Quark Masses:
Minimal Renormalon Subtracted Mass and Results from Lattice QCD

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Outline

- Original motivation
- The *minimal renormalon-subtracted* (MRS) mass [arXiv:1712.04983].

Javad Komijani
Nora Brambilla
Antonio Vairo

- Results for all quark masses except top [arXiv:1802.04248].

A. Bazavov, C. Bernard, N. Brown, C. DeTar, A.X. El-Khadra,
E. Gámiz, Steven Gottlieb, U.M. Heller, J. Komijani,
A.S. Kronfeld, J. Laiho, P.B. Mackenzie, E.T. Neil, J.N. Simone,
R.L. Sugar, D. Toussaint, R.S. Van de Water

Fermilab Lattice and MILC Collaborations

Ur Motivation

- From HQET (or other approaches to the $1/m_h$ expansion):

1 for B, $-\frac{1}{3}$ for B^*

$$M_{H_J} = m_h + \bar{\Lambda} + \frac{\mu_\pi^2}{2m_h} - d_J \frac{\mu_G^2(m_h)}{2m_h}$$

- Strategy: vary m_h within lattice QCD and use this formula to determine m_h, $\bar{\Lambda}$, μ_π^2, and $\mu_G^2(m_b)$ [*cf.*, arXiv:hep-ph/0006345].

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- M_{H_J}: mass of spin-J meson
- m_h: mass of heavy quark
- $\bar{\Lambda}$: energy of gluons and light quarks
- $\frac{\mu_\pi^2}{2m_h}$: kinetic energy of heavy quark
- d_J: 1 for B, $-\frac{1}{3}$ for B^*
- $\frac{\mu_G^2(m_h)}{2m_h}$: spin-orbit interaction

- Strategy: vary m_h within lattice QCD and use this formula to determine m_h, $\bar{\Lambda}$, μ_π^2, and $\mu_G^2(m_b)$ [*cf.*, arXiv:hep-ph/0006345].

Mass in QFT

What's a Quark Mass?

- You can't put a quark on a scale and weigh it.
- Need definition, preferably regularization-independent, in QFT.
- Natural candidate is the "perturbative pole mass." Alas, ambiguous:
  - physics—infrared gluons need to find a sink;
  - mathematics—obstruction to Borel summation of the perturbative series;
  - theorists' jargon—infrared renormalon;
  - numbers— $m_{\mathrm{b,pole}}/\bar{m}_b = (1, 1.093, 1.143, 1.183, 1.224)$.

$$\bar{m}_h \equiv m_{h,\overline{\mathrm{MS}}}(\bar{m}_h)$$

Short-Distance Definitions

- Usual work-around is to use a “short-distance” mass.
- The $\overline{\text{MS}}$ mass in dimensional regularization, $m_{h,\overline{\text{MS}}}(\mu)$; $\bar{m}_h \equiv m_{h,\overline{\text{MS}}}(\bar{m}_h)$:
  - spoils HQET power counting: $m_{\text{pole}} - \bar{m}_h \propto \alpha_s(\bar{m}_h)\bar{m}_h$.
- Other definitions subtract out infrared part at a new scale ν_f:
  - “kinetic mass” (Uraltsev) via a Wilsonian renormalization;
  - “renormalon subtracted mass” (Pineda) subtracts out renormalon at ν_f;
  - “MSR mass” (Hoang, Jain, Scimemi, Stewart) similarly, at $\nu_f = \bar{m}_h$.
- The new scale satisfies $1\ \text{GeV} < \nu_f < m_h$; often need yet another for $\alpha_s(\mu)$.

Pole Mass vs. $\overline{\text{MS}}$ Mass

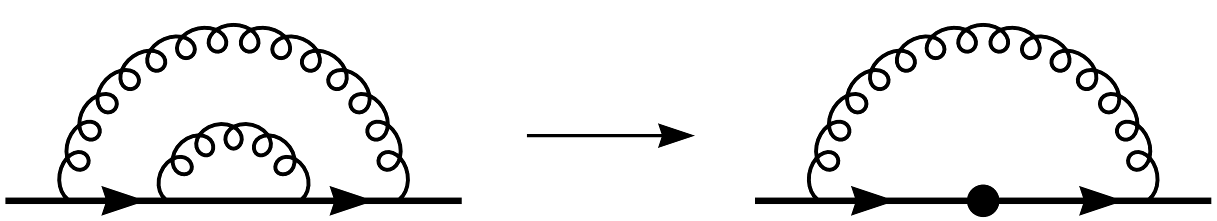
- Consider the relation between the pole mass and the $\overline{\text{MS}}$ mass:

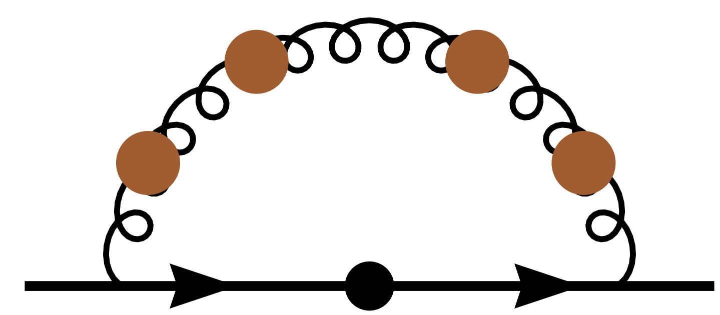
$$m_{\text{pole}} = \bar{m}\left(1 + \sum_{n=0}^{N} r_n \alpha_{\text{g}}^{n+1}(\bar{m}) + \text{O}(\alpha_{\text{g}}^{N+2})\right)$$

  where α_{g} is a scheme for α_{s} that simplifies the algebra.

- The r_n are infrared finite and gauge independent [hep-ph/9805215].

- The low ($\Lambda \ll l < m_h$) loop-momentum parts of self-energy diagrams cause the n^{th} coefficient to grow like $n!$





Factorial Growth

- Remarkably, most info on the growth still comes from the β function:

$$r_n \sim R_n = R_0(2\beta_0)^n \frac{\Gamma(n+1+b)}{\Gamma(1+b)}, \quad n \geq 0$$

$$b = \frac{\beta_1}{2\beta_0^2} = \frac{231}{645} \text{ for } (n_f = 4)$$

only the overall normalization R_0 does not. Hence name "renormalon."

- Formula for R_n is exact in the α_g coupling scheme; in other UV schemes, a series of terms in powers of $1/n$ appear on RHS, still multiplied by $R_0(2\beta_0)^n$.

Leading Renormalon Normalization

- Newly discovered formula [arXiv:1701.00347]:

$$R_0 = \sum_{k=0}^{\infty} r'_k \frac{\Gamma(1+b)}{\Gamma(2+k+b)} \frac{1+k}{(2\beta_0)^k}$$

$$r'_k = r_k - 2\left[\beta_0 k r_{k-1} + \beta_1 (k-1) r_{k-2} + \cdots + \beta_{k-1} r_0\right] \quad \longleftarrow k! \text{ terms cancel}$$

- We re-write the relation between the pole mass and the $\overline{\text{MS}}$ mass:

$$m_{\text{pole}} = \bar{m} + \bar{m} \sum_{n=0}^{\infty} [r_n - R_n] \alpha_{\text{g}}^{n+1}(\bar{m}) + \bar{m} \sum_{n=0}^{\infty} R_n \alpha_{\text{g}}^{n+1}(\bar{m})$$

and truncate the first sum, as usual, but carry out the second sum analytically.

