The Dirac field

In Quantum Field Theory (QFT) both bosons and fermions are quanta of a field

Field and corresponding wave function obey the same equation

SM assumes spin ½ fermions obey Dirac equation (not proven for neutrinos)

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0 \qquad x = (x_0, x_1, x_2, x_3)$$

$$\psi(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

The 4 elements of the bi-spinor correspond to particle and antiparticle and, for each the two possible polarisation (or helicity) states: S_z=+1/2 and S_z=–1/2

Dirac matrices can have different representations. We take

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Dirac covariants

For monochromatic plane wave $\psi(x) = u e^{-ip^\mu x_\mu}$ **Dirac equation becomes** $(\gamma_\mu p^\mu - m) u = 0$

The bispinor conjugate of $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ is $\bar{u} = u^+ \gamma^0 = \begin{pmatrix} u_1^* & u_2^* & -u_3^* & -u_4^* \end{pmatrix}$

Satisfying equation $\bar{u}(\gamma_\mu p^\mu + m) = 0$

With two bispinors *a* e *b* (corresponding to equal or different particles) and the Dirac matrices 5 covariants can be built.

Nature uses only two of them in the interactions: vector and axial currnets

$\bar{a}b$	scalar
$\bar{a}\gamma_5 b$	pseudoscalar
$\bar{a}\gamma_\mu b$	vector
$\bar{a}\gamma_\mu \gamma_5 b$	axial vector
$\frac{1}{2\sqrt{2}}\bar{a}(\gamma_\alpha\gamma_\beta - \gamma_\beta\gamma_\alpha) b$	tensor

E.M.	Weak	QCD
V	V & A	V

Meaning of the four components

The bi-spinor can be written with two spinor components φ and χ, corresponding to particle and antiparticle

$$\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

Three possible choices of the two elements of φ and the two of χ depending of the quantity we want define

1. Polarisation. The two projections of spin 1/2 on a physically defined (magnetic field, electric field) z axis

$$\varphi^+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \varphi^- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

are eigenstates of the 3rd spin component

$$\frac{1}{2}\sigma_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\frac{1}{2}; \quad \frac{1}{2}\sigma_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2}$$

2. Helicity. the component of the spin on the direction of velocity (particle not at rest)
Photon helicity corrsponds to circular polarisation of light

Helicity opertor is $\frac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{|p|}$

Chirality and Helicity

3. Chirality states are the eigenstates of γ_5

Bi-spinors can be written as $y = y_L + y_R$

$$y_L = \frac{1}{2}\left(1 - g_5\right) y, \qquad g_5 y_L = - y_L$$

$$y_R = \frac{1}{2}\left(1 + g_5\right) y, \qquad g_5 y_R = + y_R$$

Left and Right bi-spinor are the states of negative and positive chirality
Not of positive and negative helicity

γ_5 does not commute with the mass term of the Hamiltonian

CC weak interactions couple to Left fields only

Both have positive (+) and negative (–) helicity components

For *E*>>*m*

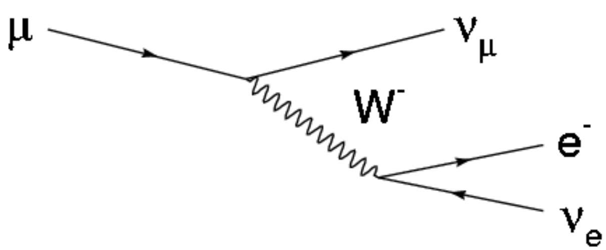
$$\varphi \approx \frac{m}{E}\varphi_L^{+} + \varphi_L^{-} \qquad \chi \approx \chi_L^{+} + \frac{m}{E}\chi_L^{-}$$

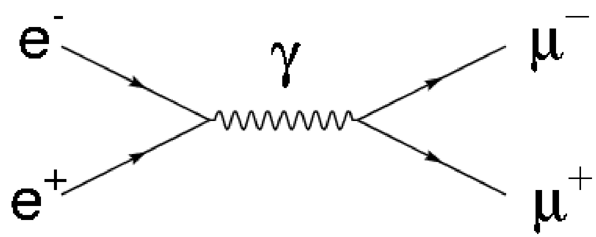
If m=0 neutrinos are pure h=– states, antineutrinos pure h=+ states

But $m \neq 0$ and $E >> m$, then neutrinos have a small (m/E) "wrong" helicity component

Feynman Diagrams

Feynman diagrams are pictorial representations of AMPLITUDES of particle reactions, i.e scatterings or decays. Use of Feynman diagrams can greatly reduce the amount of computation involved in calculating a rate or
cross section of a physical process, e.g.
muon decay: $\mu^- \rightarrow e^- \nu_e \nu_\mu$ or $e^+e^- \rightarrow \mu^+\mu^-$ scattering.





Like electrical circuit diagrams, every line in the diagram has a strict mathematical interpretation. Unfortunately the mathematical overhead necessary to do complete calculations with this technique is large and there is not enough time in this course to go through all the details.

Feynman Diagrams

Each Feynman diagram represents an AMPLITUDE (M).

Quantities such as cross sections and decay rates (lifetimes) are proportional to $|M|^2$.
The transition rate for a process can be calculated using time dependent perturbation theory using Fermi's Golden Rule:

$$transition\ rate = \frac{2\pi}{\hbar} |M|^2 \times (phase\ space)$$

In lowest order perturbation theory M is the fourier transform of the potential

The differential cross section for two body scattering (e.g. pp→pp) in the CM frame is:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2} \frac{q_f^2}{v_i v_f} |M|^2$$

q_f=final state momentum
v_f= speed of final state particle
v_i= speed of initial state **particle**

The decay rate (Γ) for a two body decay (e.g. $K^0 \rightarrow \pi^+\pi^-$) in CM is given by:

$$\Gamma = \frac{S|\vec{p}|}{8\pi\hbar m^2 c} |M|^2$$

m=mass of parent
p=momentum of decay particle
S=statistical factor (fermions/bosons)

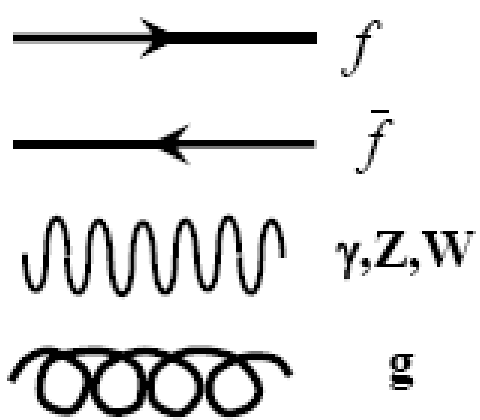
In most cases $|M|^2$ cannot be calculated exactly.
Often M is expanded in a power series.
Feynman diagrams represent terms in the series expansion of M.

Feynman Diagrams

- Invented by Richard Feynman in the `40s to describe particle interactions in space-time
- Each diagram represents a contribution to the total amplitude of the process considered

{ Feynman diagrams
Feynman rules } ⇒ **M**

- Particles are represented by lines:



