Relativity I

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- Introduction to Einstein's special relativity
- Consequences: relativity of simultaneity, time and space dilatation
- Space-time and Lorentz transforms
- Four-vectors formalism

Why Special Relativity

Why do we need special relativity to describe the behavior of elementary particles in accelerators?

- Because elementary particles typically have large kinetic energies and speeds, often approaching c, the speed of light. In such regime, classical mechanics no longer applies.
- Because reactions leading to creation or destruction of particles may occur. The rest energy, proportional to the mass of the particles, must be taken into account in the global balance of energies.

Einstein's special relativity is the theory that properly describe the kinematics and energetics of elementary particles in accelerators.

Einstein's Relativity Principle

Special relativity is based upon Einstein's *Relativity Principle*:

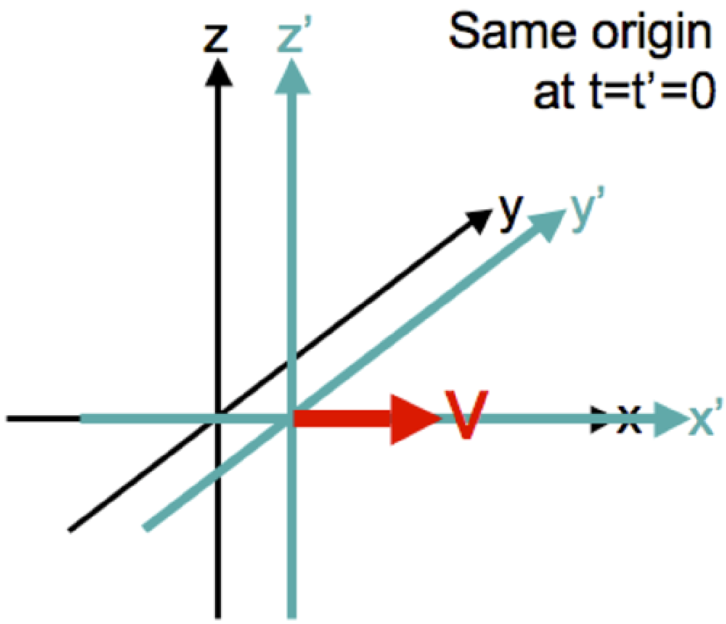
- *The laws of physics are valid in all inertial reference frames,* and
- *The speed of light in vacuum is the same in all inertial reference frames, independently upon the speed of the source*

Such principle is not compatible with Galilean transforms (valid only as limit case for speeds $v << c$), and with Newton's idea of absolute time.

We need to introduce *Lorentz transforms* and the concept of *space-time*.

Reminder: Galilean Transforms

Transformation rules between an inertial system $\mathcal{S}$ (described by x, y, z, t) and another one, $\mathcal{S}'$, (described by x', y', z', t') traveling with speed V along x with respect to S:



$$\begin{cases} x' &= x - Vt \\ y' &= y \\ z' &= z \\ t' &= t \end{cases}$$

(origins are assumed to coincide at $t = 0$). The *velocity addition rules* follows: a particle with velocity $\mathbf{v}$ in $\mathcal{S}$ has velocity $\mathbf{v}' = \mathbf{v} - \mathbf{V}$ in $\mathcal{S}'$

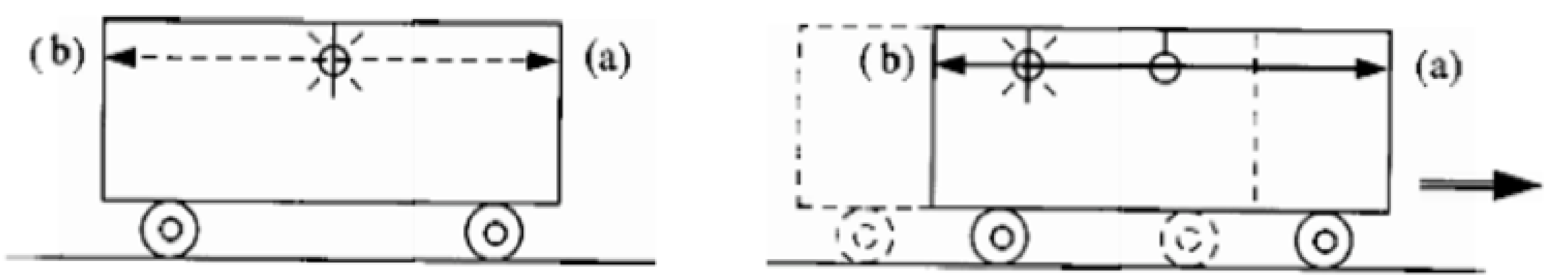
The inverse transform from $\mathcal{S}'$ to $\mathcal{S}$ is obtained reversing the sign of V.

