Relativity II

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- Relativistic mechanics: the energy-momentum four-vector
- Relativistic kinematics
- Examples: scattering, collisions, decays
- Experimenally observed quantities

Relativistic Mechanics

Let us consider a particle moving with velocity $\mathbf{v}$ in an inertial frame $\mathcal{S}$. The time dt in $\mathcal{S}$ and the time $d\tau$ in a reference frame moving with the particle are connected by

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt = \frac{dt}{\gamma}$$

The quantity $d\tau$ is called *proper time*.

The *proper velocity* $\mathbf{u}$ is thus defined as

$$\mathbf{u} \equiv \frac{d\mathbf{r}}{d\tau} = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \frac{d\mathbf{r}}{dt} = \gamma \mathbf{v}$$

Note that $d\mathbf{r}$ refers to the frame $\mathcal{S}$, but the proper velocity $\mathbf{u}$ differs from the usual definition of velocity, $\mathbf{v} = d\mathbf{r}/dt$, by a factor γ

Proper time and velocity

What is the rationale behind the introduction of these quantities?

- The proper time $d\tau$ is *invariant* (or scalar), by construction
- One may define a four-vector: the *four-velocity* $u^\mu = \dfrac{dx^\mu}{d\tau}$, where:

$$u^0 = \frac{dx^0}{d\tau} = c\frac{dt}{d\tau} = \frac{c}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma c$$

$u = (\gamma c, \gamma \mathbf{v})$ transforms according to Lorentz rules, by construction: it is a four-vector, divided by an invariant

Note that dx^μ/dt follows transformation laws that are actually more complex than those for the proper velocity $dx^\mu/d\tau$!

Relativistic momentum

In Classical Mechanics one defines the *momentum* $\mathbf{p}$:

$$\mathbf{p} = m\mathbf{v}, \qquad \frac{d\mathbf{p}}{dt} = \mathbf{F}$$

What is the equivalent of $\mathbf{p}$ in Relativistic Mechanics? A good candidate for the space part is

$$\mathbf{p} = m\mathbf{u} = \frac{m\mathbf{v}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

(note that m is an invariant)

Both Newton's second law and the conservation of momentum are still valid if we use the relativistic expression for the four-vector momentum

Energy-momentum four-vector

$\mathbf{p}$ is the space part of a four-vector: what is p^0?

$$p^0 = mu^0 = \frac{mc}{\sqrt{1-\frac{v^2}{c^2}}} \equiv \frac{E}{c}$$

where $E = \gamma mc^2$ plays the role of *relativistic energy*. If $v = 0$, we obtain the famous Einstein's formula for the energy of a particle at rest:

$$E_0 = mc^2$$

What is the relation between relativistic energy and classical kinetic energy?

$$E - E_0 \simeq \frac{1}{2}mv^2 + \frac{3}{8}m\frac{v^4}{c^2} + ..., \qquad \frac{v}{c} << 1$$

For an isolated system *the energy-momentum four-vector* p^μ *is conserved*

Energy-momentum four-vector (2)

The square module of the energy-momentum four-vector is of course a Lorentz invariant and is related to the mass of the particle via:

$$p_\mu p^\mu = -(p^0)^2 + \mathbf{p} \cdot \mathbf{p} = -m^2 c^2$$

alternatively: $E^2 - p^2 c^2 = m^2 c^4$, from which one obtains $E(p)$:

$$E = \sqrt{m^2 c^4 + p^2 c^2} = c\sqrt{m^2 c^2 + p^2}$$

Classical case: $p << mc$ and $E(p) \simeq mc^2 + \dfrac{p^2}{2m}$.

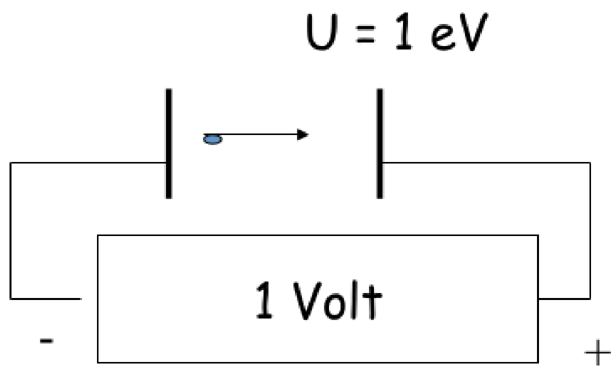
Ultrarelativistic case: $p >> mc$ and $E(p) \simeq pc$

The latter expression is exactly true, $E(p) = pc$, in the case of massless $(m = 0)$ particles traveling at the speed of light: the *photons*.