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$$R_0 = \sum_{k=0}^{\infty} r'_k \frac{\Gamma(1+b)}{\Gamma(2+k+b)} \frac{1+k}{(2\beta_0)^k} = 0.535 \pm 0.010 \; (n_f = 3)$$

$$r'_k = r_k - 2\left[\beta_0 k r_{k-1} + \beta_1 (k-1) r_{k-2} + \cdots + \beta_{k-1} r_0\right] \quad \longleftarrow k! \text{ terms cancel}$$

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Renormalon-a-Ding-Dong

- Use the technique of Borel resummation, one finds

$$\mu \sum_{n=0}^{\infty} R_n \alpha_{\rm g}^{n+1}(\mu) = \frac{R_0}{2\beta_0}\mu \int_0^{\infty} dz \, \frac{e^{-z/(2\beta_0 \alpha_{\rm g}(\mu))}}{(1-z)^{1+b}}$$
$$\equiv \mathscr{I}(\mu)$$

- The integrand has a branch point at $z = 1$. That's the (leading) ambiguity!
- Our suggestion:
  - Break the integral into an unambiguous part $z \in [0,1]$ and a totally ambiguous part $z \in [1,\infty)$.

Minimal Renormalon Subtraction

arXiv:1712.04983

- Splitting the integral (Brambilla, **Komijani**, ASK, Vairo):

$$\mathcal{J}(\mu) = \mathcal{J}_{\rm MRS}(\mu) + \delta m$$

$$\mathcal{J}_{\rm MRS}(\mu) = \frac{R_0}{2\beta_0}\mu \int_0^1 dz\, \frac{e^{-z/[2\beta_0\alpha_{\rm g}(\mu)]}}{(1-z)^{1+b}}$$

$$\delta m = \frac{R_0}{2\beta_0}\mu \int_1^\infty dz\, \frac{e^{-z/[2\beta_0\alpha_{\rm g}(\mu)]}}{(1-z)^{1+b}}$$

$$= -(-1)^b \frac{R_0}{2\beta_0}\Gamma(-b)\,\mu\, \frac{e^{-1/[2\beta_0\alpha_{\rm g}(\mu)]}}{[2\beta_0\alpha_{\rm g}(\mu)]^b}$$

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$$= -(-1)^b \frac{R_0}{2\beta_0}\Gamma(-b)\Lambda_{\overline{\text{MS}}}$$

Minimal Renormalon Subtraction

arXiv:1712.04983

- Minimal renormalon-subtracted (MRS) mass (scheme independent):

$$m_{\mathrm{MRS}} \equiv m_{\mathrm{pole}} - \delta m$$
$$= \bar{m}\left(1 + \sum_{n=0}^{\infty} [r_n - R_n]\,\alpha_{\mathrm{g}}^{n+1}(\bar{m})\right) + \mathscr{J}_{\mathrm{MRS}}(\bar{m})$$

$$\mathscr{J}_{\mathrm{MRS}}(\bar{m}) = \frac{R_0}{2\beta_0}\bar{m}e^{-1/[2\beta_0\alpha_{\mathrm{g}}(\bar{m})]}\Gamma(-b)\gamma^{\star}\left(-b, -[2\beta_0\alpha_{\mathrm{g}}(\bar{m})]^{-1}\right)$$

- This function is easy enough to evaluate.
- NB: MRS mass has same asymptotic series as the pole mass!
- Just as good a solution of the pole condition, without as bad behavior.

Perturbation Theory

- The first four r_n are known:

		R_n	
one loop [NPB 183 (1981) 384]:	$r_0 = \frac{C_F}{\pi} = 0.4244$	0.5350	
2 loops [ZPC 48 (1990) 673]:	$r_1 = 1.0351$	1.0691	$(n_f = 3)$
3 loops [2+1 papers, '99, '00]:	$r_2 = 3.6932$	3.5966	$(n_f = 3)$
4 loops [arXiv:1606.06754]:	$r_3 = 17.4358$	17.4195	$(n_f = 3)$

- The 5-loop mass anomalous dimension is known [arXiv:1402.6611].
- The 5-loop Callan-Symanzik beta function is known [arXiv:1606.08659].

Remarks

- MRS mass is a short-distance mass: subtract off long-range δm.
- No new scale: trim long-range field at $1/m_h$, not $1/\nu_f$.
- Numerically very stable: $m_{b,\mathrm{MRS}}/\bar{m}_b = (1.157, 1.133, 1.131, 1.132, 1.132)$.
  $m_{t,\mathrm{MRS}}/\bar{m}_t = (1.0687, 1.0576, 1.0573, 1.0574, 1.0574)$
- Makes HQET formula unambiguous (to order $1/m_h$):

$$M_{H_J} = m_{h,\mathrm{MRS}} + \bar{\Lambda}_{\mathrm{MRS}} + \frac{\mu_\pi^2}{2m_h} - d_J \frac{\mu_G^2(m_h)}{2m_h}$$

