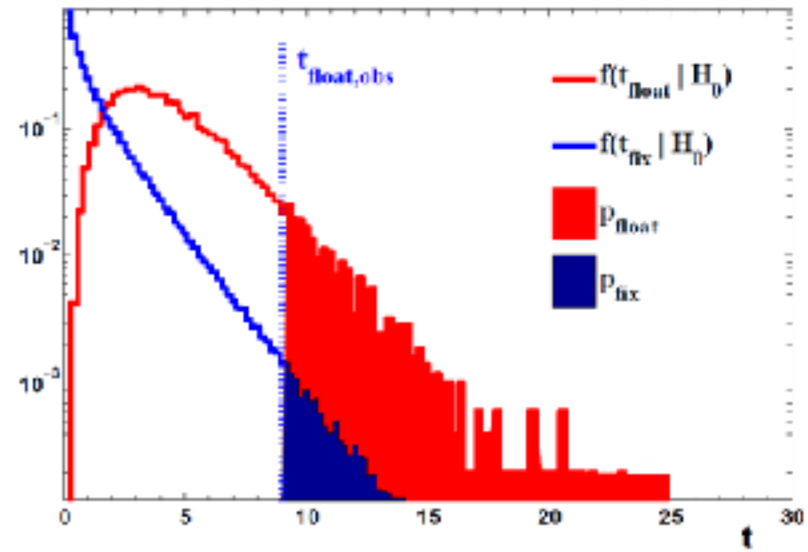
$$\text{The } \chi_1^2 \text{ distribution is pushed to the right}$$



trial#

- Assume a maximal local fluctuation at mass $\hat{m} = 30$
- The observed q_0 is given by

$$q_{0,obs} = -2ln\frac{L(\mu=0)}{L(\hat{\mu}s(m)+b)}$$



$$p_{fix} = \int_{q_{0,obs}}^{\infty} f(q_{0,fix} | H_0) dq_{0,fix}$$

$$p_{float} = \int_{q_{0,obs}}^{\infty} f(q_{0,float} | H_0) dq_{0,float}$$

$$trial\# = \frac{p_{float}}{p_{fix}}$$

Can we calculate analytically the floating mass p-value

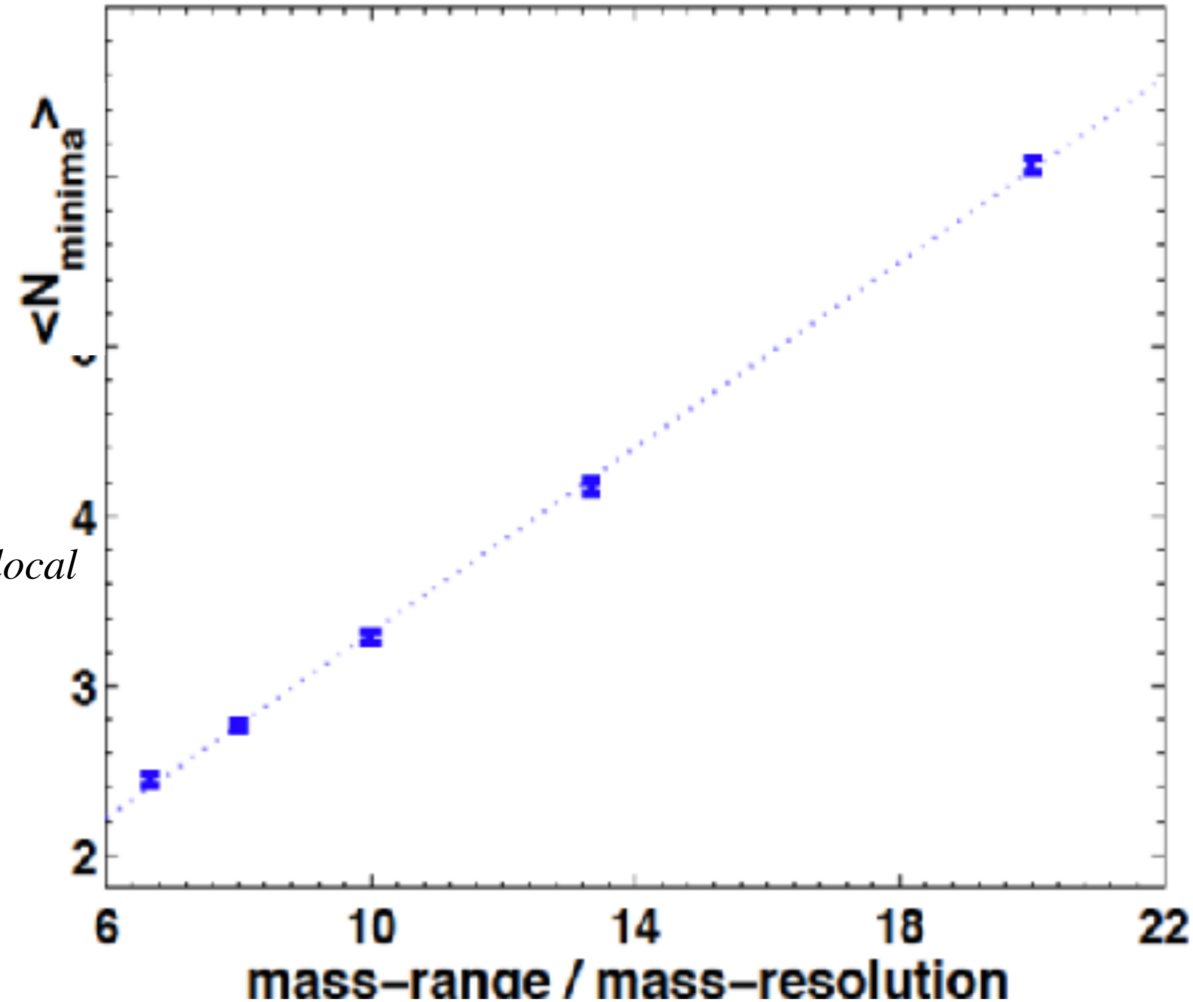
(Wrong) Thumb Rule

$$\langle N_{minima} \rangle \sim \frac{Mass\ Range}{Mass\ Resolution}$$

$$Trial\,\# \sim \langle N_{minima} \rangle$$

$$Trial\,\# \stackrel{?}{=} \langle N_{minima} \rangle\, p_{local}$$

The answer is NO



The right question :
What is the probability to have a fluctuation as or bigger than the observed one ***ANYWHERE*** *in the mass search range?*

Let θ be a nuisance parameter undefined under the null hypothesis.

$$\text{Define } q(\hat{\theta}) = max_{\theta}(q(\theta))$$

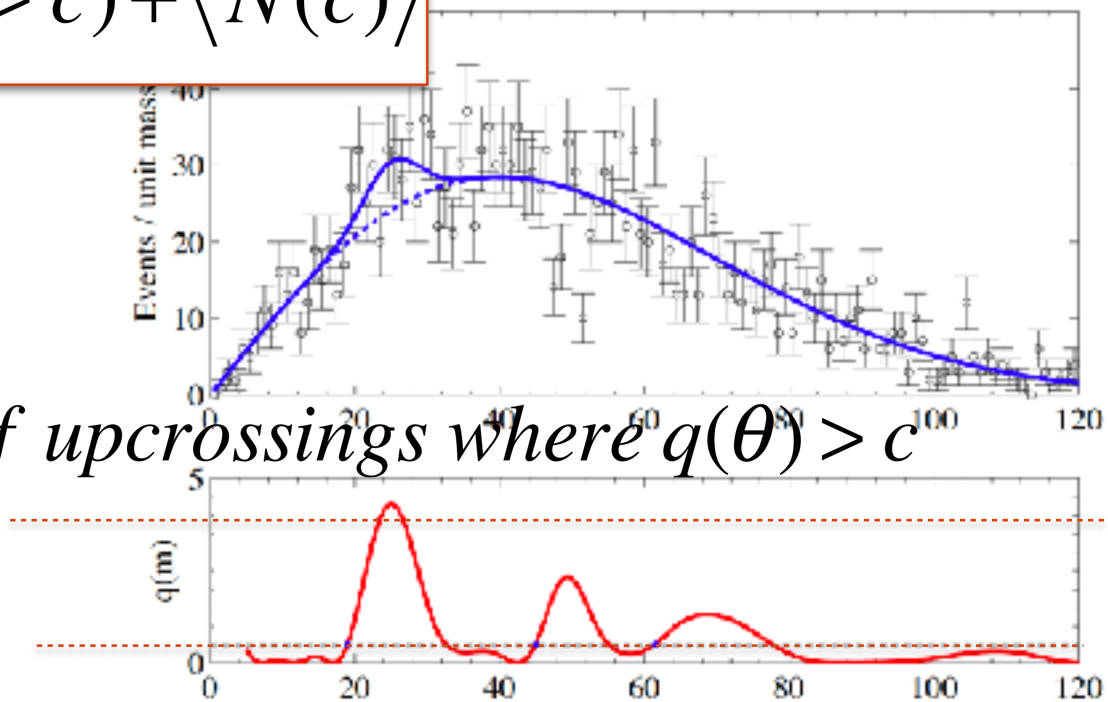
Davies (1987) finds, for $c >> 1$

$$P\left(q(\hat{\theta}) > c\right) \sim P(\chi_1^2 > c) + \langle N(c) \rangle$$

$\langle N(c) \rangle$ = *Number of upcrossings* $q(\theta) > c$

Davies Formula

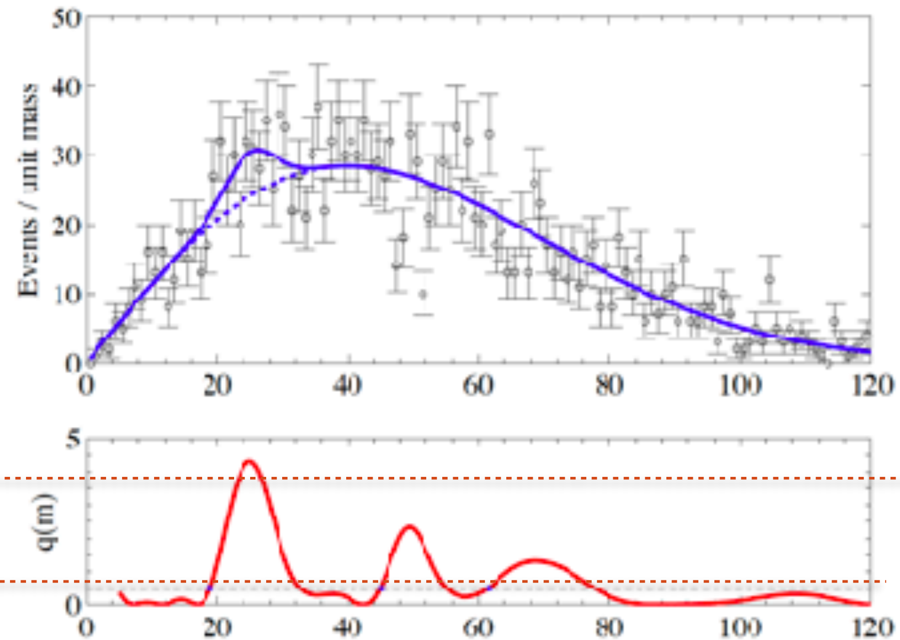
$$P\left(q(\hat{\theta}) > c\right) \sim P(\chi_1^2 > c) + \langle N(c) \rangle$$



$$\langle N(c) \rangle = Number\ of\ upcrossings\ where\ q(\theta) > c$$

$$for\ c >> 1 \rightarrow \langle N(c) \rangle << 1$$

Making Davies Formula Accessible



$$\langle N(c)\rangle \ll 1$$

$$\langle N(c)\rangle \sim e^{-c/2}$$

$$P(q(\hat{\theta})>c) \sim P(\chi_1^2 > c) + \langle N(c_0)\rangle \frac{\langle N(c)\rangle}{\langle N(c_0)\rangle}$$

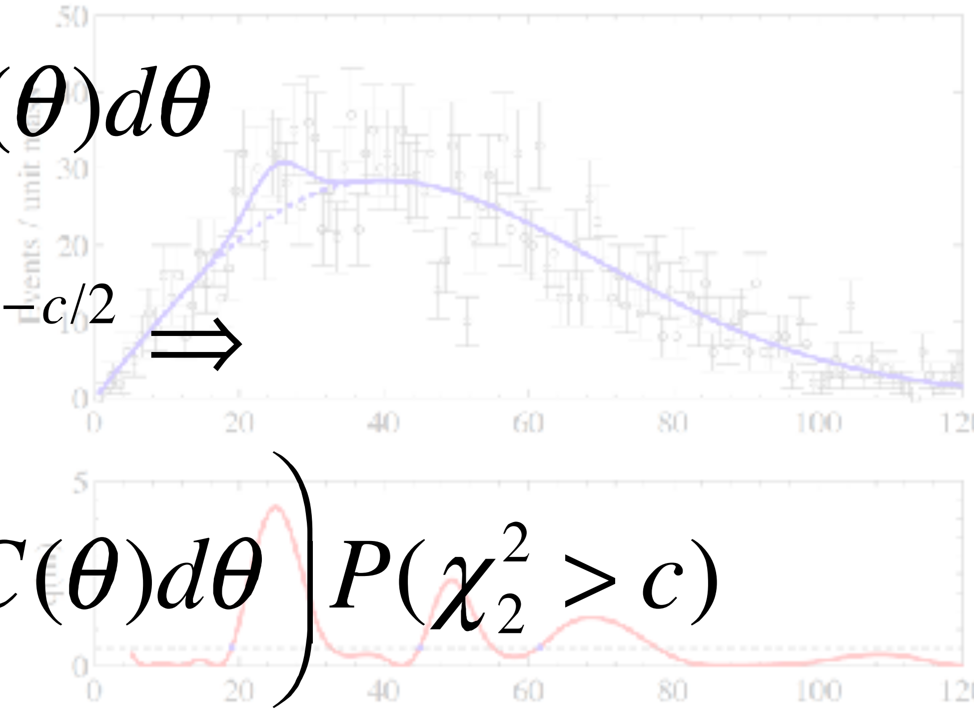
$$P(q(\hat{\theta})>c) \sim P(\chi_1^2 > c) + \langle N(c_0)\rangle e^{-(c-c_0)/2}$$