Units for high-energy physics

- In relativistic physics, it is convenient to express masses as *energy*$/c^2$ and momenta as *energy*$/c$

- Energies are typically in units of *electronvolts*, eV, or multiples of eV:
  1 eV $=$ energy acquired by an electron crossing a 1 V potential difference



  1 eV $= (1.602 \times 10^{-19}$ C) (1 V) $= 1.602 \times 10^{-19}$ J

- Typical multiples used in particle physics:
  1 MeV $= 10^6$ eV, 1 GeV $= 10^9$ eV, 1 TeV $= 10^{12}$ eV

- Mass of an electron in energy units: 0.511 MeV$/c^2$
  Mass of a proton: 938.2 MeV$/c^2$; of a neutron: 939.5 MeV$/c^2$

Units for high-energy physics (2)

Using GeV for energies and 1 fm=10^{-15} m (approximately the size of a proton) for lengths:

Quantity	SI Units	HEP Units
Length	10^{-15} m	1 fm
Energy	1.602×10^{-10} J	1 GeV
Mass	1.78×10^{-27} Kg	1 GeV$/c^2$
$\hbar$	1.055×10^{-34} J·s	6.59×10^{-25} GeV·s
c	2.998×10^{8} m/s	2.998×10^{23} fm/s
$\hbar c$	3.162×10^{-26} J·m	0.1975 GeV·fm

One often sets $c = 1$ so masses and momenta are also measured in GeV.

Since $\frac{e^2}{\hbar c} = \alpha \simeq \frac{1}{137}$, fine structure constant, $e^2 = \alpha \hbar c = 1.44$ MeV·fm

Relativistic kinematics: summary

The energy-momentum four-vector p^μ for a particle of mass m moving with velocity $\mathbf{v}$ is

$$p^\mu = (\frac{E}{c}, \mathbf{p}) = (m\gamma c, m\gamma\mathbf{v}), \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \qquad \beta = \frac{v}{c}$$

with Lorentz invariant norm $p^\mu p_\mu = -p_0^2 + |\mathbf{p}|^2 = -m^2c^2$.

For a photon of wave vector $\mathbf{k}$ and frequency $\nu = \omega/2\pi$ ($\omega = ck$):

$$p^\mu = (\frac{E}{c}, \mathbf{p}) = (\frac{\hbar\omega}{c}, \hbar\mathbf{k}) = (\hbar k, \hbar\mathbf{k})$$

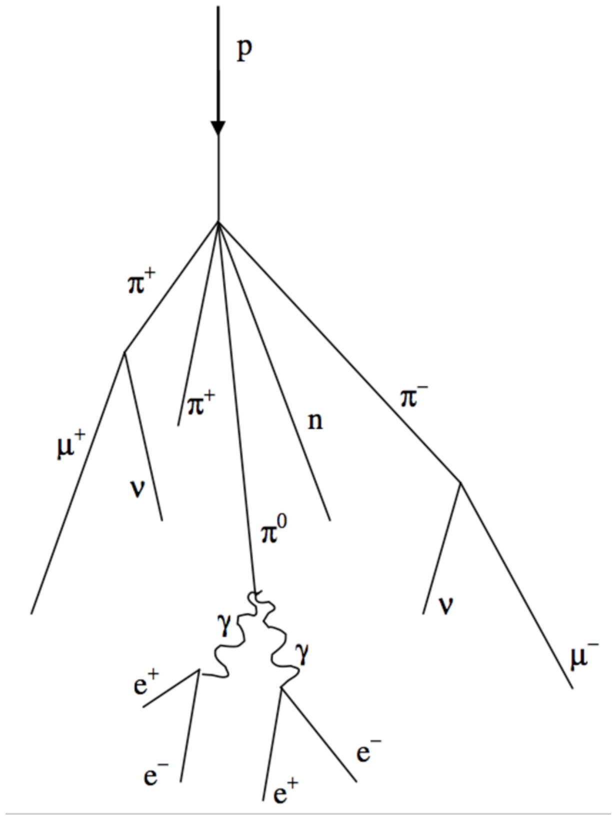
with Lorentz invariant norm $p^\mu p_\mu = 0$.

Time Dilatation in action (2)

A typical energy of a muon produced in the high atmosphere is $E \simeq 50$ GeV. Its mass is $m_\mu = 106$ MeV, so $\gamma = E/mc^2 \simeq 500$, $\beta \simeq 1$.

