Towards an analytical threshold function of the two lepton pair production in gamma-gamma collisions

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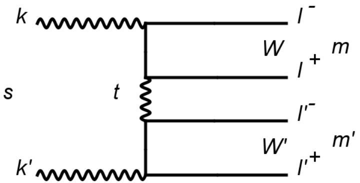
Introduction



- An analytical expression of the asymptotic cross-section ($s = \infty$), based on the 45 years old Kessler group factorization formula, using a specific parametrization, obtained by Wilfrid da Silva and I, and presented at Photon2007.
- Try now to get an analytical description for $s > 1 GeV^2$, with reasonable accuracy.
- Start with a toy model keeping $\ln * \ln$ terms.
- Use exact expressions keeping $\ln * \ln$ terms.
- Will try to conclude with QED and will not discuss consequences for the QCD $\alpha^2\alpha_s^2$ and α_s^4 Born terms.

The ingredients

Factorization formula

$$\frac{d\sigma}{dtdW^2dW'^2} = \frac{W^2W'^2}{8\pi^3s^2t^2}\left[(1+ch^2\theta)\sigma_T\sigma'_T + sh^2\theta(\sigma_T\sigma'_L + \sigma_L\sigma'_T) + ch^2\theta\sigma_L\sigma'_L\right]$$

$\gamma\gamma^* \to l\,\bar{l}$ transverse and longitudinal cross-section

$$\sigma_T = \frac{4\pi\alpha^2\beta W^2}{(W^2+t)^2}\left(\frac{3-\beta^4+2t^2/W^4}{2\beta}\ln\frac{1+\beta}{1-\beta} + 2\frac{t}{W^2} + \beta^2 - 2 - \frac{t^2}{W^4}\right)$$

$$\sigma_L = \frac{16\pi\alpha^2\beta t}{(W^2+t)^2}\left(1 - \frac{1-\beta^2}{2\beta}\ln\frac{1+\beta}{1-\beta}\right)$$

where $\beta = \sqrt{1-\frac{4m^2}{W^2}}$ and $ch\theta = 2\omega - 1$ with $\omega = \frac{st}{(W^2+t)(W'^2+t)}$

Introducing $x = \frac{t}{W^2+t}$ and $x_0 = \frac{t}{4m^2+t}$ and x', x'_0 similarly

Kinematical boundaries

Given by $x_0x'_0 \geqslant xx' \geqslant \frac{t}{s}$ implying $w = \frac{st}{(4m^2+t)(4m'^2+t)} \geqslant 1$

A toy model

Modulo a factor $f(\omega) = (1 - \frac{1}{\omega} + \frac{1}{2\omega^2})$, when $\omega \to \infty$ it can be interpreted in terms of a central exchanged photon probing the QED partonic content of the two real photons :

QED parton model

$$\frac{d\sigma}{dtdxdx'} = 2\frac{4\pi\alpha^2}{t^2}\left[l(x,t) + \bar{l}(x,t)\right]\left[l'(x',t) + \bar{l}'(x',t)\right]$$

- $l(x,t) = \frac{\alpha}{2\pi}\{[x^2 + (1-x)^2]2\chi + (1-x)^2(1 - th^4\chi)\chi - 4x(1-x)(1 - th^2\chi)\chi$
  $-[1 - 8x(1-x)]th\chi - (1-x)^2(1 - th^2\chi)th\chi\}$
- and $\chi(\beta) = \frac{1}{2}\ln\frac{1+\beta}{1-\beta}$ with $\beta = \sqrt{1 - \frac{4m^2}{W^2}}$

Keep only $\chi(\beta)\chi(\beta')$ terms, under the assumption of $t \ll 4m^2$, $W^2 \gg 4m^2$ and $W'^2 \gg 4m'^2$.