Gross Vitells Formula

Davies Formula $P\left(q(\hat{\theta}) > c\right) \sim P(\chi_1^2 > c) + \langle N(c) \rangle$

$$\langle N(c) \rangle \approx \frac{e^{-c/2}}{\sqrt{2\pi}} \int_\theta C(\theta) d\theta$$

$$P(\chi_2^2 > c) \xrightarrow[c \to \infty]{} e^{-c/2} \Rightarrow$$



$$\langle N(c) \rangle \approx \left(\frac{1}{\sqrt{2\pi}} \int_\theta C(\theta) d\theta \right) P(\chi_2^2 > c)$$

$$\langle N(c) \rangle = \mathcal{N} P(\chi_2^2 > c)$$

$$P(q(\hat{\theta}) > c) \sim P(\chi_1^2 > c) + \mathcal{N} P(\chi_2^2 > c)$$

Trial

$$P(\chi_1^2 > c) \xrightarrow{c>>1} \sqrt{\frac{2}{c}} \frac{e^{-c/2}}{\Gamma\left(\frac{1}{2}\right)}$$

$$P(\chi_2^2 > c) \xrightarrow{c>>1} e^{-c/2}$$

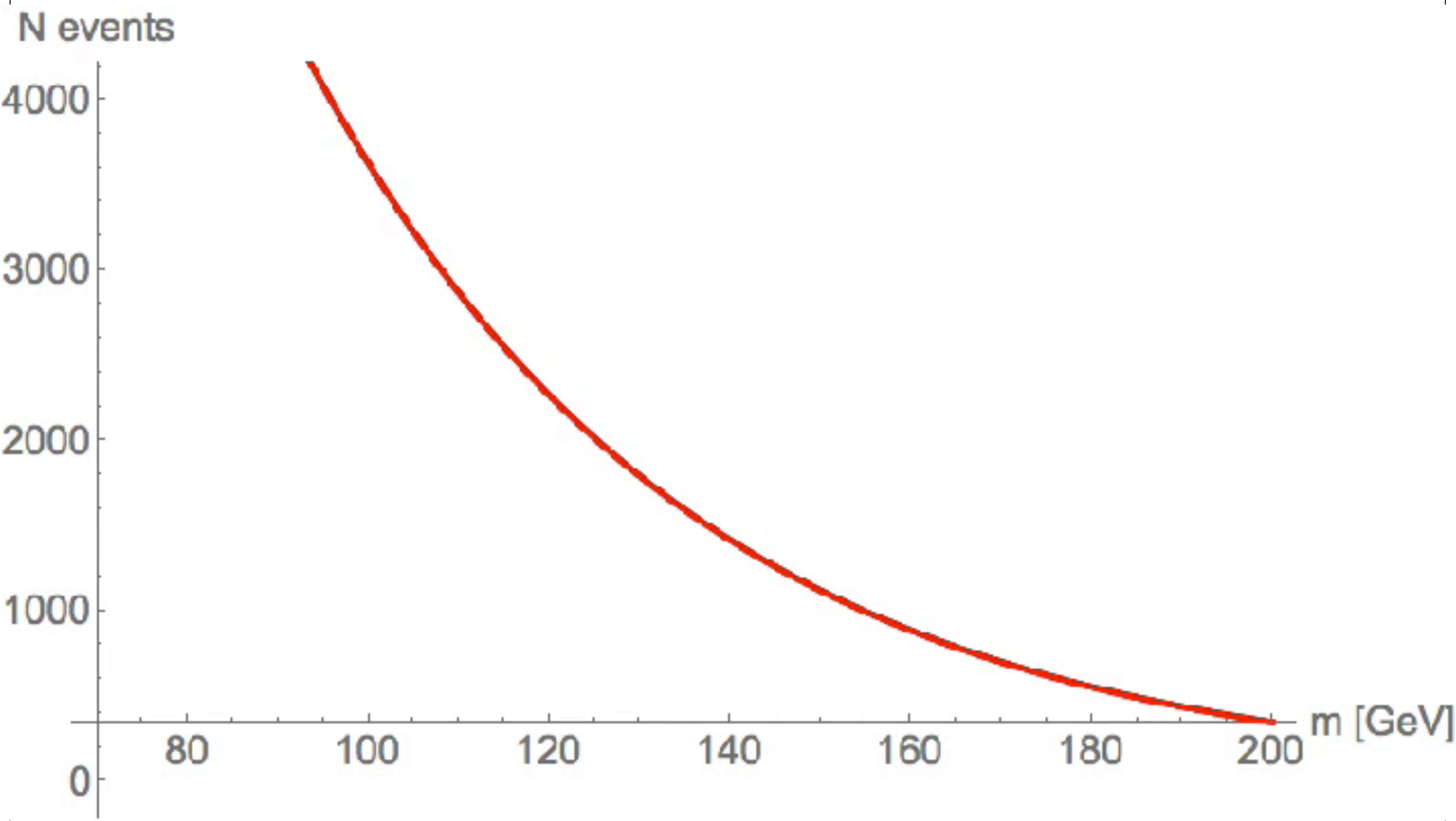
$$trial\,\# = \frac{P(q(\hat{\theta}) > c)}{P(q(\theta) > c)} \approx$$

$$\approx 1 + \mathcal{N} \frac{P(\chi_2^2 > c)}{P(\chi_1^2 > c)} \Rightarrow$$

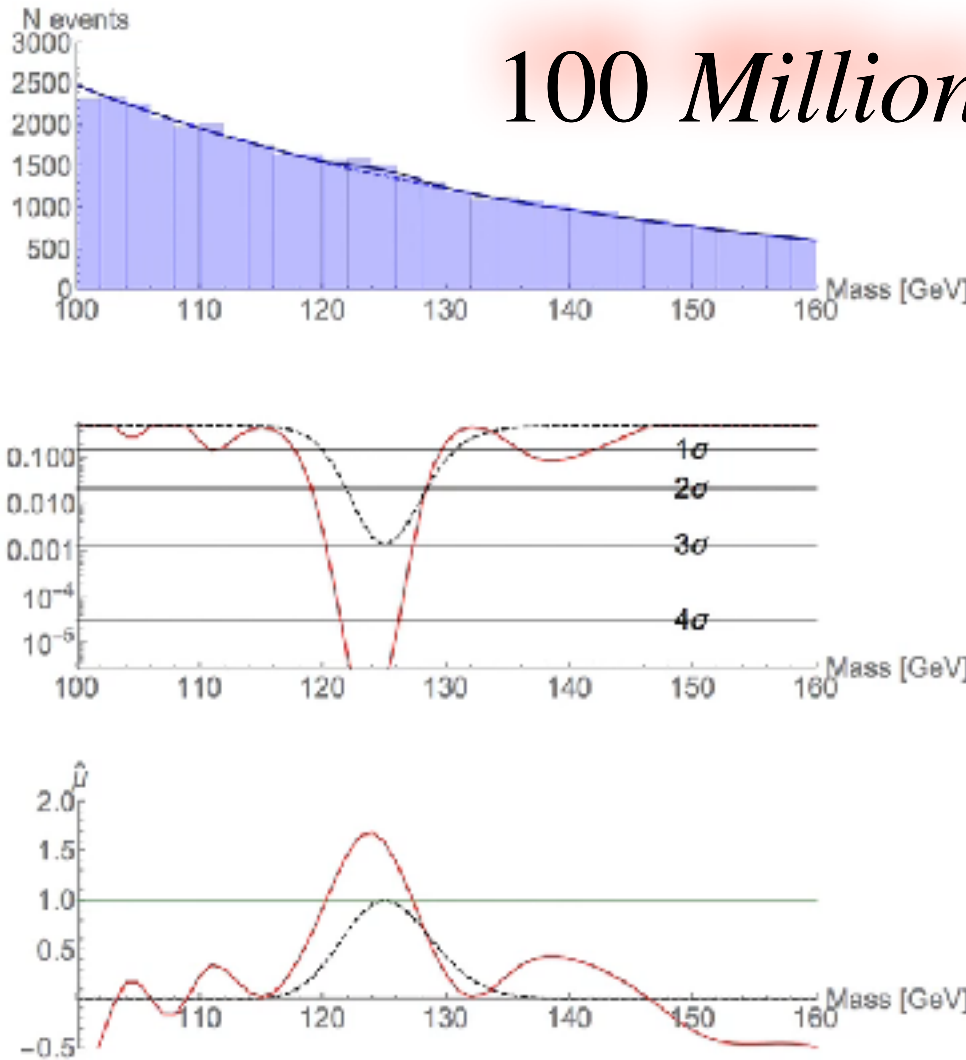
$$trial\,\# \approx 1 + \mathcal{N} \sqrt{\frac{c}{2}} \Gamma(1/2) \Rightarrow$$

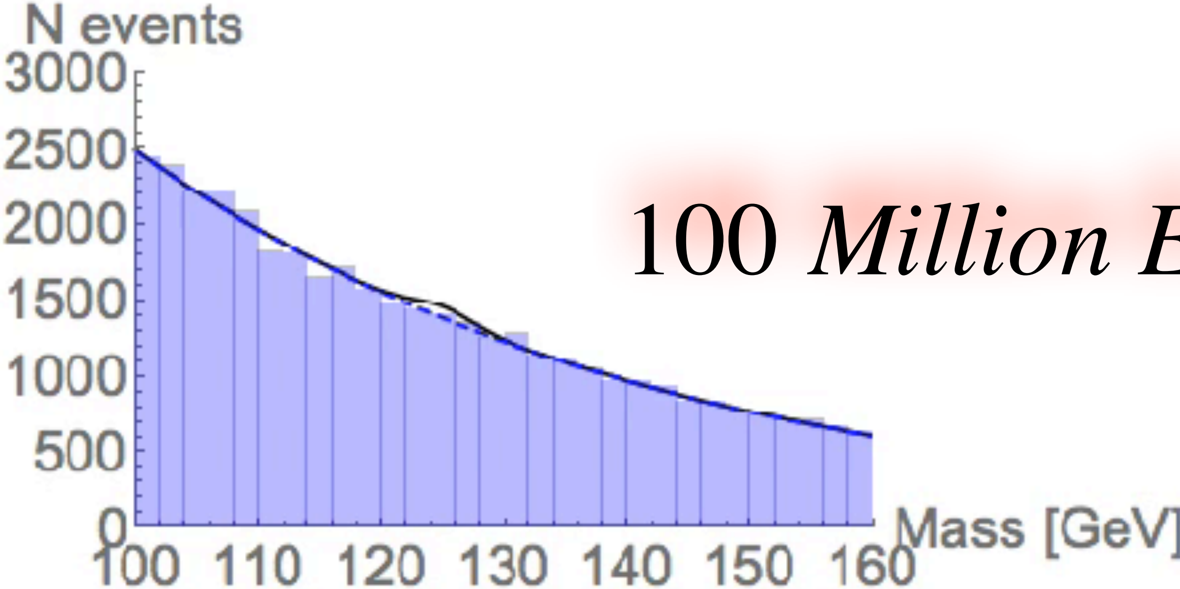
$$trial\,\# \approx 1 + \sqrt{\frac{\pi}{2}} \mathcal{N} Z_{fix}$$

What is Going On?


N events
4000
3000
2000
1000
0
80
100
120
140
160
180
200
m [GeV]