Consequences of Relativity Principle

Einstein's Relativity Principle has rather surprising consequences, that can be demonstrated on the basis of simple *gedankenexperimente* ("though experiments"):

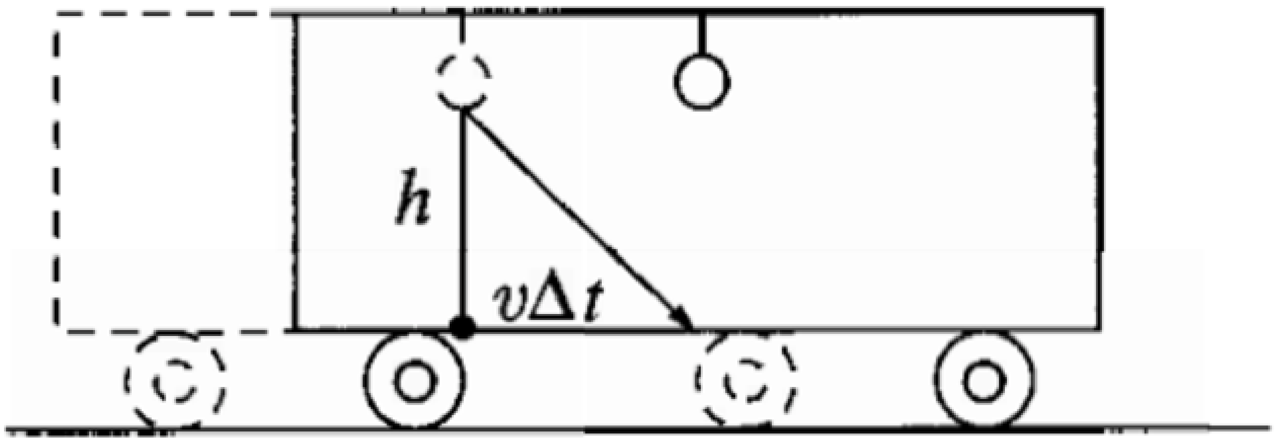
- **Relativity of simultaneity:** *Two events that are simultaneous (happen at the same time) in an inertial system may not be simultaneous in another inertial system*

In the reference frame of the moving observer, the ray of light hits simultaneously the two walls; in the laboratory (observer at rest) reference frame, this does not happen.



Consequences of Relativity Principle (2)

- **Time Dilatation:** *A moving clock runs slower*



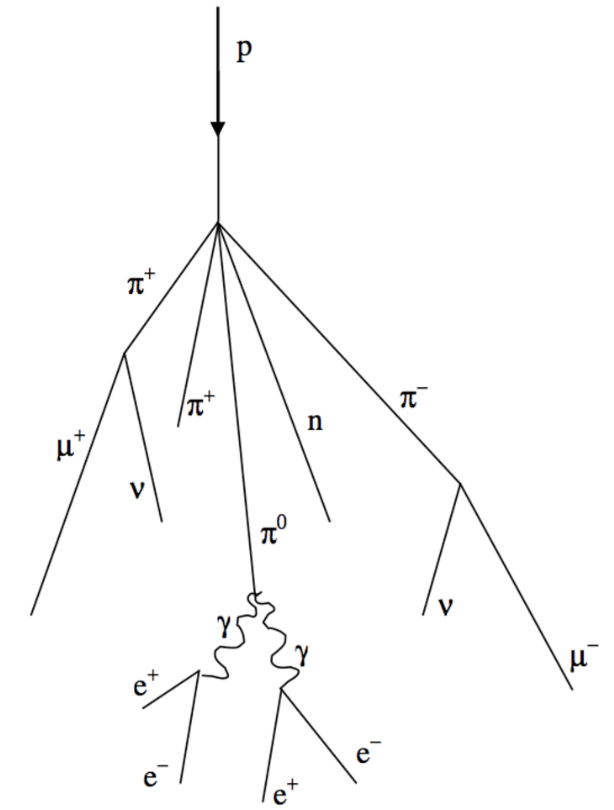
Let us consider a ray of light that hits the floor. This happen after $\Delta t' = h/c$ in the reference frame of the moving observer, in $\Delta t = \sqrt{h^2 + (V\Delta t)^2}/c$ in the reference frame for the fixed observer. Thus:

$$\Delta t' = \sqrt{1 - V^2/c^2}\Delta t \equiv \gamma^{-1}\Delta t < \Delta t$$

where we have introduced the factor $\gamma = 1/\sqrt{1 - V^2/c^2}$.

Time Dilatation in action: cosmic rays

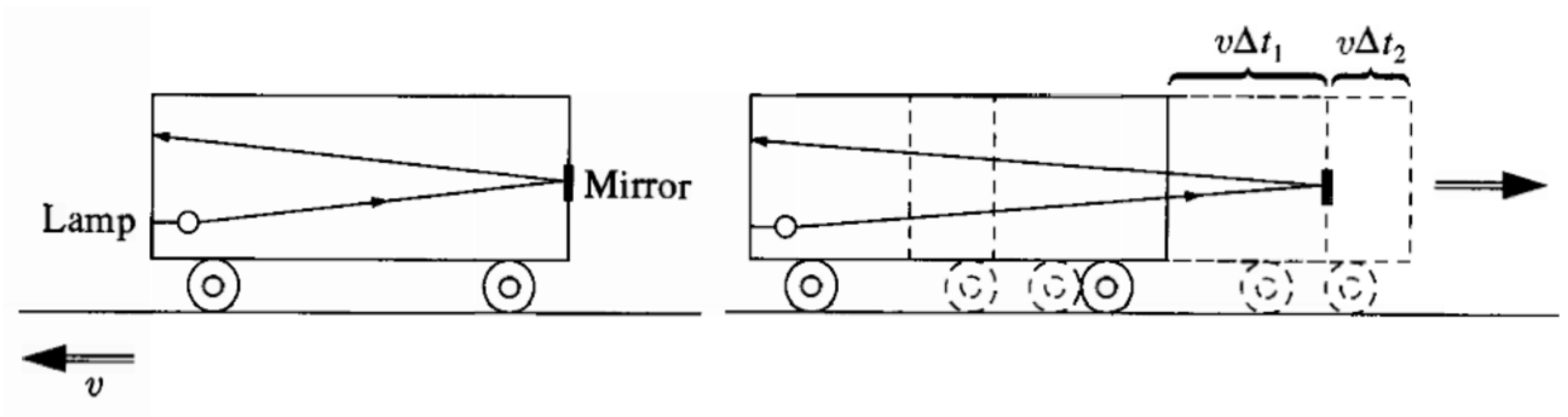
Muons (μ^+, μ^-): components of cosmic rays, generated by heavier particles (mesons) in the high atmosphere (~ 15 Km)
Muons' energy is such that $v \sim c$. The half life of a muon is $\tau_0 = 2.2\mu\text{s}$. During such time, a muon with speed c travels a distance:
$$s = c\tau_0 = 2.2 \times 10^{-6}\text{s} \cdot 3 \times 10^8\text{m/s} = 660\text{m}.$$
The flux of muons is however easily measurable at ground level. How is that possible?