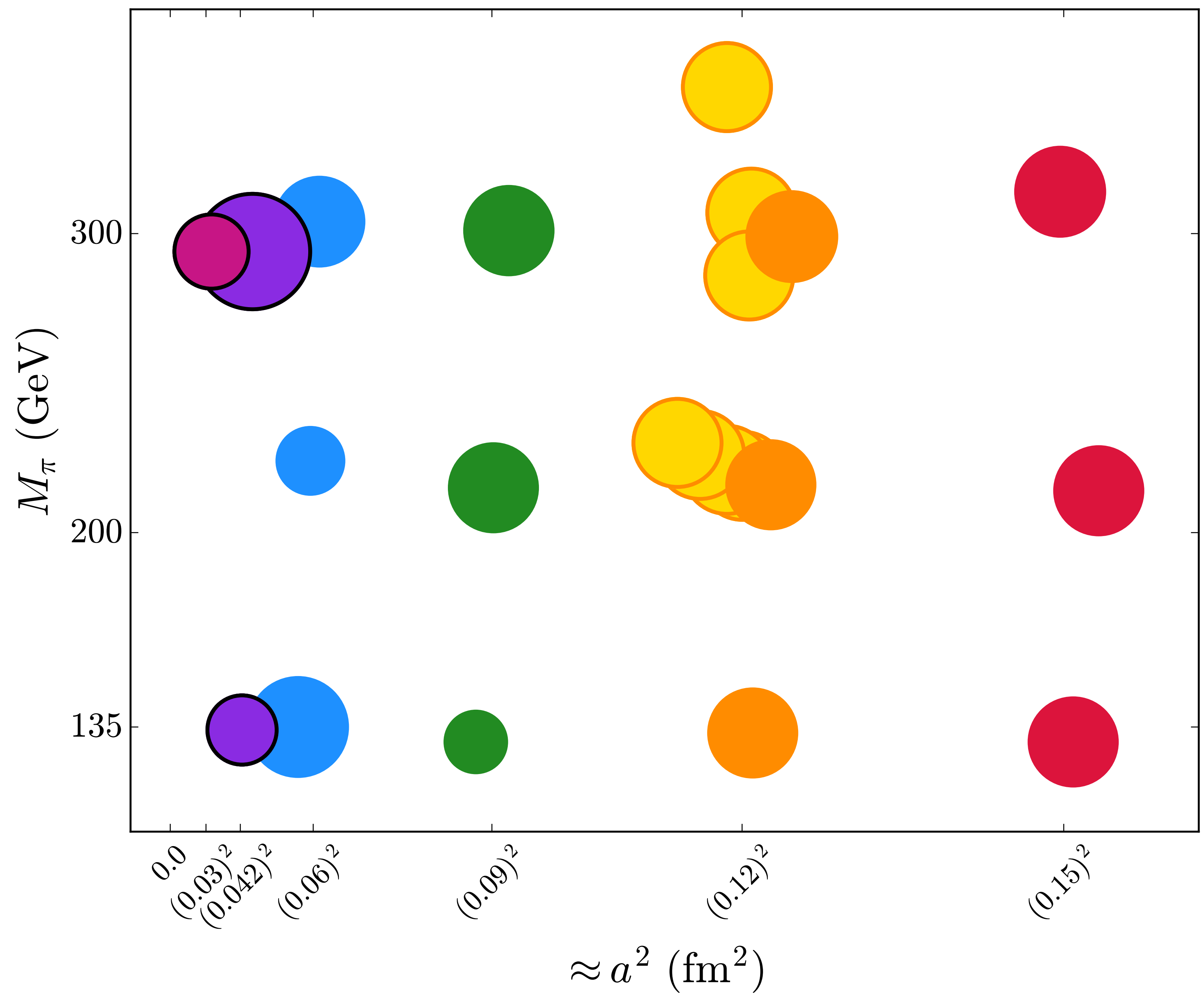
- Next step: fit this formula to lattice-QCD data!

High Performance
Computing & Analysis

MILC HISQ Ensembles

arXiv:1212.4768 + update in arXiv:1712.09262

- 2+1+1 sea quarks;
- 24 ensembles
- 5 w/ $M_\pi = 135$ MeV;
- down to $a = 0.03$ fm;
- typically 1000×4 samples;
- $M_\pi L > 3.2$, often > 5;
- up to $144^3 \times 288$.