Toy model

- $l(x,t) = \frac{\alpha}{2\pi} ln\frac{W^2+t}{4m^2+t} = \frac{\alpha}{2\pi} ln\frac{x_0}{x}$ introducing $v = \frac{x}{x_0}$ and remembering $w = \frac{s}{t}x_0x'_0$
- compute $\frac{d\sigma}{dt} = \frac{8\alpha^4}{\pi st}\int_{\frac{1}{w}}^{1} dv\frac{\ln v}{v}\int_1^{wu} d\omega \ln\frac{\omega}{wu} = \frac{8\alpha^4}{\pi st}F(w)$
- with $F(w) = w - 1 - \ln w - \frac{1}{2}\ln^2 w - \frac{1}{6}\ln^3 w = wG(w)$
- keeping $f(\omega)$, we get $G_f = 1 - \frac{3}{2\omega} + \frac{1}{2\omega^2} - \frac{1}{\omega}\left[\frac{1}{2}\ln\omega + \frac{3}{4\ln^2\omega} + \frac{1}{12}\ln^3\omega + \frac{1}{24}\ln^4\omega\right]$

Toy model results

Kinematics and notations

- $t_0 = 4mm'$, $s_0 = 4(m+m')^2$ and $s_1 = 4(m-m')^2$.
- Then for $w = 1$, $t_\pm = t_0 e^{\pm 2\zeta}$ with $th\zeta = \sqrt{\frac{s-s_0}{s-s_1}}$.
- Or with $u = th\zeta$ we have $w(t) = \frac{s(1-u^2)}{s_0 - s_1 u^2}$ with $1 \leq w \leq \frac{s}{s_0}$.

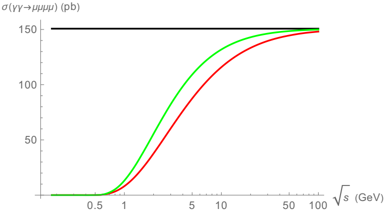
$$\sigma = \frac{16\alpha^4}{\pi s} \int_0^{u_0} \frac{2du}{1-u^2} wG(w) = \frac{16\alpha^4}{\pi} \int_0^{u_0} \frac{2du}{s_0 - s_1 u^2} G(w) \simeq \frac{16\alpha^4}{\pi} \frac{1}{\sqrt{s_0 s_1}} \ln \frac{1+\sqrt{\frac{s_1}{s_0}}\, u_0}{1-\sqrt{\frac{s_1}{s_0}}\, u_0} G(s/s_0)$$

Approximate threshold function

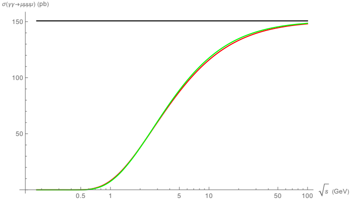
$$\sigma(s) = \sigma(\infty) \frac{\ln \frac{1+\sqrt{\frac{s_1}{s_0}}\sqrt{\frac{s-s_0}{s-s_1}}}{1-\sqrt{\frac{s_1}{s_0}}\sqrt{\frac{s-s_0}{s-s_1}}}}{\ln \frac{1+\sqrt{\frac{s_1}{s_0}}}{1-\sqrt{\frac{s_1}{s_0}}}} G(s/s_0)$$

Toy model results : $\gamma\gamma \to 4\mu$ and $\gamma\gamma \to 2\mu 2e$

$\gamma\gamma \to 4\mu$

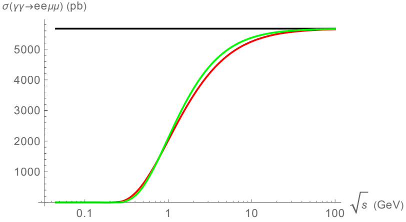


with $G(\omega)$

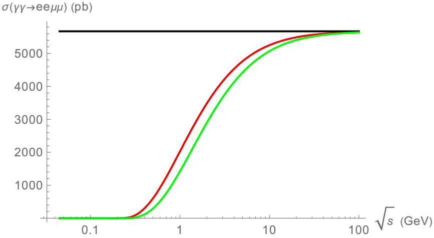


with $G_f(\omega)$

$\gamma\gamma \to 2\mu 2e$



with $G(\omega)$



with $G_f(\omega)$

Turning to the exact expressions : log*log term

We use now the exact expression of the argument of the log terms. The choice of good variables makes the approach easier. With $\chi(\beta) = \frac{1}{2} \ln \frac{1+\beta}{1-\beta}$ we have to compute :

$$\frac{d\sigma}{dtdxdx'} = \frac{32\alpha^4}{\pi t^2}\chi(\beta)\chi(\beta') \tag{1}$$