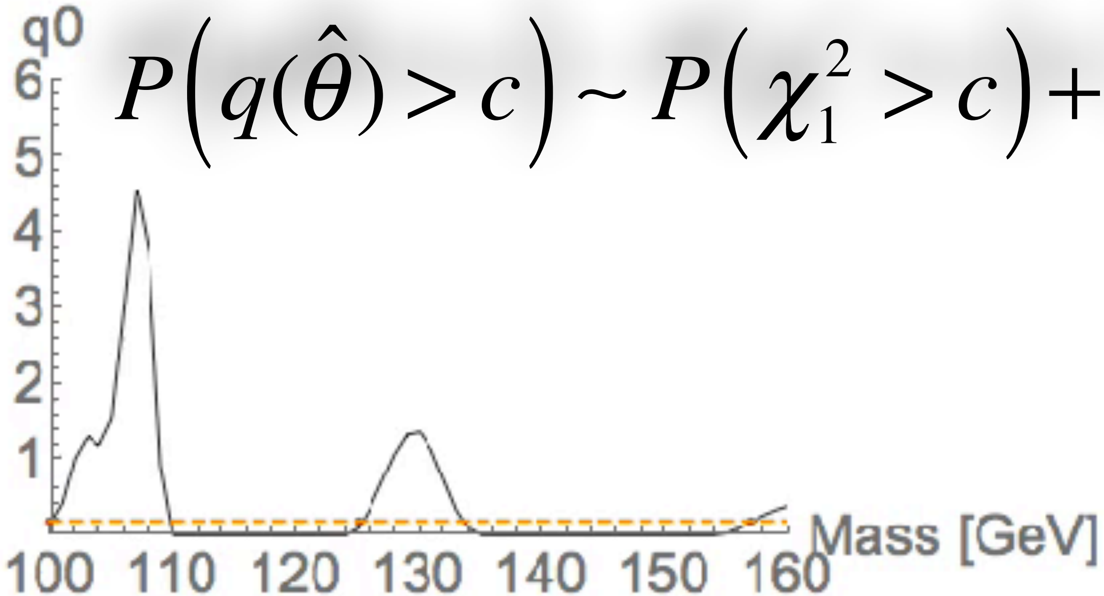
100 *Million Experiments*

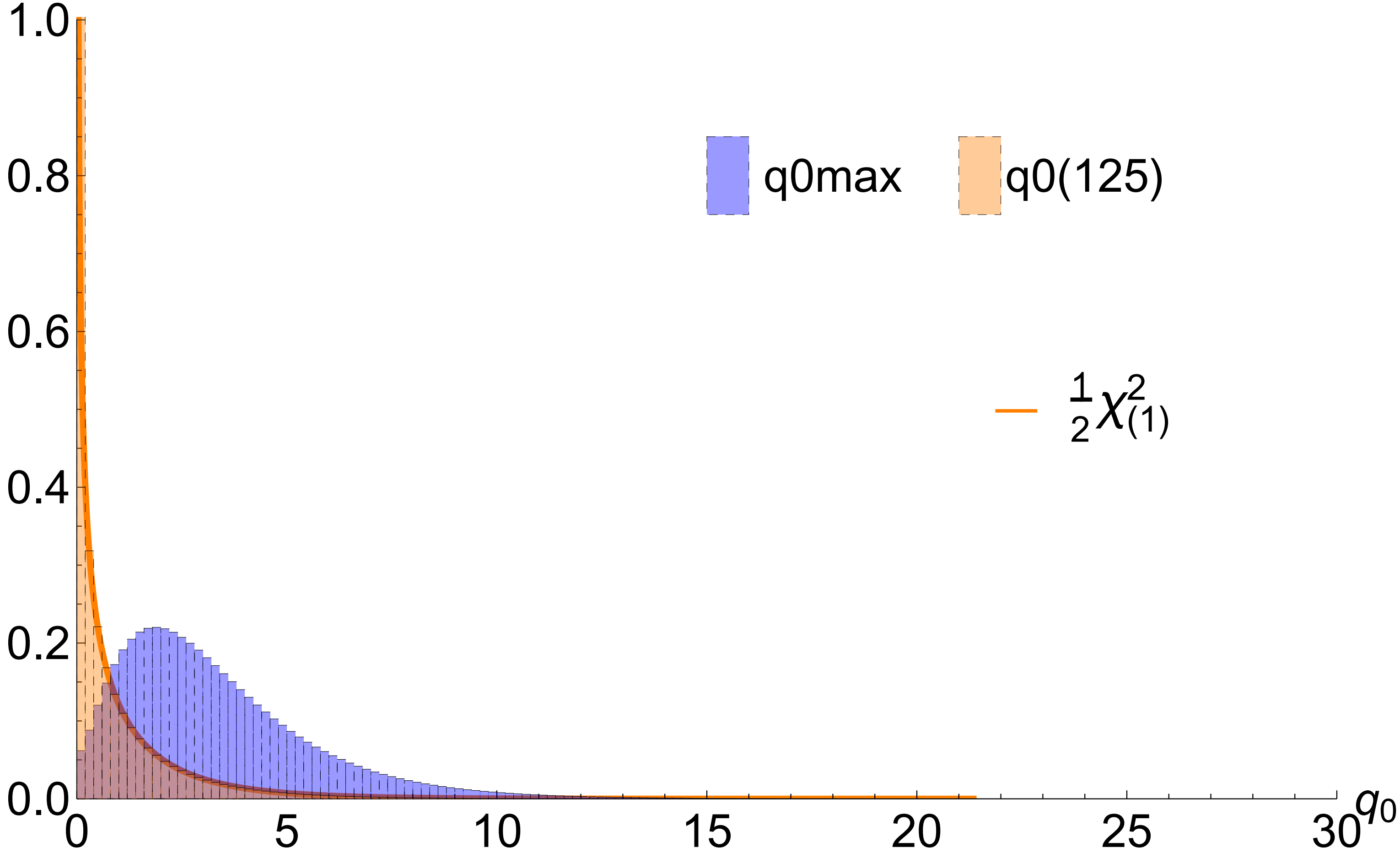




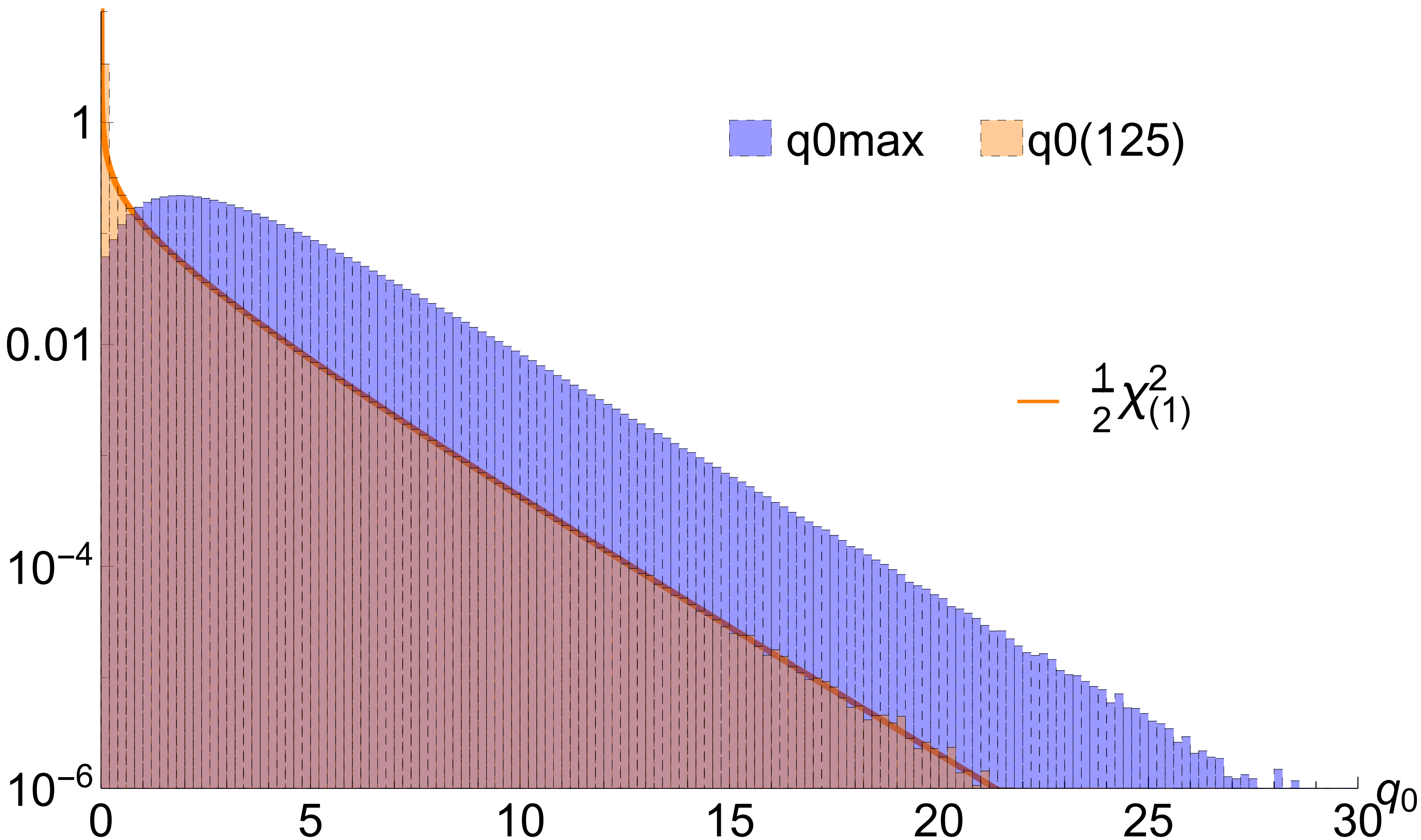
100 *Million Experiments*

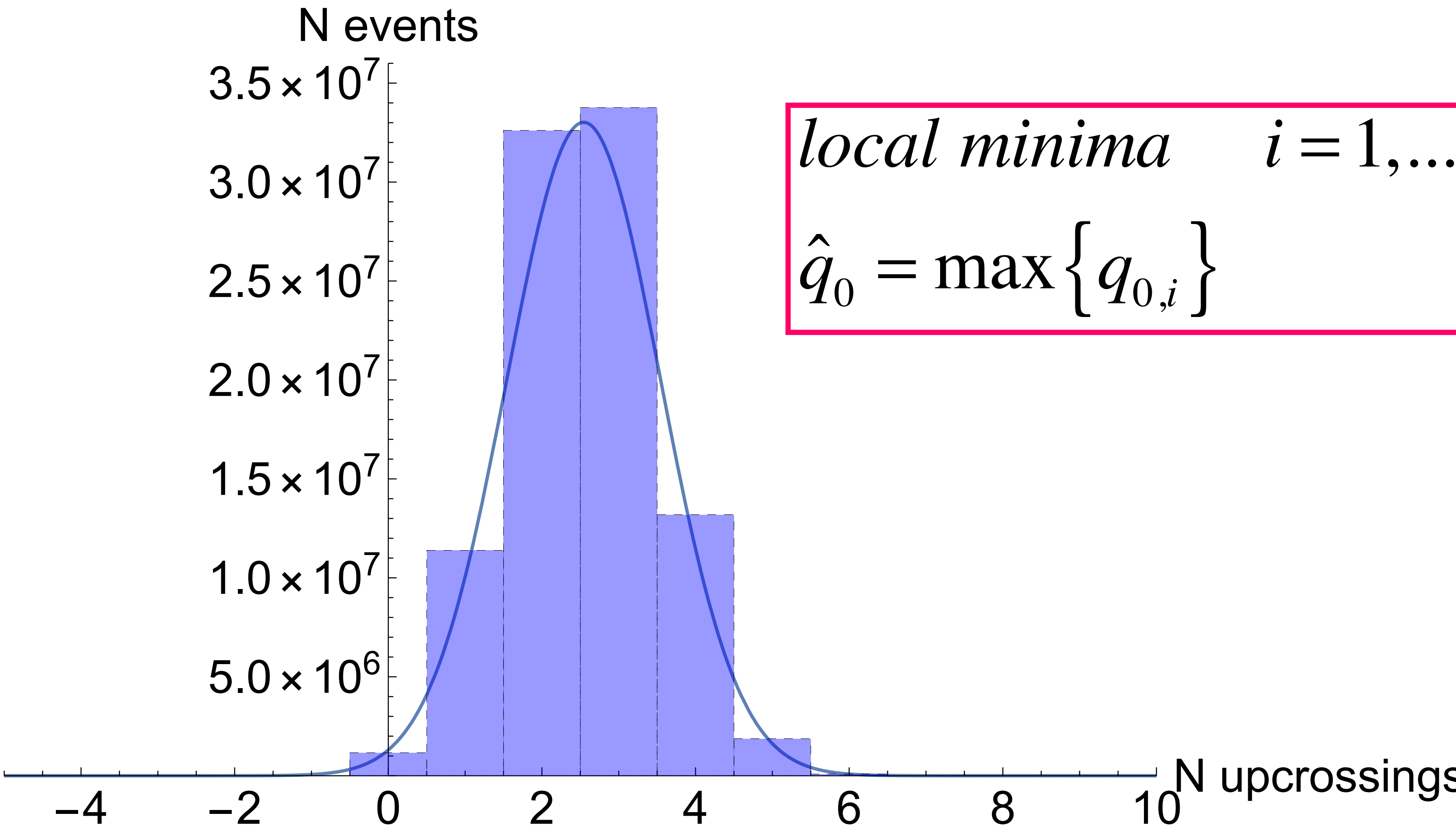
$$P\left(q(\hat{\theta}) > c\right) \sim P\left(\chi_1^2 > c\right) + \langle N(c_0)\rangle e^{-(c-c_0)/2}$$





q0max
q0(125)
$\frac{1}{2}\chi^2_{(1)}$
q_0
0.0
0.2
0.4
0.6
0.8
1.0
0
5
10
15
20
25
30


q0max
q0(125)
$\frac{1}{2}\chi^2_{(1)}$
1
0.01
10^{-4}
10^{-6}
0
5
10
15
20
25
30
q_0

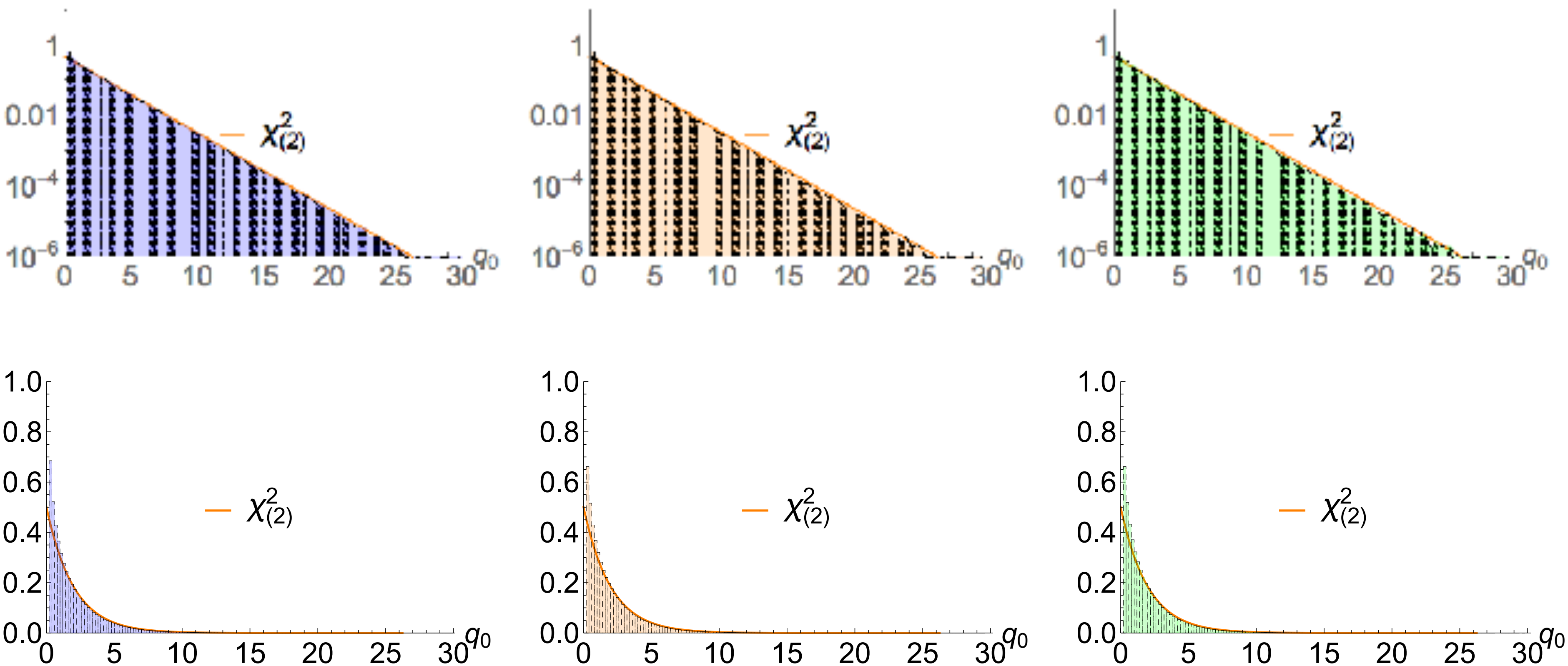


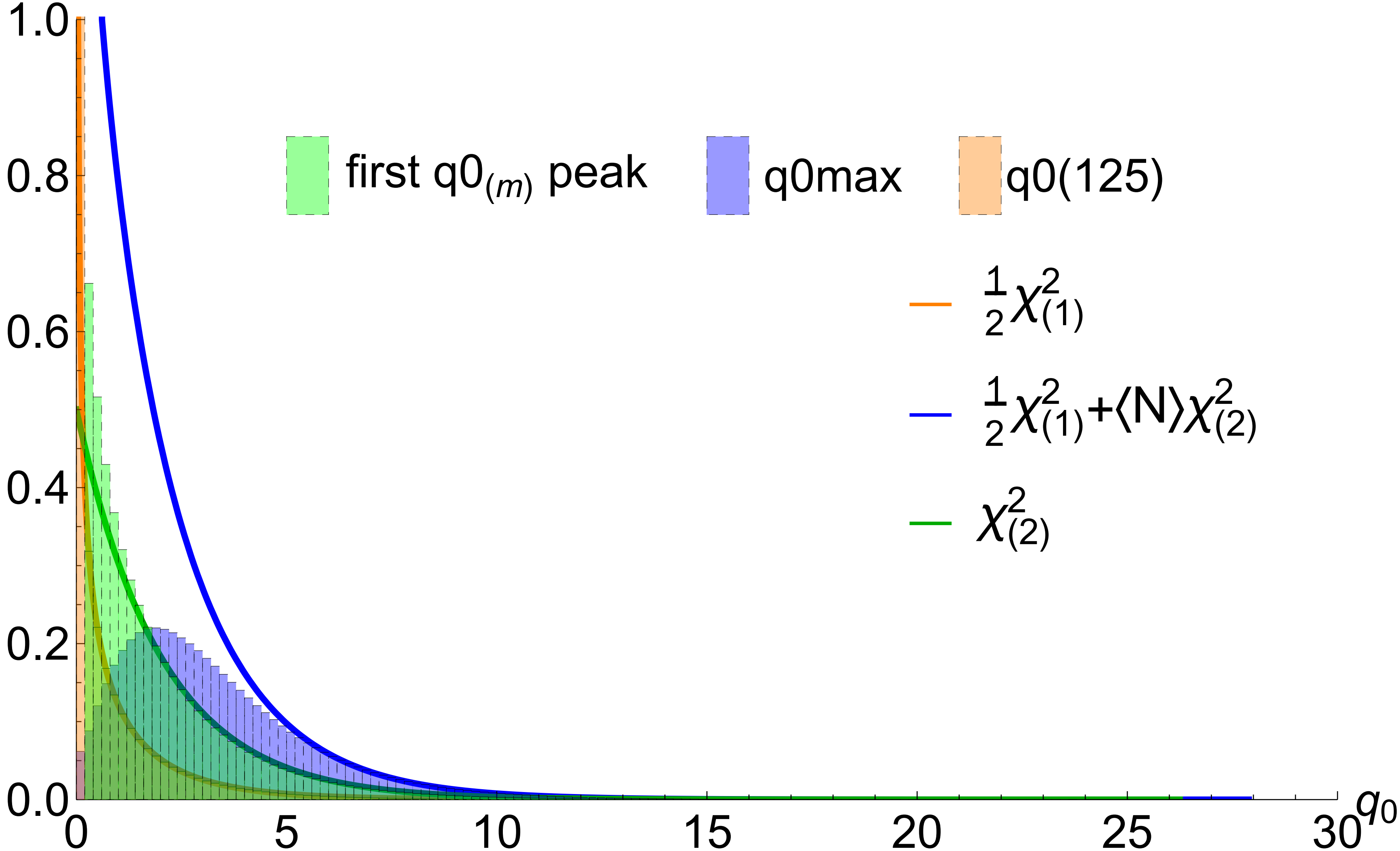
local minima $\quad i = 1, \ldots\ldots, n$

