Quartet structure of self-conjugate nuclei

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Outline

An exact tratment of the isovector proton-neutron pairing Hamiltonian in self-conjugate nuclei

Exact approaches to the $T = 1$ pairing Hamiltonian

- R.W. Richardson, Phys. Rev. **144**, 874 (1966)
  H.-T. Chen and R.W. Richardson, Phys. Lett. B **34**, 271 (1971)
  H.-T. Chen and R.W. Richardson, Nucl. Phys. A **212**, 317 (1973)
- Feng Pan and J.P. Draayer, Phys. Rev. C **66**, 044314 (2002)
- J. Links, H.-Q. Zhou, M.D. Gould, and R.H. McKenzie, J. Phys. A **35**, 6459 (2002)
- J. Dukelsky, V.G. Gueorguiev, P. Van Isacker, S. Dimitrova, B. Errea, and S. Lerma H., Phys. Rev. Lett. **96**, 072503 (2006)

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Why a new approach?

- To provide a clearer insight into the general features of the $T = 1$ pairing Hamiltonian in an even-even $N = Z$ system
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  is quartetting present in the exact eigenstates of this Hamiltonian?

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Exact eigenstates of the like-particle pairing Hamiltonian

$$H = \sum_{i=1}^{\Omega} \epsilon_i \mathcal{N}_i - g \sum_{i,i'=1}^{\Omega} P_i^\dagger P_{i'}$$

$$\mathcal{N}_i = \sum_{\sigma} a_{i\sigma}^\dagger a_{i\sigma}, \qquad P_i^\dagger = a_{i+}^\dagger a_{i-}^\dagger, \qquad (P_i^\dagger)^\dagger = P_i$$

The eigenstates and eigenvalues

$$|\Psi\rangle = \prod_{\nu=1}^{N} B_\nu^\dagger |0\rangle, \qquad B_\nu^\dagger = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_\nu} P_k^\dagger, \qquad E^{(\Psi)} = \sum_{\nu=1}^{N} E_\nu$$

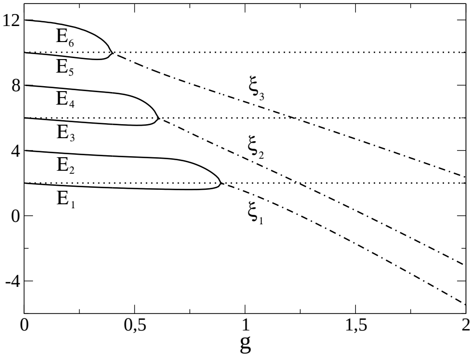
The equations

$$1 - \sum_{k=1}^{\Omega} \frac{g}{2\epsilon_k - E_\nu} + \sum_{\nu' \neq \nu}^{N} \frac{2g}{E_{\nu'} - E_\nu} = 0$$

R.W. Richardson, Phys. Lett. **3**, 277 (1963)
R.W. Richardson and N. Sherman, Nucl. Phys. **52**, 221 (1964)
R.W. Richardson, Phys. Rev. **141**, 949 (1966)

Numerical results

Pair energies for a system of
12 particles over 12 equispaced levels



M.S., PRC 75, 054314 (2007)

The $T = 1$ pairing Hamiltonian

$$H^{(iv)} = \sum_{i=1}^{\Omega} \epsilon_i \mathcal{N}_i - g \sum_{i,i'=1}^{\Omega} \sum_{\tau=-1}^{1} P^\dagger_{i\tau} P_{i'\tau}$$

$$\mathcal{N}_i = \sum_{\sigma=\mp} \sum_{\tau=-1}^{1} a^\dagger_{i\sigma\tau} a_{i\sigma\tau}, \qquad P^\dagger_{i\tau} = [a^\dagger_{i+} a^\dagger_{i-}]^{T=1}_\tau, \qquad (P^\dagger_{i\tau})^\dagger = P_{i\tau}$$

$$\tau = -1 \quad (pp), \qquad \tau = 0 \quad (pn), \qquad \tau = +1 \quad (nn)$$

First case: 2 protons and 2 neutrons

Ansatz:

$$|\Psi\rangle = [B_1^\dagger B_2^\dagger]^{T=0}|0\rangle, \qquad B_{\nu\tau}^\dagger = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_\nu} P_{k\tau}^\dagger$$

$$\begin{aligned} H^{(iv)}|\Psi\rangle &= (E_1 + E_2)|\Psi\rangle \\ &+\left(1 - g\sum_k \frac{1}{2\epsilon_k - E_1} - \frac{g}{E_2 - E_1}\right)[P^\dagger B_2^\dagger]^{T=0}|0\rangle \\ &+\left(1 - g\sum_k \frac{1}{2\epsilon_k - E_2} - \frac{g}{E_1 - E_2}\right)[P^\dagger B_1^\dagger]^{T=0}|0\rangle \end{aligned}$$

