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Quantum Mechanics II

Schroedinger equation

Wave-particles wave function governed by

$$i\hbar \frac{\partial}{\partial t}\psi(t, \mathbf{x}) = \hat{H}\psi(t, \mathbf{x})$$

where $\hat{H}$ is the "Hamiltonian" $(E + V)$ and

$$\begin{aligned} \vec{p} &\to -\imath\hbar \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = -\imath\hbar\vec{\nabla} \\ E &\to \imath\hbar\frac{\partial}{\partial t} \end{aligned}$$

Example of a free particle possible solution (1D):

$$\begin{aligned} \hat{H} &= \frac{p^2}{2m} \\ \psi(t, x) &= \exp\left(\frac{\imath}{\hbar} \left(px - \frac{p^2}{2m} t \right) \right) \end{aligned}$$

(verify as an exercise)

Time independent

Schroedinger equation

(From here on assume always 1D, i.e. $\mathbf{x} \to x$)

Under the general assumption

$$\psi(t,x) = \phi(t)\chi(x)$$

the Schroedinger equation can be split into:

$$\imath\hbar\frac{\partial}{\partial t}\phi(t)\chi(x) = -\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\phi(t)\chi(x) + V(t,x)\phi(t)\chi(x)$$

if time independent potential $V(t,x) = V(x)$

$$\chi(x)\ \imath\hbar\frac{\partial}{\partial t}\phi(t) = \phi(t)\left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\chi(x) + V(x)\chi(x)\right)$$

Time independent

Schroedinger equation

$$\frac{1}{\phi(t)}\ \imath\hbar\frac{\partial}{\partial t}\phi(t) = \frac{1}{\chi(x)}\left(-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\chi(x) + V(x)\chi(x)\right)$$

$$A(t) = B(x) \quad \Rightarrow A(t) = E = B(x)$$

Which means the time independent Schoredinger equation satisfies the eigenvalue problem:

$$-\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2}\chi(x) \quad + \quad V(x)\chi(x) = E\chi(x)$$

$E \rightarrow$ "Eigenvalue" $\qquad \chi(x) \rightarrow$ "Eigenfunction"

Eigenvalue problem

$$\hat{H}\chi(x) = E\chi(x)$$

Axiomatic formulation

1. The state of a quantum mechanical system is completely specified by its wave function

$$\psi(t, \mathbf{x}), P \propto |\psi(t, \mathbf{x})|^2, \quad \int dx \; dt \; |\psi(t, \mathbf{x})|^2 = 1$$

   The wavefunction must be continuous and finite.

2. To every observable in classical mechanics there corresponds a linear, Hermitian ($H^\dagger = H$) operator in quantum mechanics (ex: $E \to -\imath\hbar\vec{\nabla}$).

Axiomatic formulation

3. Measurements of observables $\hat{O}$ only outputs corresponding eigenvalues o_i, where by definition

$$\hat{O}\psi_i = o_i\psi_i$$

Note that a general he state doesn't need to be an eigenstate, but in general it is a superposition

$$\psi \sum_i c_i\psi_i.$$

The wavefunction immediately "collapses" into the corresponding eigenstate Ψ_i

4. If a system is in a state ψ, then the average value of the observable corresponding to $\hat{O}$ is given by

$$< O >= \int_{-\infty}^{\infty} \psi^* \hat{O}\psi \; d\tau$$

Axiomatic formulation

5. The wavefunction of a system evolves in time according to the time-dependent Schroedinger equation

$$\hat{H}\psi(\mathbf{r}, t) = i\hbar\frac{\partial\psi}{\partial t}$$

   (which must be accepted as a postulate)

6. The total wavefunction of a system of identical (fermions)bosons must be (anti)symmetric with respect to the interchange of all coordinates of one with those of another. Electronic spin must be included in this set of coordinates.

Klein Gordon equation

What is a generic relativistic equation for a scalar field? Start from Schroedinger:

$$\hat{H}\phi = E\phi,$$

where now use the relativistic relation:

$$E = \sqrt{p^2c^2 + m^2c^4}$$

and, to avoid $\sqrt{\dots}$, "square" it

Klein-Gordon equation

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\phi = 0$$

where $\Box = \partial_\mu\partial^\mu$, and $\partial_\mu = \frac{\partial}{\partial x_\mu}$

Dirac equation

Dirac, 1928: is there a way to "linearize" KG equation? Introduce a set of 4×4 matrices γ_μ such that

$$\{\gamma_\mu, \gamma_\nu\} = 4g_{\mu\nu}$$

where

$$g_{\mu\nu} = diag(1, -1, -1, -1)$$

and, to avoid $\sqrt{\ldots}$, "square" it

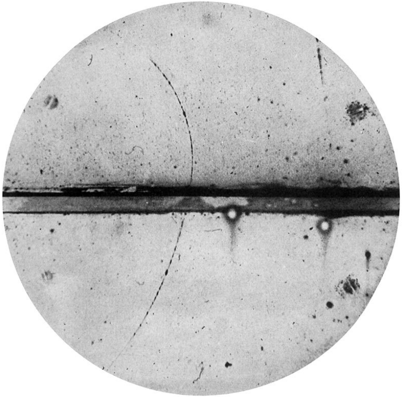
Klein-Gordon equation

$$(\imath\gamma^\mu\partial_\mu - m)\,\psi = 0$$

Predicts antimatter!

Positron discovery

Anderson, 1932



Cloud chamber photograph by C. D. Anderson of the first positron ever identified. A 6 mm lead plate separates the upper and lower halves of the chamber. The deflection and direction of the particle's ion trail indicate the particle is a positron .

Feynman diagrams

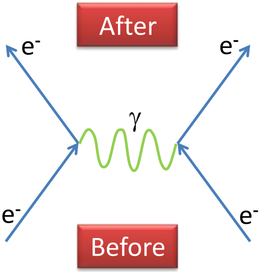
Richard Feynman designed a way to illustrate interactions between particles through exchange particles. His idea is simple:

- Straight lines represent particles before and after the interaction
- Wavy lines connect the straight lines and represent the particle exchange
- The charge must be conserved at each junction
- Particle lines point in the same direction for both attraction and repulsion
- The direction of the lines does **not** show the direction of the particles

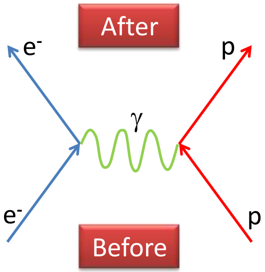
Calculational tool, not a picture of the phenomenon!

Feynman diagrams

The exchange particle responsible for electromagnetic interactions is a photon. So, Feynman's diagrams for e^- - e^- and e^- - p interactions are:



electron – electron repulsion



electron – proton attraction

$$\propto \frac{e^2}{q^2}$$