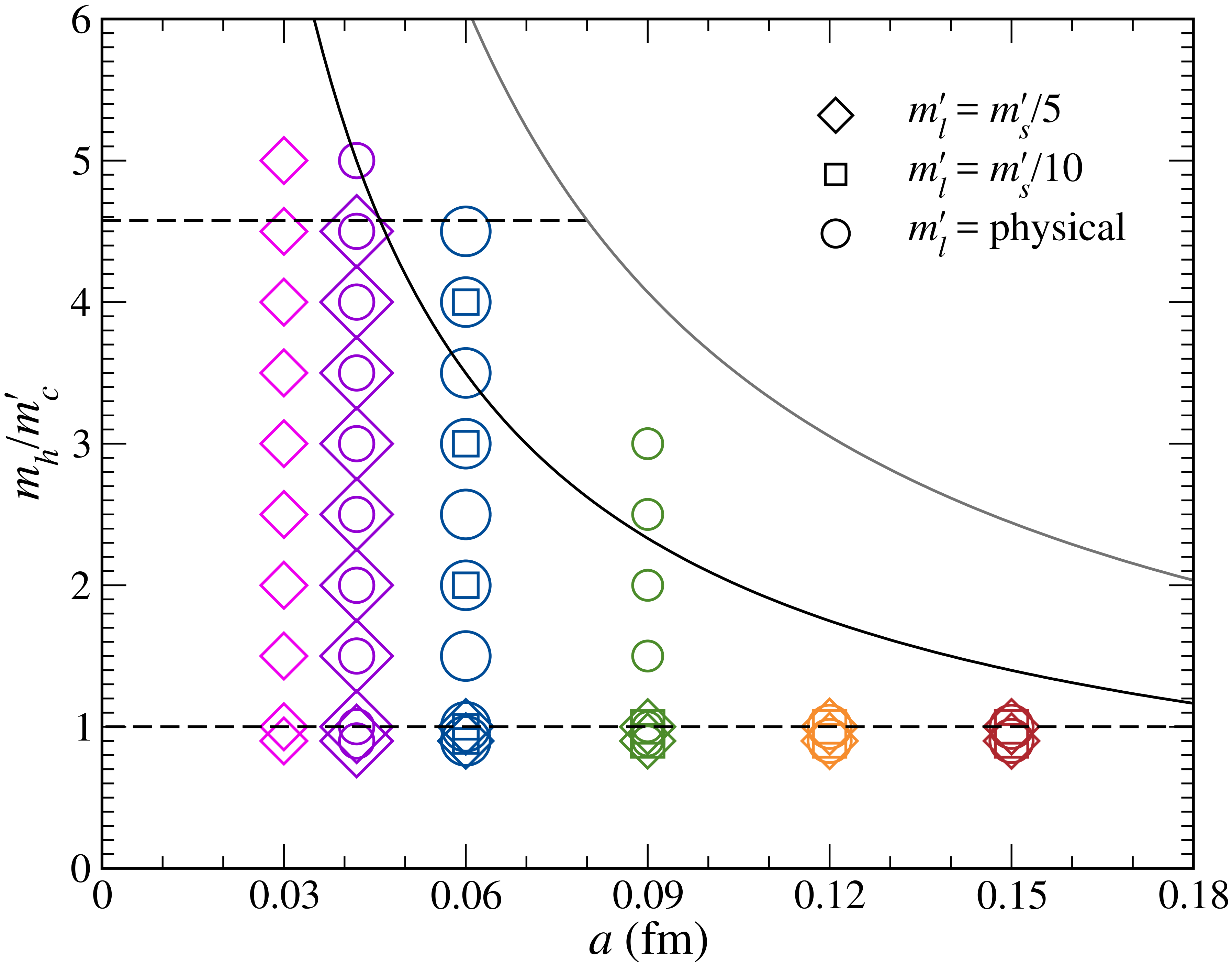
HISQ Ensembles: 2+1+1

MILC, arXiv:1212.4768 + further runs

a (fm)	size	$am'_l/am'_s/am'_c$	# confs	# sources	notes
≈ 0.15	$16^3\times 48$	0.0130/0.065/0.838	1020	4	
≈ 0.15	$24^3\times 48$	0.0064/0.064/0.828	1000	4	
≈ 0.15	$32^3\times 48$	0.00235/0.0647/0.831	1000	4	physical
≈ 0.12	$24^3\times 64$	0.0102/0.0509/0.635	1040	4	
≈ 0.12	$32^3\times 64$	0.00507/0.0507/0.628	1020	4	also 24^3, 40^3
≈ 0.12	$48^3\times 64$	0.00184/0.0507/0.628	999	4	physical
≈ 0.12	$24^3\times 64$	0.0102/0.03054/0.635	1020	4	$m'_s < m_s$
≈ 0.12	$24^3\times 64$	0.01275/0.01275/0.640	1020	4	$m'_s = m'_l$
≈ 0.12	$32^3\times 64$	0.00507/0.0304/0.628	1020	4	$m'_s < m_s$
≈ 0.12	$32^3\times 64$	0.00507/0.022815/0.628	1020	4	$m'_s < m_s$
≈ 0.12	$32^3\times 64$	0.00507/0.012675/0.628	1020	4	$m'_s \ll m_s$
≈ 0.12	$32^3\times 64$	0.00507/0.00507/0.628	1020	4	$m'_s = m'_l$
≈ 0.12	$32^3\times 64$	0.0088725/0.022815/0.628	1020	4	$m'_s < m_s$
≈ 0.09	$32^3\times 96$	0.0074/0.037/0.440	1005	4	
≈ 0.09	$48^3\times 96$	0.00363/0.0363/0.430	999	4	
≈ 0.09	$64^3\times 96$	0.0012/0.0363/0.432	484	4	physical
≈ 0.06	$48^3\times 144$	0.0048/0.024/0.286	1016	4	
≈ 0.06	$64^3\times 144$	0.0024/0.024/0.286	572	4	
≈ 0.06	$96^3\times 192$	0.0008/0.022/0.260	842	6	physical
≈ 0.042	$64^3\times 192$	0.00316/0.0158/0.188	1167	6	
≈ 0.042	$144^3\times 288$	0.000569/0.01555/0.1827	429	6	physical
≈ 0.03	$96^3\times 288$	0.00223/0.01115/0.1316	724	4	

Heavy-Quark Masses

- always $0.9m_c$, m_c;
- up to $5m_c$;
- omit $am_c \geq 0.9$ from heavy-quark fits (need < $\pi/2$);
- omit 0.15 fm in base fit;
- 492 data points (498 w/ 0.15 fm).



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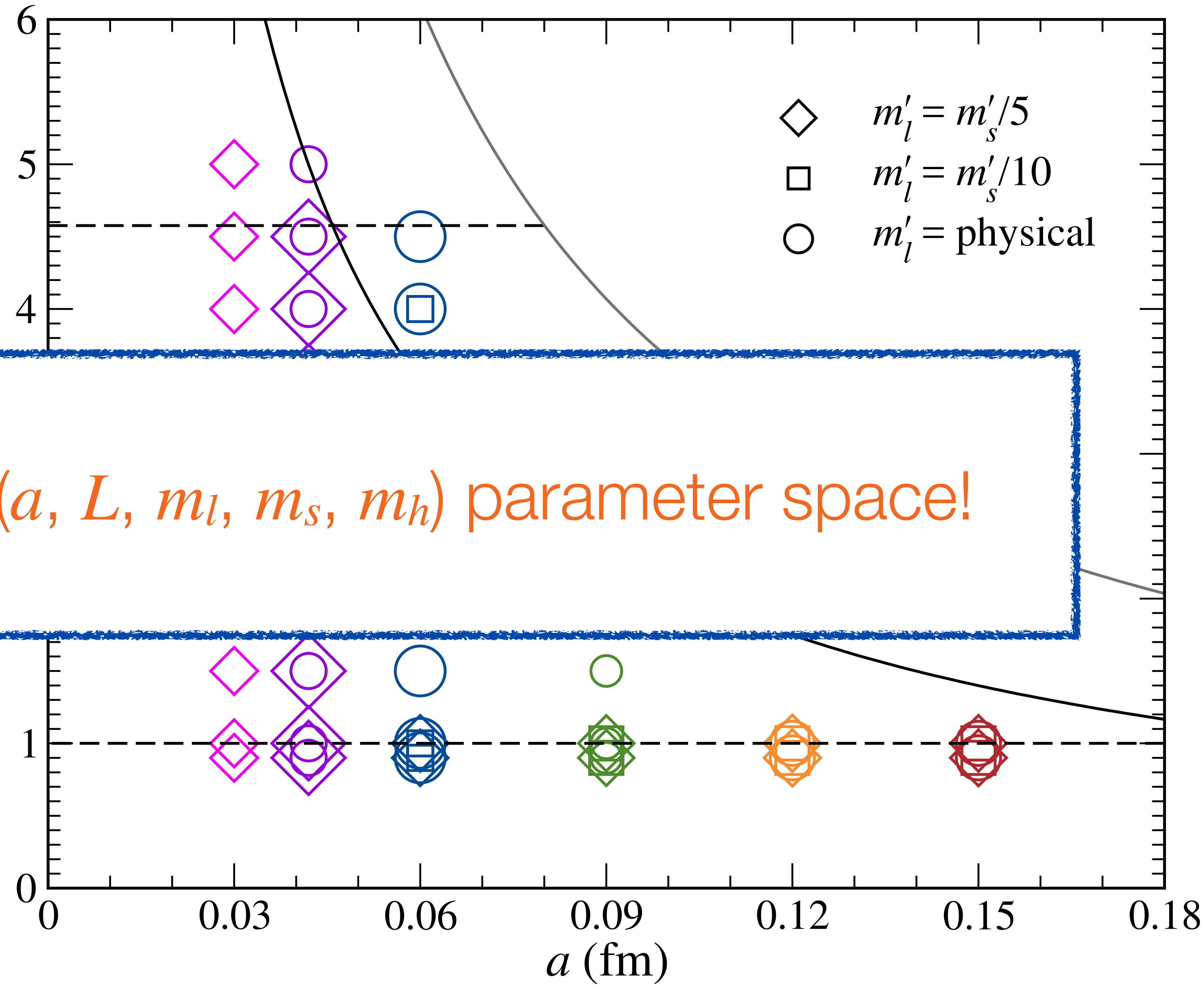

Huge slab in (a, L, m_l, m_s, m_h) parameter space!

HQET ⊕ Symanzik EFT ⊕ χPT Fits

- As noted, the slab of parameter space (5-dimensional) is huge.
- The raw statistical precision of the simulation data is
  - 0.04–1.4% for heavy-light meson decay constants;
  - 0.005–0.12% for heavy-light meson masses.
- It is insufficient to have a simple function to fit the dependence on (a, m_l, m_s, m_h).
- Functional form follows power-counting and builds in leading chiral logs and HQET anomalous dimension.