The half life of a muon is $\tau_0 = 2.2 \times 10^{-6}$ s in its reference frame.

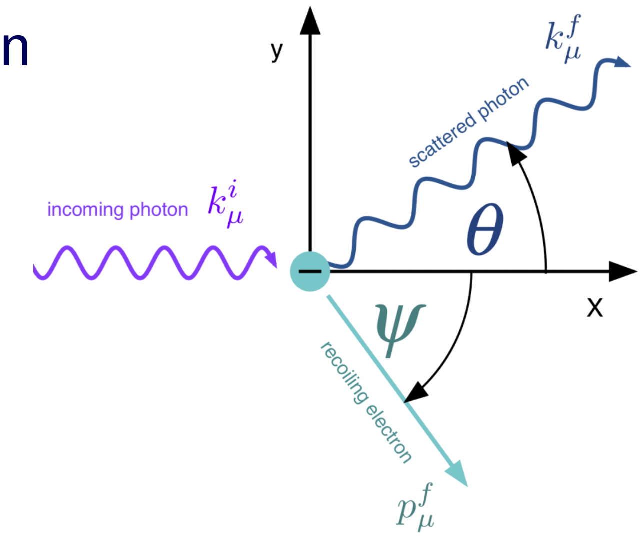
In our reference frame: $\tau = \gamma\tau_0 \simeq 1.1$ ms.
In this time, the muon travels a distance:
$s \simeq c\tau = (1.1 \times 10^{-3}\text{s})(3 \times 10^8\text{m/s})$=330 km



Relativistic kinematics: collisions

During a collision process, *the sum of energy-momentum four-vectors of all particles* $P^\mu = \sum_i p_i^\mu$ is conserved.

A simple example: a photon hitting an electron at rest (*Compton scattering*).



$$\begin{aligned} \hbar k + mc &= \hbar k' + E/c \\ \hbar \mathbf{k} &= \hbar \mathbf{k}' + \mathbf{p} \end{aligned}$$

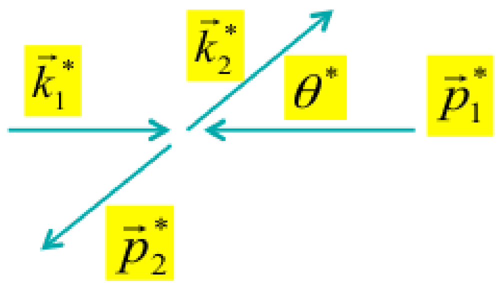
One derives $\mathbf{p} = \hbar(\mathbf{k} - \mathbf{k}')$ and $E = \hbar(k - k')c + mc^2$. Using $E^2 = m^2c^4 + p^2c^2$, one finds $\hbar k k'(1 - \cos\theta) = mc(k - k')$. In terms of the wavelength $\lambda = \dfrac{2\pi}{k}$, one can write $\boxed{\dfrac{1}{\lambda'} - \dfrac{1}{\lambda} = \dfrac{mc}{\hbar}(1 - \cos\theta)}$

In this case, the nature of the particles does not change in the collision

Relativistic kinematics: collisions (2)

A more general case: a neutrino hits an electron and produces a muon

$$\nu(k_1) + e(p_1) \to \nu(k_2) + l(p_2)$$



In the following, we use units in which $\hbar = 1$ and $c = 1$ **and the sign convention** $p^2 = p_0^2 - \mathbf{p}^2 = -p_\mu p^\mu$ **for square module of four-vectors**

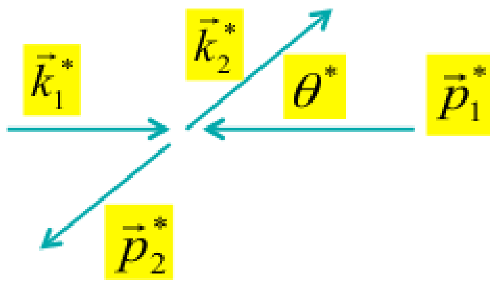
$k_i = (\omega_i, \mathbf{k}_i)$ with $\omega_i^2 - \mathbf{k}_i^2 = \epsilon^2$ (very small!)
$p_1 = (E_1, \mathbf{p}_1)$ with $E_1^2 - \mathbf{p}_1^2 = m^2$; $p_2 = (E_2, \mathbf{p}_2)$ with $E_2^2 - \mathbf{p}_2^2 = m_\mu^2$ (mass of the muon). In the laboratory (LAB) reference frame:

$$p_1 + k_1 = p_2 + k_2 \longrightarrow \begin{cases} \omega_1 + E_1 & = & \omega_2 + E_2 \\ \mathbf{k}_1 + \mathbf{p}_1 & = & \mathbf{k}_2 + \mathbf{p}_2 \end{cases}$$