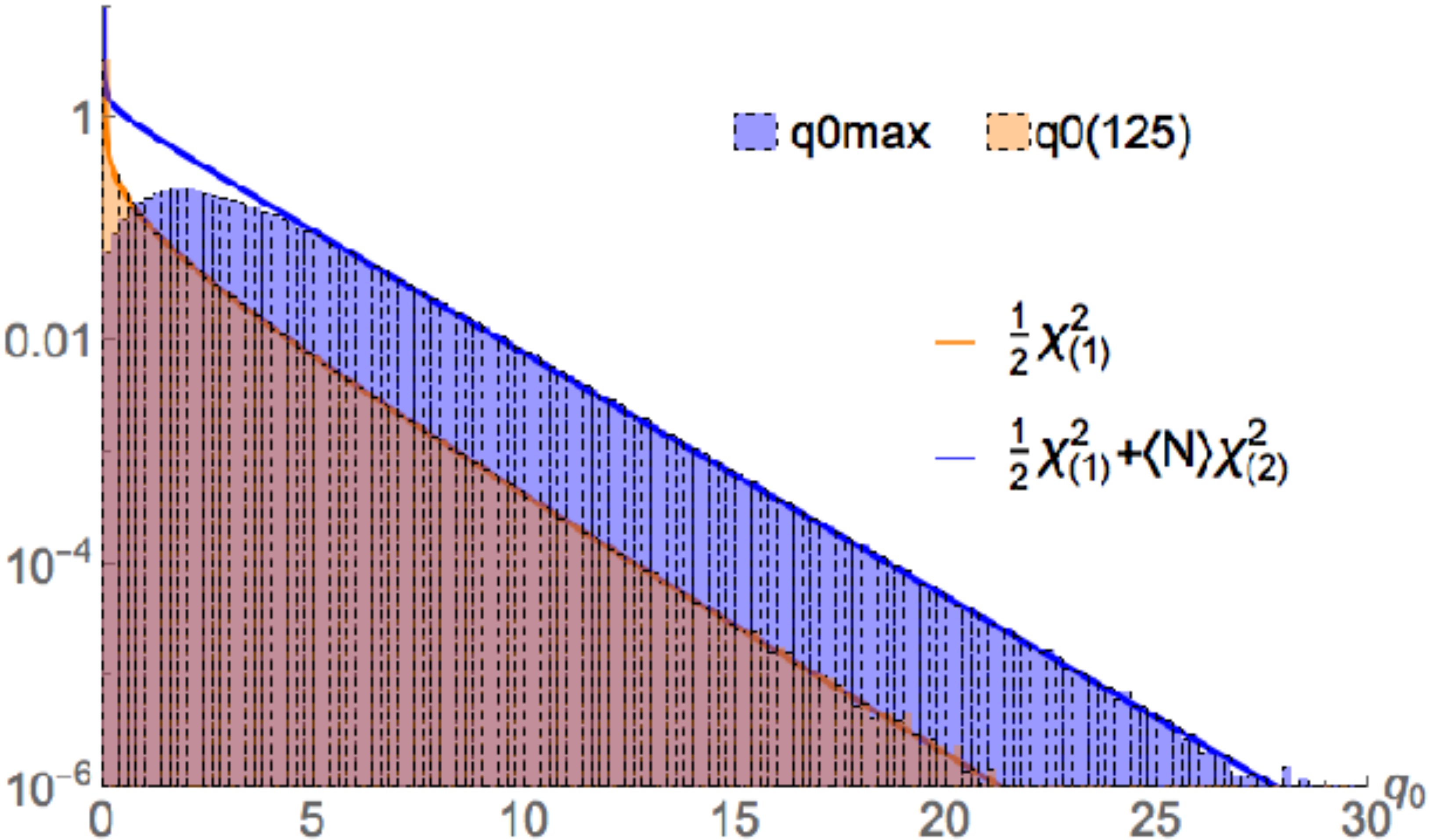
$$\hat{q}_0 = \max\left\{ q_{0,i} \right\}$$

$$\forall i \quad q_{0,i} \sim \chi^2_2$$

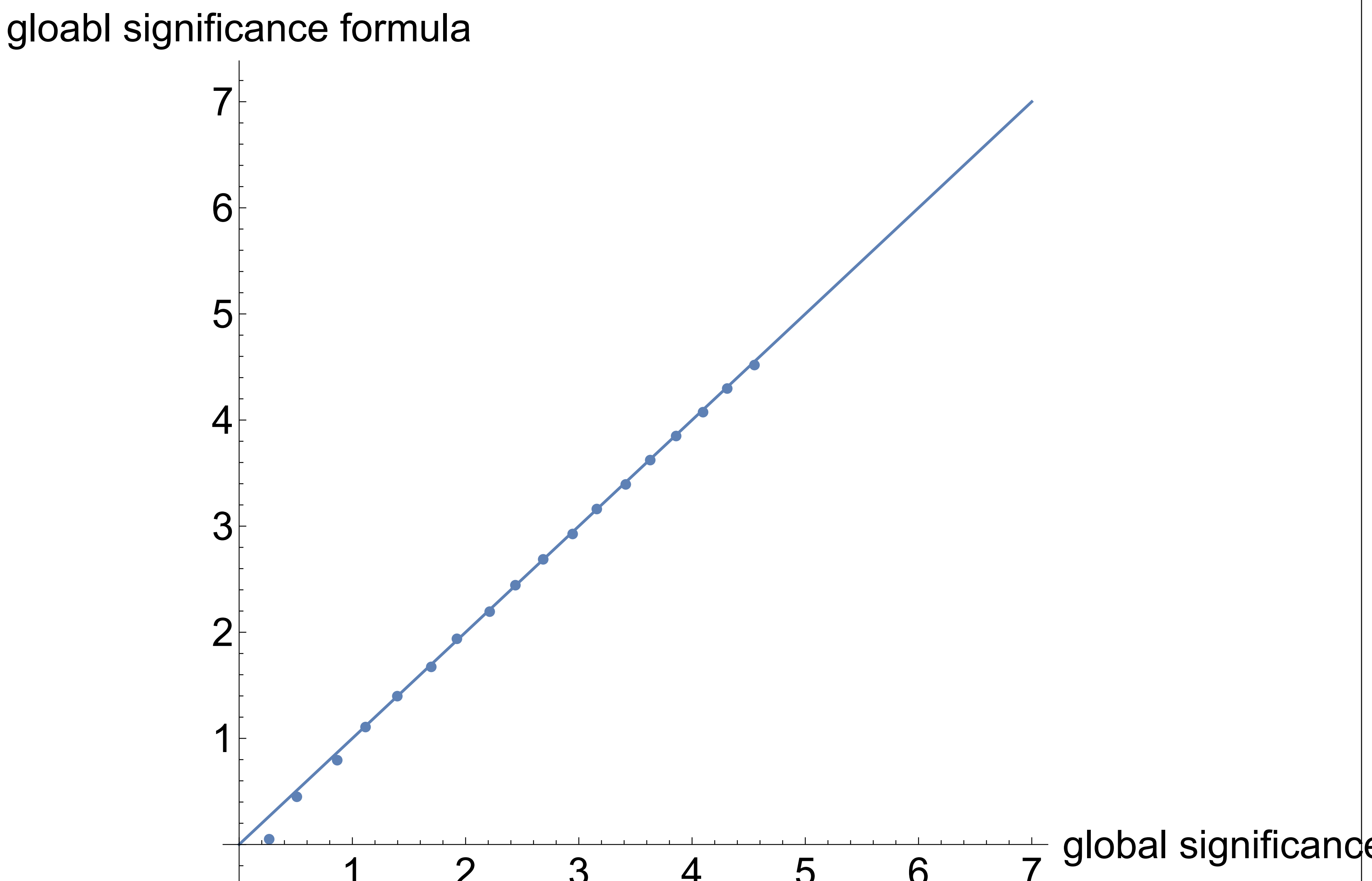
$q_{0,1}$ $\qquad$ $q_{0,2}$ $\qquad$ $q_{0,3}$




first $q0_{(m)}$ peak
q0max
q0(125)
$\frac{1}{2}\chi^2_{(1)}$
$\frac{1}{2}\chi^2_{(1)}+\langle N\rangle\chi^2_{(2)}$
$\chi^2_{(2)}$
1.0
0.8
0.6
0.4
0.2
0.0
0
5
10
15
20
25
30
q_0



q0max
q0(125)
$\frac{1}{2}\chi^2_{(1)}$
$\frac{1}{2}\chi^2_{(1)}+\langle N\rangle\chi^2_{(2)}$
1
0.01
10^{-4}
10^{-6}
0
5
10
15
20
25
30
q_0


gloabl significance formula
1
2
3
4
5
6
7
global significance
1
2
3
4
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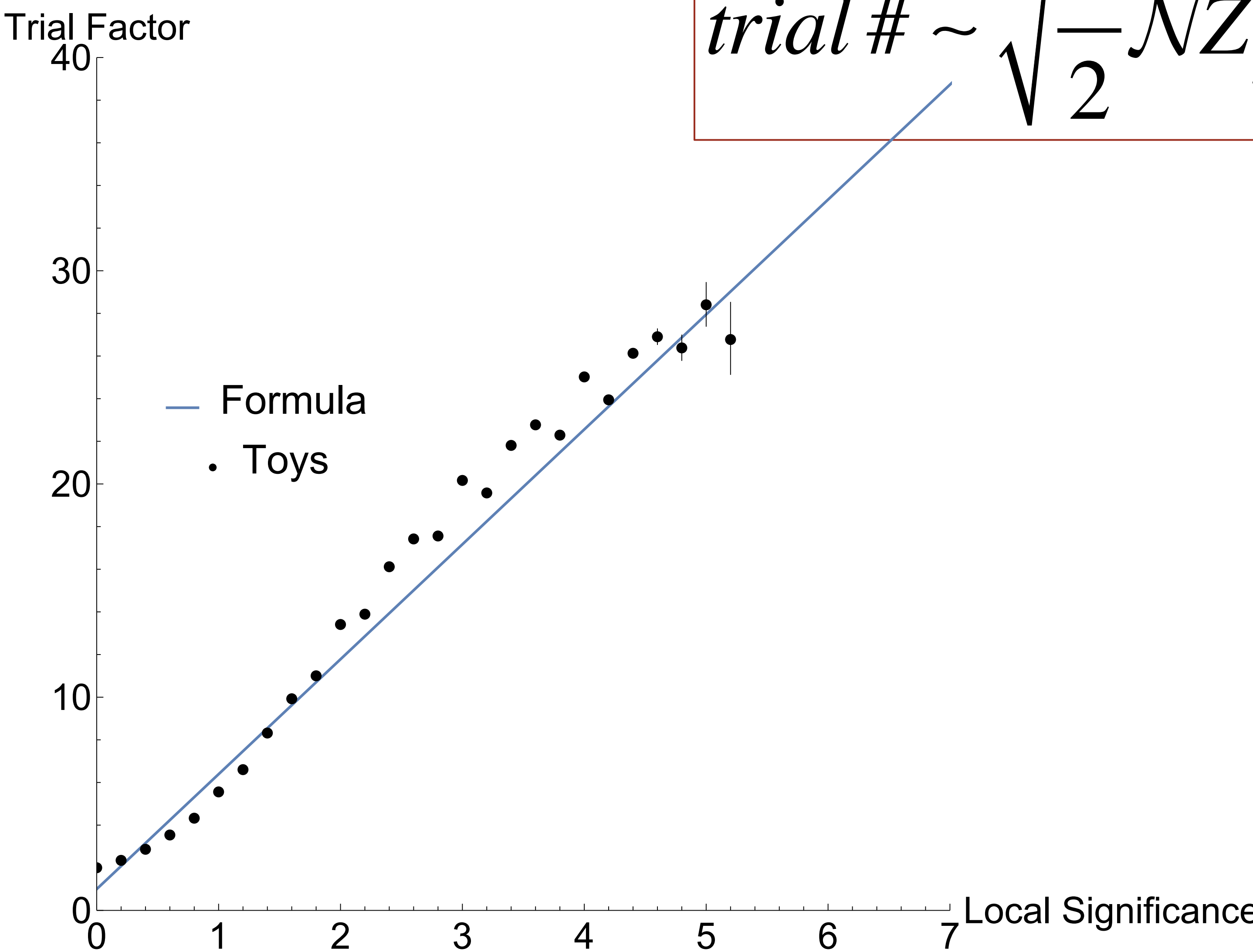
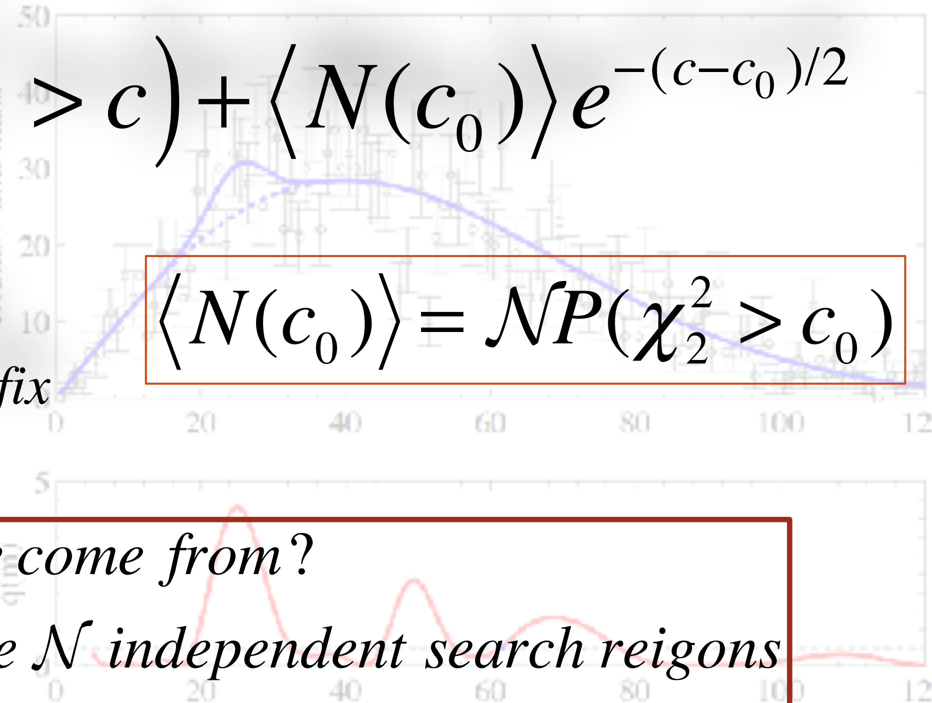
6
7

$$trial\,\# \sim \sqrt{\frac{\pi}{2}}\mathcal{N}Z_{fix}$$



Why Trial#~Z_{fix}?