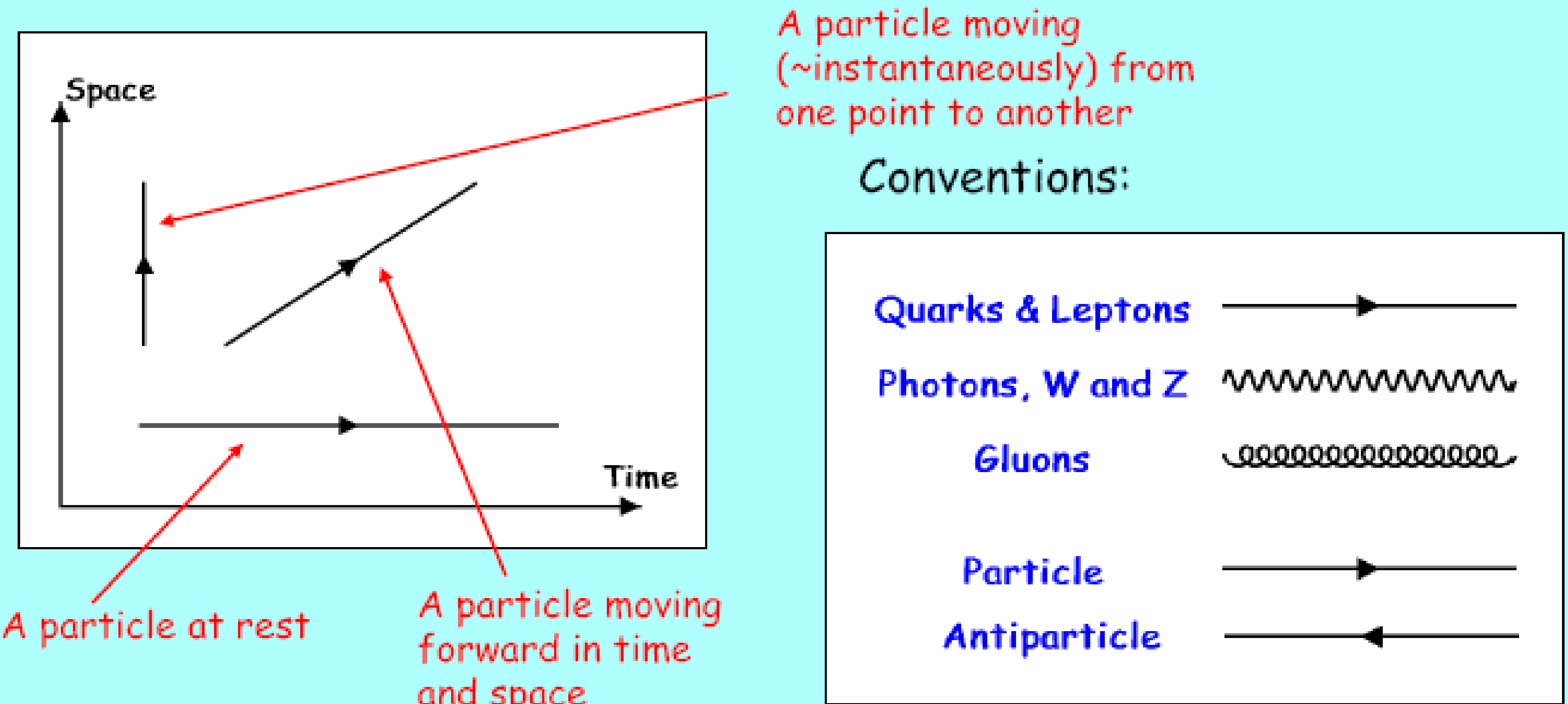
time →

• Particles go forward in time

•Antiparticles go backwards in time

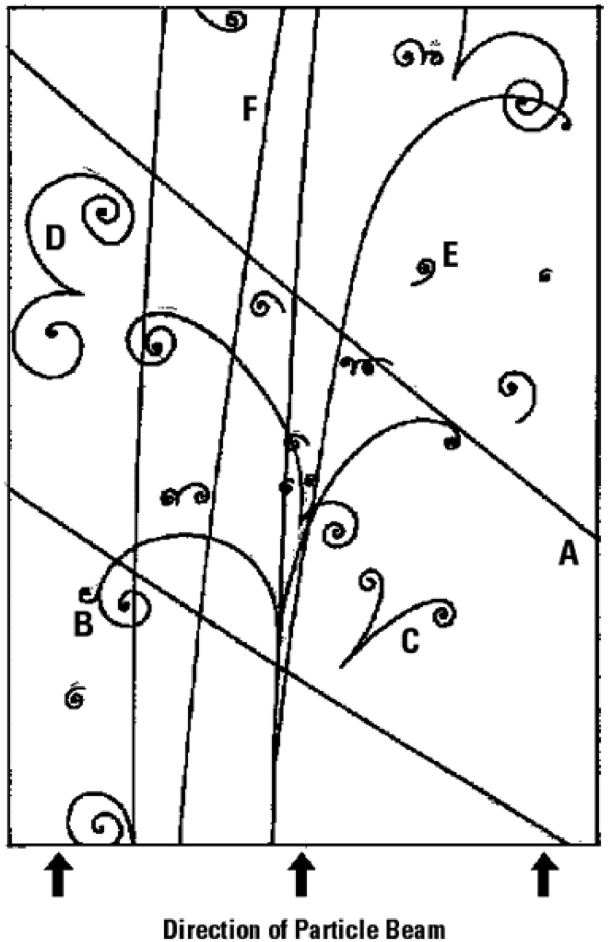
Feynman Diagrams

Feynman Diagrams are like circuit diagrams - they show what is connected to what but not the detailed momentum vectors - lengths and angles are not relevant.

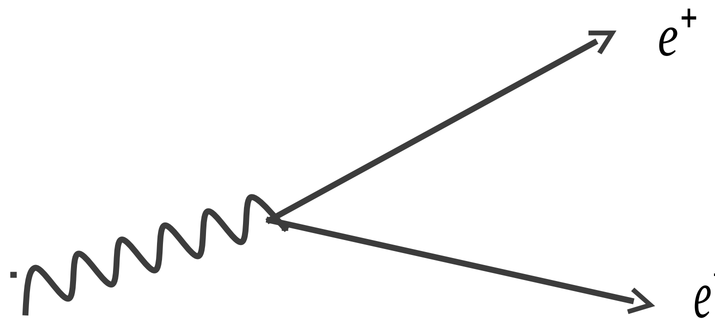


Antiparticles

We have seen evidence, through the Bubble Chamber, that a photon can pair-produce an electron and a positron.



What does this look like in a Feynman diagram?

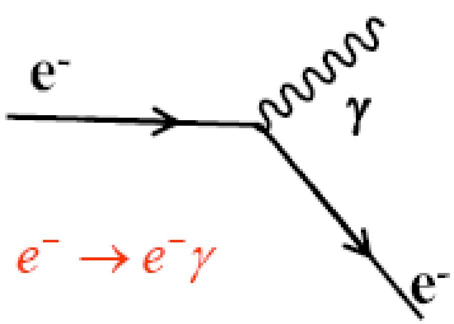


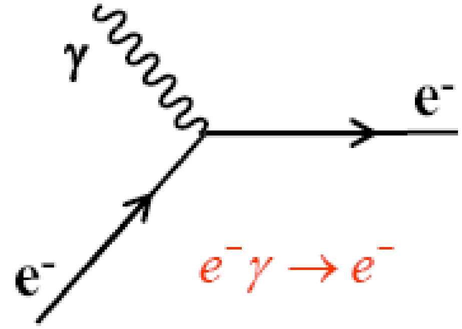
WRONG!
Since every vertex has a arrowhead pointing toward it and one leaving

electric charge is not conserved.

Vertices

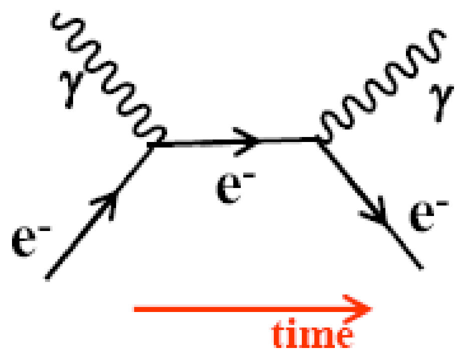
- Lines connect into vertices, which are the building blocks of Feynman diagrams





- Charge, lepton number and baryon number are always conserved at a vertex
- Lowest order diagram has two vertices

Ex: Compton scattering



Virtual processes

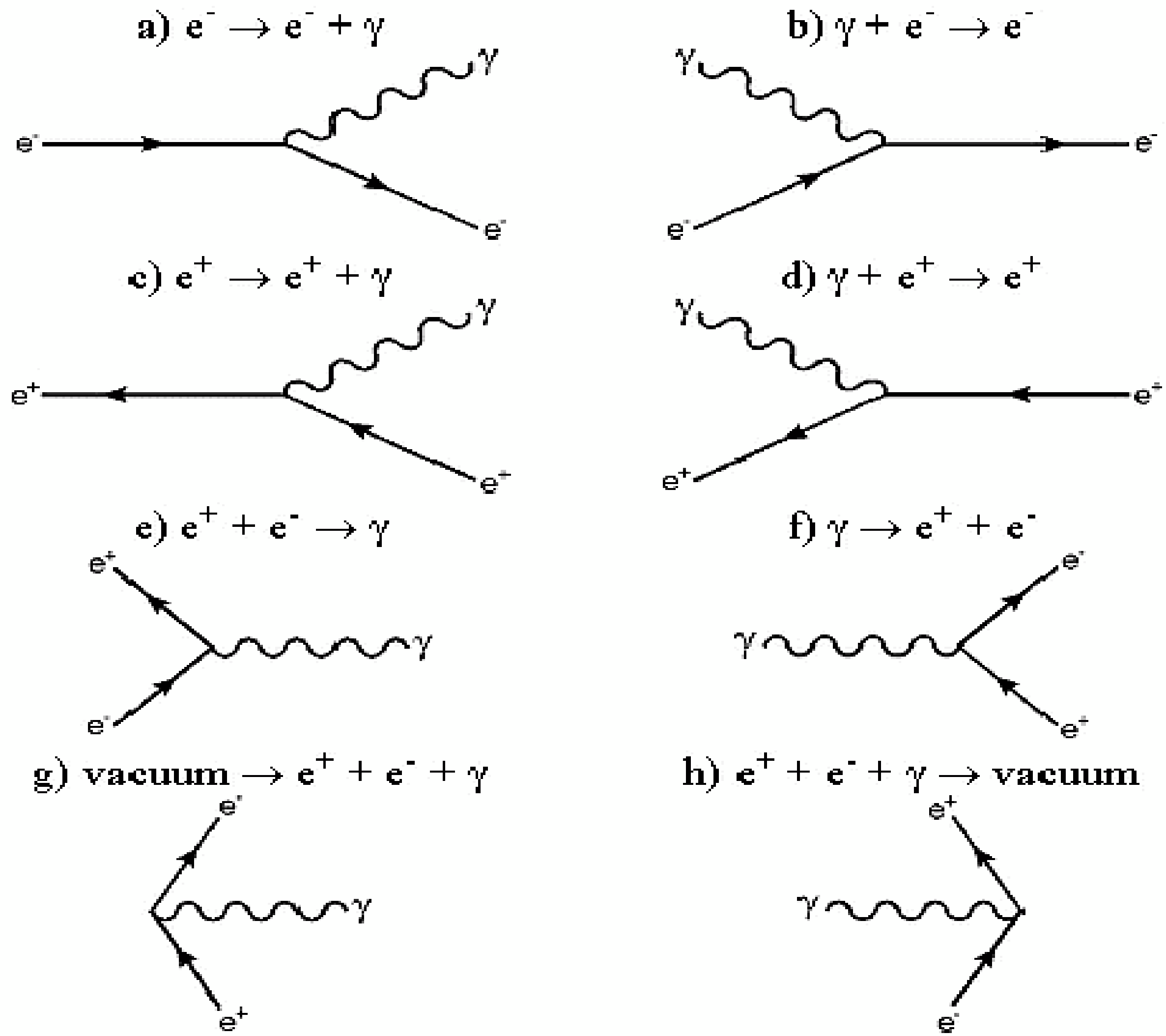
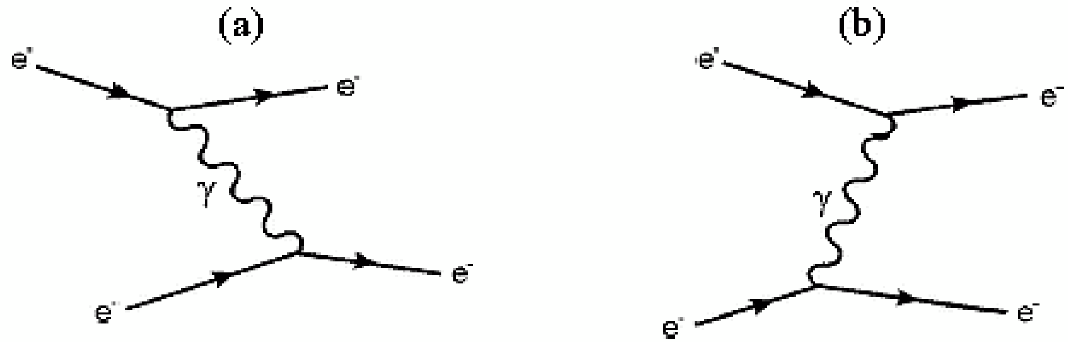


Figure 6: Feynman diagrams for basic processes involving electron, positron and photon

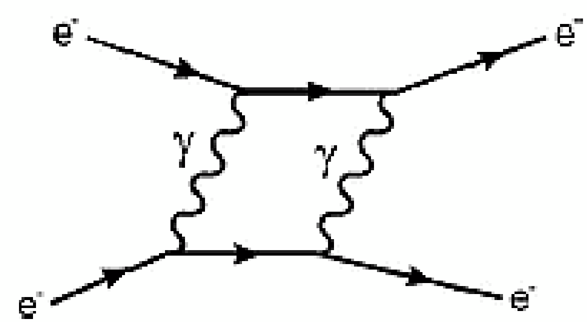
Real processes

- For a real process there must be energy conservation ⟶ it has to be a combination of virtual processes.



Electron-electron scattering, single γ exchange

- Any real process receives contributions from all possible virtual processes.



- Points at which 3 or more particles meet are called **vertices**
Each vertex corresponds to a term in the transition matrix element which includes the structure and strength of the interaction.