Results

Quark Masses

- We now fit the HQET formula (with numerous nuisance terms not shown):

$$M_{H_x} = m_{h,\mathrm{MRS}} + \bar{\Lambda}_{\mathrm{MRS}} + \frac{\mu_\pi^2}{2m_h} - 3\,\frac{\mu_G^2(m_h)}{2m_h}$$

$m_{h,\mathrm{MRS}} =$ $m_{h,\mathrm{MRS}}$

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$$m_{h,\mathrm{MRS}} = \frac{m_{r,\overline{\mathrm{MS}}}(\mu)\, am_h}{m_{h,\overline{\mathrm{MS}}}(\mu)\, am_r} m_{h,\mathrm{MRS}}$$

$1 + \mathrm{O}(a^2)$

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$$m_{h,\mathrm{MRS}} = \frac{m_{r,\overline{\mathrm{MS}}}(\mu)\, am_h}{m_{h,\overline{\mathrm{MS}}}(\mu)\, am_r} m_{h,\mathrm{MRS}}$$

$$= m_{r,\overline{\mathrm{MS}}}(\mu) \frac{\bar{m}_h}{m_{h,\overline{\mathrm{MS}}}(\mu)} \frac{m_{h,\mathrm{MRS}}}{\bar{m}_h} \frac{am_h}{am_r},$$

1 + O(a^2)

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$$= m_{r,\overline{\mathrm{MS}}}(\mu) \frac{\bar{m}_h}{m_{h,\overline{\mathrm{MS}}}(\mu)} \frac{m_{h,\mathrm{MRS}}}{\bar{m}_h} \frac{am_h}{am_r},$$