Solution of the LEE problem



$$P\left(q(\hat{\theta}) > c\right) \sim P\left(\chi_1^2 > c\right) + \left\langle N(c_0)\right\rangle e^{-(c-c_0)/2}$$

$$trial\,\# \sim 1 + \sqrt{\frac{\pi}{2}}\mathcal{N} Z_{fix}$$

$$\left\langle N(c_0)\right\rangle = \mathcal{N} P(\chi_2^2 > c_0)$$

Where does the Z dependence come from?

View the results as if there are $\mathcal{N}$ independent search reigons

In each one there is a χ_2^2 distribution of $q_0(\mu, m)$

The mass is a dof even though it is undefined under the null

$$\sigma_{\hat{m}} \sim \frac{1}{Z} \Rightarrow \frac{\Delta m}{\sigma_{\hat{m}}} \sim Z$$

Why trial#~Z

$$Var(m) = \left[-E\left(\frac{\partial^2 \log \mathcal{L}}{\partial m^2} \right) \right]^{-1}$$

$$n \sim Poiss\left(\mu s(m) + b\right) \approx e^{-\left(\mu s(m)+b\right)} \left(\mu s(m) + b\right)^n$$

$$\log \mathcal{L} = -\mu s(m) - b + n \log\left(\mu s(m) + b\right)$$

$$\frac{\partial \log \mathcal{L}}{\partial m} = -\mu \frac{\partial s(m)}{\partial m} + n \frac{\mu}{\mu s(m) + b} \frac{\partial s(m)}{\partial m}$$

$$\frac{\partial^2 \log \mathcal{L}}{\partial m^2} = -\mu \frac{\partial^2 s(m)}{\partial m^2} + n \frac{\mu}{\mu s(m) + b} \frac{\partial^2 s(m)}{\partial m^2} - n \frac{\mu^2}{\left(\mu s(m) + b\right)^2} \left(\frac{\partial s(m)}{\partial m} \right)^2$$

$$E\left[n\right] = \mu s(m) + b$$

$$E\left[\frac{\partial^2 \log \mathcal{L}}{\partial m^2} \right] = -\frac{\mu^2}{\mu s(m) + b} \left(\frac{\partial s(m)}{\partial m} \right)^2$$

$$Var\left[m\right] \sim \frac{1}{\mu} \sim \frac{1}{Z} \Rightarrow \sigma_{\hat{m}} \sim \frac{1}{Z} \Rightarrow trial\# \sim \frac{range}{\sigma_{\hat{m}}} \sim Z$$

A real life example

$$P(q_0 > u) \le E[N_u] + P(q_0(0) > u)$$

$$E[N_u] = \mathcal{N}_1 e^{-u/2}$$

$$\mathcal{N}_1 \cong \langle N_{u_0} \rangle e^{u_0/2}$$

$$P(q_0 > u) = \mathcal{N}_1 e^{-u/2} + \frac{1}{2} P(\chi_1^2 > u)$$

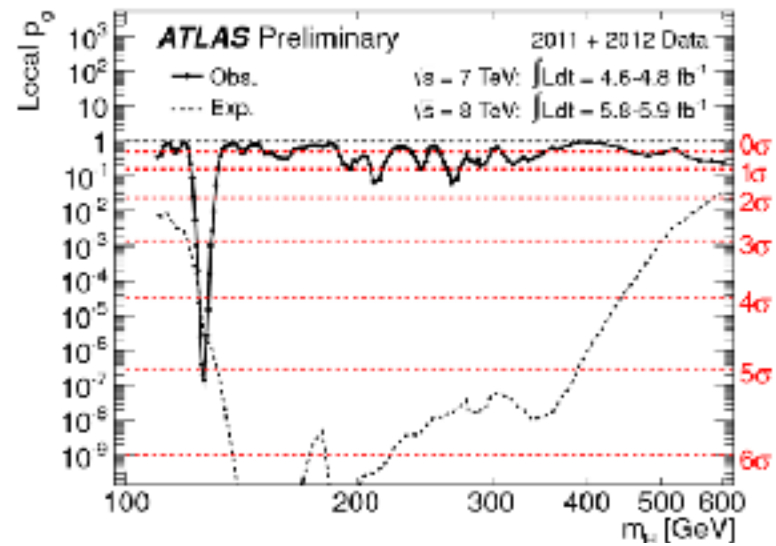
$$p_{global} = \mathcal{N}_1 e^{-u/2} + p_{local}$$

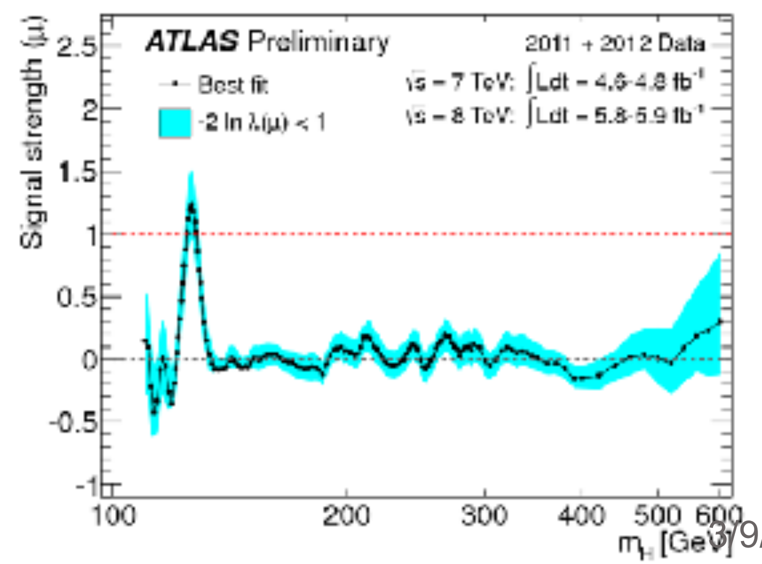
$$p_{global} = \left\langle N_{u_0} \right\rangle e^{\frac{u_0 - u}{2}} + p_{local}$$

$$N_{u_0=0} = 9 \pm 3$$

$$p_{global} = 9 \cdot e^{-25/2} + O(10^{-7}) = 3.3 \cdot 10^{-5}$$

$5\sigma \rightarrow 4\sigma$ trial#~100





Example: The 750 GeV Resonance

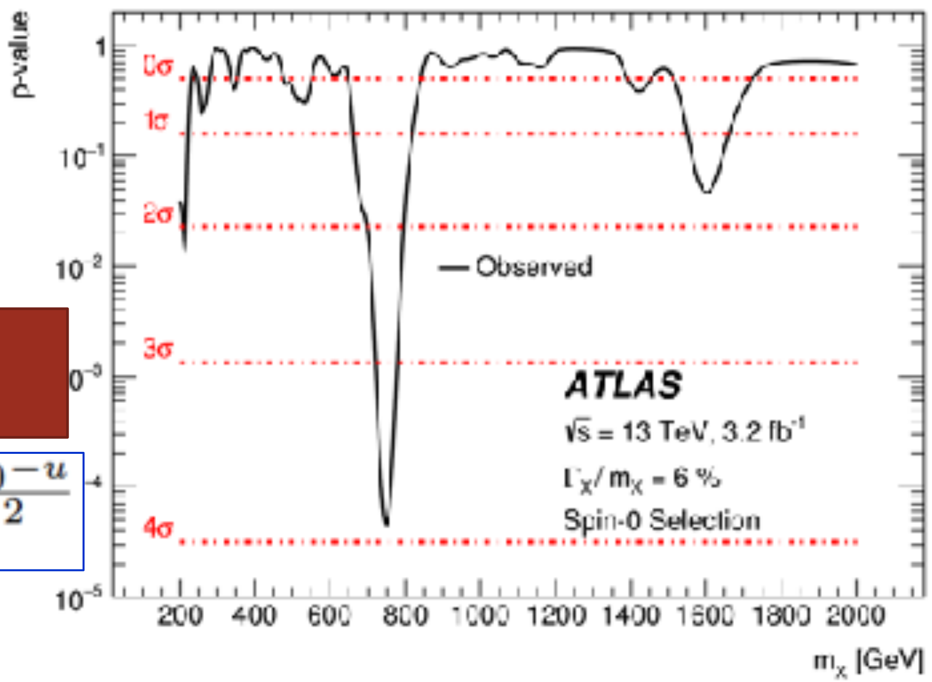
Spin 0 2015

Largest significance

$m_x \sim 750GeV, \Gamma_x \sim 45GeV(6$

Local Z = **3.9σ**

Any peak with Z>3.8σ
with m=500-2000 will draw our attention



$$P_{global}(u) \approx p_{local}(u) + E(n_{u_0})e^{\frac{u_0-u}{2}}$$

$$p_{local} = 5 \cdot 10^{-5}$$

$$u_0 = 0$$

$$n_{u_0} = 7 \pm 2.6$$

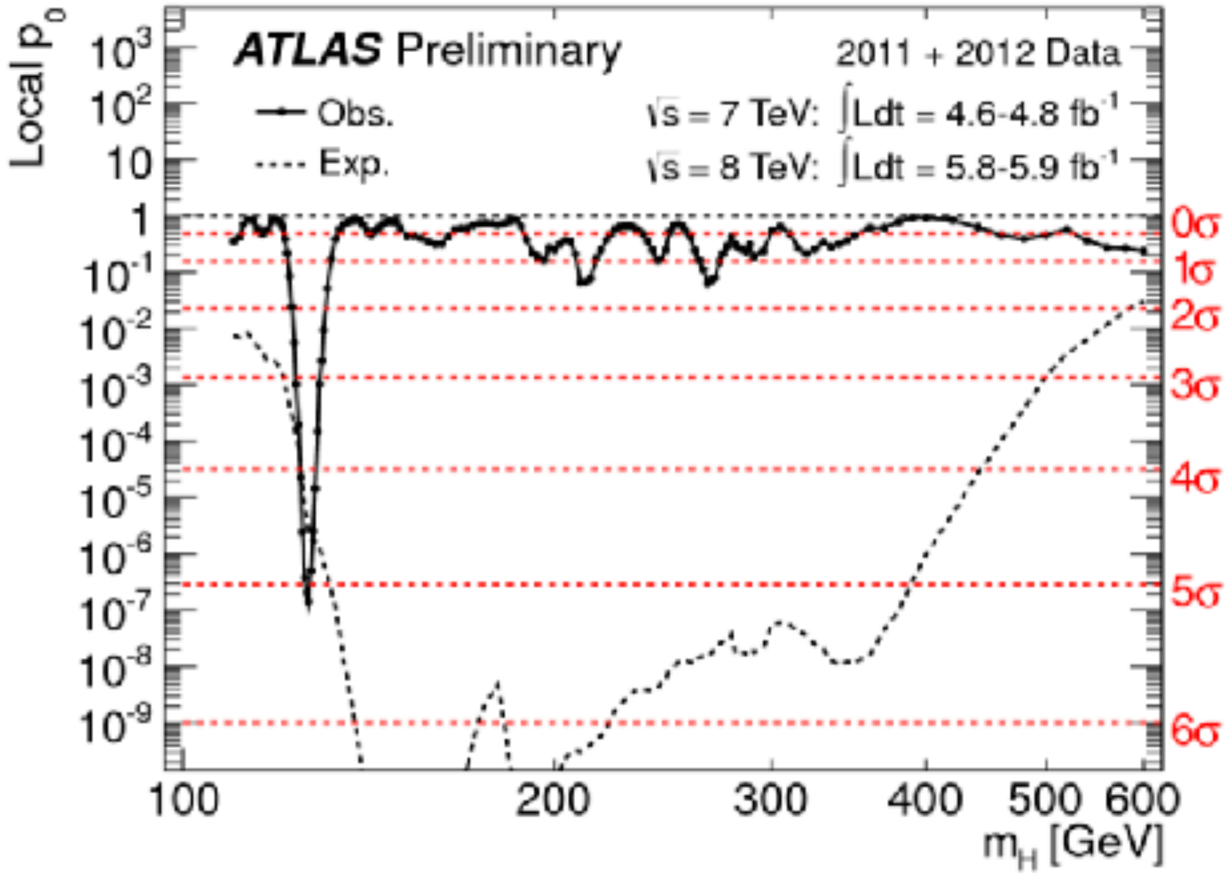
$$u = Z^2 = 3.9^2 = 15.2$$

$$p_{global} = 5 \cdot 10^{-5} + (7 \pm 2.6)e^{-15.2/2} = (2.2 - 4.8)10^{-3}$$

$$Z_{global} \sim 2.7 \pm 0.1\sigma$$

The LEE is even stronger when you consider another dimension (the width range (0-10%m) should also be taken into account)

Here us a result of a p0 scan of the measured Higgs mass



1. Can you try to estimate the signal strength at the peak (mH=125 GeV)
2.What is p0 value when Z=0 sigma?
3. What is the number of upcrossings at u=0 (with the error)?

4.remember the formulae: $p_{global} = \langle N_{u_0} \rangle e^{\frac{u_0 - u}{2}} + p_{local}$

what is p_{local}?