-The nr. of vertices in a diagram is called **order**

-Each vertex has an associated probability proportional to a **coupling constant**, usually denoted as a. In the em processes this is:

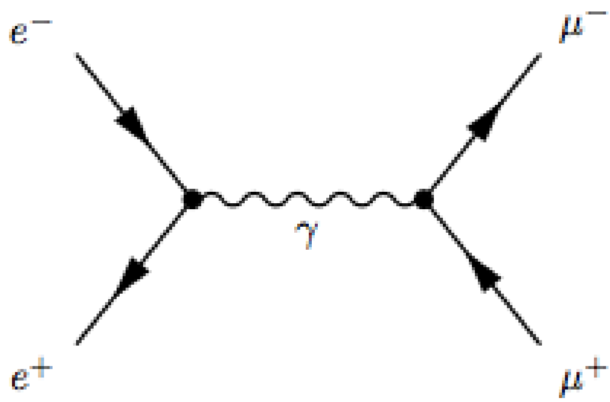
$$\alpha_{em} \approx \frac{1}{137}$$

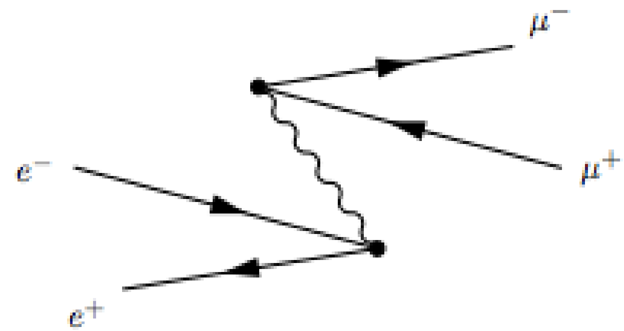
-For the real processes, a diagram of order n gives a contribution of order α^n

-Providing that α is small enough, higher order contributions to many real processes can be neglected

Time ordering

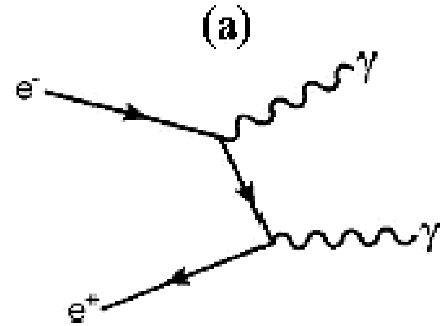
A Feynman diagram represents all possible time orderings of the possible vertices, so the positions of the vertices within the graph are arbitrary. Consider the following two diagrams for $e^+ + e^- \rightarrow \mu^+ + \mu^-$:

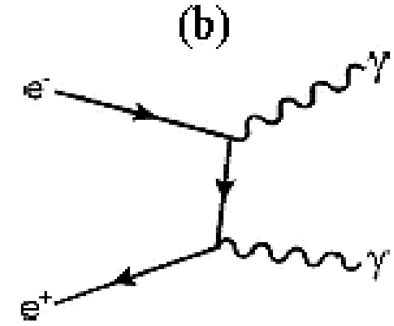




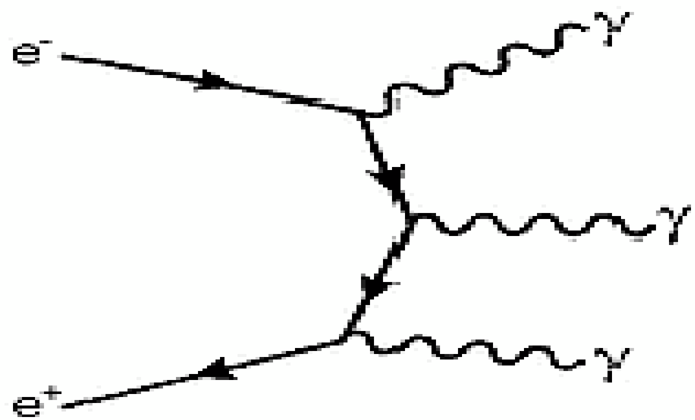
In the left diagram it appears that the incoming particles annihilated to form a virtual photon, which then split to produce the outgoing particles. On the right diagram it appears that the muons and the photon appeared out of the vacuum together, and that the photon subsequently collided with the electron and positron, leaving nothing. Changing the position of the internal vertices does not affect the Feynman diagram – it still represents the same contribution to the amplitude. The left side and right side just represent different time-orderings, so each is just a different way of writing the same Feynman diagram.

-Diagrams which differ only by time-ordering are usually implied by drawing only one of them





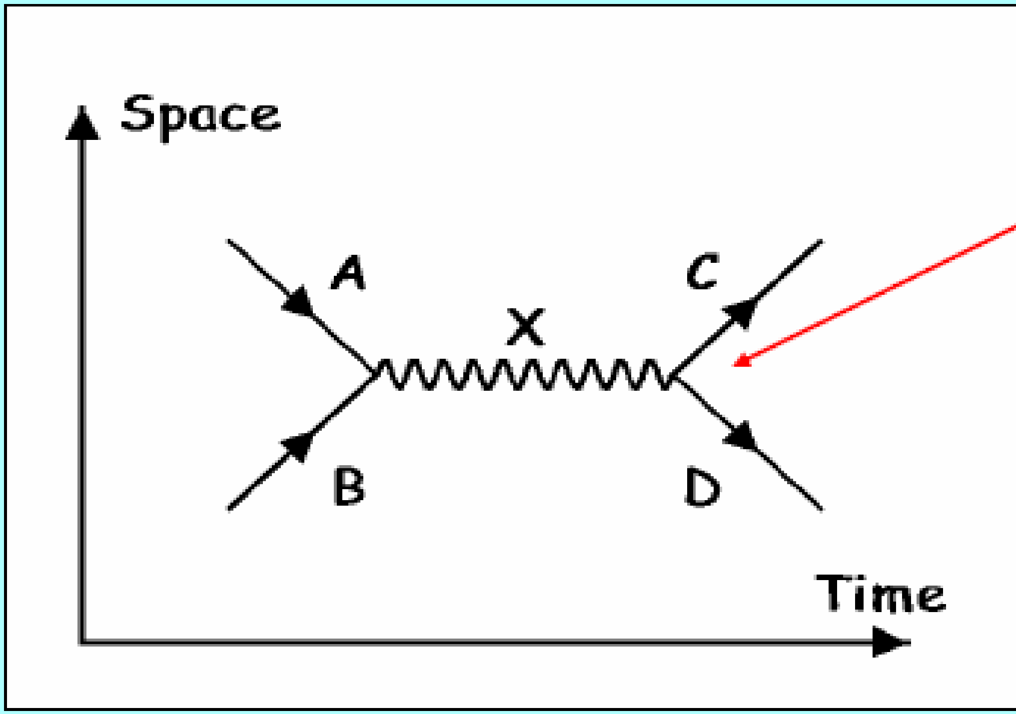
-This kind of process:



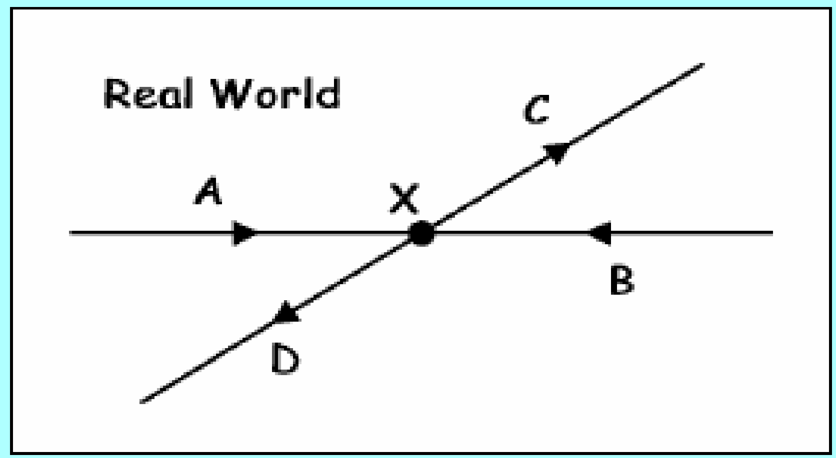
-Implies 3!=6 different time orderings

Annihilation diagrams

Annihilation/Formation Diagram. Particles A and B collide to form particle X which later decays to C and D.

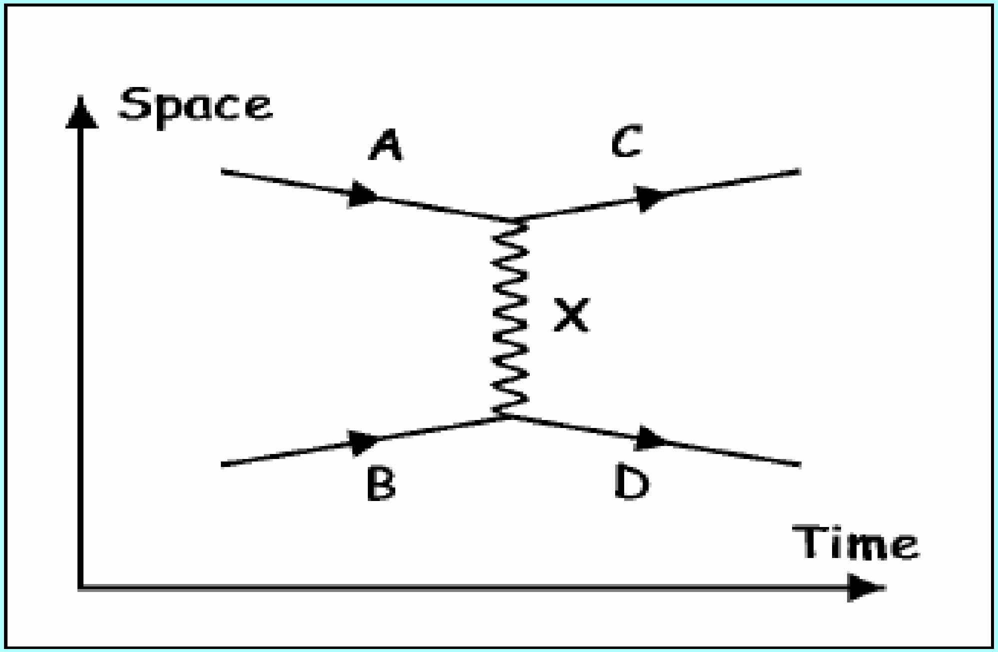


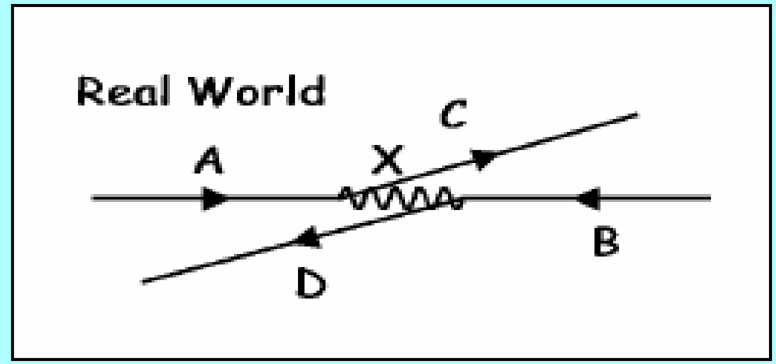
At each vertex, electric charge must be conserved and, except in Weak Interactions, quark or lepton flavours.

