UNIVERSITY OF BIRMINGHAM

A systematic approach to the realization of quantum optical systems

Joe Bentley

Haixing Miao

In collaboration with:

Hendra Nurdin, Naoki Yamamoto, Yanbei Chen, Rana Adhikari, Denis Martynov

University of Birmingham
Institute for Gravitational Wave Astronomy

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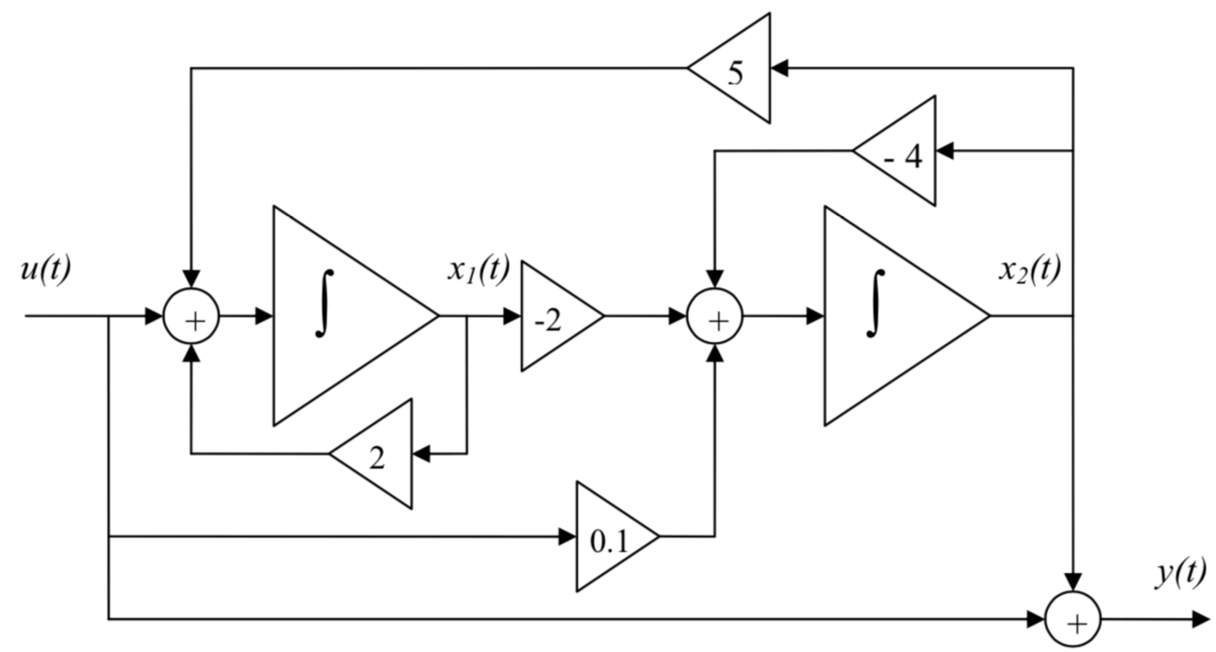
Realizing classical systems

- For *classical systems* it is easy to realize *arbitrary* state-space representations using integrators and feedback

$$\begin{aligned} \frac{dx(t)}{dt} &= \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0.1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + u(t), \end{aligned}$$



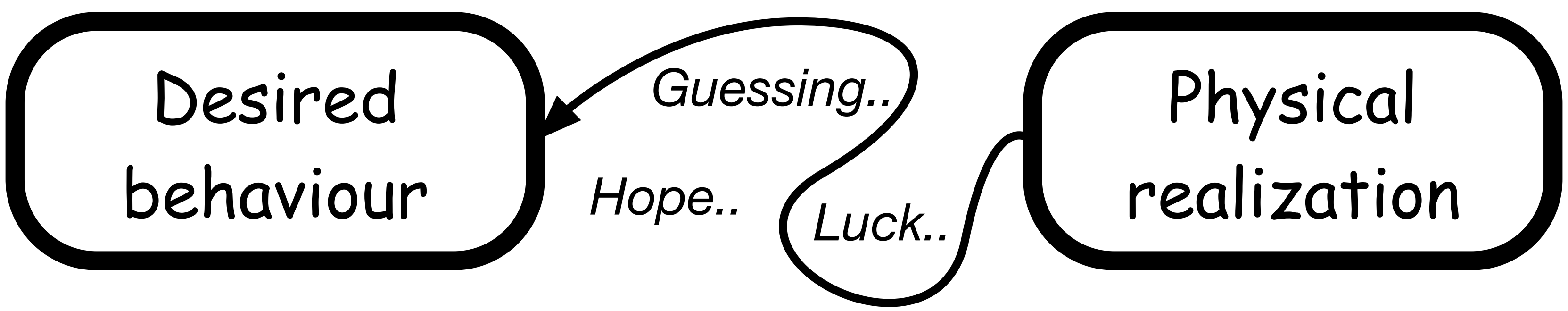
- For *quantum systems* most state-space representations not physically possible