5.Find the global p-value

6. What is the corresponding significance taking the Look Elsewhere Effect into account?

7. Calculate the trial factor

The 2D LEE

Define the Problem

- Let $n = \mu s(m,\Gamma) + b$
- m, Γ are nuisance parameters undefined under the null hypothesis $\mu = 0$
- What is the pdf of

$$\hat{q}_0 \equiv q_0(\hat{m},\hat{\Gamma}) = -2\ln\frac{L(\mu=0)}{L(\hat{\mu},\hat{m},\hat{\Gamma})} = \max_{m,\Gamma} q_0(m,\Gamma)$$

under the null hypothesis

Define the Problem

- To generalize the problem, let Θ be the nuisance parameter, undefined under the null hypothesis, and let us try to find out the pdf of

$$\hat{q}_0 \equiv q_0(\hat{\theta}) = -2\ln\frac{L(\mu=0)}{L(\hat{\mu},\hat{\theta})} = \max_{\theta} q_0(\theta)$$

for which we want to calculate

$$p - value = P\left(\max_{\theta}\left[q_0(\theta)\right] \geq u\right), \quad u = Z^2$$

Chi Squared Random Field

- For fixed θ $\quad q_0(\theta) = -2\ln\frac{L(\mu=0)}{L(\hat{\mu},\theta)} \sim \chi_1^2$
- $q_0(\theta)$ is a chi squared random field over the space of θ
  (a random variable indexed by a continuous parameter(s))
- We are interested in

$$\hat{q}_0 \equiv q_0(\hat{\theta}) = -2\ln\frac{L(\mu=0)}{L(\hat{\mu},\hat{\theta})} = \max_{\theta} q_0(\theta)$$

for which we want to calculate

$$p-value = P\left(\max_{\theta}\left[q_0(\theta)\right] \geq u\right),\ u = Z^2$$

Chi Squared Random Field

- We are only interested in positive signals (downward fluctuations of the background are not considered as an evidence against the background)

$$q_0(\theta) = \begin{cases} -2\log\frac{L(\mu=0)}{L(\hat{\mu},\theta)} \\ 0 \end{cases}$$

$$q_0(\theta) \sim \frac{1}{2}\chi_1^2$$

[H. Chernoff, Ann. Math. Stat. 25, 573578 (1954)]

Chi Squared Random Field

- We are only interested in positive signals
(downward fluctuations of the background are not considered as an evidence against the background)

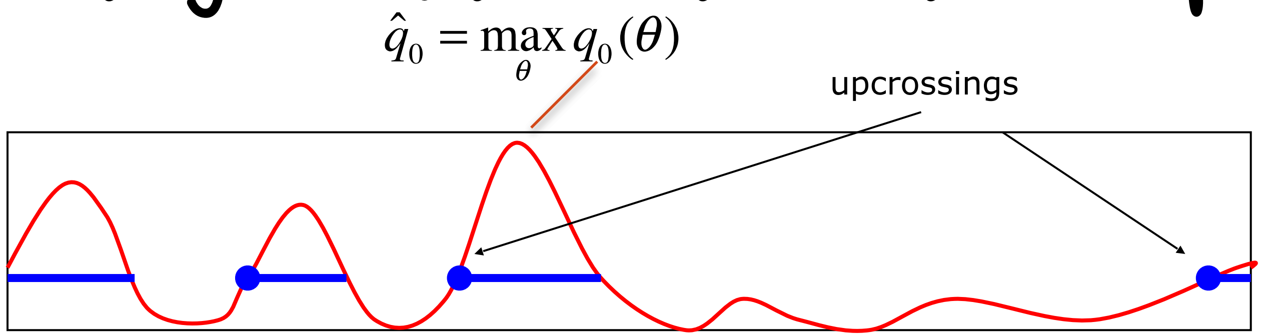
$$q_0(\theta) = \begin{cases} -2\log\dfrac{L(\mu=0)}{L(\hat{\mu},\theta)} \\ 0 \end{cases} \qquad q_0(\theta) \sim \frac{1}{2}\chi_1^2$$

[H. Chernoff, Ann. Math. Stat. 25, 573578 (1954)]

- $q_0(\theta) = \left(\dfrac{\hat{\mu}(\theta)}{\sigma}\right)^2$ $\hat{\mu}(\theta)$ is a Gaussuan Random Field over θ

1-D Random Fields

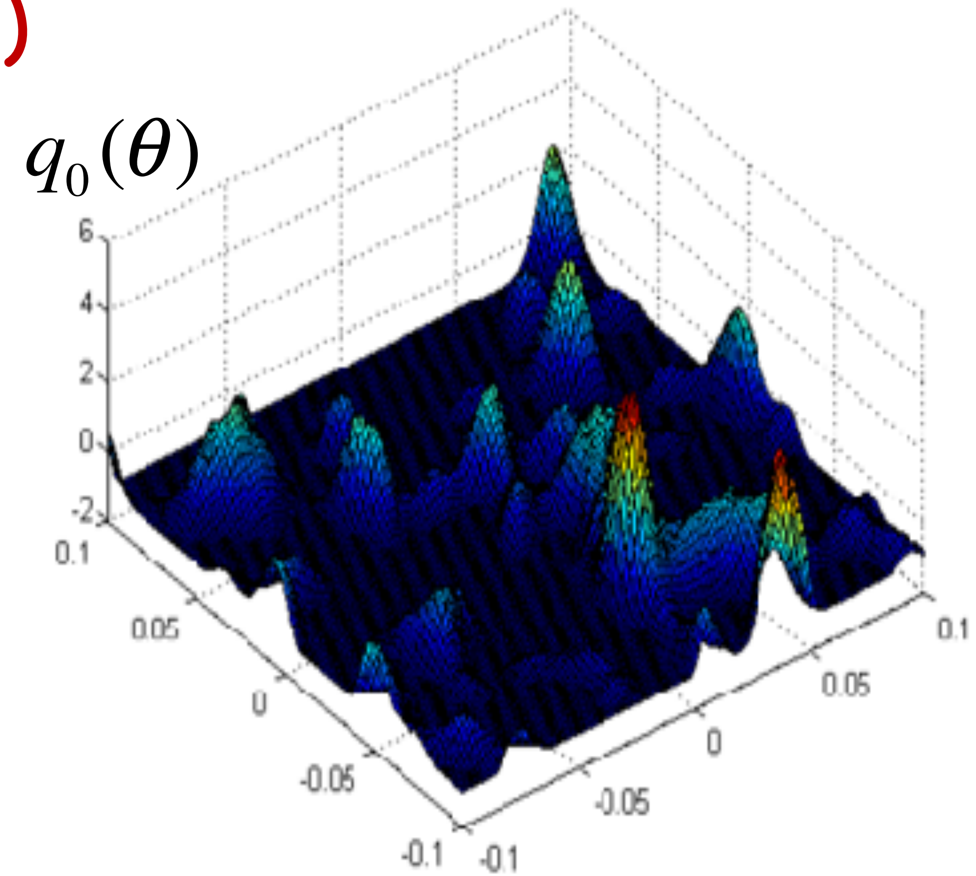
- In 1-D points where the field becomes larger than u are called upcrossings.

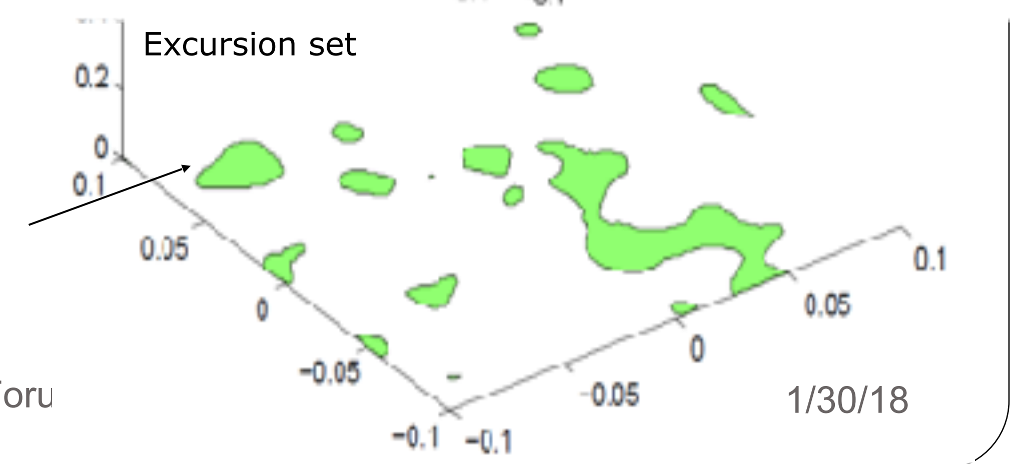


- The probability that the global maximum is above the level u is called exceedance probability. (p-value of $q_0(\hat{\theta})$) $p = P\left(\max_\theta [q_0(\theta)] \geq u\right), \quad u = Z^2$

Random fields (>1 D)

- The set of points where the field has values larger then some number u is called the **excursion set A_u above the level u.**





Random fields

- Fortunately, quite a lot of statistical literature on the properties of random fields in D- dimensions
- Applications in Cosmology, Brain mapping, Oceanography …

[3] R.J. Adler and A.M. Hasofer, *Level Crossings for Random Fields*, Ann. Probab. 4, Number 1 (1976), 1-12.

[4] R.J. Adler, *The Geometry of Random Fields*, New York (1981), Wiley, ISBN: 0471278440.

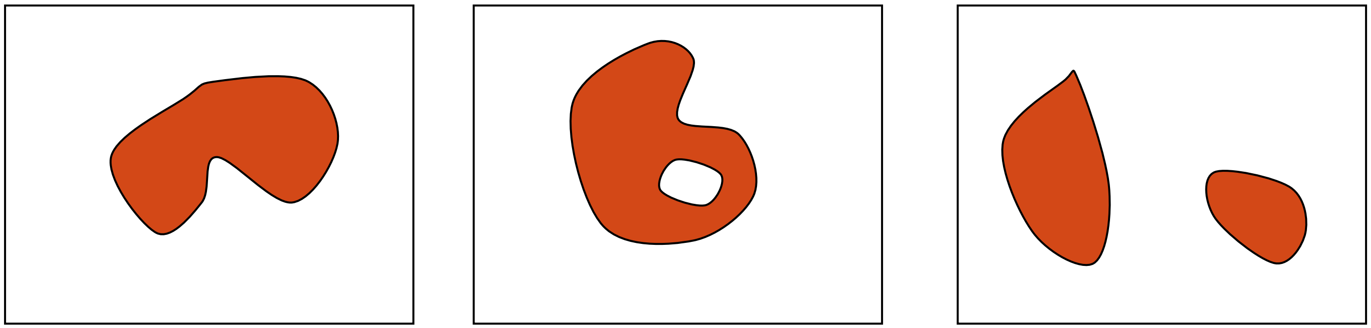
[5] K.J. Worsley, S. Marrett, P. Neelin, A.C. Vandal, K.J. Friston and A.C. Evans, *A Unified Statistical Approach for Determining Significant Signals in Location and Scale Space Images of Cerebral Activation*, Human Brain Mapping 4 (1996) 58-73.

[6] R.J. Adler and J.E. Taylor, *Random Fields and Geometry* , Springer Monographs in Mathematics (2007). ISBN: 978-0-387-48112-8.

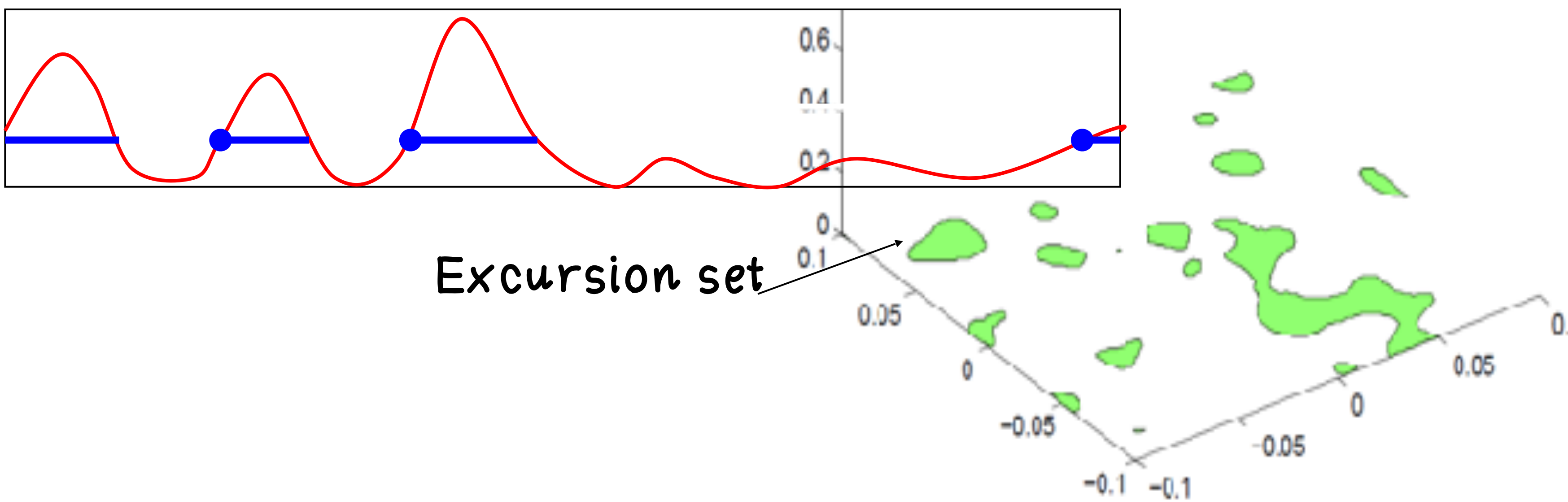
[9] J. Taylor, A. Takemura and R.J. Adler, *Validity of the expected Euler characteristic heuristic*, Ann. Probab. 33 (2005) 1362-1396.

Euler characteristic

- Number of disconnected components minus number of 'holes'



φ=1 φ=0 φ=2



The n-dimensional case

[R.J. Adler and J.E. Taylor, *Random Fields and Geometry* (2007), Springer Monographs in Mathematics]

- The upcrossings formula is a special case of a more general result which gives the expectation of the Euler characteristic of the excursion set of a random field over a general n-dimensional manifold

$$E\left[\vartheta(A_u)\right] = \sum_{d=0}^{D} \mathcal{N}_d \rho_d^s(u)$$

A_u is the excursion set of the field above a level u

(set of points where $q_0(\theta)>u$)

$\varphi(A_u)$ is it's Euler characteristic

ρ_d are 'universal' functions

(depend only on the level u and s, number of poi)

Adleret. al. Formula

$$E\left[\vartheta(A_u)\right]=\sum_{d=0}^{D}\mathcal{N}_d\rho_d^s(u)$$

n is the Dimension
(number of Nuisance parameters
undefined under the null hypothesis)

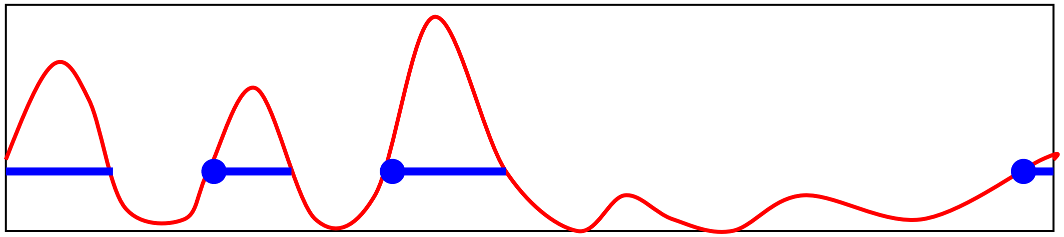
- For a Chi Squared field with
  S parameters of interest and dimension D

$D=1, s\ poi$	$D=2, s\ poi$	$D=2, s=1$
$\rho_0(u)=P(\chi_1^2>u)$	$\rho_0(u)=P(\chi_2^2>u)$	$\rho_0(u)=P(\chi_2^2>u)$
$\rho_1(u)=u^{(s-1)/2}e^{-u/2}$	$\rho_1(u)=u^{(s-1)/2}e^{-u/2}$	$\rho_1(u)=e^{-u/2}$
$-$	$\rho_2(u)=u^{(s-2)/2}\left(u-(s-1)\right)e^{-u/2}$	$\rho_2(u)=\sqrt{u}e^{-u/2}$

$$1D:\ E\left[\vartheta(A_u)\right]=\frac{1}{2}P(\chi_1^2>u)+\mathcal{N}_1e^{-u/2}$$

$$2D:E\left[\vartheta(A_u)\right]=\frac{1}{2}P(\chi_2^2>u)+\left(\mathcal{N}_1+\mathcal{N}_2\sqrt{u}\right)e^{-u/2}$$