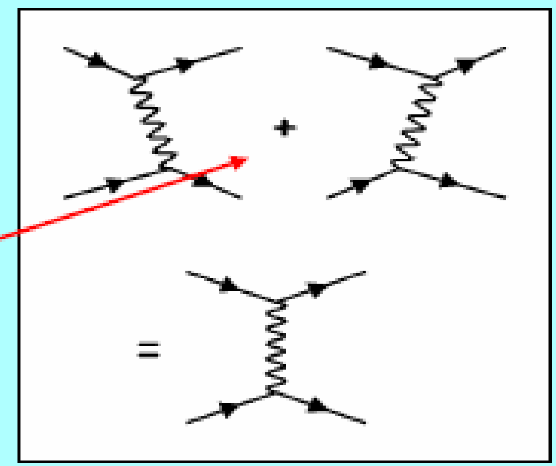


Exchange diagrams

Exchange Diagram. Particles A scatters off particle B by exchanging particle X. Particle A becomes particle C and B becomes D.



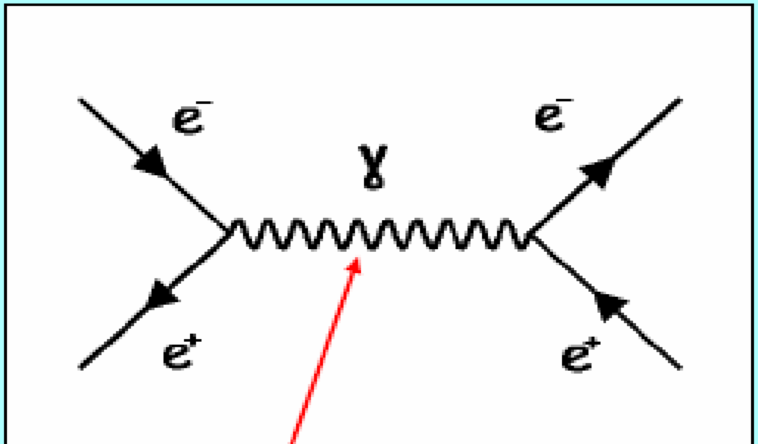




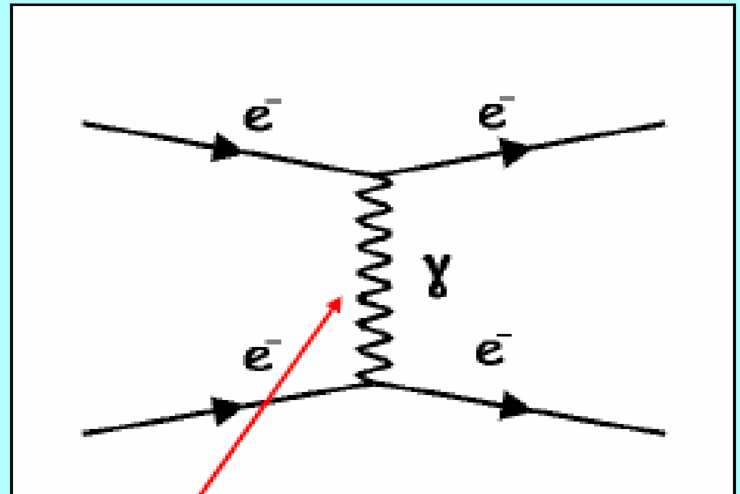
We don't know if A emitted X and B absorbed it or vice versa.

Virtual particles

In both previous cases particle X is 'virtual' and the time it exists is governed by the uncertainty principle $\Delta E\ \Delta t \sim \hbar$. The mass of particle X is usually not its rest mass.



If and electron and positron annihilate, X is a photon (γ) with zero charge, zero momentum and energy $2E_e$ and hence an apparent mass of $2E_e/c^2$.



If two electrons scatter, X is a photon (γ) with zero charge, momentum $2p_e$ and zero energy and hence an apparent imaginary mass of $\sqrt{-p_e^2/c^2}$.

$$E^2 = p^2c^2 + m^2c^4$$

Virtual particles

The photon that carries the information in the last example was emitted from a free electron.

This certainly violates the conservation of mass-energy (inertia). However, Heinsenberg's Uncertainty Principle can be recasted into:

> Measurements of the energy of a particle or of an energy level are subject to an uncertainty.

The energy measurement must be completed within a certain time interval, Δt.

The uncertainty in the energy, ΔE is related to Δt through:

$$\Delta E \cdot \Delta t \geq \frac{h}{4\pi}$$

So the photon can exist, energy can be created (!), as long as it does not last longer than

$$\Delta t \geq \frac{h}{4\pi}\frac{1}{\Delta E}$$

Virtual particles

Since the photon is very quickly absorbed by the other electron, the energy “borrowed” is returned, (in time).

Particles like this photon are called virtual particles.

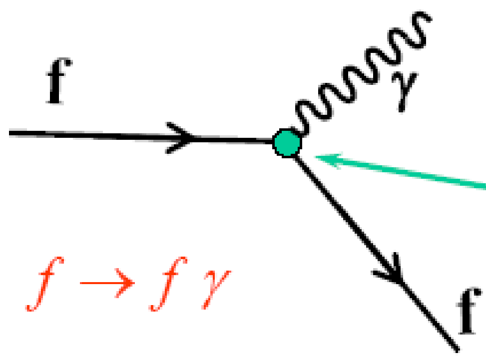
Virtual particles are particles that cannot be observed because they are absorbed as soon as they are created. They may violate conservations laws.

Feynman Diagrams: couplings

- A coupling constant (multiplicative factor) is associated with each vertex
- Value of constant depends on type of interaction

Electromagnetic

$f \to f\,\gamma$



Q_f = fermion charge
(in units of electron charge)

$$Q_f \sqrt{\alpha} = Q_f \sqrt{\frac{e^2}{4\pi}}$$

For example, Compton scattering off an electron:

$$Diagram \propto (coupling)^2 \propto \alpha$$

$$\sigma \propto |Diagram|^2 \propto \alpha^2 \propto e^4$$

- Total four-momentum is conserved at a vertex
- Can move particle from initial to final state by replacing it with its antiparticle

$f \to f\,\gamma$ becomes $f\bar{f} \to \gamma$

Feynman Diagrams: QED coupling

$$\alpha_{\mathsf{EM}} \equiv \frac{e^2}{4\pi\epsilon_0 \hbar c} \approx \frac{1}{137}.$$

is dimensionless. It is convenient to choose g_{EM} such that

$$\alpha_{\mathsf{EM}} = \frac{g_{\mathrm{EM}}^2}{4\pi}.$$

In other words the coupling constant g_{EM} is a dimensionless measure of the $|e|$ where e is the charge of the electron. The size of the coupling between the photon and the electron is

$$-g_{\mathrm{EM}} = -\sqrt{4\pi\alpha_{\mathsf{EM}}}.$$

The electromagnetic vertex factor for any other charged particle f with charge Q_f times that of the proton is then

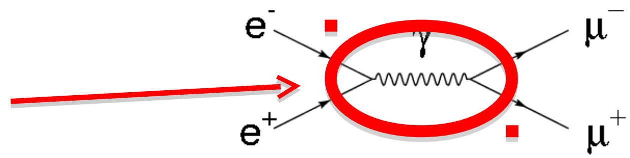
$$\boxed{g_{\mathrm{EM}} Q_f}$$

So, for example, the electromagnetic vertex factor for an electron is of size $-g_{\mathrm{EM}}$ while for the up quark it is of size $+\frac{2}{3}g_{\mathrm{EM}}$.

Feynman Diagrams: propagator

For each internal line – that is each **virtual particle** – we associate a **propagator factor**. The propagator tells us about the contribution to the amplitude from a particle travelling through space and time (integrated over all space and time). For a particle with no spin, the **Feynman propagator** is a factor

$$\boxed{\frac{1}{Q \cdot Q - m^2}}$$



where $Q \cdot Q = E_Q^2 - \mathbf{q} \cdot \mathbf{q}$ is the four-momentum-squared of the internal virtual particle[4].

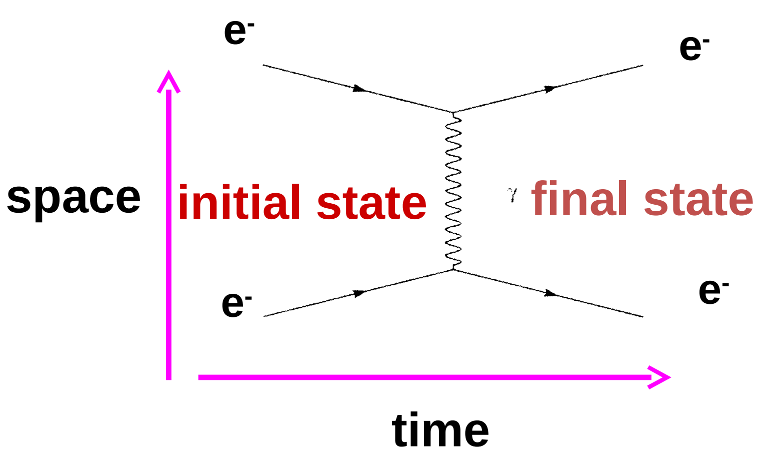
These intermediate particles are called **virtual particles**. They do **not** satisfy the usual relativistic energy-momentum constraint $Q \cdot Q = m^2$. For an intermediate virtual particle,

$$\boxed{Q \cdot Q = E_Q^2 - \mathbf{q} \cdot \mathbf{q} \neq \mathbf{m^2}.}$$

Such particles are said to be **off their mass-shell**.

Feynman diagrams

External particles in Feynman diagrams do always individually satisfy the relativistic energy-momentum constraint $E^2 - p^2 = m^2$, and for these particles we should therefor not include any propagator factor. The external lines are included in the diagram purely to show which kinds of particles are in the initial and final states.