Need to conserve $[x_i, x_j]$

Current limits of our quantum realization techniques

Say we want to build quantum system with a *desired transfer function* or general behaviour…

Current methods



New method

Quick intro to state-space representation

Used to dealing with frequency-domain transfer functions

$$y(s) = G(s)u(s)$$

y outputs u inputs

Control theorists prefer time-domain *state-space representation*

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

y outputs u inputs

x internal system state

A system dynamics matrix

B input coupling matrix

C output coupling matrix

D "Direct feed" matrix

(everything linear here)

State-space degeneracy

- **(A, B, C, D) -> G(s)** is *many-to-one* mapping
    - *Many* state-space reps, *(even non-physical ones)*, correspond to *one* transfer function **G(s)**
- Therefore, actual (A, B, C, D) gives physical insight
    - *bijection* exist between *(A, B, C, D)* and *full Hamiltonian for system*

Finding the state-space rep for transfer func.

Identify desired transfer function **(freq. domain)**

↓

Find a *state-space* representation **(time domain)**

↓

Find the physically realizable one

Example: tuned cavity

$$G(i\omega) = \frac{i\omega - \gamma}{i\omega + \gamma}$$

$$\dot{\vec{x}} = A\vec{x} + B\vec{u}$$

$$\vec{y} = C\vec{x} + D\vec{u}$$

No unique mapping from G to (A, B, C, D)!

Need to ensure that

$$d[x_i, x_j] = 0$$

Constrains (A, B, C, D)

Constraints on (A, B, C, D) for physical realizability

It can be shown that $d[x_i, x_j] = 0$ *(quantum îto product)*

implies that
$$AJ + JA^{\dagger} + BJB^{\dagger} = 0$$
$$JC^{\dagger} + BJD^{\dagger} = 0$$

if we use cavity mode operators (annihilation & creation)

$x_i = a, \quad x_j = a^{\dagger}$ and $[a, a^{\dagger}] = 1$

then $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ J is *"commutation matrix"*

Example: unstable filter

- Unstable filter = optomechanical device w/ *negative dispersion* $\phi \propto -\Omega\tau$

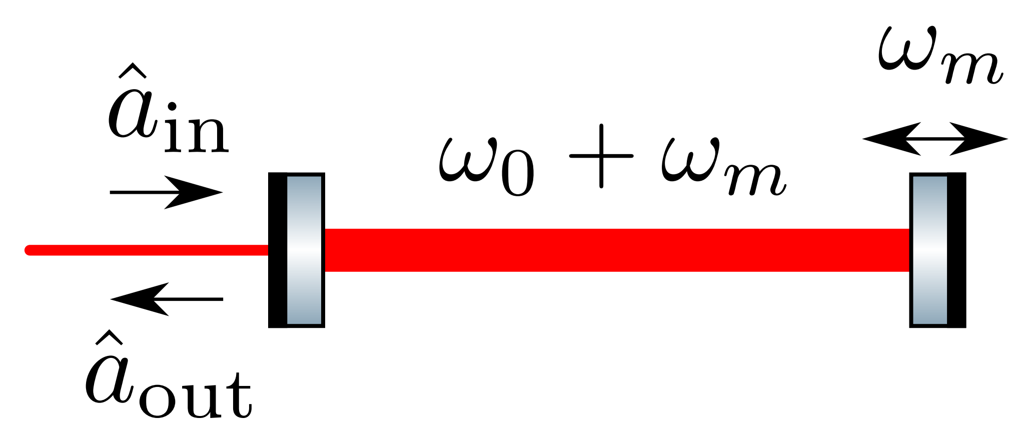
- Can be used to broaden bandwidth of GW detector *without sacrificing sensitivity*

$$G(s) = \frac{s+2}{s-2}$$

opp. sign to tuned cavity

Known physical realization:

cavity coupled to mirror via off-resonant pump $\omega_0 + \omega_m$



Finding realizable state-space representation

Guess a *state-space representation*, not necessarily physical

$$\begin{bmatrix} \dot{a} \\ \dot{a}^\dagger \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \begin{bmatrix} u \\ u^\dagger \end{bmatrix}$$

$$\begin{bmatrix} Y \\ Y^\dagger \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \begin{bmatrix} u \\ u^\dagger \end{bmatrix}$$