Relativistic kinematics: collisions (3)

In the Center of Mass (CM) reference frame:

$$\begin{cases} \omega_1^* + E_1^* & = & \omega_2^* + E_2^* \\ \mathbf{k}_1^* + \mathbf{p}_1^* & = & \mathbf{k}_2^* + \mathbf{p}_2^* \end{cases}$$



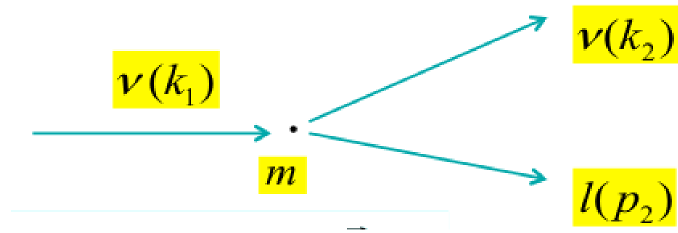
The norm is conserved and is a Lorentz invariant:

$$s = (k_1 + p_1)^2 = (\omega_1^* + E_1^*)^2 - (\mathbf{k}_1^* + \mathbf{p}_1^*)^2 \longrightarrow \sqrt{s} = \omega_1^* + E_1^*$$

because $\mathbf{k}_1 + \mathbf{p}_1 = 0$ (we are in the CM). $\sqrt{s}$ is the maximum energy that can be transformed into mass: $s = (\omega_2 + E_2)^2 \geq m_\mu^2$.
In general, the sum over masses in the final state, $\sum_f m_f \leq \sqrt{s}$.
$M = \sqrt{s}$ is also called the *effective* or *invariant* mass of a process.

Fixed target vs colliding targets

Assuming that the target is fixed:



$$
\begin{aligned}
s &= (k_1+p_1)^2 = k_1^2 + p_1^2 + 2k_1\cdot p_1 = \epsilon^2 + m^2 + 2\omega_1 E_1 - 2\mathbf{k}_1\cdot\mathbf{p}_1 \\
&= \epsilon^2 + m^2 + 2\omega_1 m
\end{aligned}
$$

$\epsilon \sim 0$ and at high energies, m is also negligible: $\boxed{\sqrt{s} \simeq \sqrt{2\omega_1 m}}$

The production of a muon (mass: $m_\mu = 106$ MeV) is possible if

$$\epsilon^2 + m^2 + 2\omega_1 m \geq m_\mu^2 \longrightarrow \omega_1 \geq \frac{m_\mu^2 - m^2}{2m} = \frac{11200 - 0.26}{1.02}\text{MeV} \simeq 11\text{GeV}$$

Fixed target vs colliding targets (2)

Let us assume now a head-to-head collision as in the picture below.

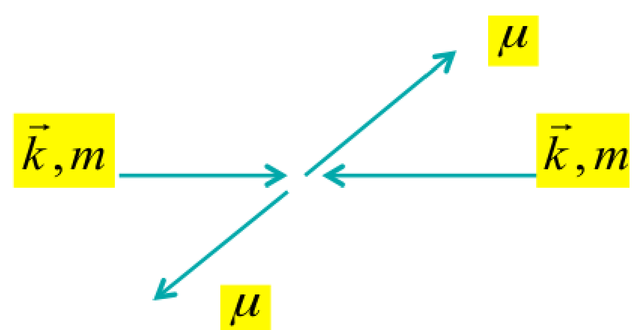
$$k_1 = (\omega_1 = \sqrt{\mathbf{k}_1^2 + m_1^2}, \mathbf{k}_1), \qquad k_2 = (\omega_2 = \sqrt{\mathbf{k}_2^2 + m_2^2}, \mathbf{k}_2)$$

$$s = (k_1 + k_2)^2 = (\omega_1 + \omega_2)^2 - (\mathbf{k}_1 + \mathbf{k}_2)^2 = (\omega_1^2 + \omega_2)^2$$

because $\mathbf{k}_1 + \mathbf{k}_2 = 0$. At high energies $\boxed{\sqrt{s} = \omega_1 + \omega_2 \simeq 2|\mathbf{k}_1|}$

Pairs of muons can be produced from an electron and a positron if

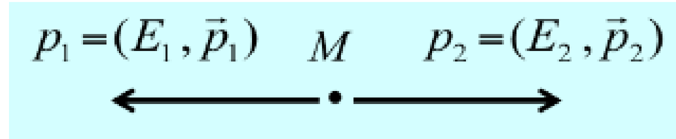
$$2|\mathbf{k}_1| \geq 2m_\mu \longrightarrow |\mathbf{k}_1| \sim m_\mu = 106\text{MeV}$$



Note the difference wrt the previous case!