being

$$P_\tau^\dagger = \sum_k P_{k,\tau}^\dagger$$

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being

$$P_\tau^\dagger = \sum_k P_{k,\tau}^\dagger$$

Second case: 4 protons and 4 neutrons

Ansatz:

$$|\Psi\rangle = d_1[B_1^\dagger B_2^\dagger]^0[B_3^\dagger B_4^\dagger]^0|0\rangle + d_2[B_1^\dagger B_3^\dagger]^0[B_2^\dagger B_4^\dagger]^0|0\rangle + d_3[B_1^\dagger B_4^\dagger]^0[B_2^\dagger B_3^\dagger]^0|0\rangle$$

$$\begin{aligned}
H^{(iv)}|\Psi\rangle &= (E_1+E_2+E_3+E_4)|\Psi\rangle \\
&+\Big(d_1 - \sum_k \frac{g\cdot d_1}{2\epsilon_k - E_1} - \frac{g\cdot d_{123}}{E_2-E_1} - \frac{g\cdot d_{12}}{E_1-E_4} - \frac{g\cdot d_{13}}{E_1-E_3}\Big)[P^\dagger B_2^\dagger]^0[B_3^\dagger B_4^\dagger]^0|0\rangle \\
&+\Big(d_1 - \sum_k \frac{g\cdot d_1}{2\epsilon_k - E_2} - \frac{g\cdot d_{123}}{E_1-E_2} - \frac{g\cdot d_{12}}{E_2-E_3} - \frac{g\cdot d_{13}}{E_2-E_4}\Big)[P^\dagger B_1^\dagger]^0[B_3^\dagger B_4^\dagger]^0|0\rangle \\
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&+\cdots(8\ \text{similar terms})
\end{aligned}$$

being: $d_{ij} \equiv d_i + d_j$, $d_{123} \equiv d_1 + d_2 + d_3$

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Solution for 4 protons and 4 neutrons

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$$H^{(iv)}|\Psi\rangle = (E_1 + E_2 + E_3 + E_4)|\Psi\rangle + \sum_{i=1}^{12} F(i)[P^\dagger B_{i_1}^\dagger]^0[B_{i_2}^\dagger B_{i_3}^\dagger]^0|0\rangle$$

Variables:

$$E_1, E_2, E_3, E_4, d_2, d_3$$

$|\Psi\rangle$ is an eigenstate of $H^{(iv)}$ if a set of these variables exists such that

$$F(i) = 0 \qquad i = 1, 2, \cdots, 12$$

This set of variables must be such that

$$\min\left(\sum_{i=1}^{12}[F(i)]^2\right) = 0$$

This in turn implies that

$$\frac{\partial}{\partial E_\nu}\left(\sum_{i=1}^{12}[F(i)]^2\right) = 0, \qquad \frac{\partial}{\partial d_k}\left(\sum_{i=1}^{12}[F(i)]^2\right) = 0$$

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The general recipe for an even-even $N = Z$ system

- Define

$$B^\dagger_{\nu\tau} = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_\nu} P^\dagger_{k\tau}$$

- Construct the states $|s\rangle$, product of $B^\dagger_{\nu\tau}$'s arranged into $T = 0$ quartets,

$$|s\rangle = \prod_{q=1}^{N_q} [B^\dagger_{\nu(1,q,s)} B^\dagger_{\nu(2,q,s)}]^0 |0\rangle,$$

  such that $\{|s\rangle\}$ be invariant under the interchange of any two pairs.
- Expand $|\Psi\rangle$ into this basis: $\quad |\Psi\rangle = \sum_{s=1}^{N_s} d_s |s\rangle$
- Solve the set of equations

$$d_s - \sum_{k=1}^{\Omega} \frac{g \cdot d_s}{2\epsilon_k - E_\nu} - \sum_{\nu' \neq \nu}^{(1,2N_q)} \frac{g \cdot S_{\nu'\nu}(s)}{E_{\nu'} - E_\nu} = 0, \qquad S_{\nu'\nu}(s) = \sum_t I(t, \nu', \nu, s) d_t$$

  This guarantees that

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Comparison with the like-particle case

The Richardson approach for the like-particle pairing

$$|\Psi\rangle = \prod_{\nu=1}^{N} B_\nu^\dagger |0\rangle, \qquad B_\nu^\dagger = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_\nu} P_k^\dagger$$

$$1 - \sum_{k=1}^{\Omega} \frac{g}{2\epsilon_k - E_\nu} + \sum_{\nu' \neq \nu}^{N} \frac{2g}{E_{\nu'} - E_\nu} = 0$$

The present approach for the proton-neutron $T = 1$ pairing

$$|\Psi\rangle = \sum_{s=1}^{N_s} d_s \prod_{q=1}^{N_q} [B^\dagger_{\nu(1,q,s)} B^\dagger_{\nu(2,q,s)}]^0 |0\rangle, \qquad B^\dagger_{\nu\tau} = \sum_{k=1}^{\Omega} \frac{1}{2\epsilon_k - E_\nu} P^\dagger_{k\tau}$$