The half life of muons refers to the *reference frame of the muon* (that is: moving with it). In a laboratory reference frame (fixed on the Earth):
$$\tau = \tau_0 / \sqrt{1 - v^2/c^2} >> \tau_0!$$

Consequences of Relativity Principle (3)

- **Length contraction:** *A moving object becomes shorter* (only in the direction of velocity)

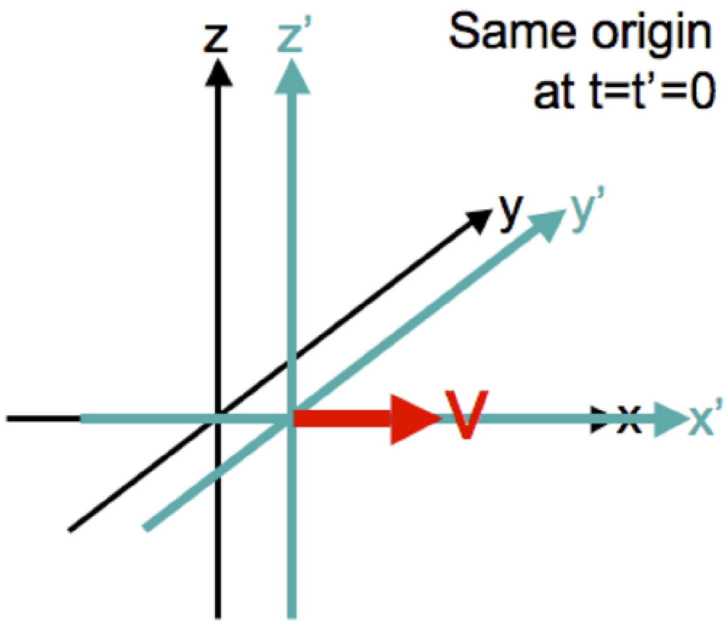


A light ray is reflected by the wall in a time $\Delta t' = 2\Delta x'/c$ for the moving observer, $\Delta t = \Delta t_1 + \Delta t_2$, where $\Delta t_1 = (\Delta x + v\Delta t_1)/c$, $\Delta t_2 = (\Delta x - v\Delta t_2)/c$, for the fixed observer. One finds $\Delta t = 2\gamma^2 \Delta x/c$. Finally, since $\Delta t = \gamma \Delta t'$:

$$\Delta x' = \gamma \Delta x = \frac{1}{\sqrt{1 - V^2/c^2}} \Delta x$$

Lorentz Transforms

Transformation rules between an inertial system $\mathcal{S}$ (described by x, y, z, t) and a $\mathcal{S}'$ one (described by x', y', z', t') traveling with speed V along x with respect to S:



$$\begin{cases} x' &= \gamma(x - Vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma(t - \frac{V}{c^2}x) \end{cases}$$

(origins are assumed to coincide at $t = 0$)

In the limit $\gamma = 1$, $V/c << 1$, we recover Galilean transforms. The inverse transform from $\mathcal{S}'$ to $\mathcal{S}$ is obtained by reversing the sign of V.

Velocity addition rules

Let us assume that a particle travels in $\mathcal{S}$ for a distance dx in a time dt: $v_x = dx/dt$. In $\mathcal{S}'$ it travels for a a distance $dx' = \gamma(dx - V dt)$ in a time $dt' = \gamma\left(dt - (V/c^2)dx\right)$. It follows that:

$$v'_x = \frac{dx'}{dt'} = \frac{\gamma(dx - V dt)}{\gamma\left(dt - (V/c^2)dx\right)} = \frac{v_x - V}{1 - v_x V/c^2}$$

The is *Einstein's velocity addition rule*. In the limit $V/c << 1$, it reduces to the usual (Galilean) rule: $v'_x = v_x - V$. If $v_x = c$, also $v'_x = c$. In no case can one exceed the speed of light c.