Decay of a particle into two particles

An unstable particle decays into two particles. In the CM:



$$0 = \mathbf{p}_1 + \mathbf{p}_2, \qquad p = |\mathbf{p}_1| = |\mathbf{p}_2|$$

$$M = E_1 + E_2 = \sqrt{m_1^2 + p^2} + \sqrt{m_2^2 + p^2}, \qquad M \geq m_1 + m_2$$

From $E_1 = \sqrt{m_1^2 + p^2}$ one finds $p^2 = E_1^2 - m_1^2$ and $E_2 = M - E_1 = \sqrt{m_2^2 - m_1^2 + E_1^2}$. After some algebra, one finds

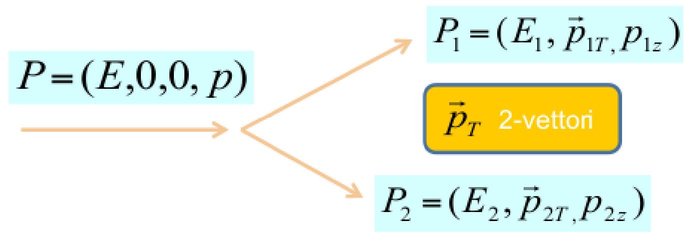
$$E_1 = \frac{M^2 + m_1^2 - m_2^2}{2M}, \qquad E_2 = \frac{M^2 + m_2^2 - m_1^2}{2M},$$

$$p = \frac{\sqrt{\left(M^2 - (m_1 - m_2)^2\right)\left(M^2 - (m_1 + m_2)^2\right)}}{2M}$$

Example: $\pi^- \to \mu^- + \nu_\mu$, $m_\pi = 140$ MeV, $E_\mu \simeq \dfrac{m_\pi^2 + m_\mu^2}{2m_\pi} \simeq 110$ MeV

Decay of a particle into two particles (2)

As before, in the laboratory frame, with particle momentum along z:



$$\begin{aligned} p &= p_{1z} + p_{2z} \\ 0 &= \mathbf{p}_{1T} + \mathbf{p}_{2T} \\ E &= E_1 + E_2 \end{aligned}$$

Let us use Lorentz transforms between LAB and CM. Note that: $\gamma = E/M$; $\beta = \sqrt{\gamma^2 - 1}/\gamma = p/E$

$$E_1 = \gamma\left(E_1^{CM} + \beta p_{1z}^{CM}\right); \qquad E_2 = \gamma\left(E_2^{CM} + \beta p_{2z}^{CM}\right)$$

$$p_{1z} = \gamma\left(p_{1z}^{CM} + \beta E_1^{CM}\right); \qquad p_{2z} = \gamma\left(p_{2z}^{CM} + \beta E_2^{CM}\right)$$