Found using *"Controllable Canonical Form"*

Transform matrices to obey

$$AJ + JA^\dagger + BJB^\dagger = 0$$
$$JC^\dagger + BJD^\dagger = 0$$

(I omitted details but it is easy to do)

Find *physically realizable form*

$$\begin{bmatrix} \dot{a} \\ \dot{a}^\dagger \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + 2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ u^\dagger \end{bmatrix}$$

$$\begin{bmatrix} Y \\ Y^\dagger \end{bmatrix} = 2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ a^\dagger \end{bmatrix} + \begin{bmatrix} u \\ u^\dagger \end{bmatrix}$$

Final steps

From this (A, B, C, D) can find (H, L)

H internal system Hamiltonian

L linear coupling matrix

(Usually) have a clear physical realization

<u>For our state-space find</u> $(\hbar = 1)$

$H = 0$ (chose *rotating frame* w.r.t resonant freq.)

$$L = \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} \implies H_{int} = \frac{\sqrt{\gamma_2}}{2}(ab + a^\dagger b^\dagger)$$

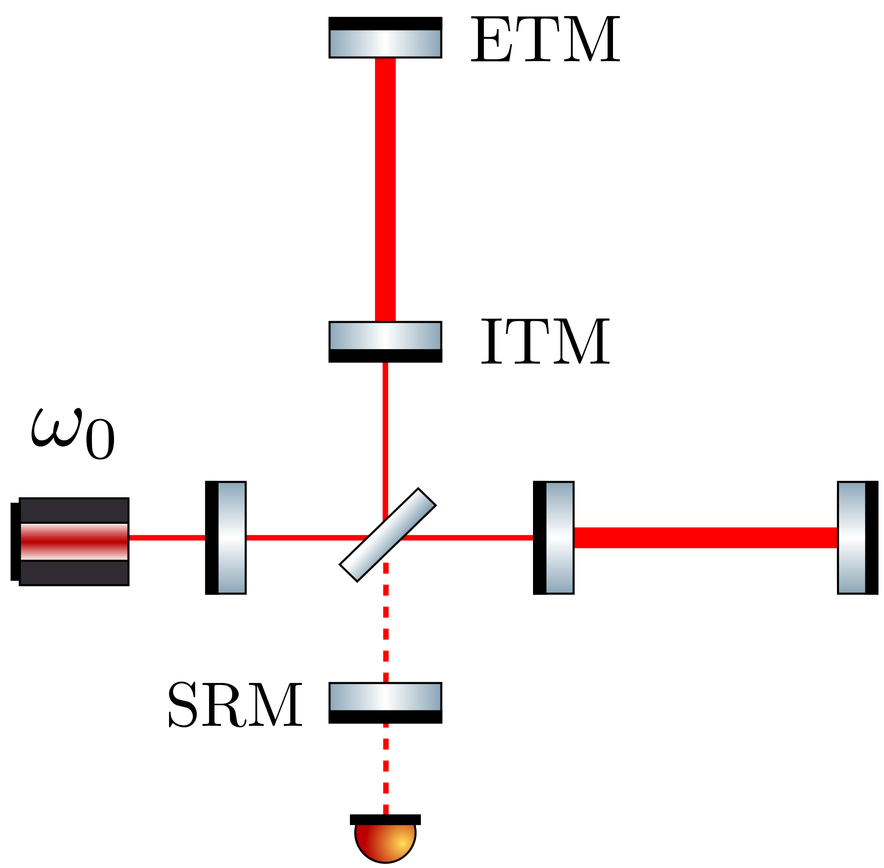
Same physics as unstable filter

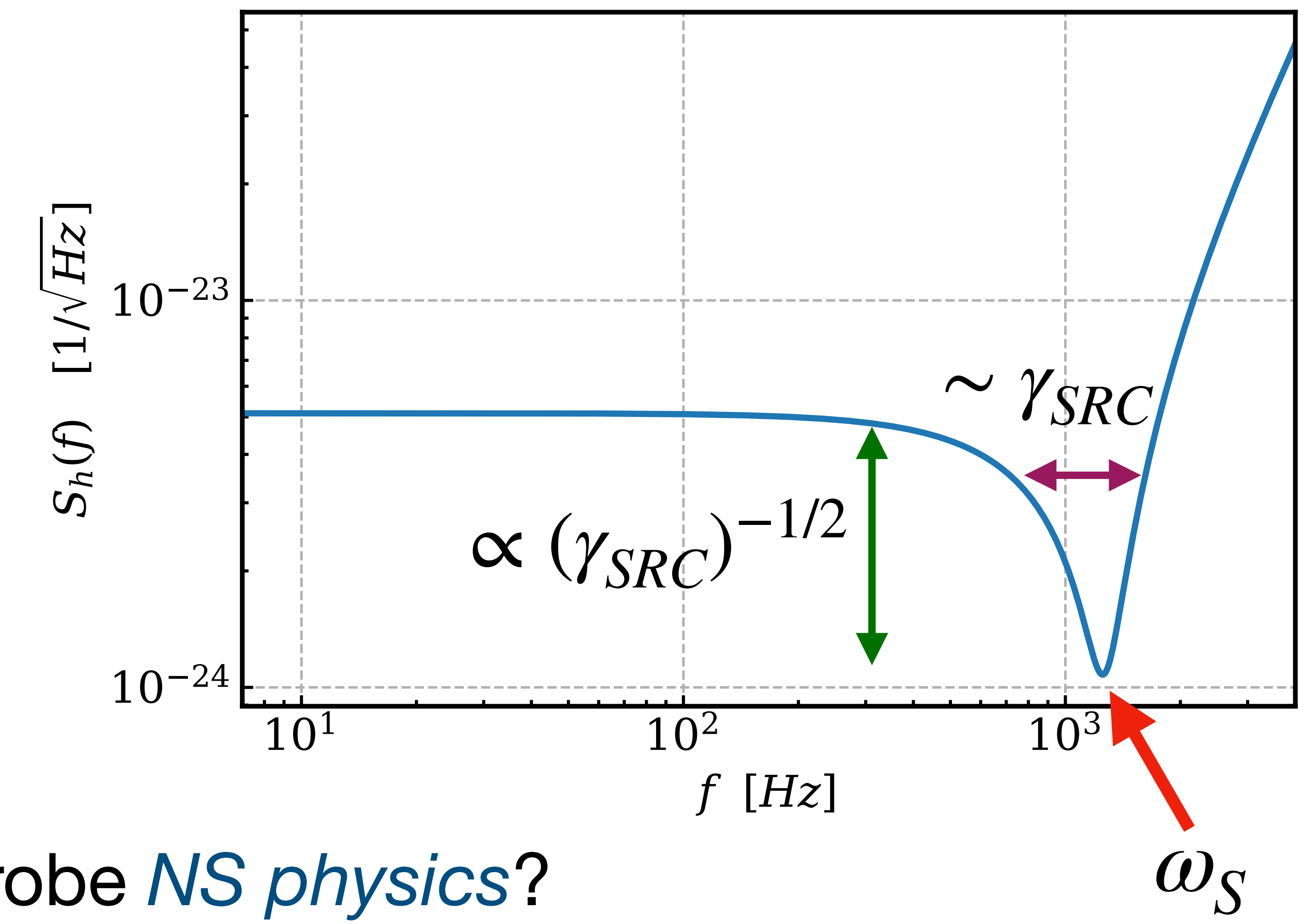
adiabatically eliminated high frequency auxiliary mode

(e.g. mechanically suspended mirror)