Einstein's second Relativity Principle can actually be reformulated as follows:

- *For all inertial reference frames, there is a finite limit speed c for physical objects*

The concept of Space-Time

- Let us define an *event* as a point in the space (x, y, z) at time t (in a given inertial reference frame)

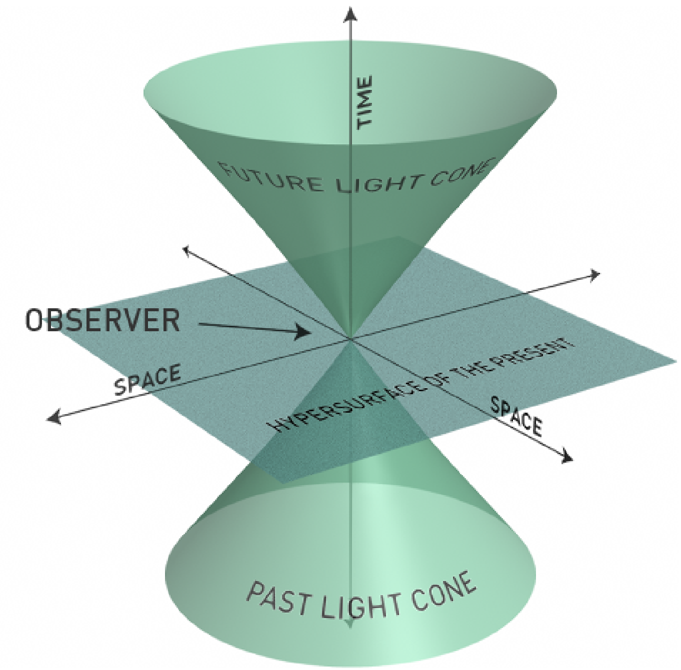
- Let us represent an event as a vector in a four-dimensional space: a *four-vector*. It is convenient to use (ct, x, y, z) as dimensionally consistent coordinates

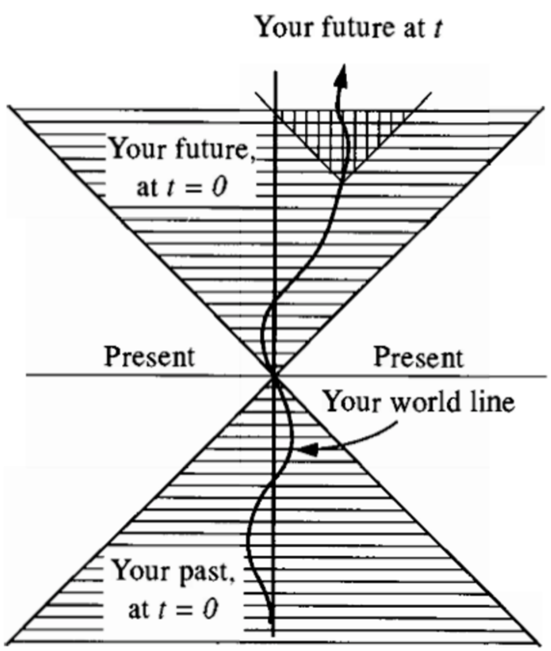
- A moving particle describes a line in the space-time, called *world line*

- Light rays passing through origin at $t = 0$ define a surface called *light cone*.

Light cone

In the figure, a typical example of a light cone, projected over two space coordinates and with the time axis in the vertical direction (*Minkowski diagram*)