First we consider :

$$\int_{\frac{t}{sx_0}}^{x_0'} dx' \int_{\frac{t}{sx'}}^{x_0} dx\chi(\beta)\chi(\beta') \tag{2}$$

Changing variables, using $u = \frac{s}{t}x_0 x'$, we get :

$$\frac{t}{s}\int_1^w \frac{du}{u}\chi(\beta')\int_1^u d\omega\chi(\beta) \tag{3}$$

where

$$\beta^2 = \frac{u-\omega}{u-\omega x_0} \text{ and } \beta'^2 = \frac{w-u}{w-ux_0'} \tag{4}$$

Since

$$\frac{\omega}{u} = \frac{1-\beta^2}{1-\beta^2 x_0} = \frac{x}{x_0} \text{ and noting } \beta_1^2 = \frac{u-1}{u-x_0} \tag{5}$$

the integration by parts on ω gives :

$$\int_1^u d\omega\chi(\beta) = -\chi(\beta_1) + \frac{u}{\sqrt{x_0}}\chi(\beta_1\sqrt{x_0}) \tag{6}$$

Turning to the exact expressions : keeping the leading term

Considering now :

$$\frac{t}{s}\int_1^w du\chi(\beta')\left[\frac{1}{\sqrt{x_0}}\chi(\beta_1\sqrt{x_0}) - \frac{1}{u}\chi(\beta_1)\right] \tag{7}$$

to a good approximation we put $\beta_1 = \beta_0$. Note that β_0 is the β_1 value for $u = w$.

$$\beta_0{}^2 = \frac{w-1}{w-x_0} \text{ and similarly } \beta_0'{}^2 = \frac{w-1}{w-x_0'} \tag{8}$$

Since

$$\frac{u}{w} = \frac{1-\beta'^2}{1-\beta'^2 x_0'} = \frac{x'}{x_0'} \tag{9}$$

we get in a similar way as above :

$$\int_1^w du\chi(\beta') = -\chi(\beta_0') + \frac{w}{\sqrt{x_0'}}\chi(\beta_0'\sqrt{x_0'}) \tag{10}$$

We keep the leading term :

$$\sqrt{x_0}\chi(\beta_0\sqrt{x_0})\sqrt{x_0'}\chi(\beta_0'\sqrt{x_0'}) \tag{11}$$

which gives the correct expression when $s \to \infty$

$$\sqrt{x_0}\chi(\sqrt{x_0})\sqrt{x_0'}\chi(\sqrt{x_0'}) \tag{12}$$

and goes to 0 at threshold.

The last integration

$$\sigma = \frac{32\alpha^4}{\pi} \int_{t_-}^{t_+} \frac{dt}{t^2} \sqrt{x_0}\chi(\beta_0\sqrt{x_0})\sqrt{x_0'}\chi(\beta_0'\sqrt{x_0'})$$

Noting that :

$$X_0 = x_0\beta_0^2 = \frac{w-1}{\frac{w}{x_0}-1} \text{ and } X_0' = x_0'\beta_0'^2 = \frac{w-1}{\frac{w}{x_0'}-1} \tag{13}$$

which implies

$$\left(\frac{1}{X_0}-1\right)\left(\frac{1}{x_0'}-1\right) = \left(\frac{1}{x_0}-1\right)\left(\frac{1}{X_0'}-1\right) \tag{14}$$

The following parametrization

$$x_0 = \frac{sh^2\eta_0}{sh^2\eta} \text{ and } x_0' = \frac{ch^2\eta_0}{ch^2\eta} \text{ with} th\eta_0 = \frac{m'}{m} \tag{15}$$

$$\text{satisfies } \left(\frac{1}{x_0}-1\right)m'^2 = m^2\left(\frac{1}{x_0'}-1\right) \text{ and implies } \left(\frac{1}{X_0}-1\right)m'^2 = m^2\left(\frac{1}{X_0'}-1\right) \tag{16}$$

which allows to write similarly :

$$X_0 = \frac{sh^2\eta_0}{sh^2\psi} \text{ and } X_0' = \frac{ch^2\eta_0}{ch^2\psi} \tag{17}$$