QED Rules

At each vertex there is a coupling constant

$\sqrt{\alpha}$, α= 1/137=fine structure constant

Quantum numbers are conserved at a vertex

e.g. electric charge, lepton number

“Virtual” Particles do not conserve E, p

virtual particles are internal to diagram(s)

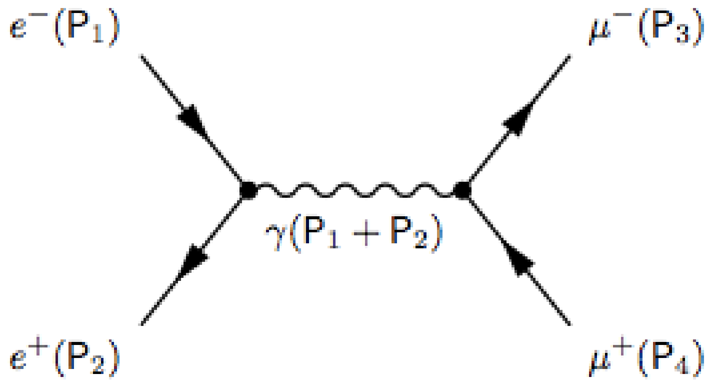
for γ’s: $E^2 - p^2 \neq 0$ (off “mass shell”)

in all calculations we integrate over the virtual particles 4-momentum (4d integral)

Photons couple to electric charge

no photons only vertices

Feynman Diagrams: example



calculate the photon's energy-momentum four-vector Q_γ from that of the electron P_1 and the positron P_2. Four momentum is conserved **at each vertex** so the photon four-vector is $Q_\gamma = P_1 + P_2$. Calculating the momentum components in the zero momentum frame:

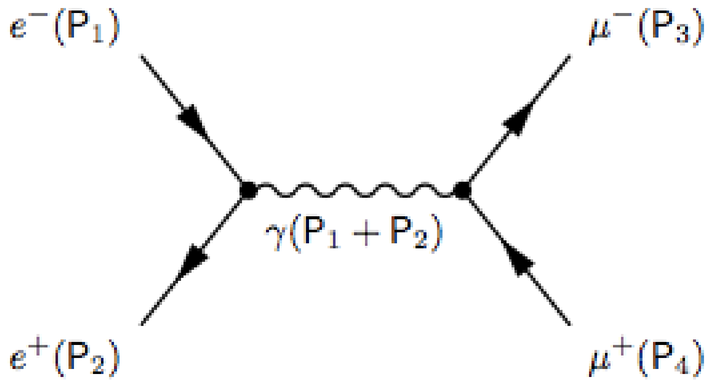
$$P_1 = (E, \mathbf{p}), \quad P_2 = (E, -\mathbf{p}). \tag{2}$$

Conserving energy and momentum at the first vertex, the energy-momentum vector of the internal photon is

$$Q_\gamma = (2E, \mathbf{0}).$$

So this *virtual* photon has more energy than momentum.

Feynman Diagrams: example



The propagator factor for the photon in this example is then

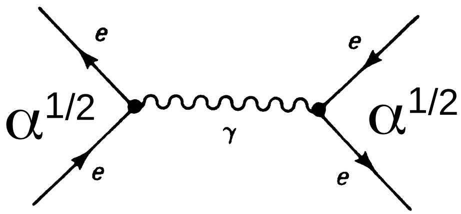
$$\frac{1}{(2E)^2 - m_\gamma^2} = \frac{1}{4E^2}.$$

The contribution to $\mathcal{M}_{fi}$ from this diagram is obtained my multiplying this propagator by two vectex factors each of size g_{EM}. The modulus-squared of the matrix element is then

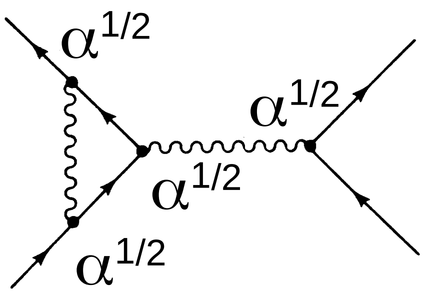
$$|\mathcal{M}_{fi}|^2 = \left|\frac{g_{\mathrm{EM}}^2}{4E_e^2}\right|^2.$$

Higher Order Contributions

Bhabha scattering: $e^+e^- \rightarrow e^+e^-$



Amplitude is of order α^1.



Amplitude is of order $\alpha^{1/2}$.

Since α_{QED} =1/137, higher order diagrams should be corrections to lower order diagrams.

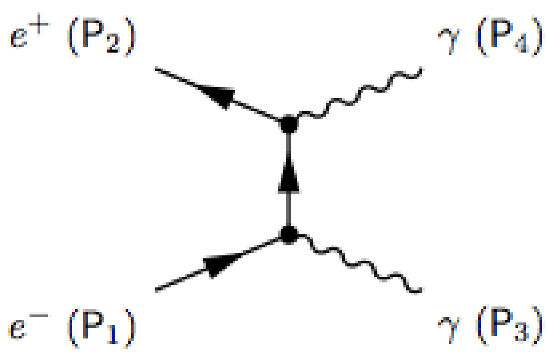
This expansion in the coupling constant works for QED since α_{QED} =1/137

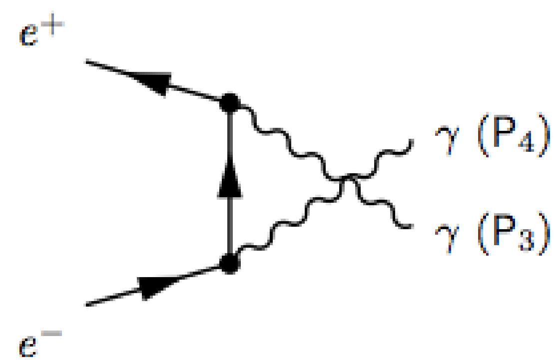
Does not work well for QCD where α_{QCD} » 1

Summing Amplitudes

On the other hand, changing the way in which the lines in a diagram are connected to one another does however result in a new diagram. Consider for example the process $e^+ + e^- \rightarrow \gamma + \gamma$

In the two diagrams above the outgoing photons have been swapped. There is no way to move around the vertices in the second diagram so that it is the same as the first. The two diagrams therefore provide separate contributions to $\mathcal{M}_{fi}$, and must be added.

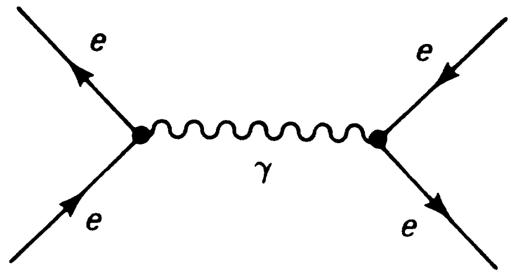


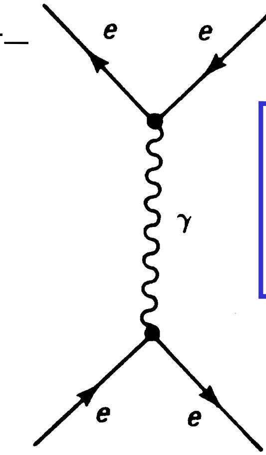


Summing Amplitudes

For a given order of the coupling constant there can be many diagrams

Bhabha scattering: e^+e^-–



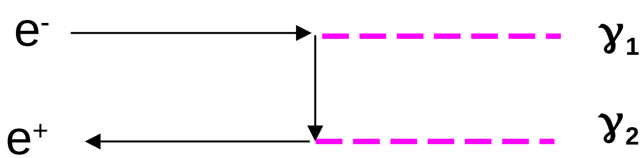


amplitudes can interfere constructively or destructively

Must add/subtract diagram together to get the total amplitude

total amplitude must reflect the symmetry of the process

$e^+e^-\rightarrow\gamma\gamma$ identical bosons in final state, amplitude symmetric under exchange of γ_1, γ_2.



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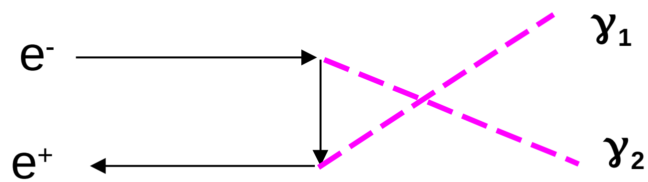
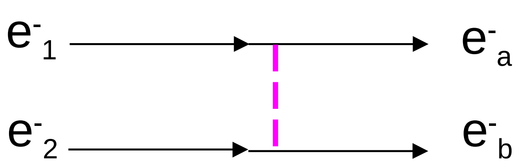


Moller scattering: $e^-e^-\rightarrow e^-e^-$ identical fermions in initial and final state

amplitude anti-symmetric under exchange of (1,2) and (a,b)



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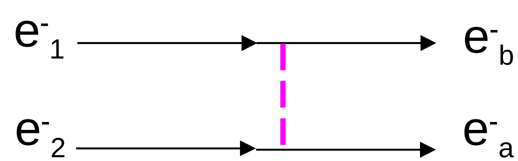
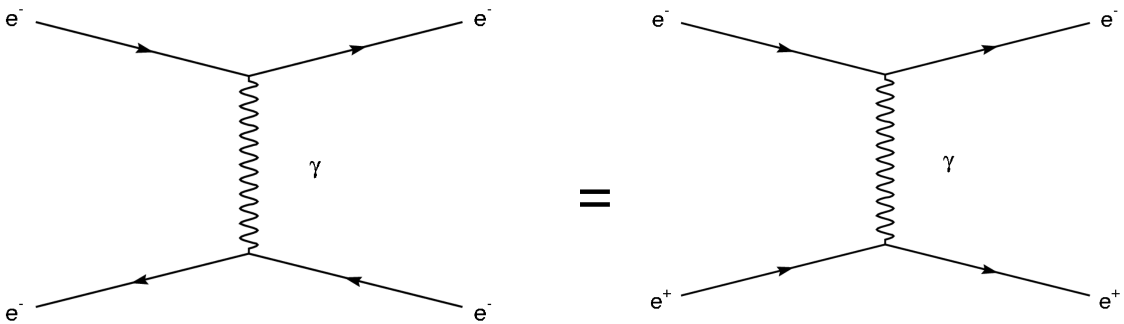
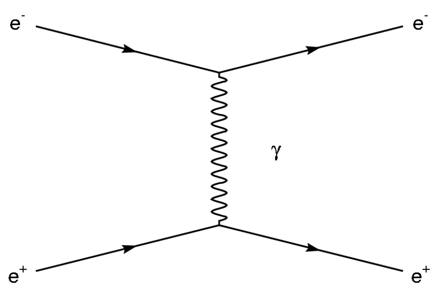


Summing Amplitudes

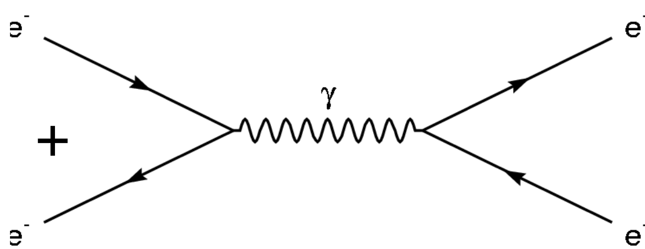


Note that an electron going backward in time is equivalent to a positron going forward in time.