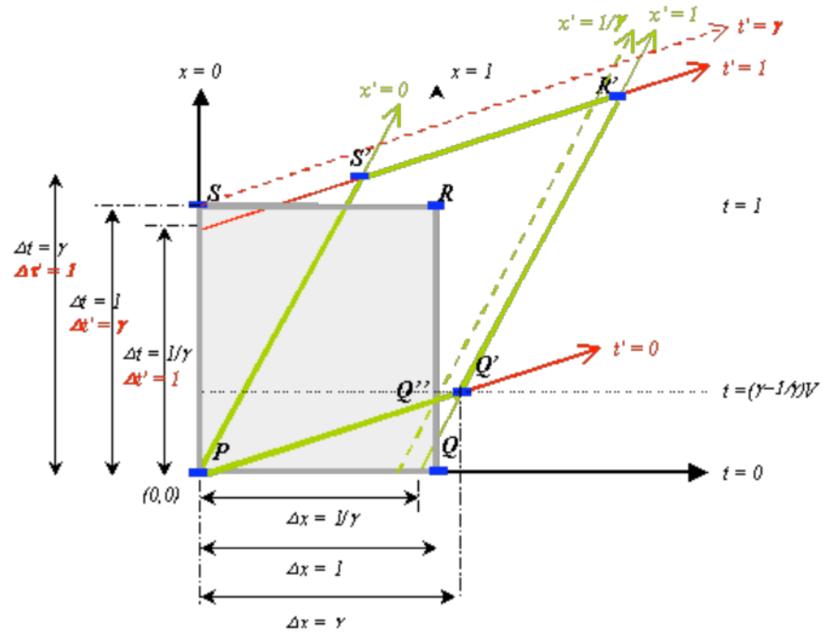




The world line of a physical object always stays "inside" the cone; at time $t = 0$ it goes from the "cone of the past" (below) to the "cone of the future" (above); the tangent to the curve in any point is always "inside" the cone (because the speed $v < c$). This is called a "time-like" world line.

Lorentz transforms in space-time

- Lorentz transforms are generalized rotations in space-time, that modify the relative directions of the axis, expand or contract them.



- Contrary to usual rotations, they do not leave the usual square module of vectors: $x^2 + y^2 + z^2 + (ct)^2$, unchanged. The invariant quantity is $I = x^2 + y^2 + z^2 - (ct)^2$: the *square module of four-vectors*

- Lorentz transforms move a point over the set of points with constant I: an hyperbole in t and x, a rotation hyperboloid in t, x, y, with asymptotes on the light cone.

Events in space-time

- Let us consider two events in space-time:

$$\mathbf{x}_1 = (ct_1, x_1, y_1, z_1), \quad \mathbf{x}_2 = (ct_2, x_2, y_2, z_2)$$

  and their four-vector difference (*interval*): $\Delta\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$. Depending upon the value of $I = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2$, we can distinguish the interval into

  – *Type space*: $I > 0$. The two events may be simultaneous in some reference frame
  – *Type light*: $I = 0$. The two events are "connected" by a ray of light
  – *Type time*: $I < 0$. The two events cannot be simultaneous in any reference frame

Four-vector formalism

Let us introduce notations: $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$

$$\begin{cases} (x^0)' & = & \gamma(x^0 - \beta x^1) \\ (x^1)' & = & \gamma(x^1 - \beta x^0) \\ (x^2)' & = & x^2 \\ (x^3)' & = & x^3 \end{cases}$$

where $\beta = V/c$. Lorentz transforms in matrix form:

$$\begin{pmatrix} (x^0)' \\ (x^1)' \\ (x^2)' \\ (x^3)' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

In general, *a four-vector is an object whose components follow Lorentz transforms*

Four-vector formalism (2)

Alternatively, Lorentz transforms may be written as:

$$(x^i)' = \sum_{j=0}^{3} \Lambda^i_j x^j, \qquad i = 0 \div 3$$

where Λ^i_j is the matrix earlier defined (the reason for "high" and "low" indices will be clarified soon; note that the matrix has unit determinant)

It is easily verified that such transform conserves the square module I of four-vectors:

$$I = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^0)^2$$

and in general, the four-vector analogue of the scalar product:

$$\mathbf{x} \cdot \mathbf{y} \equiv x^1 y^1 + x^2 y^2 + x^3 y^3 - x^0 y^0$$

Covariant and contravariant indices

It is practical to introduce *covariant* components:

$$x_0 = -ct, \quad x_1 = x, \quad x_2 = y, \quad x_3 = z$$

in addition to those (known as *contravariant*) already introduced. The only difference is in the sign of the time component. The square module and scalar product of four-vectors become:

$$I = \sum_{i=0}^{3} x_i x^i, \qquad \mathbf{x} \cdot \mathbf{y} = \sum_{i=0}^{3} x_i y^i = \sum_{i=0}^{3} x^i y_i$$

The *Einstein convention* is used: repeated indices are understood to be summed. In all physical quantities, covariant indices are summed with contravariant indices. This guarantees both the correct form and the correct invariance properties with respect to a change of reference frame.