Proof of concept: coupled cavity resonance

Coupling of tuned SRC and arms leads to resonance splitting



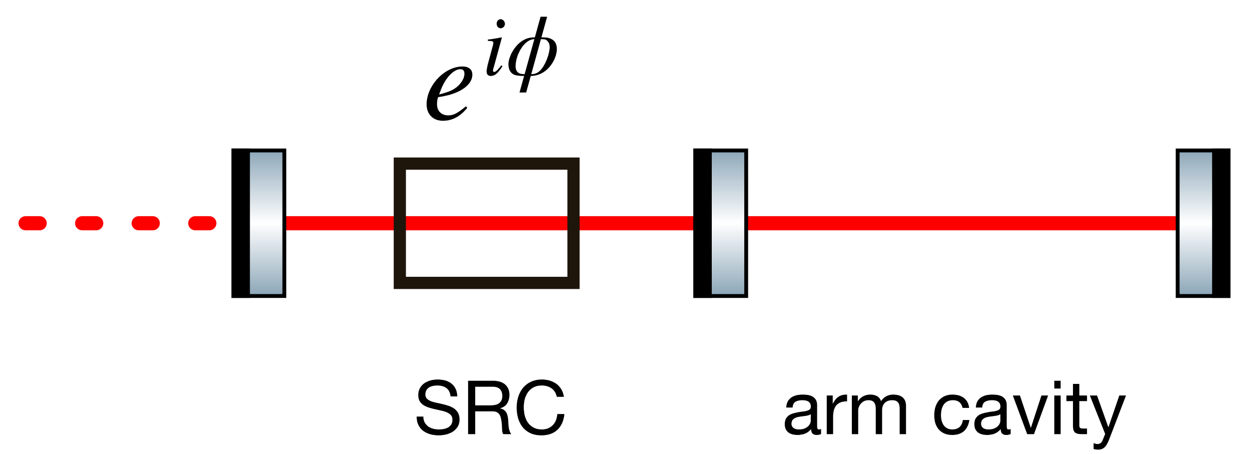


Maybe we can tune ω_S to probe *NS physics*?

Would like to boost width of dip *without sacrificing depth*

Broadening with frequency-dependent phase

Imagine a "black box" w/ freq. dependent phase $\phi(\Omega)$



Found phase that broadens dip while *keeping depth same*

$$\phi(\Omega) = -\frac{\tau_1(\Omega^2 - \omega_s^2)(\gamma_1 - \gamma_1')}{\gamma_1 \Omega}.$$

Controls *amount* of broadening

In pole-zero form

$$G(i\Omega) = e^{i\phi} \approx \frac{2\gamma_1\Omega - i\tau_1(\Omega^2 - \omega_s^2)(\gamma_1 - \gamma_1')}{2\gamma_1\Omega + i\tau_1(\Omega^2 - \omega_s^2)(\gamma_1 - \gamma_1')}$$

Next steps to find physical realization for black box

1. From pole zero form find a *state-space representation*

   $$(A, B, C, D)$$

2. Transform to *physically realizable state-space rep.*

   $$(A', B', C', D') \quad \text{satisfying} \quad \begin{aligned} A'J + J(A')^\dagger + B'J(B')^\dagger &= 0 \\ J(C')^\dagger + B'J(D')^\dagger &= 0 \end{aligned}$$

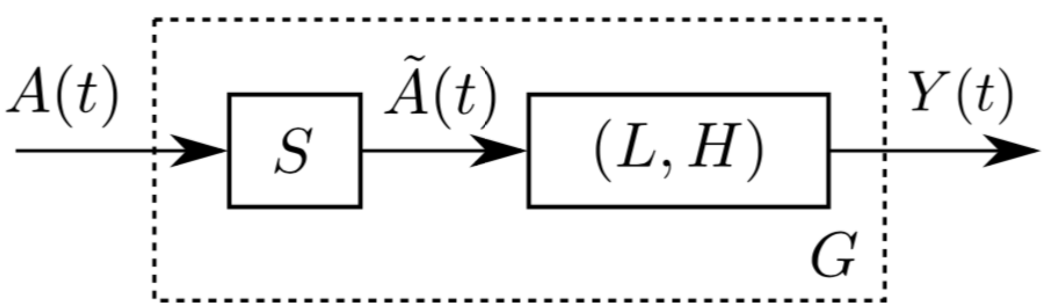
3. Find *physical realization*

Step 3 not as trivial as unstable filter…
…because transfer func. *second order* in Ω