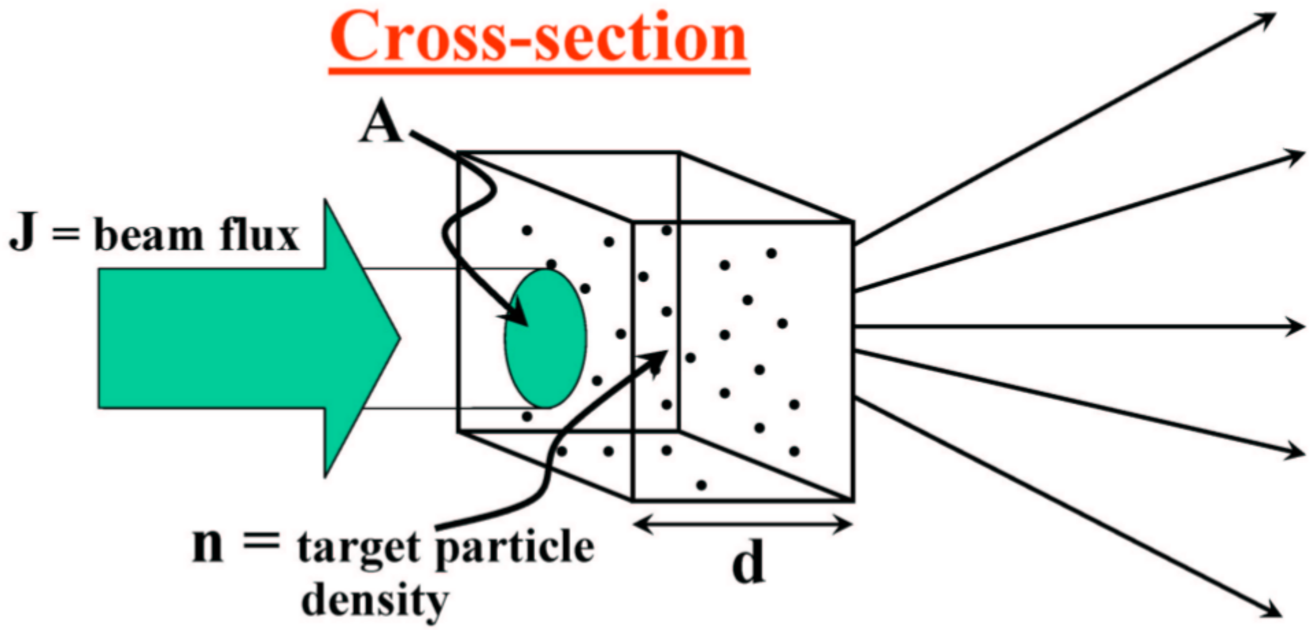
$$\mathbf{p}_{1T} = \mathbf{p}_{1T}^{CM}; \qquad \mathbf{p}_{2T} = \mathbf{p}_{2T}^{CM}$$

Observables: cross sections, decay rates

- For scattering processes, the relevant quantity to be measured in experiments is the *cross section* σ. The cross section has the units of a surface (m^2, or cm^2; also used in high-energy physics, the *barn*, 10^{-28} m^2, or 100 fm^2)

- For decay processes, the relevant quantity to be measured in experiments is the *decay rate* λ. The decay rate has the units of an inverse of a time

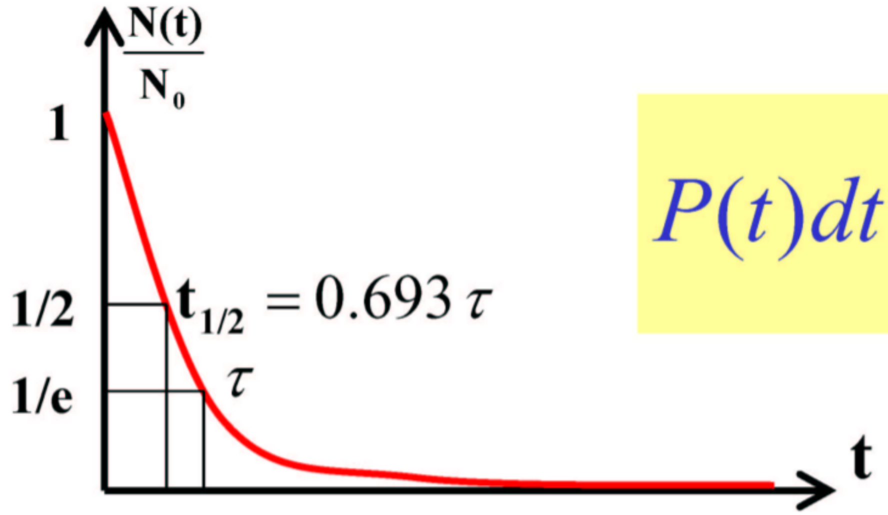
Both quantities are related to the *probability* that the considered process occurs



Definition of the (total) cross section: the number of scattering events during time dt is $dN = \sigma(JAnd)dt$, where J is the number of incident particles per unit time per unit surface, A, n, d as in the picture.

The quantity $L = JAnd$ is known as the *luminosity* (units: $[\ell]^{-2}[t]^{-1}$).

Decay time and Lifetime



$$P(t)dt = \frac{1}{\tau} e^{-t/\tau} dt$$

If the decay rate is λ, there are $dN = N(t)\lambda dt$ particle that decay in time dt. This leads to the $N(t) = N_0 e^{-\lambda t}$ law.

We define $\tau = \lambda^{-1}$ as the *mean life*. τ refers to the reference frame where the particle is at rest, that is, to the *proper time* of the particle! See the discussion about muons and cosmic rays.

For a particle with speed $v = \beta c$, $\tau' = \gamma\beta\tau$ in the LAB reference frame

Mean life and decay width

If a particle has a finite lifetime with mean life τ, i.e. it decays with probability $1/\tau$, we can define the *decay width* Γ as $\boxed{\Gamma = \frac{\hbar}{\tau}}$.