Turning to the exact expressions : relation between ψ and η

$$\text{Since} \quad \left(\frac{1}{X_0}-1\right)=\left(\frac{w}{w-1}\right)\left(\frac{1}{x_0}-1\right) \quad \text{and} \quad \frac{w-1}{w}=\frac{t}{s}\left(1-\frac{t_-}{t}\right)\left(\frac{t_+}{t}-1\right) \tag{18}$$

one can express $\frac{t}{t_0}=\tau$ as a function of $sh^2\psi$ using

$$\frac{t}{4m^2}\frac{t}{s}\left(1-\frac{t_-}{t}\right)\left(\frac{t_+}{t}-1\right)=\frac{sh^2\eta_0}{sh^2\psi-sh^2\eta_0} \tag{19}$$

Introducing $\tau_{0+}=e^{2\zeta}$ and $\tau_{0-}=e^{-2\zeta}$ and remembering that $t_0=4mm'$, $s_0=4(m+m')^2$ and $s_1=4(m-m')^2$ with $ch^2(\zeta)=\frac{s-s_1}{s_0-s_1}$ and $sh^2(\zeta)=\frac{s-s_0}{s_0-s_1}$ we get

$$(\tau-\tau_{0-})(\tau_{0+}-\tau)=\frac{s}{t_0}\frac{sh\eta_0\,ch\eta_0}{sh^2\psi-sh^2\eta_0} \quad \text{leading to} \tag{20}$$

$$\tau_\pm=ch(2\zeta)\pm sh(2\zeta)\sqrt{K(\psi)}=\frac{sh\eta_0\,ch\eta_0}{sh^2\eta-sh^2\eta_0} \quad \text{with} \quad K(\psi)=\frac{sh^2\psi-sh^2\psi_0}{sh^2\psi-sh^2\eta_0} \tag{21}$$

$$\text{where} \quad sh^2(\psi_0)=sh^2(\eta_0)+sh(\eta_0)ch(\eta_0)C(s) \quad \text{with} \quad C(s)\frac{s(s_0-s_1)}{(s-s_0)(s-s_1)} \tag{22}$$

$$e^{\psi_0}=\frac{1}{\sqrt{1-r^2}}\left[\sqrt{1+rC(s)}+\sqrt{r}\sqrt{r+C(s)}\right] \quad \text{with} \quad e^{-\eta_0}=\sqrt{\frac{1-r}{1+r}}$$

Turning to the exact expressions : the volume element

Using the η parametrization with $\sqrt{x_0} = \frac{sh\eta_0}{sh\eta}$, $\sqrt{x_0'} = \frac{ch\eta_0}{ch\eta}$ and $\frac{1}{\tau} = \frac{sh^2\eta - sh^2\eta_0}{sh\eta_0 ch\eta_0}$,

$$t_0 \frac{dt}{t^2}\sqrt{x_0}\sqrt{x_0'} = 2d\eta \tag{23}$$

Since $\psi \to \eta$ when $s \to \infty$ we will use $2d\psi$ instead. Noting that we can write :

$$\frac{1+\sqrt{X_0}}{1-\sqrt{X_0}} = \frac{(1+e^{-(\psi-\eta_0)})}{(1-e^{-(\psi-\eta_0)})}\frac{(1-e^{-(\psi+\eta_0)})}{(1+e^{-(\psi+\eta_0)})} \tag{24}$$

and

$$\frac{1+\sqrt{X_0'}}{1-\sqrt{X_0'}} = \frac{(1+e^{-(\psi-\eta_0)})}{(1-e^{-(\psi-\eta_0)})}\frac{(1+e^{-(\psi+\eta_0)})}{(1-e^{-(\psi+\eta_0)})} \tag{25}$$