1D Euler characteristic



$$E[\varphi(A_u)] = \sum_{d=0}^{n} N_d \rho_d(u)$$

In 1 dimension:

$$\varphi(A_u) = N_u + 1_{[q_0(0)>u]}$$

$$E\left[\varphi(A_u)\right] = E\left[N_u\right] + P(q_0(0) > u)$$

$$= N_0 P(\chi_1^2 > u) + N_1 e^{-u/2}$$

$$N_0 = \varphi(manifold) = 1$$

$$E\left[\varphi(A_u)\right] = P(\chi_1^2 > u) + N_1 e^{-u/2}$$

This is Davies Formula

The general case

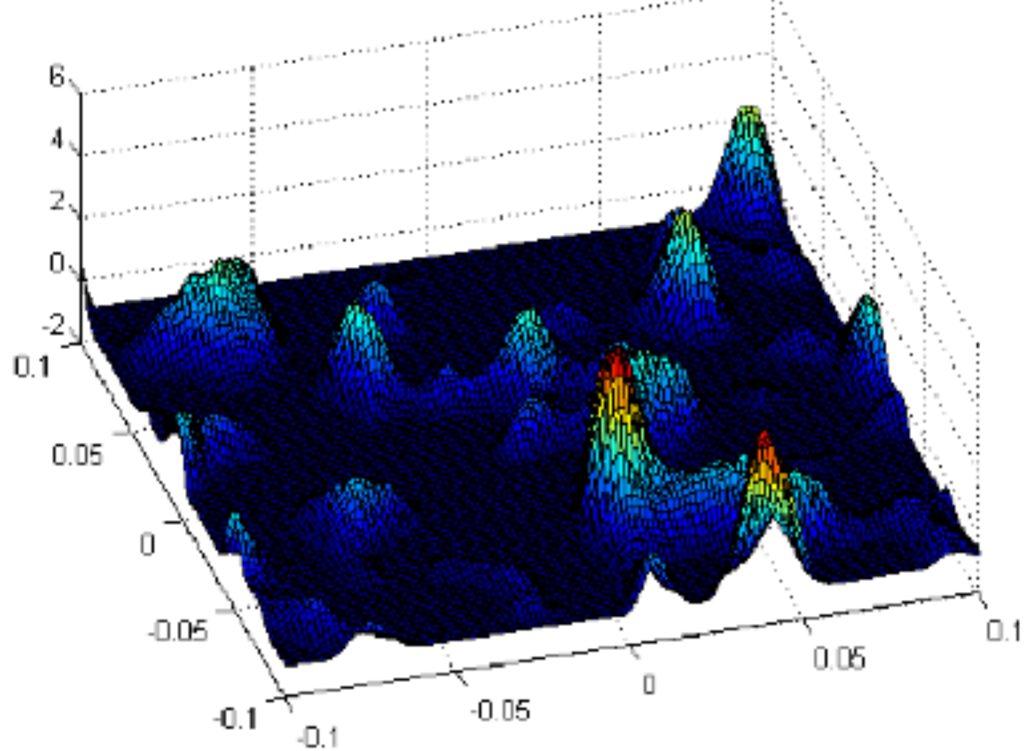
$$N_0 = \varphi(manifold)$$

$$\rho_0(u) = P(\chi_s^2 > u)$$

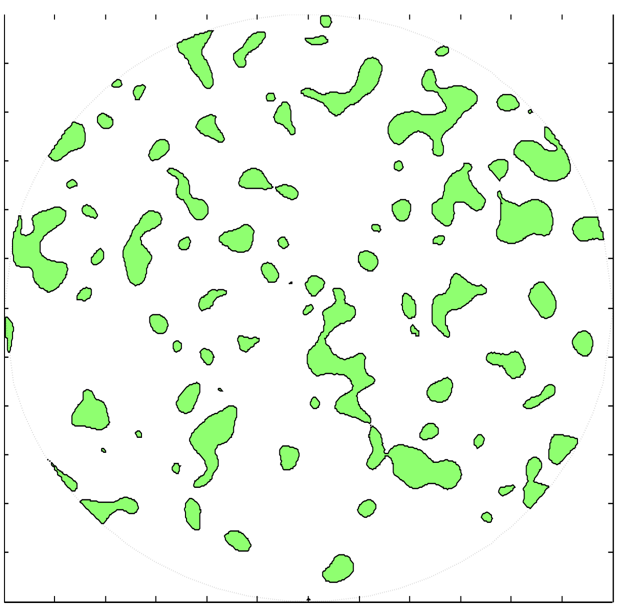
In general for high-level excursions

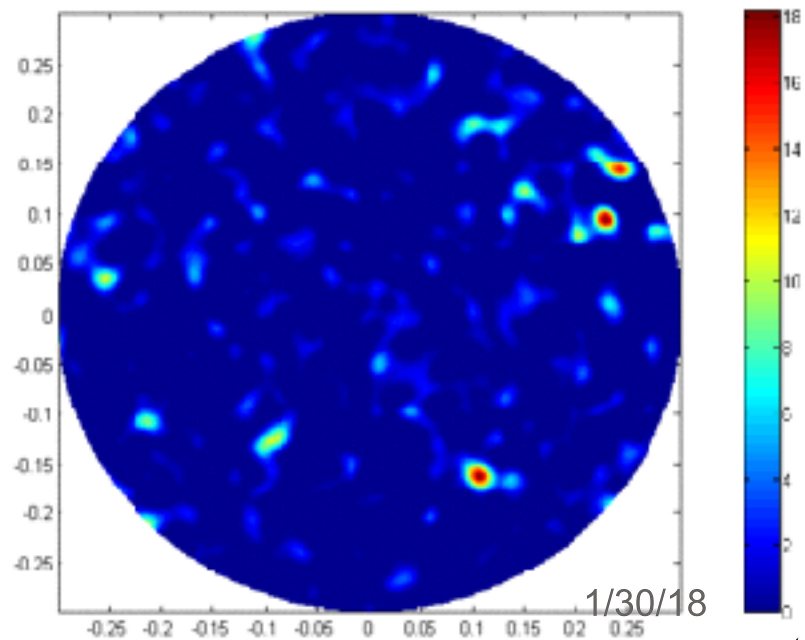
$$E\left[\varphi(A_u)\right] \xrightarrow[u \gg 1]{} P\left(\max_\theta\left[q_0(\theta)\right] \geq u\right)$$

[J. Taylor, A. Takemura, and R. J. Adler, Ann. Probab. 33, 4 (2005)]

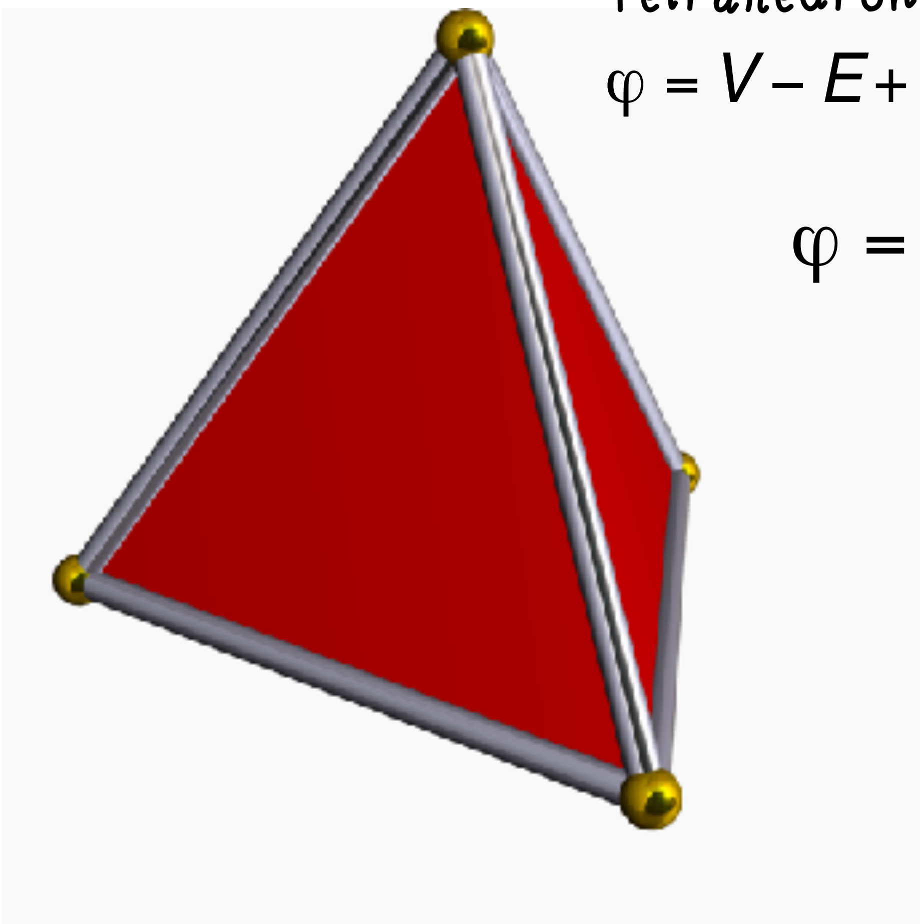


Excursion set
(u=1)





Calculation of the Euler characteristic

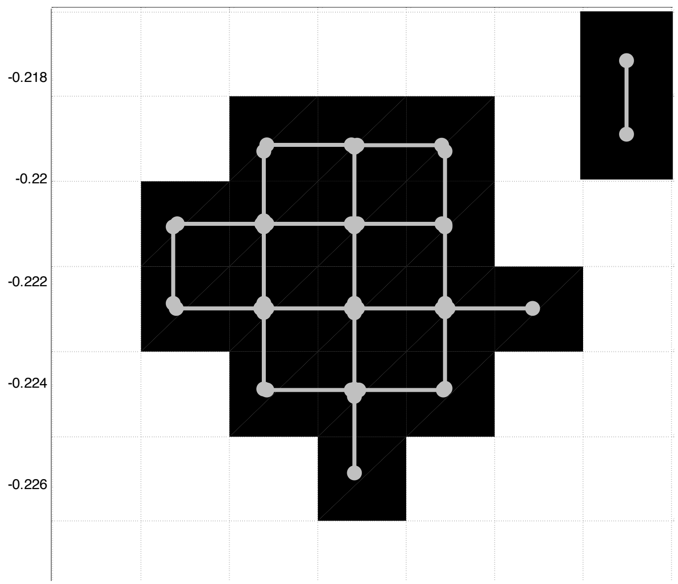


Tetrahedron

$$\varphi = V - E + F = \#vertices - \#edges + \#faces$$

$$\varphi = 4 - 6 + 4 = 2$$

Calculation of the Euler characteristic

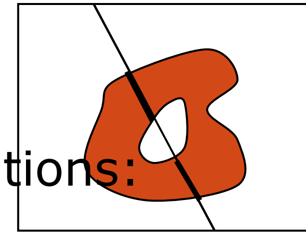


- Usually we have $q(\theta)$ calculated on a grid of points
- Calculation of the E.C. is straightforward:
- φ = #vertices - #edges + #faces
- Generalizes to higher dimensions

$$\varphi = 18(\text{points}) - 23(\text{edges}) + 7(\text{faces}) = 2$$

Slicing

- Exploit the azimuthal angle symmetry to reduce computations:



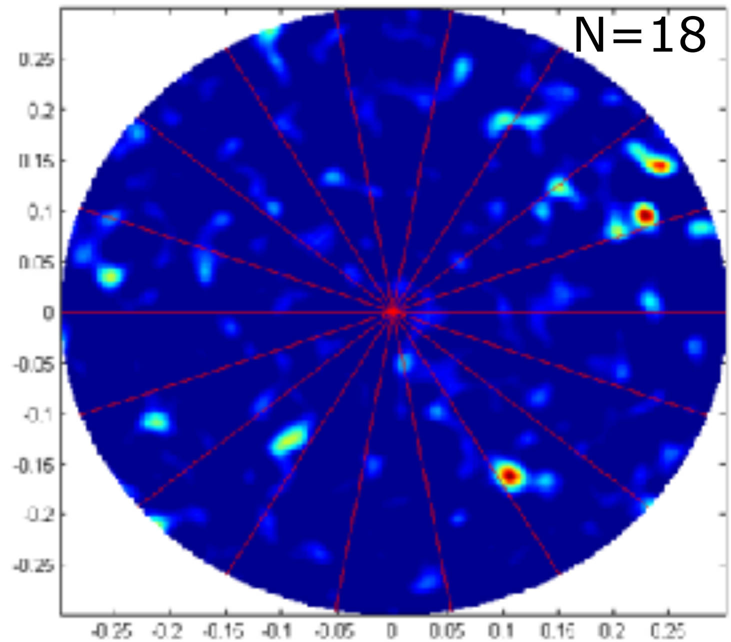
$\varphi=0=1+1-2$

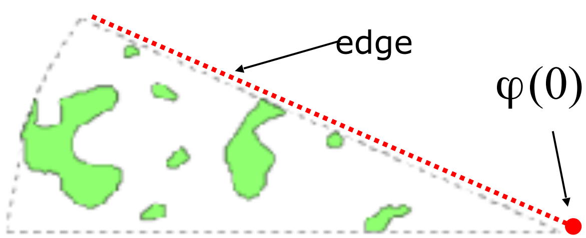
$$\varphi(A\cup B)=\varphi(A)+\varphi(B)-\varphi(A\cap B)$$

Divide to N slices:

$$\varphi=\sum_i[\varphi(\text{slice}_i)-\varphi(\text{edge}_i)]+\varphi(0)$$

$$E[\varphi]=N\times(E[\varphi(\text{slice})]-E[\varphi(\text{edge})])+\varphi(0)$$





2-d example: search for neutrino sources (IceCube)

For a chi² field in 2 dimensions:

$$E\left[\vartheta(A_u)\right] = \frac{1}{2}P(\chi^2_2 > u) + \left(\mathcal{N}_1 + \mathcal{N}_2\sqrt{u}\right)e^{-u/2}$$

Estimate E[φ] at two levels, e.g. 0 and 1, and solve for $\mathcal{N}_1$ and $\mathcal{N}_2$

From 20 bkg. Simulations:

$\langle \varphi_0 \rangle = 33.5 \pm 2$

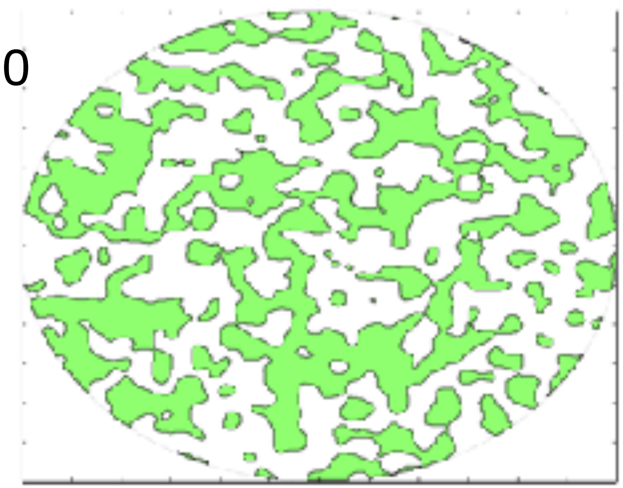
$\langle \varphi_1 \rangle = 94.6 \pm 1.3$