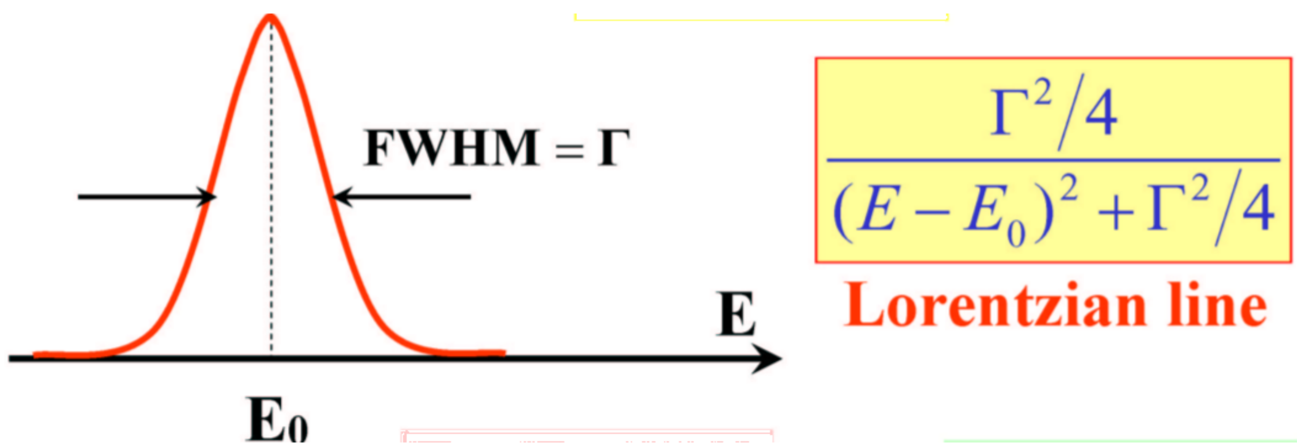
The decay width has the dimension of an energy ($\hbar =$ energy x time). In high-energy units, $\hbar = 6.58 \cdot 10^{-22}$ MeV s.

Strongly decaying particles have short lifetimes and hence large decay widths, e.g. the p(770) has $\tau = 4.4 \cdot 10{-}24$ s and $\Gamma = 151$ MeV.

Weakly decaying particles have long lifetimes and small decay widths, e.g. the K° meson has $\tau = 0.9 \cdot 10^{-10}$ s and $\Gamma = 7.3 \cdot 10^{-12}$ MeV

Mean life and decay width (2)

The decay width is a purely quantum phenomenon that can be interpreted as a manifestation of the indeterminacy principle $\Delta E \Delta t \geq \hbar$. It appears as a finite width in experiments as a function of energy:

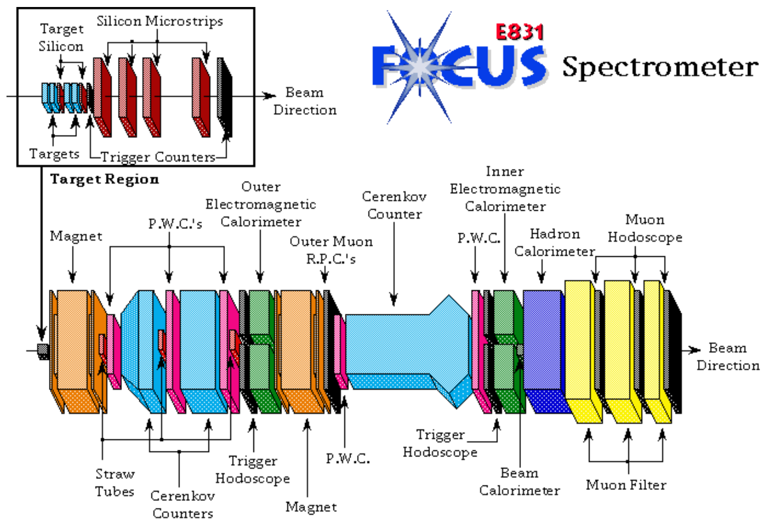


with a typical Lorentzian form (FWHM=Full Width at Half Maximum). If there are more possible decay processes with decay widths Γ_i and relative probability $B_r(i)$ such that $\sum_i B_r(i) = 1$, then $\boxed{\Gamma = \sum_i \Gamma_i}$

Invariant mass

Let us consider the decay of a particle into three (or any number)

A series of detectors can distinguish different particles, measure E and $\mathbf{p}$ of all particles