Gluing everything we get :

$$\begin{aligned} t_0 \int_{t_-}^{t_+} \frac{dt}{t^2}\sqrt{x_0}\chi(\beta_0\sqrt{x_0})\sqrt{x_0'}\chi(\beta_0'\sqrt{x_0'}) \simeq \\ \int_{\psi_0}^{\infty} 2d\psi \frac{1}{4}\left[\ln^2\left|\frac{1+e^{-(\psi-\eta_0)}}{1-e^{-(\psi-\eta_0)}}\right| - \ln^2\left|\frac{1+e^{-(\psi+\eta_0)}}{1-e^{-(\psi+\eta_0)}}\right|\right] \end{aligned} \tag{26}$$

Turning to the exact expressions : the integration variable

For an integration between 0 and 1 we just use $y = e^{-(\psi-\psi_0)}$.

$$t_0 \int_{t_-}^{t_+} \frac{dt}{t^2} \sqrt{x_0}\chi(\beta_0\sqrt{x_0})\sqrt{x_0'}\chi(\beta_0'\sqrt{x_0'}) \simeq \int_0^1 2\frac{dy}{y}\frac{1}{4}\left[\ln^2\left|\frac{1+ye^{-(\psi_0-\eta_0)}}{1-ye^{-(\psi_0-\eta_0)}}\right| - \ln^2\left|\frac{1+ye^{-(\psi_0+\eta_0)}}{1-ye^{-(\psi_0+\eta_0)}}\right|\right] \tag{27}$$

$$\text{With} \quad P(th\eta) = \int_0^1 \frac{dy}{y}\left[\ln^2\left|\frac{1+y}{1-y}\right| - \ln^2\left|\frac{1+ye^{-2\eta}}{1-ye^{-2\eta}}\right|\right] \quad \text{where} \tag{28}$$

$$P(u) = \ln^2 u \ln\frac{1+u}{1-u} - 2\ln u\,[\mathrm{Li}_2(u) - \mathrm{Li}_2(-u)] + 2\,[\mathrm{Li}_3(u) - \mathrm{Li}_3(-u)] = \Lambda_3(u) - \Lambda_3(-u) \tag{29}$$

$$\text{we get} \quad t_0 \int_{t_-}^{t_+} \frac{dt}{t^2} \sqrt{x_0}\chi(\beta_0\sqrt{x_0})\sqrt{x_0'}\chi(\beta_0'\sqrt{x_0'}) \simeq \frac{1}{2}\left[P(th\frac{\psi_0+\eta_0}{2}) - P(th\frac{\psi_0-\eta_0}{2})\right] \tag{30}$$

when $s \to \infty$ we recover $P(th\eta_0)$ inside the brackets.
So the corresponding form factor is :

$$\frac{P(th\frac{\psi_0+\eta_0}{2}) - P(th\frac{\psi_0-\eta_0}{2})}{P(th\eta_0)} \tag{31}$$

Turning to the exact expressions : the jacobian at low y

With the help of the following change of variables $y = e^{-(\psi-\psi_0)}$ and $z = e^{-(\eta-\eta_0)}$ we have :

$$\begin{aligned} \tau &= \frac{z^2(1-e^{-4\eta_0})}{(1-z^2)(1-z^2e^{-4\eta_0})} \\ K(\psi) &= \frac{(1-y^2)(1-y^2e^{-4\psi_0})}{[1-y^2e^{-2(\psi_0+\eta_0)}][1-y^2e^{-2(\psi_0-\eta_0)}]} \end{aligned} \tag{32}$$

And at low y, the jacobian behaves like :

$$\frac{dz^2}{z^2} \simeq \frac{dy^2}{a^2+y^2} \tag{33}$$

we get for large s

$$a^2 \simeq \frac{(m+m')^2}{s} \tag{34}$$

and near threshold when $s \to s_0$

$$a^2 \simeq \sqrt{\frac{4t_0}{s-s_0}} \tag{35}$$

This explicit form of the jacobian explains the previous approximation in the large s limit.