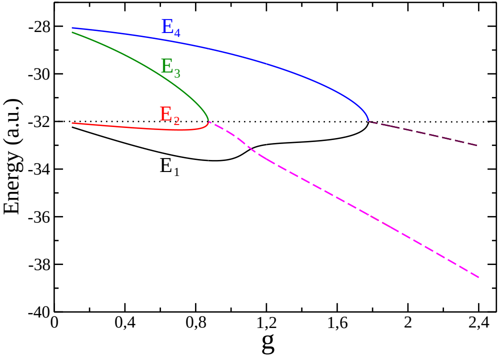
$$d_s - \sum_{k=1}^{\Omega} \frac{g \cdot d_s}{2\epsilon_k - E_\nu} - \sum_{\nu' \neq \nu}^{(1,2N_q)} \frac{g \cdot S_{\nu'\nu}(s)}{E_{\nu'} - E_\nu} = 0$$

In both cases

$$E^{(\Psi)} = \sum_{\nu} E_\nu$$

Numerical results: 2 quartets

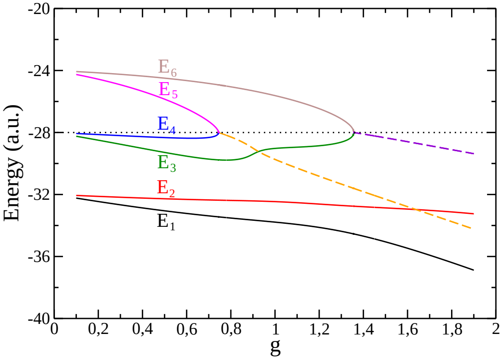
Pair energies for a system of
4 protons and 4 neutrons over 4 equispaced levels



M.S. and N. Sandulescu, in preparation

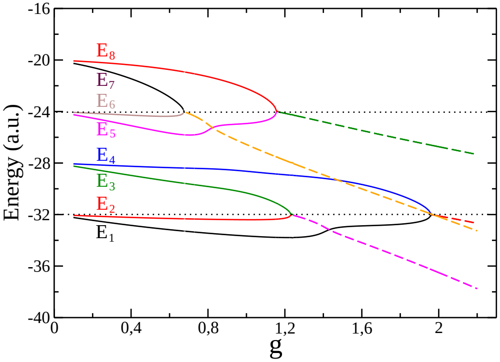
Numerical results: 3 quartets

Pair energies for a system of
6 protons and 6 neutrons over 6 equispaced levels



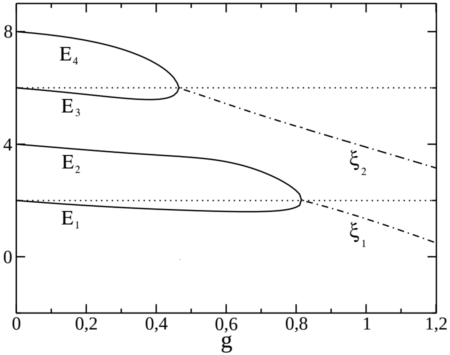
Numerical results: 4 quartets

Pair energies for a system of
8 protons and 8 neutrons over 8 equispaced levels



Richardson results for the like-particle pairing: 4 pairs

Pair energies for a system of
8 particles over 8 equispaced levels



Approximate treatments of pairing

Like-particle pairing

$$|PBCS\rangle = (B^\dagger)^N|0\rangle, \qquad B^\dagger = \sum_{k=1}^{\Omega} x(k)P_k^\dagger$$

Proton-neutron $T = 1$ pairing

$$|QCM\rangle = ([B^\dagger B^\dagger]^0)^{N/2}|0\rangle, \qquad B_\tau^\dagger = \sum_{k=1}^{\Omega} y(k)P_{k\tau}^\dagger$$

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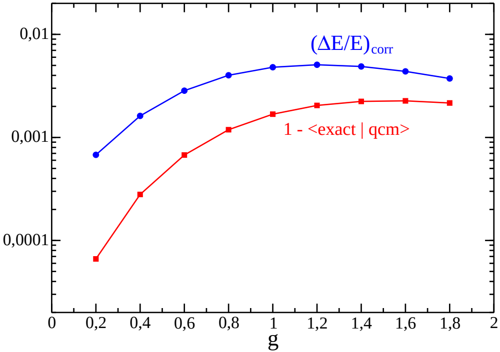
qcm vs exact results

6 protons and 6 neutrons over 6 equispaced levels



Conclusions

- We have provided an exact (and "readable") description of the exact ground state of an isovector proton-neutron pairing Hamiltonian in an even-even $N = Z$ system.
- The key role of $T = 0$ quartets in this ground state has clearly emerged.
- The analogies with the Richardson treatment of the like-particle pairing have been pointed out.