$1 + \mathrm{O}(a^2)$

lattice input

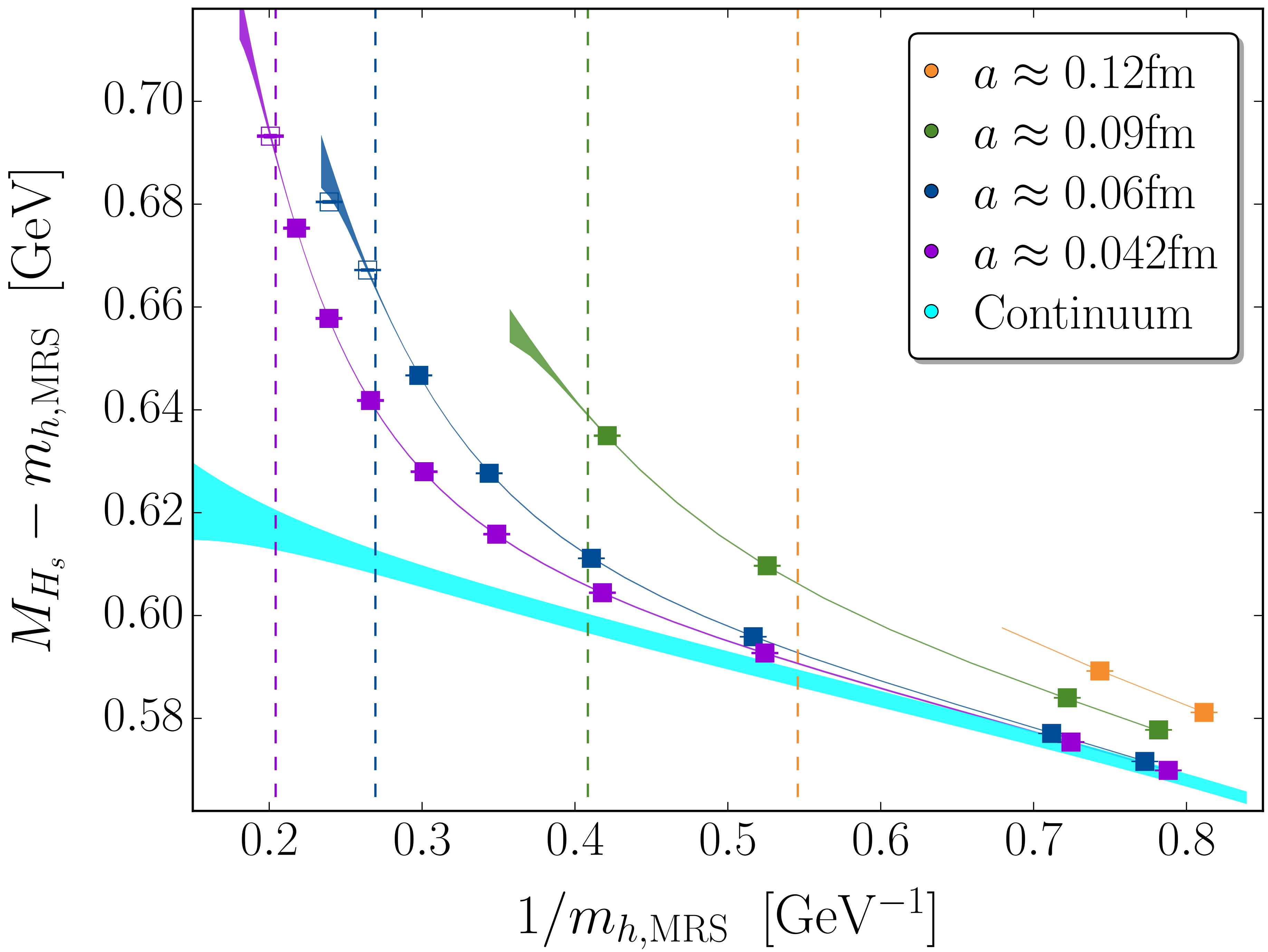
convenient fit parameter

run with anomalous dimension

MRS definition

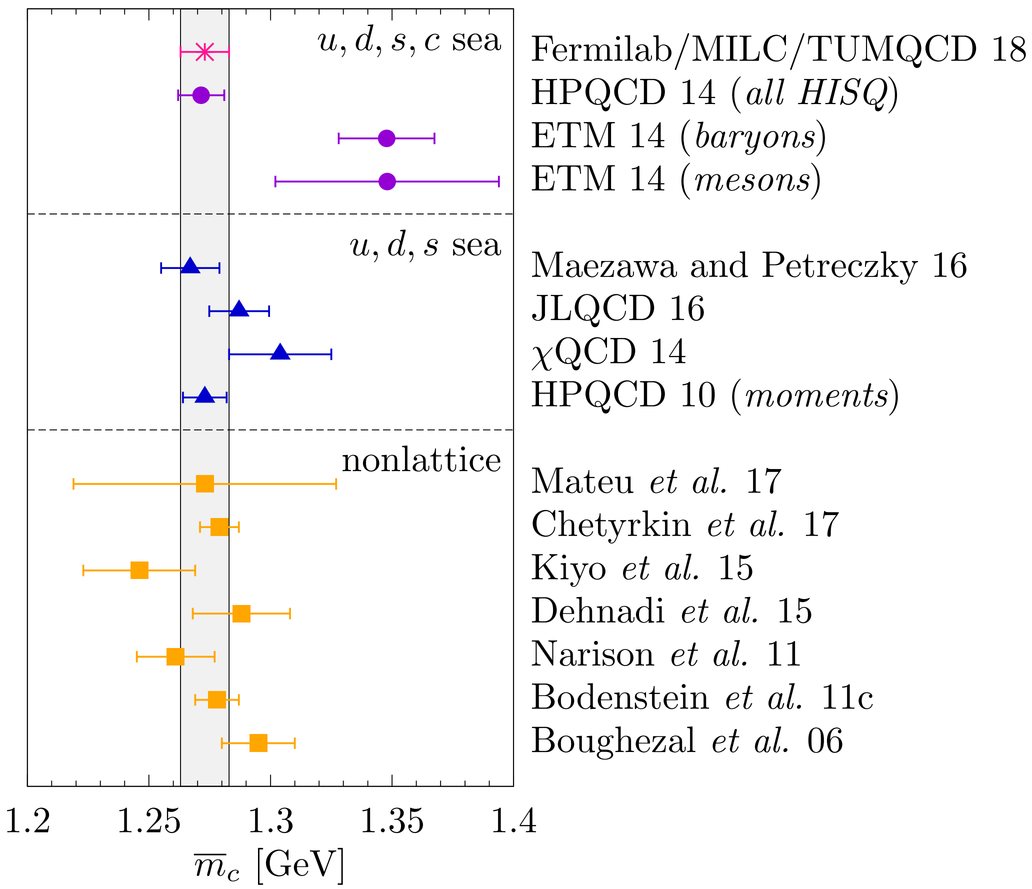
HQET Fit ⊕ Symanzik EFT ⊕ χPT

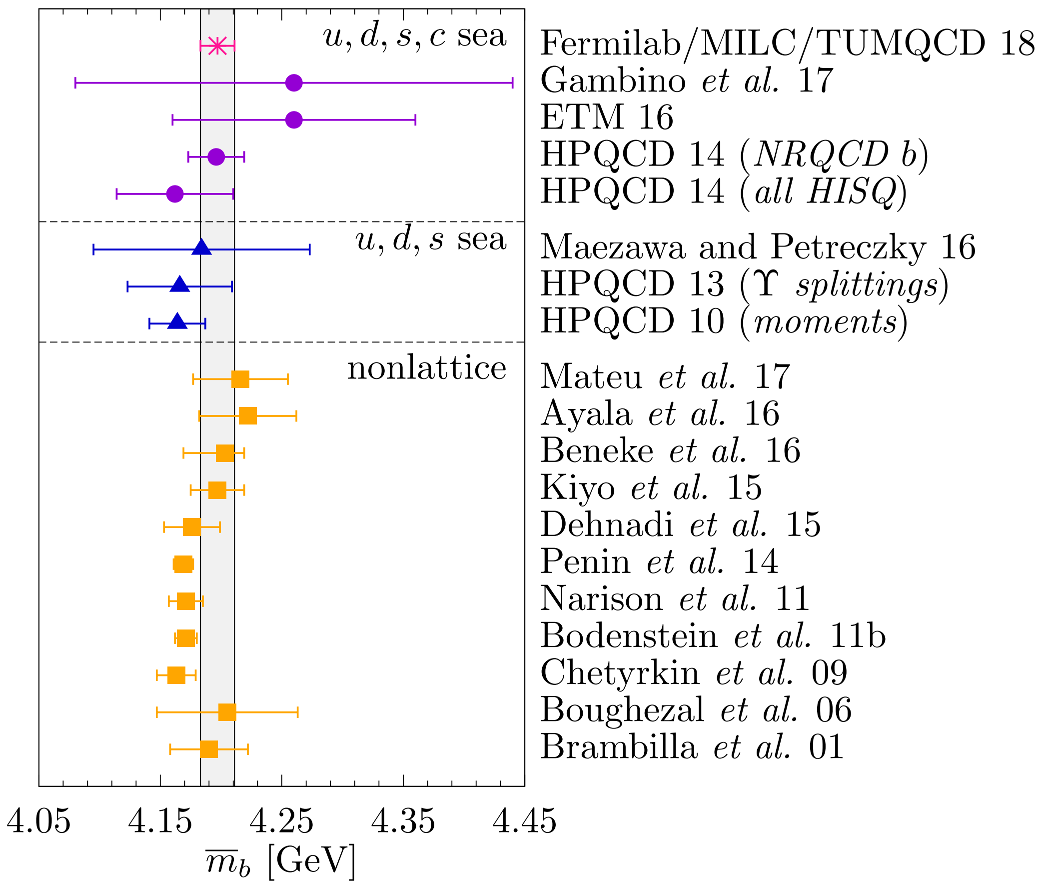
- 384 data pts;
- 77 parameters;
- $\chi^2/\text{dof} = 312/307$;
- $p = 0.3$;
- stable under fit variations;
- extra errors for FV, topology, EM.



Results & Comparisons

- Results form arXiv:1802.04248:



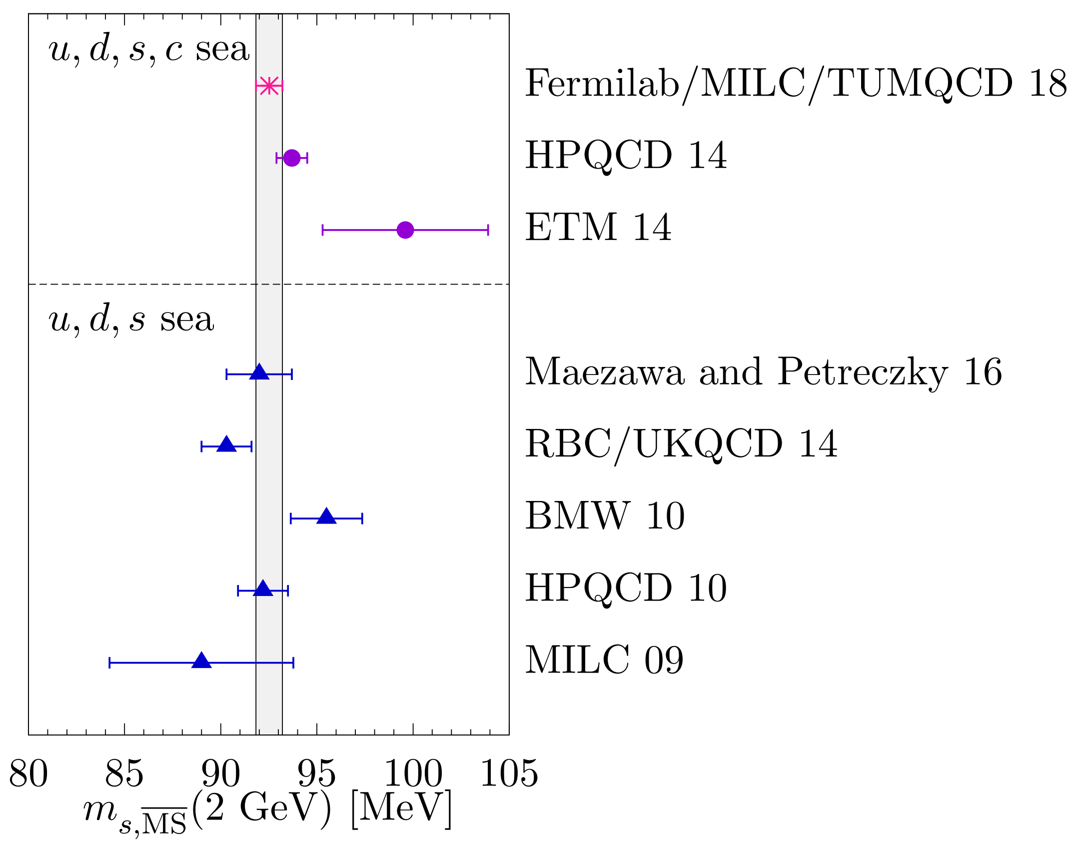


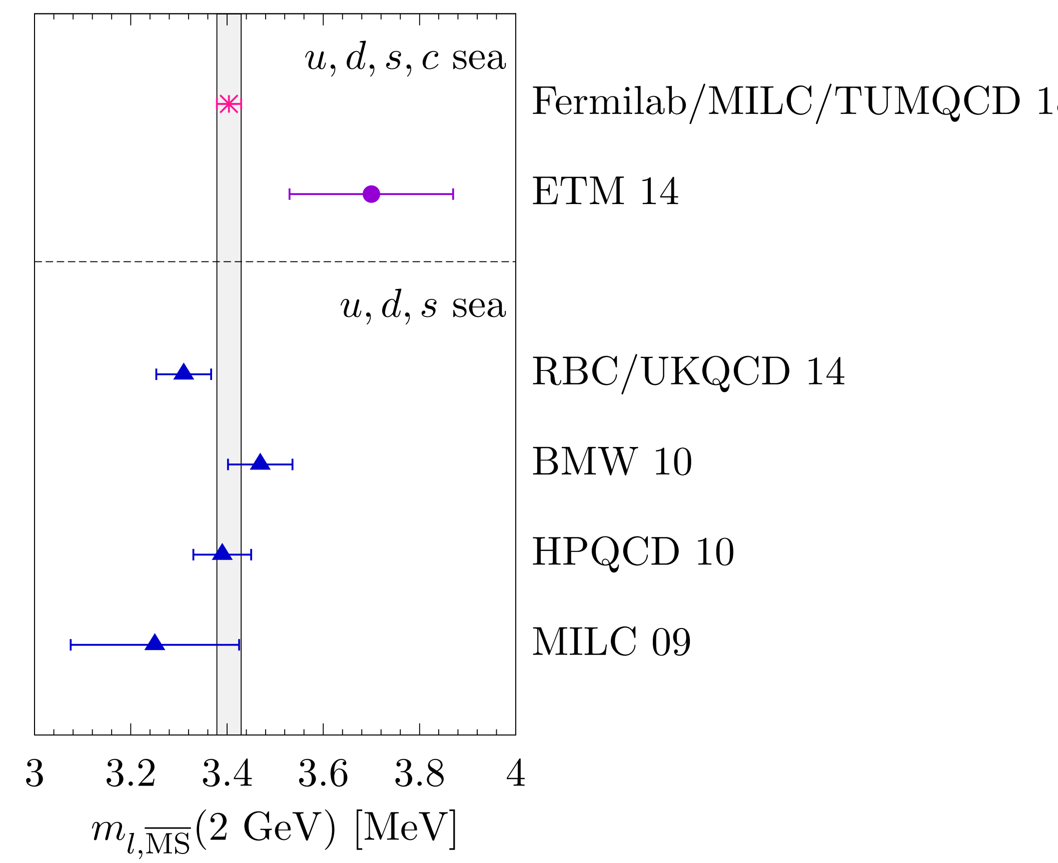
- To our knowledge, first results w/ order-α_s^5 running & order-α_s^4 matching.

- Precision: 0.3% for bottom to 0.5% for charm.

Results & Comparisons 2

- With mass ratios from light pseudoscalar mesons:





- Most precise strange and "light" quark masses to date.
- Most (~) precise quark masses for all quarks except top ($m_u > 50\sigma$).

Results & Comparisons 3

- Masses in numerical form:

$$m_{l,\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 3.402(15)_{\mathrm{stat}}(05)_{\mathrm{syst}}(19)_{\alpha_s}(04)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV}$$

$$m_{u,\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 2.130(18)_{\mathrm{stat}}(35)_{\mathrm{syst}}(12)_{\alpha_s}(03)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV}$$

$$m_{d,\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 4.675(30)_{\mathrm{stat}}(39)_{\mathrm{syst}}(26)_{\alpha_s}(06)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV}$$

$$m_{s,\overline{\mathrm{MS}}}(2\ \mathrm{GeV}) = 92.47(39)_{\mathrm{stat}}(18)_{\mathrm{syst}}(52)_{\alpha_s}(11)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV}$$

$$m_{c,\overline{\mathrm{MS}}}(3\ \mathrm{GeV}) = 983.7(4.3)_{\mathrm{stat}}(1.4)_{\mathrm{syst}}(3.3)_{\alpha_s}(0.5)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV}$$

$$m_{b,\overline{\mathrm{MS}}}(\overline{m}_b) = 4201(12)_{\mathrm{stat}}(1)_{\mathrm{syst}}(8)_{\alpha_s}(1)_{f_{\pi,\mathrm{PDG}}}\ \mathrm{MeV}$$

- Mass ratios:

$$m_c/m_s = 11.783(11)_{\mathrm{stat}}(21)_{\mathrm{syst}}(00)_{\alpha_s}(08)_{f_{\pi,\mathrm{PDG}}}$$

$$m_b/m_s = 53.94(6)_{\mathrm{stat}}(10)_{\mathrm{syst}}(1)_{\alpha_s}(5)_{f_{\pi,\mathrm{PDG}}}$$

$$m_b/m_c = 4.578(5)_{\mathrm{stat}}(6)_{\mathrm{syst}}(0)_{\alpha_s}(1)_{f_{\pi,\mathrm{PDG}}}$$

Outlook

Summary

- New approach to renormalons: may have wider applicability.
- MRS mass: a new version of the pole mass, with smaller IR sensitivity:
  - is there an analogous approach to the top mass (not with lattice QCD)?
- High statistics lattice data from MILC ensembles with
  - large volumes,
  - absolutely normalized pseudoscalar density,
  - huge slab of parameter space,
  - yield results of previously unseen precision from lattice QCD.

Thank you!

Top Quark Physics

- Can the MRS mass be identified with the mass in Pythia?
  - It all the advantages without the disadvantage.
- Is there an observable that is analogous to the heavy-light meson mass?
  - The “hadron”—i.e., the color singlet—in which the top quark sits is the “fat jet” containing all the decay products;
  - think about mass-sensitive properties of this object.
- What can be varied to separate the MRS mass from the rest of the jet?
  - The top-quark mass cannot be varied at will.