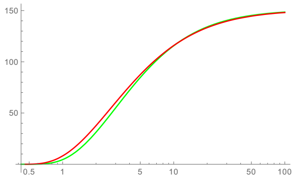
Turning to the exact expressions : modified Kummer function

Before gluing everything for a comparison with the exact expression, we just have now to compute an integral of the following type :

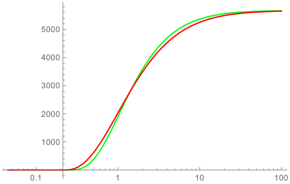
$$P(\theta, \frac{1-b}{1+b}) = \int_0^1 2y \frac{dy}{a^2+y^2} \frac{1}{4} \ln^2 \left| \frac{1+yb}{1-yb} \right| \quad \text{with} \quad e^{-i\theta} = \frac{1-iab}{1+iab} \quad \text{and} \quad u = \frac{1-b}{1+b} \tag{36}$$

$$\text{We get} \quad P(\theta, u) = \Lambda_3(u) - \frac{1}{2}\left[\Lambda_3(-ue^{i\theta}) + \Lambda_3(-ue^{-i\theta})\right] \tag{37}$$

$$\text{and define} \quad \mathrm{Li}_n(re^{i\theta}) + \mathrm{Li}_n(re^{-i\theta}) = 2\mathrm{Li}_n(r, \theta) \tag{38}$$



$\gamma\gamma \to 4\mu$



$\gamma\gamma \to 2e2\mu$

Discussion

We have already got an approximate expression corresponding to the $\frac{1}{3}5\sqrt{x_0}\chi(\beta_0\sqrt{x_0})$ piece, which might be sufficient for a reasonable limited accuracy.
Integrating the full $I(x,t)$ expression leads to :

$$\frac{1}{3}\left[\left(5\sqrt{x_0}-\frac{1}{\sqrt{x_0}}\right)\chi(\beta_0\sqrt{x_0})\ +\frac{\beta_0}{2}\,\frac{2-x_0(3+\beta_0^4)+x_0^2(3-4\beta_0^2+3\beta_0^4)}{(1-\beta_0^2x_0)^2}\right] \tag{39}$$

When $\beta_0=1$, we get as expected :

$$\frac{1}{3}\left[1+\left(5\sqrt{x_0}-\frac{1}{\sqrt{x_0}}\right)\chi(\sqrt{x_0})\right] \tag{40}$$

which was used to obtain the asymptotic cross section which can be written as :

$$\sigma_\infty=\frac{4\alpha^4}{9\pi m^2 th\eta_0}\left\{\left[\frac{25}{4}+\frac{19}{32}\left(\frac{1}{th\eta_0}-th\eta_0\right)^2\right]P(th\eta_0)+Q(th\eta_0)\right\} \tag{41}$$

with

$$Q(u)=\frac{19}{16}\left[2\left(\frac{1}{u}-u\right)\ln u-\left(\frac{1}{u}+u\right)\left(1+\ln^2 u\right)\right] \tag{42}$$

Discussion

$$\text{Keeping only} \quad \sigma_0 = \frac{25\alpha^4}{9\pi m^2} \frac{P(th\eta_0)}{th\eta_0} \tag{43}$$

$$\text{For small } th\eta_0 \text{ we obtain} \quad \frac{25\alpha^4}{18\pi m^2} \left(\ln^2 th\eta_0^2 - 2 \ln th\eta_0^2 + 4 \right) \tag{44}$$

$$\text{Instead of} \quad \frac{28\alpha^4}{27\pi m^2} \left(\ln^2 th\eta_0^2 - \frac{103}{21} \ln th\eta_0^2 + \frac{485}{63} \right) \tag{45}$$

For equal masses, once divided by 2, obtain :

$$\sigma_0 = \frac{\alpha^4}{\pi m^2} \frac{175}{36} \zeta(3) \quad \text{instead of} \frac{\alpha^4}{\pi m^2} \left(\frac{175}{36} \zeta(3) - \frac{19}{18} \right) \tag{46}$$

The expression to be tested against the numerical integration is then :

$$\frac{P(th\frac{\psi_0+\eta_0}{2}) - P(th\frac{\psi_0-\eta_0}{2})}{P(th\eta_0)} \sigma_\infty \quad \text{or better} \quad \frac{P(\theta_+, th\frac{\psi_0+\eta_0}{2}) - P(\theta_-, th\frac{\psi_0-\eta_0}{2})}{P(th\eta_0)} \sigma_\infty \tag{47}$$