M =



\+

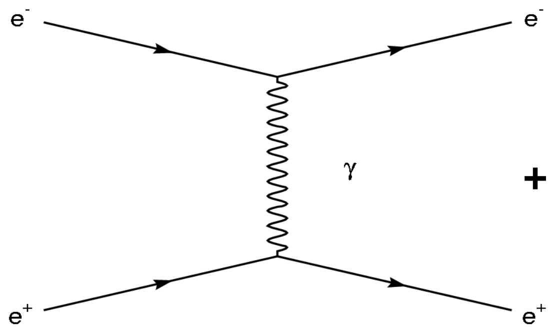


exchange **annihilation**

Transition amplitudes (matrix elements) must be summed over indistinguishable initial and final states.

Putting it Together

M =



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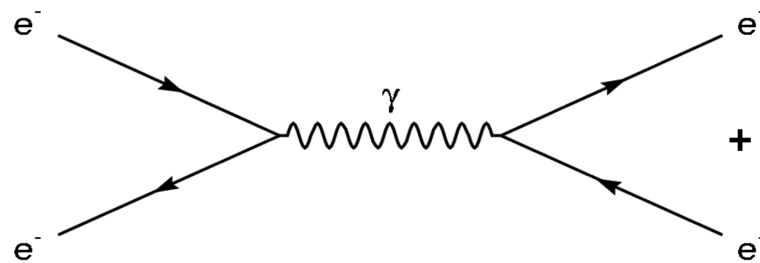

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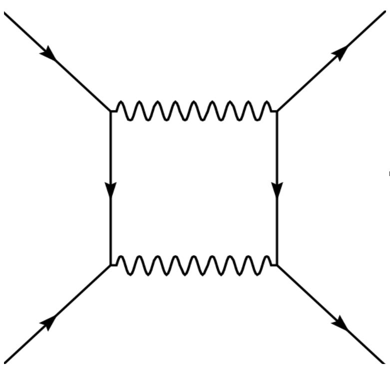

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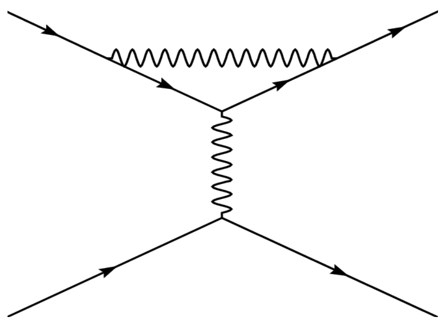

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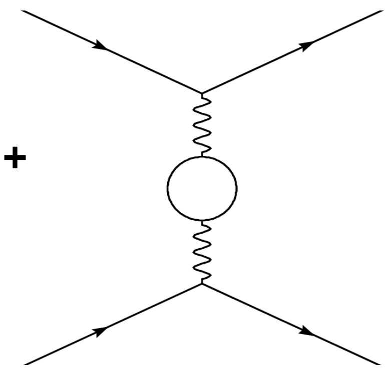

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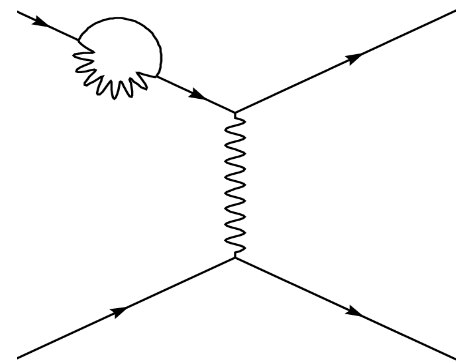
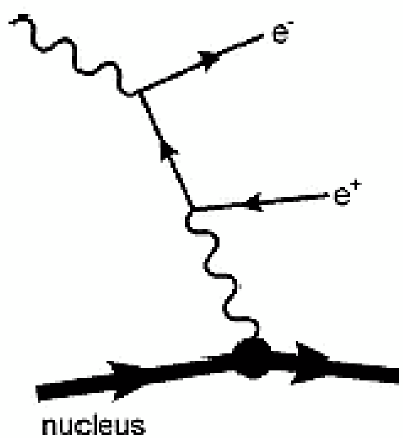
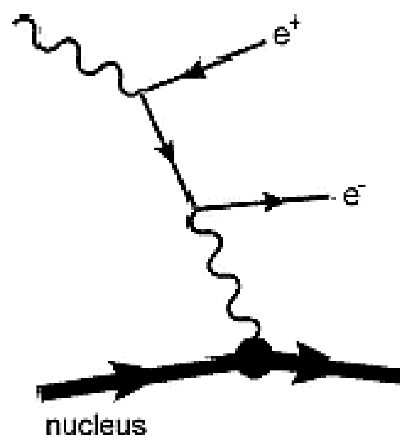


- From the order of diagrams the ratio of rates of processes can be estimated:

$$R = \frac{Rate\left(e^{+}e^{-} \rightarrow \gamma\gamma\gamma\right)}{Rate\left(e^{+}e^{-} \rightarrow \gamma\gamma\right)} = O\left(\alpha\right)$$

- For example, this ratio measured appears to be: R = 0.9 x 10^{-3}, smaller than α_{em} (but estimate is only a first order prediction)





\- For nucleus, coupling is proportional to $Z^2\alpha$, hence the rate for this process is of order $Z^2\alpha^3$

The elementary fermions

Three quark families

	Q	I_z	S	C	B	T	Mass
d	–1/3	–1/2	0	0	0	0	$4.8^{+0.7}_{-0.3}$ MeV
u	+2/3	+1/2	0	0	0	0	$2.3^{+0.7}_{-0.5}$ MeV
s	–1/3	0	–1	0	0	0	95±5 MeV
c	+2/3	0	0	+1	0	0	1.275±0.0025 GeV
b	–1/3	0	0	0	–1	0	4.18±0.03 GeV
t	+2/3	0	0	0	0	+1	173.5±1.0 GeV

Quark masses are not observable.

The mass parameters are determined from the properties (masses, couplings, …) of the hadrons. Possible only within a defined theoretical scheme.

	m(MeV)	lifetime
e^-	0.511	>5 10^{26} y
ν_e	n.a.	n.a
μ^-	105.7	2.2 μs
ν_μ	n.a.	n.a
τ^-	1776.8	291 fs
ν_τ	n.a.	n.a.

Three lepton families

The Standard Model *assumes* zero mass neutrinos

Observation of neutrino oscillations

neutrino masses are nonzero

ν_e, ν_μ, ν_τ are not mass eigenstates, but linear superpositions of them

Physics beyond the MS

Fundamental interactions

Interaction	Boson	M(GeV)	J^P
Weak CC	$W^{\pm}$	91.2	1^-
Weak NC	Z^0	80.4	1^-
El. Mag.	γ	0	1^-
Strong	8 gluons	0	1^-

Gravity does not have a microscopic theory (yet)

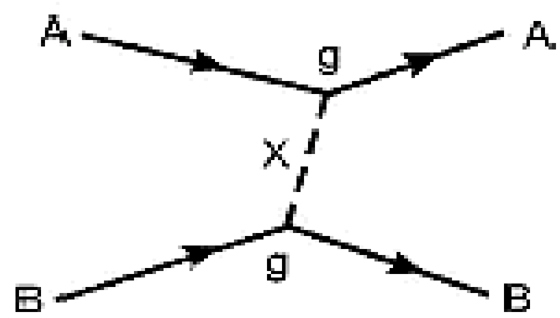
Forces binding the nuclei are not fundamental, but “tails” of QCD

The photon has no charge. Gauge symmetry U(1)

***W*s have weak and electric charge. *Z* has weak charge. Symmetry SU(2)**

Gluons have colour charges. Symmetry SU(3)

Exchange of a massive boson



-In the rest frame of particle A:

where: $E = M_A, \vec{p} = (0,0,0),$

$E' = \sqrt{p'^2 + M_A^2}, E_x = \sqrt{p'^2 + M_X^2}$

$$y = y_L + y_R$$

From this one can estimate the max distance over which X can propagate:

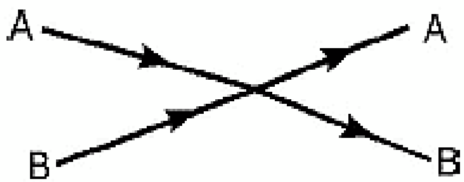
$$\Delta E = E_X + E' - M_A \geq M_X$$

This energy violation can be only for $\Delta t \approx \hbar / \Delta E$

→ The interaction range is: $r \approx \hbar c / M_X$

- For a massless exchanged particle, the interaction has an infinite range (e.g: em)

-If the exchanged particle is very heavy (like the W boson in the weak interaction), the interaction can be approximated by a **zero-range** or **point interaction**