$\Downarrow$

$\mathcal{N}_1 = 33 \pm 2$

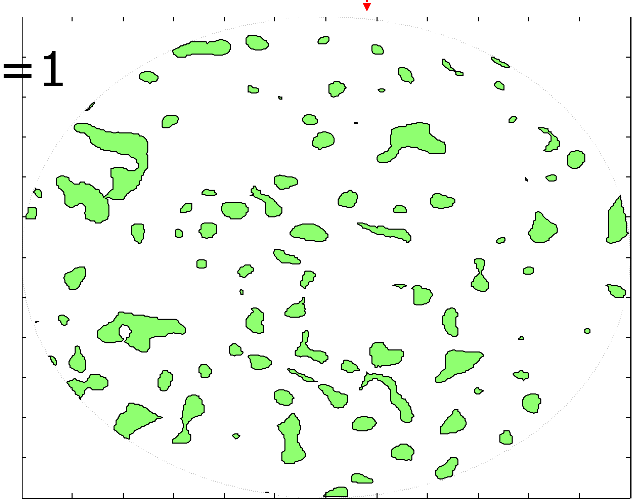
$\mathcal{N}_2 = 123 \pm 3$

u=0



φ=35

u=1



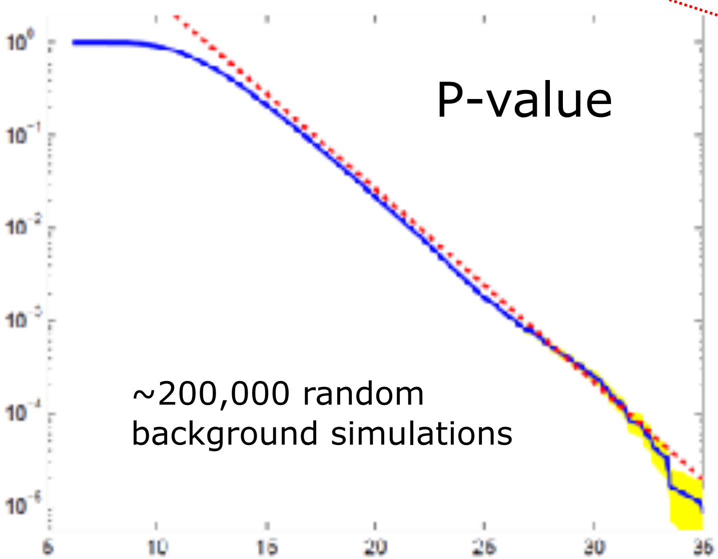
φ=95

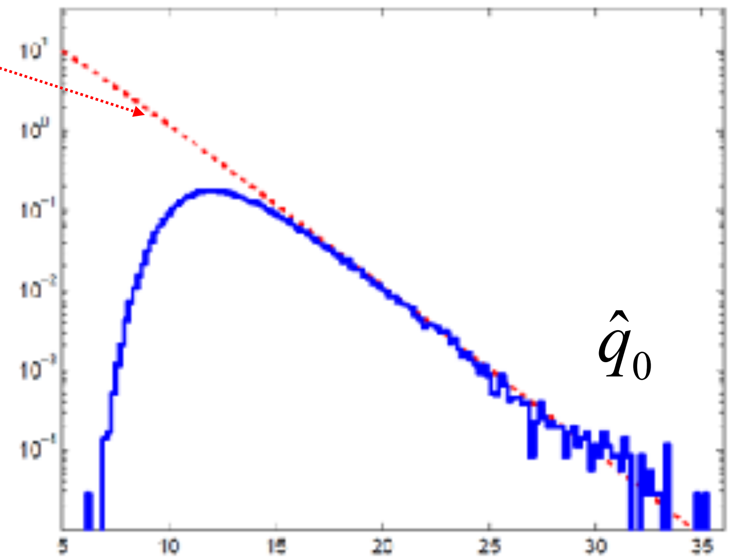
2-d example: search for neutrino sources (IceCube)

$$E\left[\vartheta(A_u)\right] = \frac{1}{2}P(\chi^2_2 > u) + \left(\mathcal{N}_1 + \mathcal{N}_2\sqrt{u}\right)e^{-u/2}$$

$\mathcal{N}_1 = 33 \pm 2$

$\mathcal{N}_2 = 123 \pm 3$





e.g.: $P(\max q_0 > 30) = (2.5 \pm 0.4)\times10^{-4}$ (estimated)

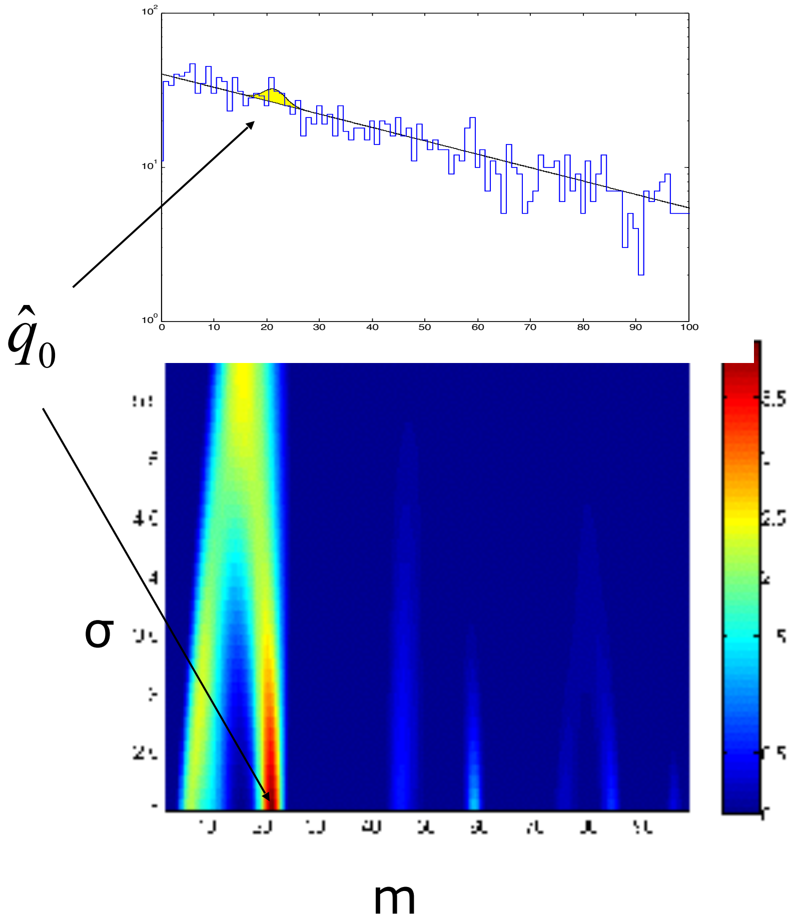
E.C. Formula : $(2.28 \pm 0.06)\times10^{-4}$

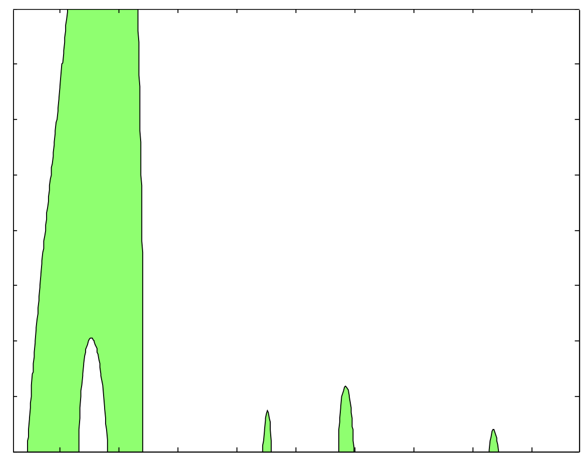
2-D exapmle #2: resonance search with unknown width

- Gaussian signal on exponential background
- Toy model : 0<m<100 ,   2<σ<6
- Unbinned likelihood:

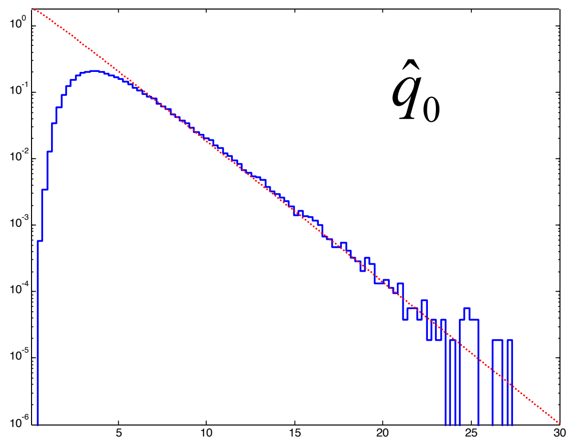
$$L = \prod_i \frac{N_s f_s(x_i) + N_b f_b(x_i)}{N_s + N_b} \times Poiss(N \mid N_s + N_b)$$

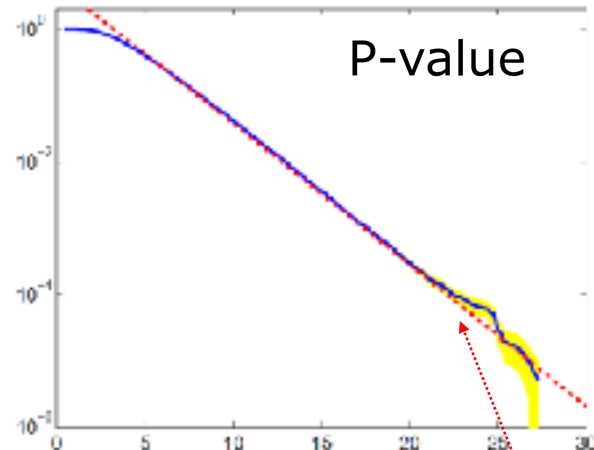
$$f_s(x;m,\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}} \qquad f_b(x) = ce^{-cx}$$





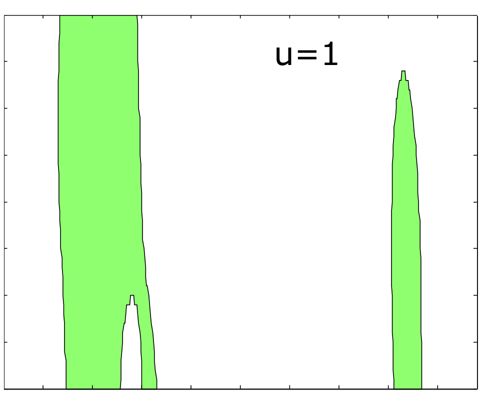
2-D exapmle #2: resonance search with unknown width



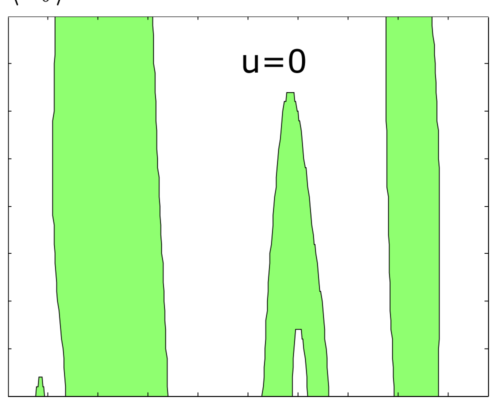


Excellent approximation above the ~2σ level

$\langle \varphi_1 \rangle = 3 \pm 0.16$



$\langle \varphi_0 \rangle = 4.5 \pm 0.2$



$$E\left[\vartheta(A_u)\right] = \frac{1}{2} P(\chi_2^2 > u) + \left(\mathcal{N}_1 + \mathcal{N}_2 \sqrt{u}\right) e^{-u/2}$$

$$\mathcal{N}_1 = 4 \pm 0.2$$

$$\mathcal{N}_2 = 0.7 \pm 0.3$$

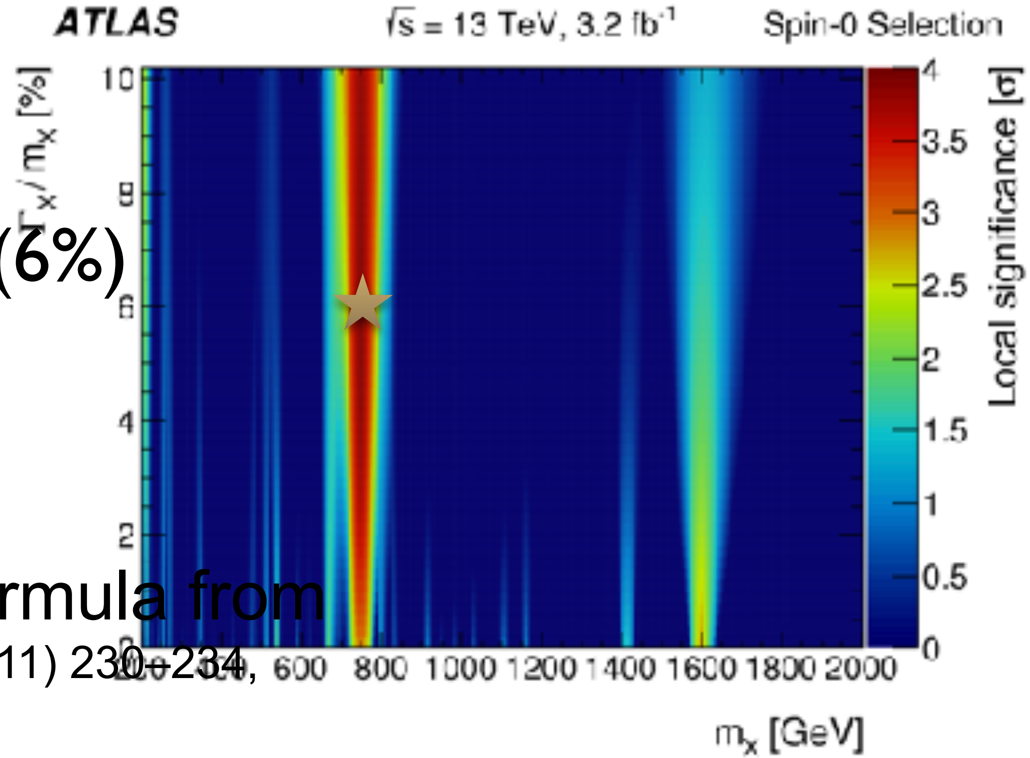
2015

2D Scan
Largest significance

$m_x \sim 750\text{GeV}, \Gamma_x \sim 45\text{GeV}(6\%)$

Local Z = **3.9σ**

m=200-2000 GeV
Γ_X/m_X=0-10%



Use toys or asymptotic formula from
O. Vitells et. al. Astropart. Phys. 35 (2011) 230-234,
arXiv:1105.4355

$$Z_{local} = 3.9\sigma$$

$$Z_{global} = 2.1\sigma$$

2.1σ is not something to write home about

Summary

$$p_{global}(s=1,D=1)\approx E\left[\vartheta(A_u)\right]=\frac{1}{2}P(\chi_1^2>u)+\mathcal{N}_1 e^{-u/2}$$

$$p_{global}(s=1,D=2)\approx E\left[\vartheta(A_u)\right]=\frac{1}{2}P(\chi_2^2>u)+\left(\mathcal{N}_1+\mathcal{N}_2\sqrt{u}\right)e^{-u/2}$$

- The procedure for estimating the p-value is simple and reliable.
- The Euler characteristic formula provides a practical way of estimating the look-elsewhere effect.
- It is easily expandable to s p.o.i and D NPs (undefined under the null hypothesis)

End of Lectures
Thank You

Eilam Gross