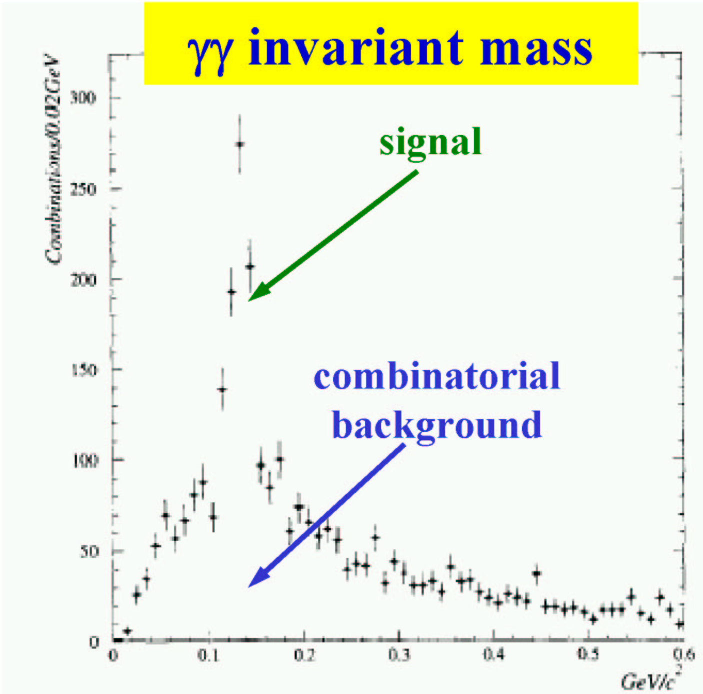


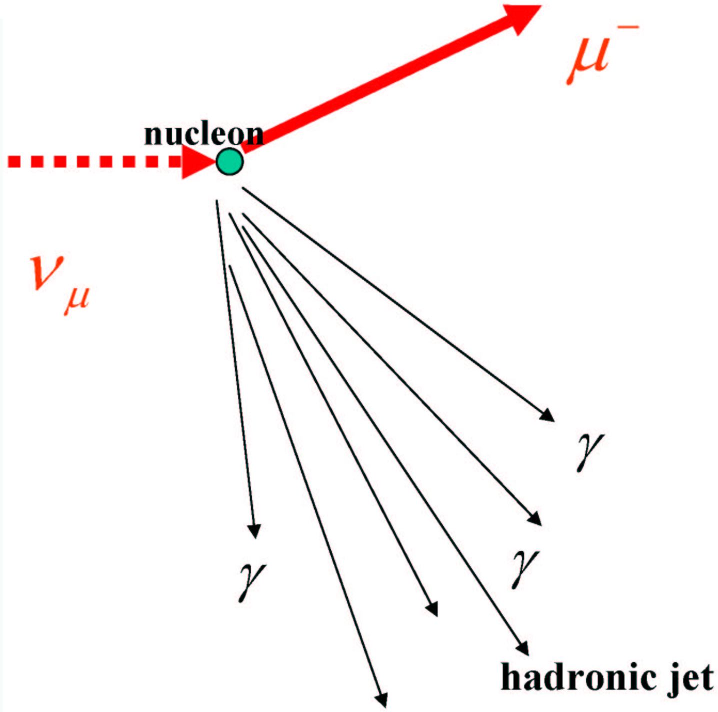
$$\begin{cases} P_1 = (E_1, \mathbf{p}_1) \\ P_2 = (E_2, \mathbf{p}_2) \\ P_3 = (E_3, \mathbf{p}_3) \end{cases}$$

Let us build the quantity $s = (E_1 + E_2 + E_3)^2 - (\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)^2$. This is a Lorentz invariant, that in the rest frame of the decaying particle can be easily evaluated: $s = M^2$, the *invariant mass*. Then one searches for peaks in histograms of invariant mass.

Example: decay of a pion into two photons, $\pi^0 \to \gamma\gamma$. Invariant mass: $M^2 = (P_{1\gamma} + P_{2\gamma})^2 = P_{1\gamma}^2 + P_{2\gamma}^2 + 2P_{1\gamma}P_{2\gamma} = 2E_1E_2(1 - \cos\theta_{12})$

Example: Data from NOMAD experiment at CERN





$$\pi^0 \rightarrow \gamma\gamma$$

Compute invariant mass $m_{\gamma\gamma}$ for all possible photon pairs

$$m_{ij} = 2E_i E_j (1 - \cos\vartheta_{ij})$$