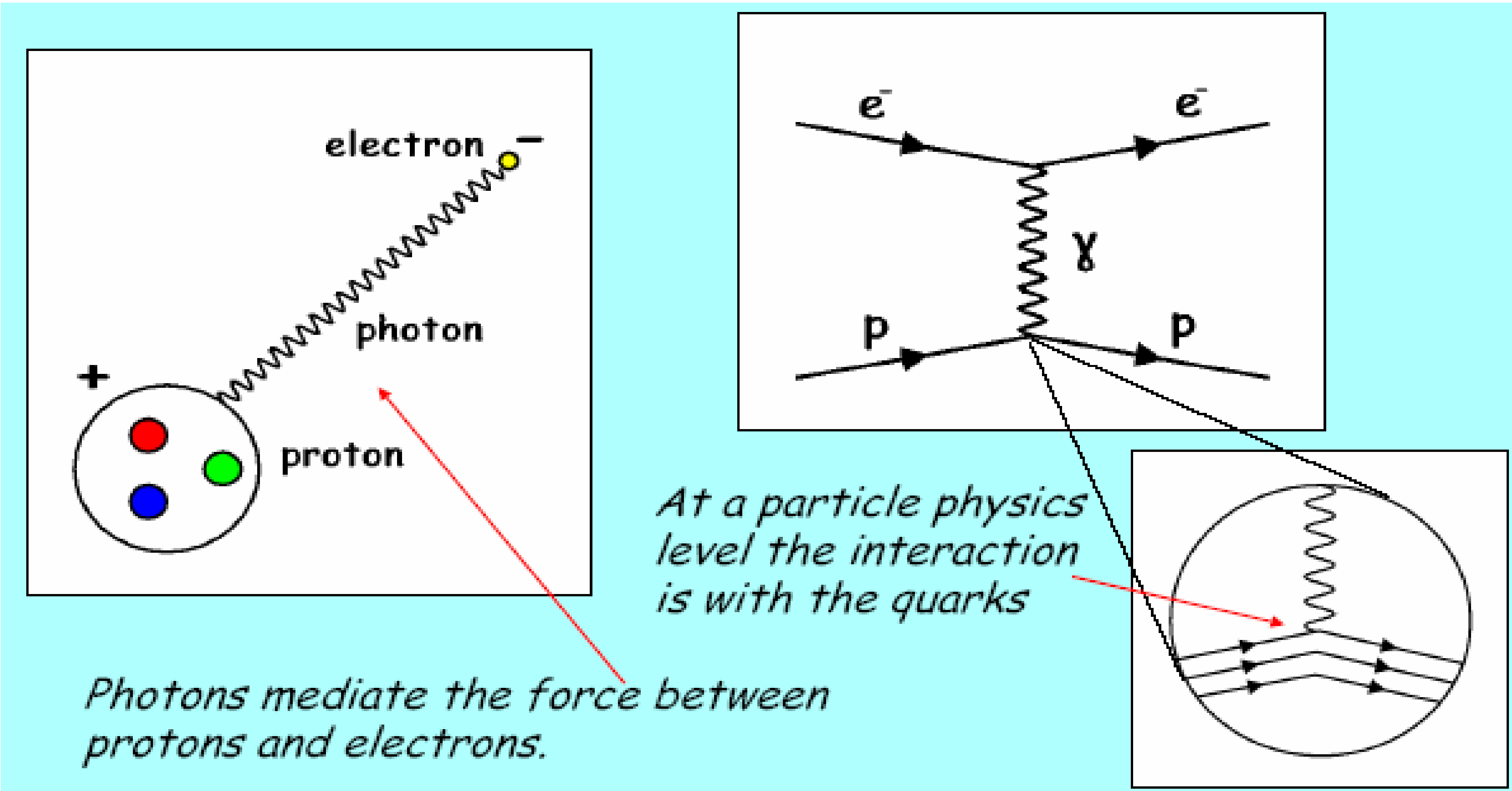


$$R_W = \hbar c / M_W = \hbar c / (80.4\, GeV/c^2) = \frac{197.3 \cdot 10^{-18}}{80.4} \approx 2 x 10^{-18} m$$

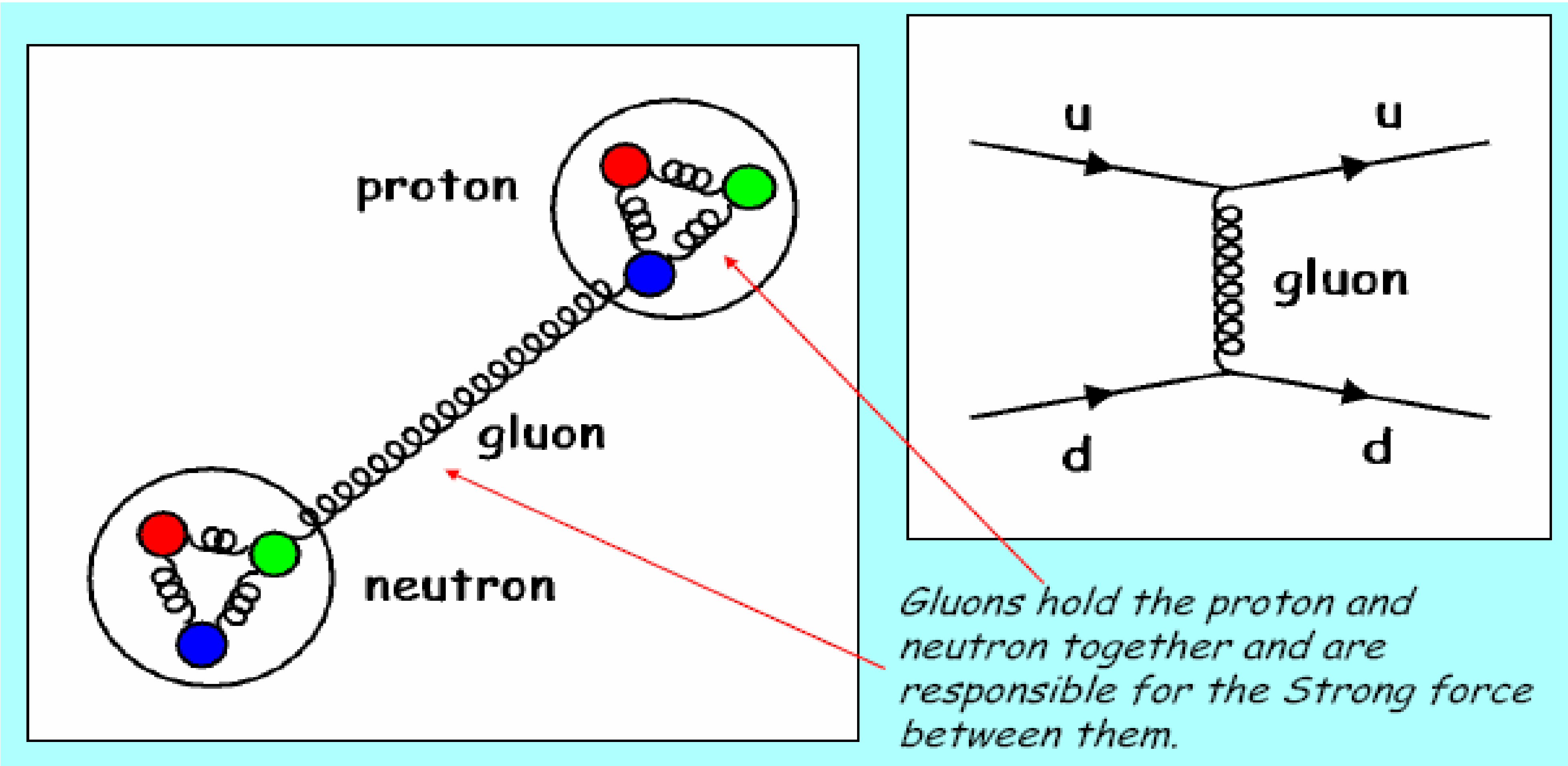
-If one consider the particle X as an electrostatic potential V(r), then the Klein-Gordon equation looks like

$$\nabla^2 V(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) = M_X^2 V(r)$$

Electromagnetism

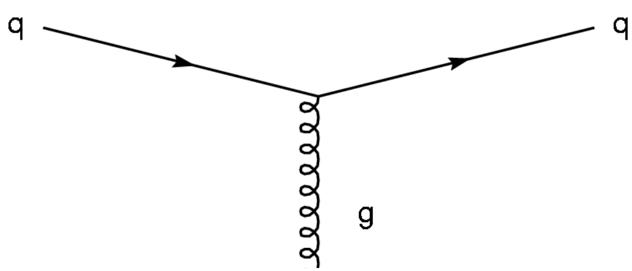


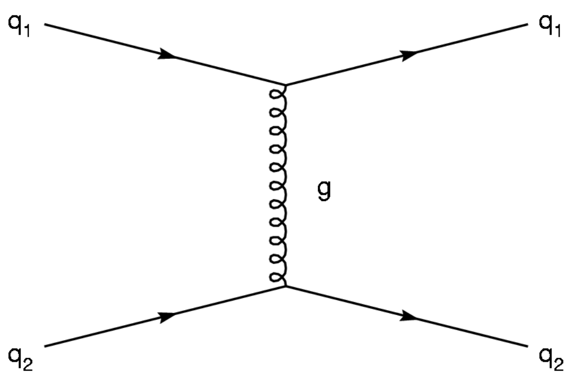
Strong Interactions

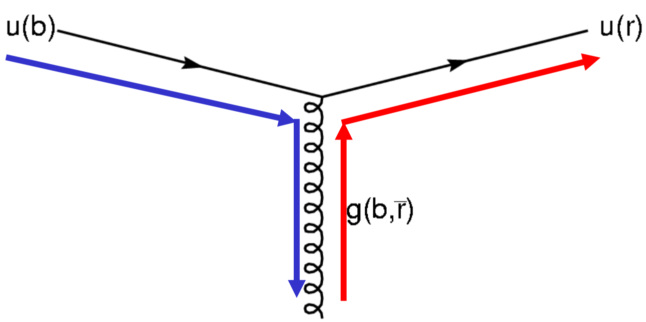


Gluons hold the proton and neutron together and are responsible for the Strong force between them.

Strong Interactions



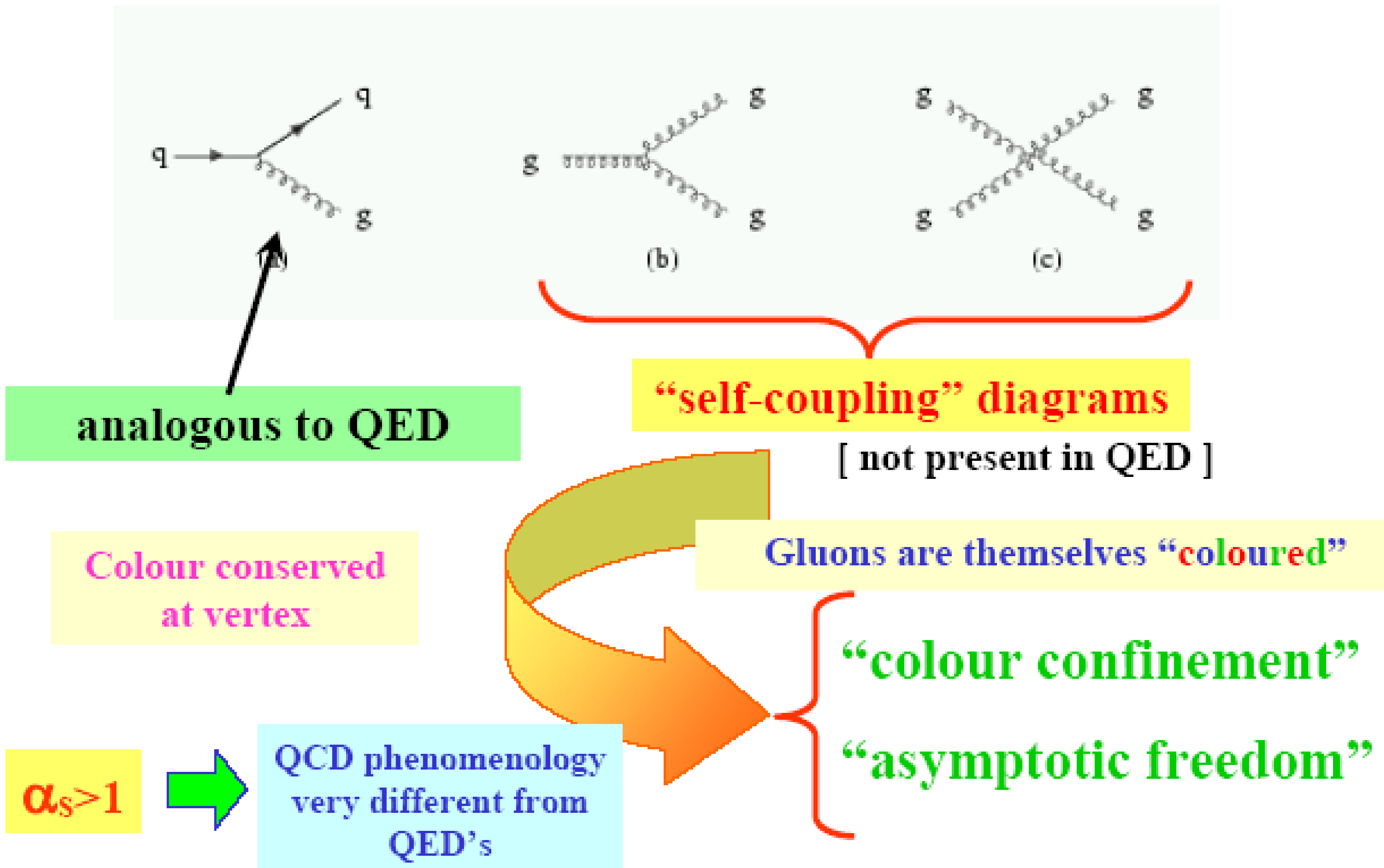




The Feynman diagrams for strong interactions look very much like those for QED.