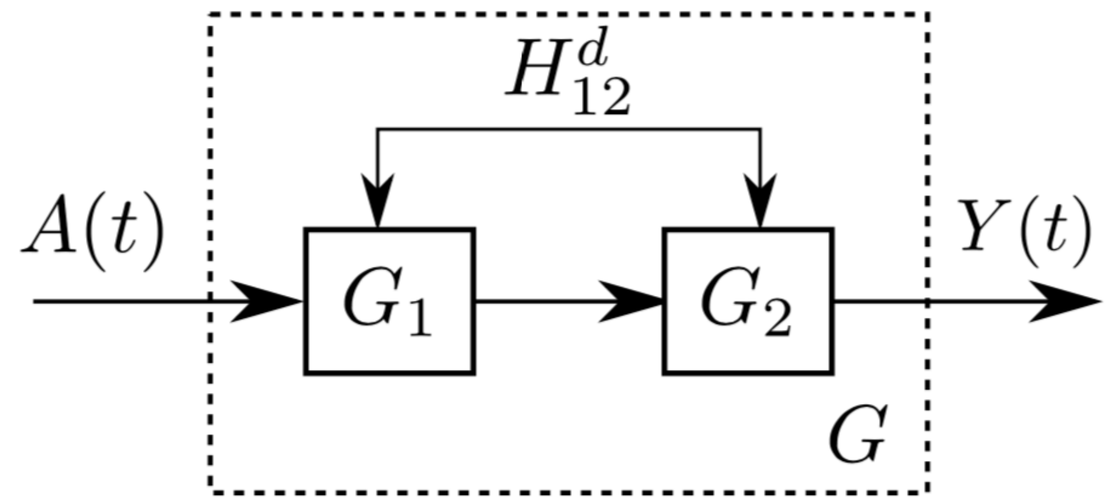
$\Longrightarrow$ **2 internal degrees of freedom**

More degrees of freedom

Single degree of freedom "generalized open oscillator"



2 degrees of freedom = 2 coupled generalized open oscillators



Main Synthesis Theorem [32]. *We can separate an n-dof generalized open oscillator G into n 1-dof (i.e. $x_0 = [q_1, p_1]^T$) generalized open oscillators $G_j, j = 1 \ldots n$ with a direct (nearest-neighbour only, i.e. G_j only coupled to $G_{j\pm1}$) interaction Hamiltonian. An illustrative example for a 2-dof generalized open oscillator is given in figure. 22.*

Theorem in [32] gives forms for H^d and G_j

[32] Hendra I. Nurdin, Matthew R. James, and Andrew C. Doherty. Network synthesis of linear dynamical quantum stochastic systems. *SIAM Journal on Control and Optimization*, 48(4):2686–2718, jan 2009.

Summary

- Have outlined a method for realizing arbitrary quantum optical systems with desired behaviour
  - Discussed for one degree of freedom but can easily be extended to more
- Have demonstrated this with unstable filter
- Will use this to find physical system that broadens bandwidth of coupled cavity resonance

Main references

[1] J. E. Gough, M. R. James, H. I. Nurdin, Squeezing components in linear quantum feedback networks, Phys. Rev. A 81 (2010) 023804–1– 023804–15.

[2] H. Nurdin, M. James, A. Doherty, Network synthesis of linear dynamical quantum stochastic systems, SIAM J. Control and Optim. 48 (4) (2009) 2686–2718.

[12] M. R. James, H. I. Nurdin, and I. R. Petersen, *H^∞ control of linear quantum stochastic systems*, IEEE Trans. Automat. Control, 53 (2008), pp. 1787–1803.

Supplementary slides

G_j and H^d

Scattering matrix

Coupling matrix

$$G_j = \left(S_j, \tilde{K}_j x_j, \underline{\tfrac{1}{2} x_j^T R_{jj} x_j}\right)$$

Free Hamiltonian
(internal dynamics)

$$\tilde{K}_k = S^{\dagger}_{n \twoheadleftarrow k+1} K_k \ \textit{for}\ k = 1, \ldots, n$$

define $S_{k \twoheadleftarrow j}$ *for* $j \leq k+1$ *as* $S_{k \twoheadleftarrow j} = \prod_{l=j}^{k} S_l = S_k \cdots S_{j+1} S_j$ *for* $j < k$, $S_{k \twoheadleftarrow k} = S_k$, *and* $S_{k \twoheadleftarrow k+1} = I_{m \times m}$, *and let* H^d *be a direct interaction Hamiltonian given by*

$$H^d = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} x_k^T \left(R_{jk}^T - \frac{1}{2i} (\tilde{K}_k^{\dagger} S_{k \twoheadleftarrow j+1} \tilde{K}_j - \tilde{K}_k^T S^{\#}_{k \twoheadleftarrow j+1} \tilde{K}_j^{\#}) \right) x_j.$$

Broadening of dip