“Geometric” Scheme for α_s

- Scheme defined by the sum of a geometric series for the beta function:

$$\beta\left(\alpha_g(\mu)\right) = -\frac{\beta_0 \alpha_g^2(\mu)}{1-(\beta_1/\beta_0)\alpha_g(\mu)}$$

supplemented with

$$\frac{1}{\alpha_g(\mu)} = \frac{1}{\alpha_{\overline{MS}}(\mu)} + b_1 + b_2 \alpha_{\overline{MS}}(\mu) + \cdots$$

- Must choose b_1, which is proportional to $\ln\left(\Lambda_g/\Lambda_{\overline{MS}}\right)$.
- One finds $b_2 = \beta_2/\beta_0 - (\beta_1/\beta_0)^2$, $b_3 = \frac{1}{2}[\beta_3/\beta_0 - (\beta_1/\beta_0)^3]$, ….
- Note that α_g is regularization independent.

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$$\frac{1}{\alpha_g(\mu)} = \frac{1}{\alpha_{\overline{\mathrm{MS}}}(\mu)} + 0 + b_2 \alpha_{\overline{\mathrm{MS}}}(\mu) + \cdots$$

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- Note that α_g is regularization independent.

Frozen Topology

- Continuum gauge fields: topological charge Q cannot change with an infinitesimal change in the gauge field.

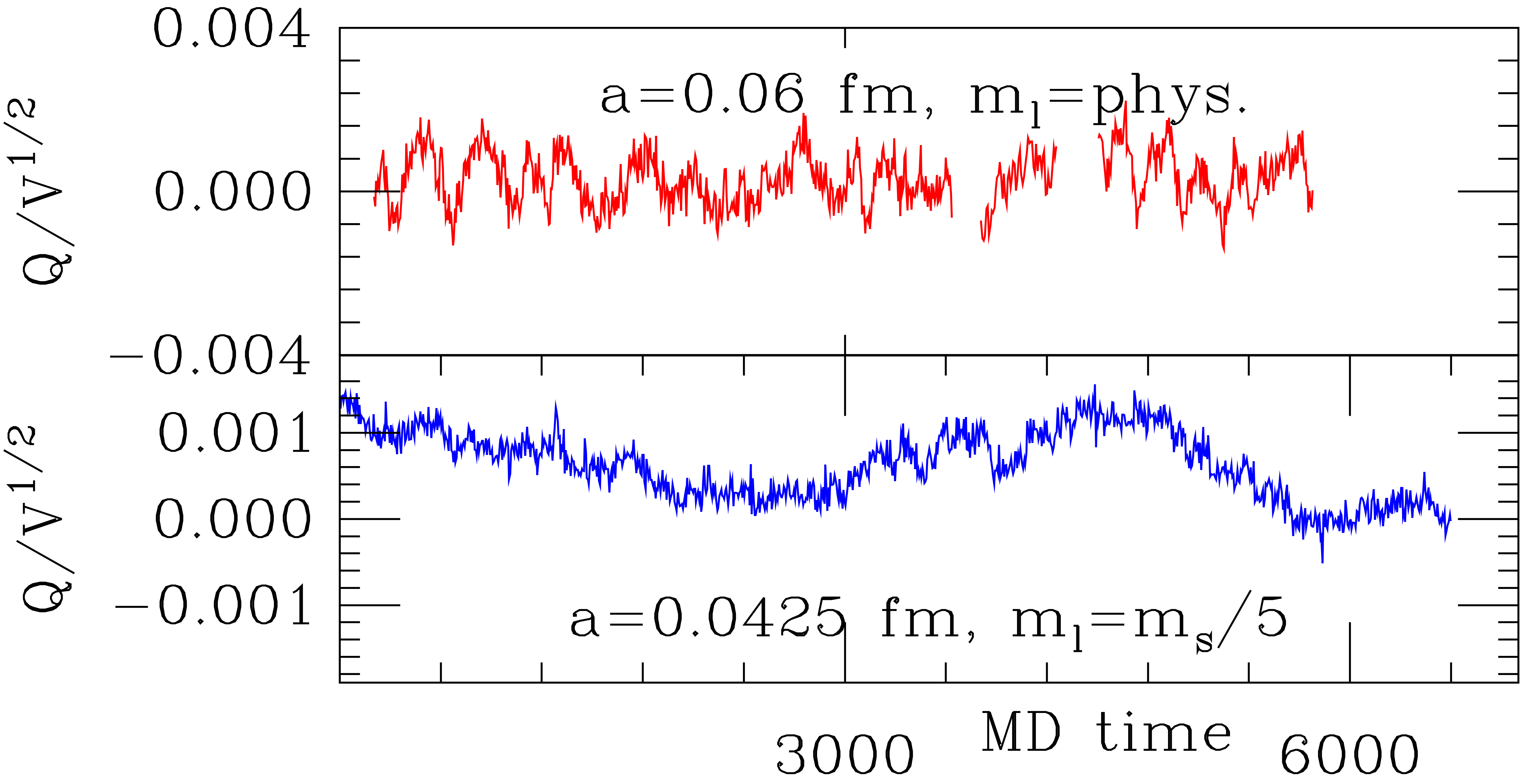
- Evolution of lattice gauge fields in CPU time consists of small steps that (in physical units) become smaller and smaller as lattice spacing $a \to 0$.

- Some reactions:

  - "Oh, my! Physics is now impossible!"—anonymous

  - "Physical quantities will suffer a systematic error, and we need to either correct for this error or account for it in our error budgets."
    —Bernard & Toussaint [arXiv:1707.05430]

Good vs. Bad Sampling



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$\chi_T = \langle Q^2\rangle / V$
in χPT, $\chi_T \propto f_\pi^2 M_\pi^2$
if $M_\pi L \sim$ const, $\chi_T V \propto f_\pi^2 LT$

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References:
Leutwyler, Smilga [PRD46 (1992) 5607];
Brower *et alia* [hep-lat/0302005];
Aoki *et alia* [arXiv:0707.0396];
Aoki, Fukaya [arXiv:0906.4852].

Typical Corrections

Bernard & Toussaint, arXiv:1707.05430

	$m_l' = m_s'/5$	m_l' = physical
$\langle Q^2 \rangle_{\text{ens}} / \langle Q^2 \rangle_{\chi\text{PT}}$	1.30	0.65
f_K/f_π	1.20508(0.00250) [–0.01271]	1.19680(0.00114) [0.00015]
aM_π	0.031147(0.000172) [–0.000707]	0.028964(0.000020) [0.000008]
aM_D	0.048858(0.000261) [–0.000552]	0.045389(0.000245) [0.000006]
af_D	0.409786(0.000391) [–0.000044]	0.400678(0.000258) [0.000001]
aM_{Ds}	0.054828(0.000068) [–0.000001]	0.053582(0.000025) [0.000000]
af_{Ds}	0.430966(0.000116) [–0.000004]	0.422041(0.000037) [0.000000]

- Must be examined ensemble by ensemble.

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Tiny, and sometimes significant.

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Note on Finite Width

- The finite width arises from an “absorptive” part in the self energy.
- No extra UV divergences here.
- The proofs of infrared finiteness and gauge independence go through if one finds the pole of the propagator in the complex plane.
- IR renormalon remains [hep-ph/9612329].
- I still hear about people trying to take the real part, basing a mass on that, and putting the width back in by hand: don’t do that.