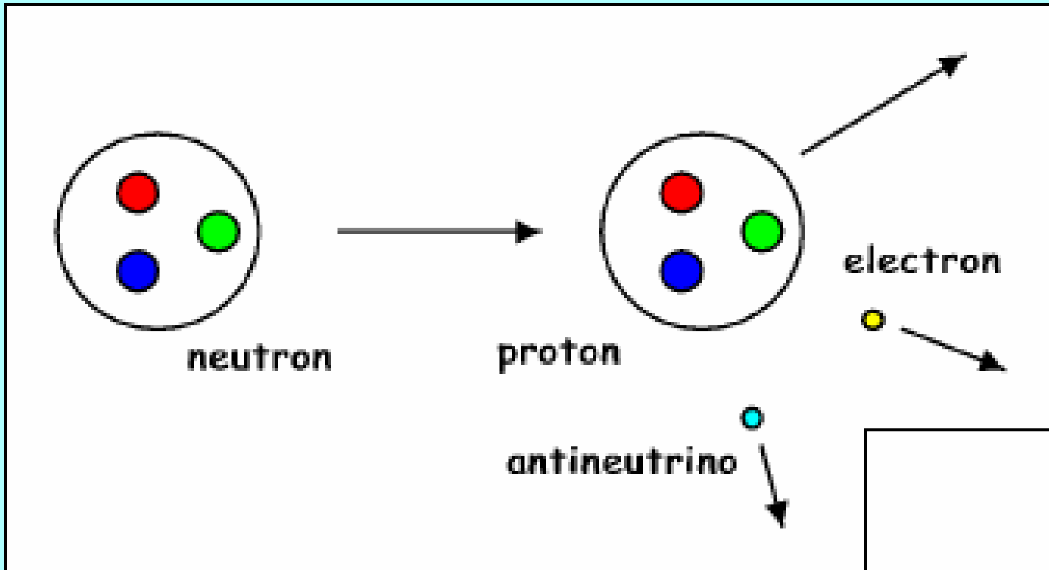
In place of photons, the quanta of the strong field are called gluons.

- The coupling strength at each vertex depends on the momentum transfer (as is true in QED, but at a much reduced level).
- Strong charge (whimsically called color) comes in three varieties, often called blue, red, and green.
- Gluons carry strong charge. Each gluon carries a color and an anti-color.

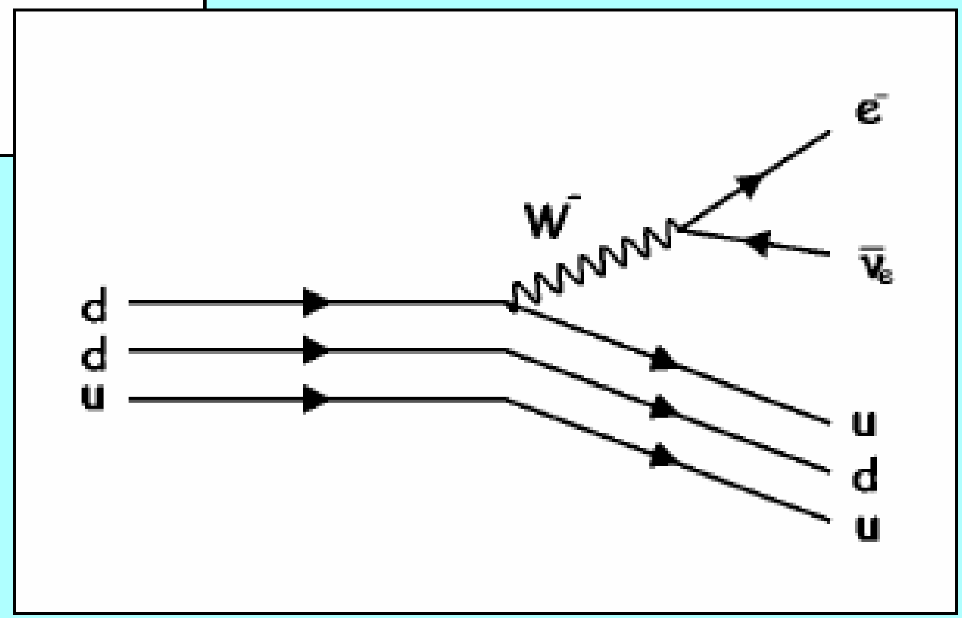


q
q
g
g
g
g
g
g
g
g
(b)
(c)
analogous to QED
"self-coupling" diagrams
[not present in QED]
Colour conserved at vertex
Gluons are themselves "coloured"
"colour confinement"
"asymptotic freedom"
$\alpha_S > 1$
QCD phenomenology very different from QED's

Weak Interactions



Beta decay $n \rightarrow p + e^- + \bar{\nu}_e$

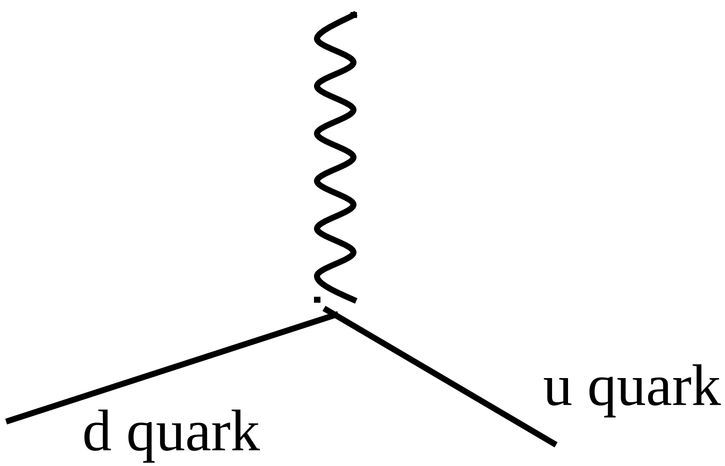


Mediated by charged W exchange

Weak Interactions

- Weak interaction involves the emission or absorption of the W & Z bosons.
- The W particle carries either a positive or a negative electric charge between particles whereas the Z particle carries no electric charge.
- The W particle is a quark-changing or a lepton-changing particle.

  Gluons and photons carry no charge and do not change the particle flavour.



The charge before is -1/3. Afterward it is +2/3.

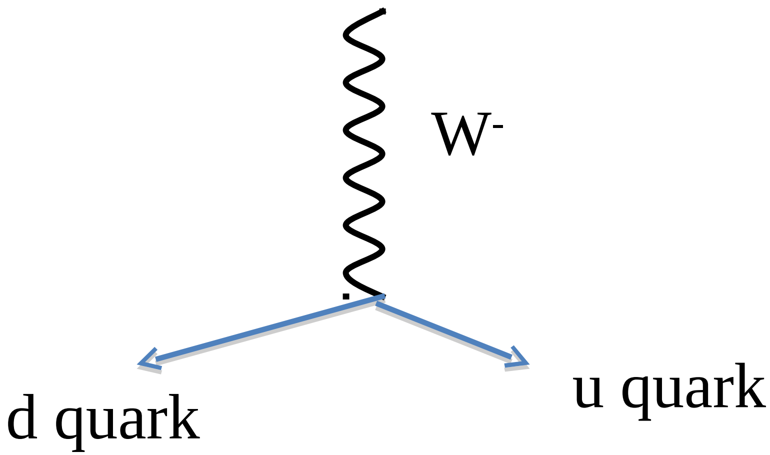
$$-1/3 = +2/3 + x$$

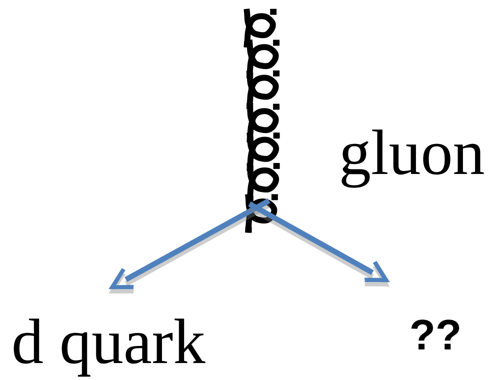
$$X = -1$$

It must be the W^-

Weak Interactions

Don't confuse this with:



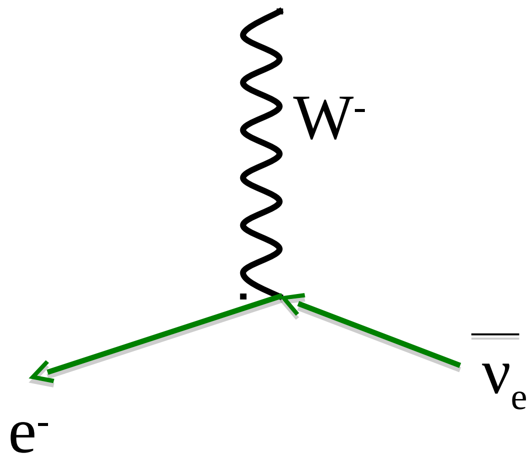


What particle is leaving the vertex?

A d quark of a different color.

This is not a weak interaction!!

Weak Interactions



Here is another example of a weak interaction.

Notice the change between lepton family.

Exercises

1) Knowing the dimensions of neutron and protons, you can estimate the mass of the particle, which is responsible for the interaction between the nucleus

Hint:

- The interaction between charged particles is carried by photons. The range of this interaction is infinite, the rest mass of photon has to be zero. The interaction between nucleons is limited to a range of about 10^{-15} m.

- The Heisemberg uncertainty relation allows fluctuation of energy for a very short time, so that ”virtual” particles can be created which are responsible for the interaction

Solution:

$r = c \cdot \Delta t$ $\Delta t = r/c = 3.34 \cdot 10^{-24}$ s

$\Delta E \Delta t \geq h/4\pi$ $\Delta E = h/(4\pi\Delta t) \sim 100$ MeV

The uncertainty relation for a distance from 10^{-15} m allows a max energy deviation of about 100 MeV.

A particle which is responsible for the interaction of two nucleons should have a mass in the order of 100 MeV but can only exist for about 10^{-24} s